A Solved and Unsolved Graph Coloring Problem

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Problem 73-11*. On Hadamard Matrices, by K. W. Schmidt (University of Manitoba, Manitoba, Winnipeg, Canada).

Let \( r \) and \( s \) denote orders of existing Hadamard matrices. Does a Hadamard matrix of order \( r \times s \) necessarily contain submatrices of order \( r \) or \( s \) which are also Hadamard matrices?

Problem 73-12, A Nonlinear Differential Equation, by Otto G. Ruehr (Michigan Technological University).

Determine the general solution of

\[
\left\{ \frac{d^2 f}{dx^2} + 2 \right\} f = \left\{ \frac{df}{dx} + 2x \right\} \left\{ \frac{df}{dx} + x \right\}.
\]

The problem arose in modeling the Helmholtz equation in two dimensions.

Problem 73-13. An Integral Inequality, by Richard Askey (University of Wisconsin).

Heisenberg’s inequality can be stated as

\[
M_0 \left\{ \int_{-\infty}^{\infty} x^2 |F(x)|^2 \, dx, \int_{-\infty}^{\infty} t^2 |\tilde{F}(t)|^2 \right\} \geq \frac{1}{4\pi} \int_{-\infty}^{\infty} |F(x)|^2 \, dx,
\]

where

\[
M_0[a, b] = \sqrt{ab}
\]

and

\[
\tilde{F}(t) = \int_{-\infty}^{\infty} F(x) e^{2\pi i xt} \, dx.
\]

By Schlömilch’s inequality,

\[
M_p \left\{ \int_{-\infty}^{\infty} x^2 |F(x)|^2 \, dx, \int_{-\infty}^{\infty} t^2 |\tilde{F}(t)|^2 \, dt \right\} \geq \frac{1}{4\pi} \int_{-\infty}^{\infty} |F(x)|^2 \, dx
\]

holds for \( p > 0 \), where \( M_p[a, b] = \left( (a^p + b^p)/2 \right)^{1/p} \). Show that even if we replace the \( 1/4\pi \) by any positive constant \( A_p \) in (1), the inequality will be invalid for any \( p < 0 \).

SOLUTIONS

Problem 72-6*. A Solved and Unsolved Graph Coloring Problem, by P. Erdős (University of Waterloo, Waterloo, Ontario, Canada).

Show that if one two-colors the edges of a \( K_{2n-1} \) (complete graph of \( 2n - 1 \) vertices), then there is a \( C_n \) (cycle of order \( n \)) of one color.

Editorial note. The proposer notes that his problem was recently solved by J. A. Bondy (University of Waterloo) who also conjectures the related result:

If \( k \leq n \) and one two-colors the edges of a \( K_{2n-1} \), then either there is a \( C_k \) of the first color or else there is a \( C_n \) of the second color.
The proposer has proved the latter result for fixed $k$ if $n > n_0(k^4)$ and notes that this weaker result was first conjectured by W. G. Brown.

Solution by R. J. FAUDREE and R. H. SCHELP (Memphis State University).

It is convenient to rephrase this problem and its solution in terms of Ramsey numbers for cycles in graphs. Let $G$ be a graph of order $m$ with edge set $E$, and let $G$ and $E$ denote the complementary graph with its edge set. $(G, \overline{G}, E, \overline{E}$ and $m$ will retain these meanings throughout the solution presented.) The Ramsey number, $R(C_k, C_n)$, for the cycles $C_k$, $C_n$ is the minimal number $m$ such that either $G$ contains a $C_k$ or $\overline{G}$ contains a $C_n$. In terms of Ramsey numbers the problem is: show $R(C_k, C_n) \leq 2n - 1$ for $k \leq n$ and $n > 3$.

A detailed solution of the problem is quite lengthy so that we shall simply present a sketch of the proof, giving only the essentials. The detailed solution of this result together with Ramsey numbers for all pairs of cycles will be published elsewhere by the authors.

Several known results are used in the proof presented and are now stated.

**Lemma 1** (Chartrand, Schuster [3]).

$$R(C_3, C_n) = \begin{cases} 
6, & n = 3, \\
2n - 1, & n > 3,
\end{cases}$$

$$R(C_4, C_n) = \begin{cases} 
6, & n = 4, \\
7, & n = 5, \\
(n + 1), & n > 5,
\end{cases}$$

$$R(C_5, C_n) = 2n - 1, \quad n \geq 2, \quad R(C_6, C_n) = 8.$$

**Lemma 2** (Erdős, Bondy [2]). $R(C_n, C_n) \leq 2n - 1, n > 3$.

**Lemma 3** (Erdős, Gallai [4]). If $G_1$ is a graph of order $n$ and size at least \((c - 1)(n - 1) + 1)/2\), then $G_1$ contains a cycle of length at least $c$.

**Lemma 4** (Bondy [1]). If $G_1$ is a graph of order $n$ and size at least $(n^2 + 1)/4$, then $G_1$ contains cycles of all lengths $j$, $3 \leq j \leq (n + 2)/2$.

In light of Lemmas 1 and 2, we know that if the bound given on $R(C_n, C_n)$ fails to hold, then $5 < k < n$. Thus we assume throughout the remainder of this solution that $m = 2n - 1$ is such that $G$ contains no $C_k$, $5 < k < n$, and $\overline{G}$ contains no $C_n$, i.e., $R(C_k, C_n) \leq 2n - 1 = m$. We shall of course eventually contradict this assumption.

As an initial step, we observe by Lemmas 3 and 4 that $|E| < (m^2 + 1)/4 = n(n - 1) + 1/2$ and unless $\overline{G}$ contains a $C_j$, $j > n$, also $|\overline{E}| < (n - 1)(m - 1) + 1)/2 = (n - 1)^2 + 1/2$. But $|E| + |\overline{E}| = (2n - 1)(2n - 2)/2 = (n - 1)^2 + n(n - 1)$, so that for $|E| < (m^2 + 1)/4$ and $|\overline{E}| < [(n - 1)(m - 1) + 1)/2$ we must have $|E| = n(n - 1)$ and $|\overline{E}| = (n - 1)^2$. Thus, consider the case where $|E| = n(n - 1)$ and $|\overline{E}| = (n - 1)^2$. By applying Lemma 4 to the graph $G \setminus \{u\}$ for $u \in G$, it can be shown that the degree of $v$ in $G$, denoted $d_u(v)$, satisfies $d_u(v) \geq n - 1$ for all points in $G$. Furthermore, applying the same lemma to $G \setminus \{u, v\}$ for $u, v \in G, u \neq v$, there exist $n$ vertices of degree $n - 1$ in $G$ all of which are adjacent. From these facts, it can be shown that $G = K_n \cup K_{n-1}$, a contradiction to $G$ containing no $C_n$. Therefore, $|E| = n(n - 1)$ and $|\overline{E}| = (n - 1)^2$ never occurs so that $|E| \geq [(n - 1) \cdot (m - 1) + 1)/2$ This gives the following lemma.
LEMMA 5. $\bar{G}$ contains a $C_{j}, j > n$.

This lemma can be strengthened to the following statement.

LEMMA 6. $\bar{G}$ contains a $C_{n+1}$.

Proof. Let $C = (x_1, x_2, \ldots, x_j, x_1)$ be a cycle of length $j > n + 1$ in $\bar{G}$. The lemma follows when we show that $\bar{G}$ contains a $C_{j-1}$ or a $C_{j-2}$. Suppose $\bar{G}$ contains neither a $C_{j-1}$ or $C_{j-2}$. Then for every $i$, $1 \leq i \leq j$, $(x_i, x_{i+2}, x_{i+3}) \in E$, where subscript addition is taken, as usual, modulo the cycle length $j$. Letting $k = 2p + 1$, when $k$ is odd, even, we then have

\[(x_1, x_4, x_6, \ldots, x_{2p}, x_{2p+3}, x_{2p+1}, x_{2p-1}, \ldots, x_3, x_1),\]

\[(x_1, x_4, x_6, \ldots, x_{2p+2}, x_{2p+1}, x_{2p-3}, \ldots, x_3, x_1),\]

is a cycle of length $k$ in $G$, a contradiction.

The remainder of the proof is specifically aimed at showing that $G$ contains a $C_{n+1}$ implies $G$ must also contain a $C_n$ or $C_1$. Hence, we pinpoint the general idea of showing that the existence of cycles $j > n$ in $\bar{G}$ implies (when $G$ contains no $C_n$) the existence of a $C_n$ in $G$.

We next make the following observation. If $\bar{G}$ contains the $n + 1$ cycle $(x_1, x_2, \ldots, x_n, x_1)$, $n \geq 7$, then $|[\{1 \leq i \leq n + 1, (x_i, x_{i+3}) \in E\}]| \geq (n + 1)/2$. To see this let $k$ be odd, $k = 2p + 1$. Then for $i$, $1 \leq i \leq n + 1$, and $(x_i, x_{i+3}) \in E$ we must have $(x_{i+2p-1}, x_{i+2p+2}) \in E$, otherwise (since $G$ contains no $C_n$ implies $(x_i, x_{j+2}) \in E$ for all $j$) we get the $k$ cycle

$(x_1, x_4, x_7, \ldots, x_{i+2p-1}, x_{i+2p+2}, x_{i+2p}, \ldots, x_{i+2}, x_i)$

in $G$. When $k$ is even the argument is similar.

One more idea is needed to prove the result. Let $C = (x_1, x_2, \ldots, x_j, x_1)$ be a cycle in $G$ and $x \in G \setminus C$. Vertex $x$ is said to be dominant (strongly dominant) to $C$ in $G$, when $|\{i \leq i \leq n, (x, x_i) \in E\}|$ dominates (strictly dominates) $j/2$.

LEMMA 7. Let $\bar{G}$ contain the $n + 1$ cycle $C = (x_1, x_2, \ldots, x_n, x_1)$, $n \geq 7$. Then:

1. If $x \in G \setminus C$ we have
   
   (a) $x$ is strongly dominant to $(x_1, x_3, \ldots, x_{n+1}, x_2, x_4, \ldots, x_n, x_1)$ in $G$ for $n$ even, and
   
   (b) $x$ is strongly dominant to $C^{(1)} = (x_1, x_3, \ldots, x_{n+1}, x_2, x_4, \ldots, x_n, x_1)$ for $n$ odd, and dominant to their union.

2. If $x, y \in G \setminus C$, $(x, y) \in E$, and $x$ is dominant to $C^{(1)}$ but not to $C^{(2)}$ in $G$, then $y$ is dominant to $C^{(2)}$ in $G$, where $C^{(1)}$ and $C^{(2)}$ are as in part 1.

Proof. The flavor of the proofs of parts 1 and 2 are similar so that we only prove part 1. Since $G$ contains no $C_n, (x_i, x_{i+2}) \in E$ for $1 \leq i \leq n + 1$. Thus in $G$ we obtain the cycles shown in part 1. Let $x_i \in C$. By the observation noted in a previous paragraph and since $n \geq 7$, there exists a $j$, $1 \leq j \leq n + 1$, such that $(x_j, x_{j+3}) \in E$ and $x_{j+1}, x_{j+2} \notin \{x_i, x_{i+1}\}$. Therefore $(x_j, x_i) \in E$ implies $(x, x_{i+1}) \in E$; otherwise $(x, x_{i+2}, \ldots, x_j, x_{j+3}, \ldots, x_i) \in E$ is an $n$ cycle in $G$. Hence $\{|i \leq j \leq n + 1, (x_i, x_j) \in E, 1 \leq i \leq n + 1\| \leq |\{|i \leq j \leq n + 1, (x_i, x_j) \in E, 1 \leq i \leq n + 1\|\|.$

THEOREM 1. $R(C_n, C_p) \leq 2p - 1, p > 3, p \geq k$.

Proof (Sketch). Suppose the theorem is false so that as before we set $p = n, m = 2n - 1$. Thus by Lemma 6, $G$ contains a $C_{n+1}$. Furthermore, by Lemma 1 we take $n \geq 7$. Let $C = (x_1, x_2, \ldots, x_{n+1}, x_1)$ be a cycle in $G$. 

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Case I. \( n \) is even.

Since \( \tilde{G} \) contains no \( C_n \), \((x_1, x_{i+1}, \ldots, x_{i+q}, x_{i+1}, \ldots, x_n, x_1) \) is an \( n + 1 \) cycle in \( G \). Let \( x \in G \setminus C \). Then by Lemma 7, part 1, \( x \) is strongly dominant to \( C \) in \( G \), so that there exists a subset \( \{x_i, x_{i+1}, \ldots, x_{i+q}\} \subseteq G \), \( q \geq (n + 1)/2 \), such that \((x, x_j) \in E \) for \( j = 1, 2, \ldots, q \). But then, since \( q \geq (n + 1)/2 \), there exists an \( i \) such that \((x, x_i), (x, x_{i-2k+2}) \in E \). Hence \((x, x_i, x_{i-2}, \ldots, x_{i-2k+2}, x) \) is a cycle of length \( k \) in \( G \), a contradiction.

Case II. \( n \) is odd.

As in Case I, \( \tilde{G} \) contains no \( C_n \) implies that \( C^{(1)} = (x_1, x_2, \ldots, x_n, x_1) \) and \( C^{(2)} = (x_2, x_3, \ldots, x_{n+1}, x_2) \) are cycles in \( G \). Let

\[
L = \{ x \in G \setminus C | x \text{ is strongly dominant to } C^{(1)} \text{ in } G \text{ and not dominant to } C^{(2)} \text{ in } G \},
\]

and

\[
R = \{ x \in G \setminus C | x \text{ is strongly dominant to } C^{(2)} \text{ in } G \text{ and not dominant to } C^{(1)} \text{ in } G \}.
\]

By Lemma 7, part 2, \( L \) and \( R \) are complete graphs in \( G \) and by Lemma 7, part 1b, \( G \setminus C = L \cup M \cup R \). If \(|L| \) or \(|R| \) is greater than or equal to \((n - 3)/2 \), then either \( L \cup C^{(1)} \) or \( L \cup C^{(2)} \) contains all \( C_n \), \( 3 \leq i \leq n - 1 \), in \( G \) and in particular a \( C_i \). Thus we assume \(|L \cup R| \leq n - 5 \) so that \(|M| \geq (n - 2) - (n - 5) = 3 \). Pick \( x, y \in M, x \neq y \). Using Lemma 7, part 1b, it can be shown by a proper choice of \( i, j, s, r \) that \((x, x_i, x_{i+2}, \ldots, x_{i+2r}, y, x_i, x_{i+2s, x_{i+2s-2}, \ldots, x_{i+2s-r, x}}) \) is a cycle of length \( k = 4 + r + s \) in \( G \), a contradiction.

Thus the supposition is false and the proof is complete.

For \( k \) odd the graph \( G = K_{p-1, p-1} \) shows that \( R(C_k, C_p) = 2p - 1, p > 3, \ p \geq k \). The remaining Ramsey numbers, where \( k \) is even, are as follows:

\[
R(C_{2k}, C_{2p}) = 2p + k - 1 \text{ for } p \geq k \geq 3,
\]

\[
R(C_{2k}, C_{2p+1}) = 2p + k + 1 \text{ for } 2p + 1 \geq 3k - 1.
\]

A detailed proof that these are the remaining Ramsey numbers will be included in the paper *All Ramsey numbers of cycles in graphs*, to be published elsewhere.

*Editorial note.* Another such paper by V. Rosta (Mathematical Institute of the Hungarian Academy of Sciences) is to appear in J. Combinatorial Theory.

REFERENCES