A Note on Generalized Degree
1 Introduction

Let $G$ be a labeled $n$-vertex graph with no loops or multiple edges. See [1] for relevant definitions. Define the generalized $k$-degree $D_k$ of $G$ to be a nonnegative integer valued function defined on the set of $k$-tuples of vertices in $V(G) \times V(G) \times \cdots \times V(G) = V(G)^k$ by

$$D_k(x_1, x_2, \ldots, x_k) = \left| \bigcup_{i=1}^{k} N(x_i) \right|$$

where $N(v)$ is the standard open neighborhood of $v$. When $k = 1$, $D_k(x)$ becomes the ordinary degree of the vertex $x$. Let $D_k$ without the $k$ arguments denote the range of the function taken as a multiset. Then $D_1$ is the ordinary degree sequence. The $n^k$ values of $D_k$ may be arranged into a symmetric $n \times n$ matrix where $n = |V(G)|$. In general one may arrange $D_k$ into a $k$-dimensional array which will again be denoted by $D_k = D_k(G)$. For simplicity denote the generalized degree of the $k$ ordered vertices $(v_1, v_2, \ldots, v_k)$ by $d_{i_1 i_2 \ldots i_k}$. As is done with degree sequences, $D_k$ is said to be graphical in case there is a graph $G$ with $D_k = D_k(G)$.

A naturally arising question is which graphs $G$ are determined by $D_k$. First notice that, clearly, $D_k(G)$ determines all $D_i(G)$ for which $1 \leq i \leq k$.

Not all graphs are determined by $D_k$. For example, consider the disjoint cycles $C_3 \cup C_2$ and the cycle $C_6$. Label the vertices of $C_3$ cyclically: 1, 2, 3. And, label the vertices of one $C_2$ with 1, 2, and the other $C_2$ with 4, 5, 6. With this labeling it can be shown that $D_2(C_3 \cup C_2) = D_2(C_6)$. Thus $C_3 \cup C_2$ and $C_6$ have the same degree sequence $D_1$, the same 2-array or matrix $D_2$, and the same $D_6$. Note that, for
example,

\[
D_2(C_3 \cup C_3) = D_2(C_6) = \begin{pmatrix}
2 & 3 & 3 & 4 & 4 & 4 \\
3 & 2 & 3 & 4 & 4 & 4 \\
3 & 3 & 2 & 4 & 4 & 4 \\
4 & 4 & 4 & 2 & 3 & 3 \\
4 & 4 & 4 & 3 & 2 & 3 \\
4 & 4 & 4 & 3 & 3 & 2
\end{pmatrix},
\]

and that the ordinary degree sequence \( D_1(C_6) = (2,2,2,2,2,2) = D_1(C_3 \cup C_3) \) is found on the diagonal of \( D_2(C_6) = D_2(C_3 \cup C_3) \).

2 A Counting Formula and Generalized Maximum Degree

Let \( I = \{(i_1, i_2, \ldots, i_k) : 1 \leq i_1 < i_2 < \cdots < i_k \leq n\} \), and \( J = \{(j_1, j_2, \ldots, j_k) : 1 \leq j_1 < j_2 < \cdots < j_k \leq n\} \). Also, let \( N_i \) denote \( N(v_i) \). The following formula gives an expression for the sum over all \( k \)-degrees in a graph \( G \).

**Proposition 1**

\[
\sum_{i} d_{i_1 i_2 \cdots i_k} = \sum_{j=1}^{k} \binom{n-j}{k-j} (-1)^{k+1} \sum_{i=1}^{n} d_i^j.
\]

**Proof:**

\[
\sum_{i} d_{i_1 i_2 \cdots i_k} = \sum_{I} \left| \bigcup_{j=1}^{k} N_i \right|
\]

\[
= \sum_{I} \left( \sum_{j=1}^{k} |N_i| - \sum_{1 \leq i < m \leq n} |N_i \cap N_m| + \cdots \right)
\]

\[
+ (-1)^{k+1} \sum_{j=1}^{k} \left| \bigcap_{i=1}^{j} N_i \right| + \cdots
\]

\[
+ (-1)^{k+1} \left( \bigcup_{i=1}^{n} N_i \right).
\]

In the first term, when summed over \( I \), each \( N_i \) appears \( \binom{n-1}{k-1} \) times, since each \( N_i \) appears with \( k-1 \) others, and there are \( n-1 \) others from which to choose. Thus, the first term contributes \( \binom{n-1}{k-1} \sum_{i=1}^{n} \binom{d_i}{1} \). In the \( s \)-th term each \( \bigcap_{i=1}^{s} N_i \) appears \( \binom{n-s}{s} \) times. If each \( |\bigcap_{i=1}^{s} N_i| \) appeared once, the total count would be \( \sum_{i=1}^{n} \binom{d_i}{1} \), where \( \binom{d_i}{1} \) is the number of \( s \)-stars in \( G \) centered at vertex \( v_i \). Thus, the term is \( (-1)^{s+1} \binom{n-s}{s} \sum_{i=1}^{n} \binom{d_i}{1} \), and the formula is proven. \( \square \)

**Corollary 1**

\[
\sum_{1 \leq i < j \leq n} d_{ij} = (n-1) \sum_{i=1}^{n} d_i - \sum_{i=1}^{n} \binom{d_i}{2}.
\]

\( \square \)
As usual $\Delta = \Delta(G)$ denotes the maximum degree in $G$ (the maximum 1-degree). If one lets $\sigma_{i_1i_2\ldots i_n}$ denote the number of stars with pendent vertices $i_1, i_2, \ldots, i_n$, then the following proposition follows by inclusion-exclusion.

**Proposition 2**

$$d_{i_1i_2\ldots i_n} = \sum_{j=1}^{k} \sigma_{i_j} - \sum_{1\leq j<i\leq k} \sigma_{i_ji_2\ldots i_n} + \sum_{1\leq j<i\leq k} \sigma_{i_ji_2\ldots i_n} - \ldots + (-1)^{k+1} \sum_{1\leq j<i\leq k} \sigma_{i_ji_2\ldots i_n} - \ldots$$

$$+ \ldots + (-1)^{k+1} \sigma_{i_1i_2\ldots i_n}.$$

It is easy to see that all the terms (other than the last) which appear on the right hand side of the equality shown in Proposition 2 depend only on the entries of $D_{k-1}$. Thus, if $\sigma_{i_1i_2\ldots i_n} = 0$, the entries of $D_k$ are completely determined by those of $D_{k-1}$. Consequently, one has the following proposition.

**Proposition 3** $D_\Delta(G)$ determines $D_k(G)$ for all $1 \leq k \leq n$. □

### 3 Results on 2-degree

The adjacency matrix $A = A(G) = (a_{ij})$ of a graph $G$ is defined by

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is adjacent to } v_j \\ 0 & \text{if not} \end{cases}$$

**Proposition 4** The matrix $D_2$ determines and is determined by $A^2$, the square of the adjacency matrix, $A$.

**Proof:** For $A^2 = (a_{ij}^{(2)})$, $a_{ij}^{(2)}$ denotes the number of two-walks from $v_i$ to $v_j$. For distinct $i$ and $j$, this is exactly the number of neighbors common to $v_i$ and $v_j$. Further, if $i = j$, $a_{ii}^{(2)} = \text{deg}(v_i)$. Thus, since $d_i = d_{ii} = \text{deg}(v_i)$ both results follow from

$$d_{ij} = d_i + d_j - |N(v_i) \cap N(v_j)| = d_i + d_j - a_{ij}^{(2)} = a_{ii}^{(2)} + a_{jj}^{(2)} - a_{ij}^{(2)}.$$ □

Note, $a_{ij}^{(2)} = \sigma_{ij}$ for $\sigma$ as defined in the previous section.

Next, an infinite class of pairs of graphs with identical $D_2$ is shown. Now let $G$ be any graph. If $G$ has adjacency matrix $B$, then let $G_1$ be the graph with adjacency matrix

$$A_1 = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}.$$
so that \( G_1 = G \cup G \). Next, let \( G_2 \) be the graph with adjacency matrix

\[
A_2 = \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}
\]

Then, \( A_2^2 = A_2^3 \), so it follows that \( D_2(G_1) = D_2(G_2) \). Note, that \( G_2 \) is a subgraph of the complement of \( G_1 \). To see that \( G_1 \) need not be isomorphic to \( G_2 \), note that if \( G \) is a complete graph on \( n \) vertices, then \( G_1 \) is the disjoint union of two complete graphs each on \( n \) vertices while \( G_2 \) is the complete bipartite graph \( K(n,n) \) less one perfect matching.

Next, given a graph \( G \), the modified square, \( G^{(2)} \) of \( G \) is defined. It will characterize 2-regular graphs with the same \( D_2 \). Given a graph \( G \), form \( G^{(2)} \) as follows. Let \( V(G^{(2)}) = V(G) \), and \( E(G^{(2)}) = \{ uv : \text{There exists a 2-path from } u \text{ to } v \} \). Thus, \((C_3 \cup C_5)^{(2)} = (C_3 \cup C_4) = C_6 \).

**Theorem 1** If \( G \) and \( H \) are 2-regular and \( G^{(2)} = H^{(2)} \), then \( D_2(G) = D_2(H) \). Conversely, if \( D_2(G) = D_2(H) \), then \( G^{(2)} = H^{(2)} \).

**Proof:** Let \( D_2(G) = D_2(H) \). Then \( A^2(G) = A^2(H) \). Let \( V_1 \) and \( V_2 \) be distinct vertices. If \( 0 \neq A^{(2)}_1(G) = A^{(2)}_1(H) \), then there is a 2-walk from \( V_1 \) to \( V_2 \) in both \( G \) and \( H \), but this must be a 2-path. Thus \( V_1 \) must be adjacent to \( V_2 \) in both \( G^{(2)} \) and \( H^{(2)} \).

On the other hand if \( 0 = A^{(2)}_1(G) = A^{(2)}_1(H) \), then there can be no 2-path from \( V_1 \) to \( V_2 \), so they must be nonadjacent in both \( G^{(2)} \) and \( H^{(2)} \).

Next, suppose \( G \) and \( H \) are both 2-regular and hence the union of disjoint cycles.

Thus, there are integers \( s \) and \( t \) sufficiently large such that \( G = U^s_{k=2} r_k C_k \) and \( H = U^t_{k=2} n_k C_k \). Then

\[
G^{(2)} = 2r_4K_2 \cup \left( U^s_{k=2} (2r_{2k+2} + r_{2k+3}) C_{2k+1} \right) \cup \left( U^s_{k=2} n_{2k+1} C_{2k+1} \right)
\]

and

\[
H^{(2)} = 2n_4K_2 \cup \left( U^t_{k=2} 2n_{2k} C_{2k} \right) \cup \left( U^t_{k=2} (2n_{2k+2} + n_{2k+3}) C_{2k+1} \right) \cup \left( U^t_{k=2} n_{2k+1} C_{2k+1} \right).
\]

Further suppose \( G^{(2)} = H^{(2)} \). After pairing all the equal components, it can be seen that the only possible differences between \( G \) and \( H \) is that one graph may have a component \( J = C_{4p+2} \) while the other graph has two corresponding components \( K = C_{2p+1} \cup C_{2p+1} \). Consider the vertices \( 1, 2, \ldots, 4p+2 \), in \( J \) placed successively around the cycle. First, arrange the matrix of \( J \) to obtain the order of vertices to be \( 1, 2, 4, 3, 2p+1, 4, 3, \ldots, 4p+2 \). In \( K \) consider the vertices \( 1, 2, \ldots, 2p+1 \), placed successively around the first cycle and \( 2p+2, 2p+3, \ldots, 4p+2, \) around the second cycle.

Then, arrange the matrix of \( K \) so that the order of vertices in its matrix is \( 1, 3, 2, 4, \ldots, 2p+1, 2p+2, 2p+4, \ldots, 4p+2, 2p+3, \ldots, 4p+1 \). Next, assume without loss of generality that \( J \) and \( K \) are subgraphs of \( G \) and \( H \) respectively. Now letting \( A(F[L]) \) denote the matrix of the restriction of graph \( F \) to subgraph \( L \) where the order of the vertices is preserved, one obtains \( A^2(G[J]) = A^2(H[K]) \), by noting that both are given by

\[
\begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}
\]
Thus, $D_2(G)/J = D_2(H|K)$. Further, for any appropriately paired subgraphs this result is true. Noting that $a_{ij}^{(2)} = 0$ for vertices $v_i$ and $v_j$ in different components of either $G$ or $H$, it follows that $D_2(G) = D_2(H)$.

Next, it is shown that with some additional knowledge a graph can be determined from $D_2$. Let $\overline{G}$ denote the complement of $G$, and $(\overline{a}_{ij}) = D_2(\overline{G})$.

**Proposition 5** $G$ is completely determined by $D_2(G)$ and $D_2(\overline{G})$.

**Proof:** Let $i$ and $j$ be vertices in $G$, $U = V(G) - \{i,j\}$, $H = |N(i) \cap N(j)|$, $I = |U \cap (N(i) - N(j))|$, $J = |U \cap (N(j) - N(i))|$, and $K = |U - (N(i) \cup N(j))|$. Thus, $|V(G)| = H + I + J + K + 2 = n$. For graph $\overline{G}$, the complement of $G$, let $\overline{a}_{ij}$ denote $d_{\overline{ij}}$. Consider $f(i,j) = a_{ij}^{(2)} + \overline{a}_{ij}$. If $i$ is adjacent to $j$, then $f(i,j) = H + I + J + K = n - 2$. If $i$ is not adjacent to $j$, then $f(i,j) = H + I + J + K + 2 = n$. Thus

\[ a_{ij} = \frac{1}{2}[n - f(i,j)], \]

and the adjacency matrix $A$ is determined by $D_2(G)$ and $D_2(\overline{G})$.  

**Corollary 2** $A^2(G)$ and $A^2(\overline{G})$ also completely determine $G$.

### 4 Trees and 2-degree

This section is devoted to the proof of the following theorem.

**Theorem 2** If $D_2$ is graphical and if for graph $G$, $D_2(G) = D_2$, then it can be determined whether or not $G$ is a tree, and, if it is, which tree.

**Lemma 1** There is a procedure which when applied to $D_2(G)$ will determine if $G$ is a tree.

**Proof:** Let $G$ be any graph with $D_2(G) = D_2$. As a first step, check the trace of $D_2(G)$ which equals $\sum_{i=1}^{n} d_i(G) = \sum_{i=1}^{n} d_i(G)$. If this quantity is not equal to $2(n - 1)$, then $G$ is not a tree.

Assuming $\sum_{i=1}^{n} d_i(G) = 2(n - 1)$, start the indexing procedure as follows. Choose any vertex $v_1$. From $A^2(G)$ and hence from $D_2(G)$, determine the 2-neighbors of $v_1$. Two distinct vertices $v_i$ and $v_j$ will be called 2-neighbors in case $a_{ij}^{(2)} \neq 0$. Index $v_i$ with one and its 2-neighbors with zeros, where zero and one are considered to be
opposite parity. Recursively, given an indexed vertex \( v_k \), index all of its 2-neighbors with parity opposite to that of \( v_k \). A given vertex may collect more than one index. Continue indexing one step past the point of obtaining no new vertices. If any vertex has two indices including both parities one and zero, then a cycle exists. Therefore, \( G \) is not a tree.

Assuming no double parity, call the set of indexed vertices \( V_1 \). If \( V_1 = V \), then \( G \) contains an odd cycle and is not a tree. Thus, assume that \( V_1 \neq V \). Repeat the indexing procedure described above starting with an unindexed vertex \( v_2 \). Assuming again no odd cycle, the vertex set \( V_2 \) is formed from the vertices indexed on the second round. Since a tree is bipartite and connected, \( V_1 \) and \( V_2 \) partition \( V \), the vertex set of \( G \), or \( G \) is not a tree. \( \Box \)

Lemma 2 If \( T \) is a tree then it may be uniquely constructed from \( D_2(T) \).

Proof:

If \( D_2(T) \) is known, then \( A^2(T) \), the 2-walk matrix of \( T \), is also known. Form, as in the procedure of the preceding lemma, two classes of vertices, \( V_1 \) and \( V_2 \). Let \( i \in \{1, 2\} \), then if \( k \in V_i \), one may write \( V_i = \{ j : \text{there exists a 2p-walk from } j \text{ to } k \} \). Let \( r \in V_1 \) and declare \( r \) to be the root of \( T \), the only vertex on level 0. Define \( X_0 = \{ r \} \). Put \( d_r = d_r \) vertices on level 1. Call this set of vertices \( X_1 \). Now given \( X_i \) for all \( 0 \leq i \leq 2k - 1 \), form \( X_{2k} \) by making it the set of vertices whose shortest walk i.e., path, from \( r \) is of length \( 2k \). Now, having named \( X_{2k} \), form \( X_{2k+1} \) to be a vertex set with \( \sum_{j \in X_{2k}} (d_j - 1) \) vertices. By an edge-count argument this process will terminate exactly when all vertices are placed.

Connect the vertices as follows. Let \( r \) be adjacent to all vertices in \( X_1 \). Now to determine adjacent vertices between levels \( 2k - 1 \) and \( 2k \) for \( k > 0 \), find the two-walks between pairs of vertices in \( X_{2k} \). There is a 2k-path from \( r \) to each vertex in \( X_{2k} \). Thus, if \( i, j \in X_{2k} \) and \( d_{ij} = 1 \), then \( i \) and \( j \) must have a common neighbor in \( X_{2k-1} \), leading to the equivalence class of all vertices having common neighbors. Some of these vertices may have no such common neighbors, and will form singleton sets. Therefore, partition in this manner the vertices of \( X_{2k} \). Let vertices from a single class \( C \) each be adjacent to a single distinct vertex \( m \in X_{2k-1} \) which is central vertex of a two-walk to each vertex in this class. Then, \( d_{mn} = d_m = |C| + 1 \). Finally, let vertex \( j \in X_{2k} \) be adjacent to \( d_j - 1 \) vertices in \( X_{2k+1} \). Again an edge-count argument guarantees the process will terminate with all the vertices matching their degree in \( D_2 \) or \( A^{(2)} \) when the tree is completely constructed. Clearly, the tree is completely determined under this construction. \( \Box \)

References