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AN EXTREMAL PROBLEM FOR COMPLETE BIPARTITE GRAPHS

P. ERDŐS, R. J. FAUDREE, C. C. ROUSSEAU and R. H. SCHELPE

Dedicated to the memory of Paul Turán

Abstract

Define \( f(n,k) \) to be the largest integer \( q \) such that for every graph \( G \) of order \( n \) and size \( q \), \( G \) contains every complete bipartite graph \( K_{a,b} \) with \( a+b=n-k \). We obtain (i) exact values for \( f(n,0) \) and \( f(n,1) \), (ii) upper and lower bounds for \( f(n,k) \) when \( k \geq 2 \) is fixed and \( n \) is large, and (iii) an upper bound for \( f(n,|\omega|) \).

1. Introduction

Extremal graph theory, which was initiated by Turán in 1941 [4], is still the source of many interesting and difficult problems. The standard problem is to determine \( f(n, G) \), the smallest integer \( q \) such that every graph with \( n \) vertices and \( q \) edges contains a subgraph isomorphic to \( G \). It is striking that whereas Turán completely determined \( f(n, K_r) \), there is much which is as yet unknown concerning \( f(n, K_{a,b}) \). In this paper, we consider a variant of the extremal problem for complete bipartite graphs. In this variant we ask how many edges must be deleted from \( K_{a,b} \) so that the resulting graph no longer contains \( K_{a,b} \) for some pair \((a,b)\) with \( a+b=n \). Specifically, we seek to determine an extremal function \( f(n,k) \) defined as follows. For \( m \geq 1 \), let \( B_m \) denote the class of all graphs \( G \) such that \( G \supseteq K_{a,b} \) for every pair \((a,b)\) with \( a+b=m \). Then for \( n\geq k+1 \), \( f(n,k) \) is the largest integer \( q \) such that every graph \( G \) of order \( n \) and size \( \binom{n}{2} - q \) is a member of \( B_{n-k} \). In this paper we obtain exact values for \( f(n,0) \) and \( f(n,1) \), upper and lower bounds for \( f(n,k) \) when \( k \geq 1 \) is fixed and \( n \) is large, and an upper bound for \( f(n,|\omega|) \).

2. Terminology and notation

All graphs considered in this paper will be ordinary graphs, i.e. finite, undirected graphs, without loops or multiple edges.

A graph with vertex set \( V \) and edge set \( E \) will be denoted \( G(V,E) \). If \(|V|=p\) and \(|E|=q\), \( G \) is said to be of order \( p \) and size \( q \). With \( X, Y \subseteq V \), the set of edges in \( E \) of the form \( \{x,y\} \) where \( x \in X \) and \( y \in Y \) will be denoted \( E(X,Y) \). The complement of \( G \) will be denoted \( \overline{G} \).

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The size of $G$ will be given by $q(G)$. The order of the largest connected component will be given by $\mu(G)$ and the order of the smallest connected component will be given by $\eta(G)$. In particular, $\eta(G)=1$ means that $G$ contains an isolated vertex.

Let $A$ be a finite set. Then $A^k$ will denote the Cartesian product $A \times A \times \ldots \times A$ with $k$ factors and $[A]^k$ will denote the collection of $k$-element subsets of $A$.

Where $x$ is a real number, $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the greatest integer $\leq x$ and the least integer $\geq x$, respectively.

For any notation or terminology not explicitly mentioned in this section, we refer the reader to [1] or [2].

3. Calculation of $f(n, k)$ where $k$ is fixed

Our starting point is the following simple observation. If $G$ is of order $n$ and $\mu(G)>\lfloor n/2 \rfloor$, then $G \cong K_{a,b}$ with $a=\lfloor n/2 \rfloor$, $b=\lfloor n/2 \rfloor$ and so $G \in B_n$. The opposite direction is described by the following useful lemma.

**Lemma 1.** If $(V, E)$ is a graph of order $n$ such that (i) $\mu(G) \geq \lfloor n/2 \rfloor$, (ii) $\eta(G)=1$, and (iii) $q(G) \leq \lceil 2n/3 \rceil-1$, then $G \in B_n$. This result is sharp.

**Proof.** The proof is by induction on $n$. If $n=2$, then $G$ is required to be empty and the conclusion holds. Let $\mu(G)=k$. It is easy to see that the result holds if $k=1$ or 2, so we may assume that $k \geq 3$. Let $H=G-X$, where $X$ is a component of order $k$. Then $H$ is a graph of order $n-k$ and $\eta(H)=1$. Now $q(H) \leq |2n/3-k| \geq \lceil 2(n-k)/3 \rceil-1$, the second inequality being by virtue of the fact that $k \geq 3$. Also, $\mu(H) \leq \min(k, \lfloor 2n/3 \rfloor - k + 1)$. If $3k \leq n$, then $k \leq \lfloor (n-k)/2 \rfloor$ and if $3k \leq n+1$, then $\lfloor 2n/3 \rfloor - k + 1 \leq \lfloor (n-k)/2 \rfloor$. Hence, in all cases $H$ satisfies (i)–(iii) and so, by the induction hypothesis, $H \in B_{n-k}$. Since $X$ and $H$ are completely joined in $G$, it follows that $G \in B_n$.

From the remark made earlier, we know that condition (i) cannot be weakened. To see that (ii) cannot be weakened, note that if $\eta(G)>1$, then $G \cong K_{1,2}$. Finally, with $n \geq 2$, set $m=\lfloor (n+1)/3 \rfloor+1$, $k=\lfloor n/3 \rfloor+1$, $l=n-m-k$ and consider the graph $G=T_m \cup T_k \cup K_l$, where $T_m$ and $T_k$ denote arbitrary trees of orders $m$ and $k$, respectively. In this case, we have $\mu(G)=\lfloor n/2 \rfloor$, $\eta(G)=1$ and $q(G) = \lfloor 2n/3 \rfloor$. However, $G \not\cong K_{a,b}$ with $a=\lfloor 2n/3 \rfloor+1$, $b=\lfloor n/3 \rfloor-1$. This example shows that condition (iii) cannot be weakened.

With the aid of Lemma 1, we can obtain the exact value of $f(n, k)$ in case $k=0$ or 1.

**Theorem 1.** For all $n \geq 2$, $f(n, 0) = \lfloor n/2 \rfloor-1$ and for all $n \geq 3$, $f(n, 1) = \lfloor (n+1)/2 \rfloor$.

**Proof.** With $m=\lfloor n/2 \rfloor+1$, let $G=T_m \cup K_{a,m}$, where $T_m$ denotes an arbitrary tree of order $m$. Thus, $G$ is a graph of order $n$, $q(G)=\lfloor n/2 \rfloor$ and $\mu(G)=\lfloor n/2 \rfloor+1$. Since $\mu(G)>\lfloor n/2 \rfloor$, it follows that $G \in B_n$ and this example shows that $f(n, 0)=\lfloor n/2 \rfloor-1$.

To prove the inequality in the other sense, consider an arbitrary graph $G$ of order $n$ and size $q(G) \geq \lfloor n/2 \rfloor-1$. Note that such a graph must satisfy (i) $\mu(G) \geq \lfloor n/2 \rfloor$, (ii) $\eta(G)=1$, and (iii) $q(G) \geq \lfloor 2n/3 \rfloor-1$. Hence, by Lemma 1, $G \in B_n$.

With $m=\lfloor (n+1)/2 \rfloor+1$, let $G=C_m \cup K_{a,m}$, where $C_m$ denotes the cycle of order $m$. Thus, $G$ is a graph of order $n$ and size $q(G)=\lfloor (n+1)/2 \rfloor+1$. Moreover,
if $x$ is an arbitrary vertex of $G$, then $\mu(G-x) \equiv \lceil (n+1)/2 \rceil \equiv \lceil (n-1)/2 \rceil$. It follows that for each $x \in G - x \cong K_{n,n}$ with $a = \lceil (n-1)/2 \rceil$, $b = \lceil (n-1)/2 \rceil$ and so this example shows that $f(n,1) \equiv \lceil (n+1)/2 \rceil$. To prove the inequality in the other sense, consider an arbitrary graph $G$ of order $n$ and size $q(G) \equiv \lceil (n+1)/2 \rceil$. Let $x$ be a vertex of maximal degree in $G$, and let $H = G - x$. If $x$ has degree $\equiv 2$, then $q(H) \equiv \lceil (n+1)/2 \rceil - 2 \equiv \lceil (n-1)/2 \rceil - 1$. If $x$ has degree $\equiv 1$, then $G$ is the union of a collection of disjoint edges and so in this case as well $q(H) \equiv \lceil (n-1)/2 \rceil - 1$. Therefore, by the first part of this theorem, $H \in \mathcal{B}_{n-1}$ and so $G \in \mathcal{B}_{n-1}$. □

**Corollary.** Let $i(n)$ denote the largest integer $q$ such that for every graph $G$ of order $n$ and size $q$, $G$ contains every tree of order $n$. For all $n \geq 2$, $i(n) = \lceil n/2 \rceil - 1$.

**Proof.** Since each tree of order $n$ is contained in an appropriate complete bipartite graph $K_{n,n}$ with $a+b = n$, it follows that $i(n) \equiv f(n,0) = \lfloor n/2 \rfloor - 1$. On the other hand, the graph $G = (n/2)P_2$ (n even) or $G = (n-3)/2P_3 \cup P_3$ (n odd) is a graph of order $n$ and size $q(G) = \lceil n/2 \rceil$ such that $G \not\cong K_{n,n}$. (Here, $\mathcal{M}$ is used to denote the graph with $m$ components, each isomorphic to $H$.) This example shows that $i(n) \equiv \lfloor n/2 \rfloor - 1$. □

At this point, one may be tempted to conjecture that for each fixed value of $k$, $f(n,k) = n/2 + O(1)$, perhaps even exactly calculable as in the case of $k=0$ or $k=1$. In fact, we find that for all $k \equiv 2$, $n/2 + A \leq f(n,k) = n/2 + B \leq n$, where the positive numbers $A$ and $B$ depend only on $k$. Thus, there is a very striking difference between the case of $k=1$ and that of $k=2$. In order to establish the facts concerning the behavior of $f(n,k)$ when $k \equiv 2$, we shall need several preliminary results.

The following lemma uses the term suspended path. A path $x_0, x_1, \ldots, x_k$ in a graph $G$ will be called suspended if its interior vertices $x_1, \ldots, x_{k-1}$ are of degree 2 in $G$, whereas its end vertices $(x_0$ and $x_k$) have degree $\geq 2$.

**Lemma 2.** Any tree having $k$ vertices of degree 1 is the union of at most $2k-3$ edge-disjoint suspended paths.

**Proof.** The proof is left to the reader.

**Lemma 3.** Let $T$ be a tree of order $n+1$ where $n \geq 2$. There exists a vertex $x$ such that $\mu(T-x) \equiv \lceil n/2 \rceil$. Consequently, there is a partition of the components of $T-x$ into two parts such that each part has at least $\lfloor n/3 \rfloor$ vertices.

**Proof.** The proof is left to the reader.

**Lemma 4.** Let $G(V,E)$ be a connected graph of order $p$ and size $p+1-1$. With $k \equiv 2$, set $\delta = \min((k/2)/4(k-3), 1/4)$. Then, there exists $X \in \mathcal{X}^k$ such that $\mu(G-X) \equiv \lceil (1-\delta)p \rceil$.

**Proof.** Delete $l$ edges from $G$ in such a way that the resulting graph $H$ is still connected, i.e. so that $H$ is a tree. The deleted edges determine a subtree $T$ in the following way. First, we find those vertices which were incident in $G$ with one of the deleted edges and so define a set $A$. Then, we define $T$ to be the union of all paths in $H$ which join pairs of vertices from $A$. Let $A_2$ denote the vertices of $A$ which have degree 1 in $T$ and set $A_2 = A - A_1$. According to Lemma 2, $T$ is the union of
at most $2|A_2| - 3$ edge-disjoint suspended paths. The vertices of $A_2$ now subdivide these suspended paths into what we shall call elementary paths. The elementary paths may be described in the following way. The end-vertices of the elementary paths are precisely those vertices $x$ such that either (i) $x \in A$ or (ii) $\deg(x) = 2$ in $T$. Suppose that there are $r$ elementary paths $P_1, P_2, \ldots, P_r$. Since $|A| \geq 2l$, it follows that $r \leq 2|A_2| + |A_2| - 3 \leq 4l - 3$.

Note the following useful property of the construction described thus far. Suppose that $x$ is a vertex of $G$ and that it is not a vertex of $T$. Then, there is a unique path in $G$ from $x$ to $T$. If there were two such paths, then one of them would have to use one of the edges which were deleted in going from $G$ to $H$. This would put $x$ on a path in $H$ joining two vertices from $A$ and so force $x$ to belong to $T$. In light of this property, we note that the collection of elementary paths $P_1, P_2, \ldots, P_r$ may be used to define a partition $V = (V_1, V_2, \ldots, V_r)$ of the vertices of $G$ according to the following scheme. If $x$ is an end-vertex of one or more elementary paths, it is identified with an arbitrarily chosen one of those paths. If $x$ is an interior vertex of an elementary path, it is identified with that path. Finally, if $x$ is a vertex of $G$ which is not a vertex of $T$, let $w$ be the other end-vertex of the unique path from $x$ to $T$ and identify $x$ with the same elementary path as $w$.

Now we are ready to describe and put to use the crucial properties of the elementary paths. Let $u_t$ and $v_t$ be the end-vertices of the $i$th elementary path, $P_t$. Our construction insures that if $x$ is any vertex of $V_t$ other than $u_t$ or $v_t$, every path from $x$ to a vertex in $V - V_t$ contains either $u_t$ or $v_t$. In other words, by deleting $u_t$ and $v_t$ from $G$, we completely disconnect the vertices of $V_t$ from the remaining vertices of $G$. Without loss of generality, we may suppose that $|V_i| = \ldots = |V_r|$. Set $m = \min \{ \lceil r/4 \rceil, \lfloor k/2 \rfloor \}$ and consider the graph $G - X$, where $X = \{ u_t, v_t, i = 1, \ldots, m \}$. Since $|V_r| + \cdots + |V_r| = mp/r \geq \delta p$, it follows that $\mu(G - X)$ satisfies the stated bound unless $|V_i| > (1 - \delta)p$. In case $|V_i| > (1 - \delta)p$, set $B = V_i \cup \{ u_t, v_t \}$ and consider the tree $T'$ spanning the vertices of $B$. By Lemma 3, there exists a vertex $x$ of this tree such that the components of $T' - x$ can be partitioned into two parts, each of cardinality at least $|V_i| - 1)/3$. Now we may delete $x$ and either $u_t$ or $v_t$, whichever is appropriate, and so disconnect from $G$ a set of at least $|p/4|$ vertices.

In this case, for $X = \{ x, u_t \}$ or $\{ x, v_t \}$ we obtain $\mu(G - X) \geq 3|p/4|$. \hfill $\Box$

Now we are prepared to prove our theorem concerning $f(n, k)$ with $k \geq 2$.

**Theorem 2.** Let $k > 1$ be fixed and set $A = \sqrt{k/2}/16$ and $B = \sqrt{3k(k-1)/(k+1)}$. Then, for all sufficiently large $n$,

$$n/2 + A \sqrt{n} < f(n, k) < n/2 + B \sqrt{n}$$

**Proof.** Let $G(V, E)$ be a graph of order $n$ and size $q = n/2 + \Delta$, where $\Delta = A \sqrt{n}$. We wish to prove that there exists $X \subseteq V$ such that $G - X$ satisfies the conditions of Lemma 1. This will establish the lower bound for $f(n, k)$. Since $\Delta = o(n)$, it follows that the number of connected components of $G$ is at least $n - q = n/2 - o(n)$. Consequently, $\eta(G) \geq 2$. On the other hand, if $\eta(G) = 2$, then $\mu(G) = o(n)$ and so by deleting just one vertex from $G$ we obtain a graph which satisfies the conditions of Lemma 1. Hence, we now assume that $\eta(G) = 1$. Since this is the case, we may assume that $\mu(G) > (n - k)/2$, in fact $\mu(G) > (n + k)/2$ for, otherwise, we may
simply delete any \( k \) vertices from the largest component. Suppose that the largest component is of order \( p \) and size \( p+1 \). Hence, we have the bounds \( p = q = n/2 + \Delta \) and \( 1 \leq q - p + 1 \leq \Delta \). With a view toward applying Lemma 4, note that if \( \delta = \lfloor k/2 \rfloor/(4l-3) \) then \( (1 - \delta)p = (1 - \lfloor k/2 \rfloor) + \Delta \), \( (n/2 + \Delta) - n/2 = \Delta - \lfloor k/2 \rfloor/8 \). Therefore, in this case and with our choice of \( \Delta \), we have \( (1 - \delta)p \geq (n - k)/2 \). Certainly if \( \delta = 1/4 \), \( (1 - \delta)p \geq (n - k)/2 \) and so the desired result follows from Lemma 4.

The upper bound is established by the following simple construction. With \( m \) chosen to be an even integer, let \( H \) be a graph of order \( m \) which is regular of degree \( k+1 \) and \((k+1)\)-connected. An example of such a graph has vertices 0, 1, ..., \( m-1 \) with two vertices \( i \) and \( j \) joined if \( i - [(k+1)/2] \equiv j - [(k+1)/2] \) (mod \( m \)) and, if \( k+1 \) is odd, \( i \) is joined to \( i + m/2 \) for \( 1 \leq i \leq m/2 \). The fact that such a graph is, indeed, \((k+1)\)-connected was proved by Harary in [3] and the proof is also given in [1, pp. 48–49]. Set \( r = m(k+1)/2 \) and let the edges of \( H \) be \( e_1, e_2, ..., e_r \). For \( i = 1, 2, ..., r \), insert a vertex \( y_i \) subdividing \( e_i \) and make \( y_i \) adjacent to \( i-1 \) new vertices. Finally, add isolated vertices so that the resulting graph \( G(V, E) \) is of order \( n \). Thus, \( G \) is of size \( q(G) = r + (l_1 + ... + l_r) \). Without loss of generality, we may assume that \( l_1 \leq l_2 \leq ... \leq l_r \). Now make the following choices for the parameters of \( G \). Set \( m = 2 \lfloor 5kn/8(k^2 - 1) \rfloor \) and \( l_1 = ... = l_r = \lfloor 5(k-1)n/8k(k+1) \rfloor \). Then choose \( l_{r+1}, ..., l_r \) so that \( m + (l_1 + ... + l_r) = (n-k)/2 \) + 1. Let \( Y = \{ y_1, ..., y_r \} \). It is apparent that for every \( X \subseteq [V]^c \), we have \( \mu(G - X) \geq \mu(G - Y) = (n-k)/2 + 1 \). Also, we have \( q(G) = (n-k)/2 + 1 + k/l + (k-1)m/2 = n/2 + B \) for every \( \varepsilon > 0 \). Since \( \mu(G - X) \geq (n-k)/2 \) for every \( X \subseteq [V]^c \), it follows that \( G \in B_{n-k} \). This establishes the upper bound.

4. An upper bound for \( f(n, \lfloor en \rfloor) \)

At present, very little is known about \( f(n, k) \) when \( k \to \infty \) with \( n \). However, the results of the preceding section suggest that \( f(n, \lfloor en \rfloor) < (1/2 + \delta)n \) where \( \delta > 0 \) with \( e \) and this much can be proved without difficulty.

**Theorem 3.** Let \( 0 < \varepsilon < e^{-4} \) be fixed and set \( \delta = \sqrt{6} \log (1/\varepsilon) \). For all sufficiently large values of \( n \),

\[
f(n, \lfloor en \rfloor) < (1/2 + \delta)n.
\]

**Proof.** Set \( p = \lfloor 1/2 + \delta \rfloor n \), \( q = \lfloor 1/2 + \delta \rfloor n \), \( k = \lfloor en \rfloor \), \( r = q - p \), \( a = (n-k)/2 \), \( b = (n-k)/2 \), and \( c = a + p - n \). Using the probabilistic method, we shall prove the existence of a graph \( G \) of order \( n \) and size \( \leq q \) such that \( G \geq K_{a,b} \). Let \( V = \{ 1, 2, ..., n \} \), \( X = \{ 1, 2, ..., p \} \) and \( Y = \{ 1, 2, ..., n \} \). The probability space used to prove the existence of \( G \) may be described as follows. Let \( \Omega = \Omega_1 \times \Omega_2 \) where \( \Omega_1 = X^p \) and \( \Omega_2 = Y \). Each point in \( \Omega \) is given probability \( 1/|\Omega| \). A typical point in \( \Omega \) is \( \omega = (\omega_1, \omega_2) \) where \( \omega_1 = (x_1, ..., x_p) \) and \( \omega_2 = (y_1, ..., y_p) \). Corresponding to \( \omega \) there is a graph defined as follows: \( [i,j] \) is an edge in the graph for each occurrence of \( x_i = j \) or \( y_i = j \), \( k = 1, ..., r \). It is understood that any loops and/or extra edges which may be generated by the random method are simply not included in the graph so formed. If \( G \geq K_{a,b} \) then for some \( m, c \leq m \leq a \), there are disjoint sub-
sets of $X$, namely $A$ and $B$ with $|A| = m$ and $|B| = p - k - m$, such that $E(A, B) = \varphi$. Now for fixed $A$ and $B$, consider the event $E(A, B) = \varphi$. The number of points of $\Omega_1$ in this event is $(m + k)^m(p - m)^{p-k-m}p^k$ and the number of points of $\Omega_2$ in this event is $\binom{n}{2} - m(p - k - m)$. Hence, we obtain the bound

$$\text{Prob}\left(G \supseteq K_{n,k}\right) \leq \sum_{m=0}^{p} \binom{p}{m} \left(\frac{m + k}{m}\right)^m \left(\frac{p - m}{p - m}\right)^{p - k - m} \left(\frac{p}{p}\right)^k \left(1 - \frac{m(p - k - m)}{\binom{n}{2}}\right).$$

Using Stirling's formula and some elementary bounds, we find that each term in the sum is bounded by

$$(1 + 2k/n)^n (p/k)^n \left(1 - a(p - k - \alpha) / \binom{n}{2}\right)^n.$$

Substituting the values of $a, k, p$ and $r$, we find that $\text{Prob}\left(G \supseteq K_{n,k}\right) \to 0$ as $n \to \infty$ provided that $(1 + 2\epsilon)((1 + \delta)/2\epsilon)^{((1 - (1 - \epsilon)(\delta - \epsilon)/2)}^{\epsilon^2}=1$. A simple calculation shows this to be the case when $0 < \epsilon < e^{-1}$ and $\delta = \frac{1}{6} \log (1/\epsilon)$. □

5. Additional problems and results

The bound for $f(n, [en])$ provides a satisfying tie with the results for $f(n, k)$ where $k$ is fixed; still, it leaves us with more questions than answers. Among other things, the result shows that if $F(\epsilon) \overset{\text{def}}{=} \lim_{n \to \infty} f(n, [en])/n$ exists, then $\lim_{\epsilon \to 0} F(\epsilon) = 1/2$. But, does $\lim_{n \to \infty} f(n, [en])/n$ exist?

**Problem 1.** For $0 < x < 1$, does $\lim_{n \to \infty} f(n, [xn])/n$ exist?

By a variety of simple arguments, it is possible to prove bounds of the form $F_1(x) \overset{\text{def}}{=} f(n, [xn])/n \leq F_2(x)$ which hold when $0 < x < 1$ is fixed and $n$ is sufficiently large. Hence, it is at least plausible that $\lim_{n \to \infty} f(n, [xn])/n$ exists. As an example of an upper bound for $f(n, [xn])/n$, we give the following argument. Starting with the complete graph $K_n$, we wish to remove $q = [yn]$ edges $e_1, e_2, ..., e_q$ in such a way that all $K_{m,m}$ subgraphs with $m = [(1-x)n/2]$ are destroyed. Having found such a number $y$, we are assured that $f(n, [xn])/n \leq y$. Let $X_i$ denote the set of $K_{m,m}$ subgraphs which remain after $e_i$ has been removed. Clearly, $|X_i| = \binom{n}{m} \binom{n - m}{m}$. At the stage of removing the edge $e_{i+1}$ there are $|X_i|$ remaining $K_{m,m}$ subgraphs and $\left(\binom{n}{2} - i\right)$ remaining edges. Counting multiplicity, the remaining $K_{m,m}$ subgraphs contain $|X_i|m^2$ edges. It follows that there is an edge whose removal destroys at least $|X_i|m^2\binom{2}{2}$ of the subgraphs in $X_i$. By choosing such an edge for $e_{i+1}$, we obtain $|X_{i+1}| \leq |X_i| \left(1 - m^2\binom{2}{2}\right)$. Following such a procedure for $i = 1, 2, ..., q$, we obtain...
An easy calculation using Stirling’s formula allows us to conclude that if \( y \) is chosen so that \( (1-(1-x)^2/2)^y = ((1-x)/2)^{1-x}x^x \) and \( n \) is sufficiently large, then \( |X| = 0 \). As \( x \to 0 \), the upper bound for \( f(n, \lfloor xn \rfloor) \) that is obtained by this argument is quite inferior to the bound given in Theorem 3. The advantage of this argument is that it is applicable for all \( x \) satisfying \( 0 < x < 1 \).

The second problem is not concerned with the calculation of \( f(n, k) \), but is certainly related to the investigation described in this paper.

**Problem 2.** For all \( n \geq 2 \), determine the largest integer \( m = f(n) \) such that for every tree \( T \) of order \( n \), \( T_G \in B_m \).

We have obtained upper and lower bounds for \( f(n) \) and these results may be published elsewhere.

Finally, we note the following generalization of the basic problem considered in this paper.

**Problem 3.** For \( r \geq 2 \) and \( n \geq k + r \), let \( f_r(n, k) \) denote the largest integer \( q \) such that for every graph \( G \) of order \( n \) and size \( q \), \( G = K(a_1, \ldots, a_r) \) for every partition \( (a_1, \ldots, a_r) \) of \( n - k \) into \( r \) parts. Determine \( f_r(n, k) \).

The proofs given in this paper extend naturally and easily to the study of \( f_r(n, k) \). For \( r = 3 \), the induction argument used in the proof of Lemma 1 yields the following result.

**Lemma.** Let \( r \geq 3 \). If \( G \) is a graph of order \( n \) such that (i) \( \mu(G) \equiv [n/r] \) and (ii) \( \varrho(G) \equiv [2n/(r+1)] - 1 \), then \( G = K(a_1, \ldots, a_r) \) for every partition \( (a_1, \ldots, a_r) \) of \( n \).

Now we can state the following generalizations of Theorems 1, 2 and 3. The reader will find that the proofs given earlier in the paper have been so structured that they readily yield the results now stated.

**Theorem.** For all \( r \geq 2 \) and \( n \geq r, f_r(n, 0) = [n/r] - 1 \). Except for certain exceptional cases, \( f_r(n, 1) = [(n-1)/r] + 1 \) holds for all \( r \geq 2 \) and \( n \geq r + 1 \). The exceptional cases are \( f_2(4, 1) = 1 \), \( f_2(6, 1) = 2 \), \( f_3(8, 1) = 3 \) and, for \( r \geq 4 \), \( f_r(r+1, 1) = 1 \) and \( f_r(r+2, 1) = f_r(r+3, 1) = 2 \).

**Theorem.** Let \( r, k > 1 \) be fixed and set \( A = \sqrt{[k/2]/8r} \) and \( B = \sqrt{6k(k-1)/(k+1)r} \). Then, for all sufficiently large \( n \),

\[
n/r + A \sqrt{n} < f_r(n, k) < n/r + B \sqrt{n}.
\]

**Theorem.** Let \( 0 < \varepsilon < e^{-4} \) be fixed and set \( \delta = \sqrt{\varepsilon(r+1)n} \log(1/\varepsilon) \). For all sufficiently large values of \( n \),

\[
f_r(n, \lfloor xn \rfloor) < \lfloor (1/r + \delta)n \rfloor.
\]

Exactly as in the special case of \( r = 2 \), the methods used in this paper provide an effective means of studying \( f_r(n, k) \) only when \( k \ll n \). Thus, for example, the generalization of Problem 1 to consider \( f_r(n, \lfloor xn \rfloor), 0 < x < 1 \), is an important
problem about which little is known at present. With \( n \) and \( r \) fixed, \( f_s(n, k) \) is defined for \( 0 \leq k \leq n - r \), and it is worth pointing out that in addition to the \( k = 0 \) and \( k = 1 \) cases, \( f_s(n, k) \) is known exactly for \( k = n - r \). We know that \( f_s(n, n - r) = (n - r + 1)s - 1 \), where \( n = (r - 1)s + t \), \( 0 \leq t < r - 1 \). This is Turán's theorem.

REFERENCES


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