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All Triangle-Graph Ramsey Numbers for Connected Graphs of Order Six

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ABSTRACT

The Ramsey numbers $r(K_3, G)$ are determined for all connected graphs $G$ of order six.

Ramsey numbers for complete collections of small graphs have been studied by several mathematicians. Chvátal and Harary determined all Ramsey numbers for pairs of graphs each with at most four vertices [1, 2]. Clancy has produced a table of Ramsey numbers $r(F, G)$, where $F$ is of order at most four and $G$ is of order five [3]. This table is complete except for five entries. Burr has determined all diagonal Ramsey numbers for graphs of size at most six. Burr's results are quoted in the article on Ramsey theory which appeared in the "Mathematical Games" section of Scientific American, November, 1977. Ramsey numbers for small graphs have evident usefulness. They provide information which can be used in proving additional results in generalized Ramsey theory, they provide raw material for new conjectures, and they give some insight concerning the complexity of graphical Ramsey theory in general. As matters stand, completion of the table of Ramsey numbers for all graphs of order at most five seems somewhat out of reach, since this would involve, among other things, computing the classical Ramsey number $r(K_5)$. Doing the same for all graphs of order at most six seems totally hopeless in the

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absence of some extraordinary new tool. In this note, we determine \( r(K_3, G) \) for all connected graphs \( G \) of order six. On the face of it, even this project appears to be one of some complexity. One reason is that there are 112 such graphs. However, these 112 graphs can be grouped into six classes corresponding to six possible values of \( r(K_3, G) \) and arguments covering the five classes of noncomplete graphs are surprisingly simple. All proofs will be given in detail.

Let \( V = \{v_1, v_2, \ldots, v_p\} \) denote a set of vertices. Then \( [V]^2 \) denotes the set of all unordered pairs of these vertices. By a two-coloring we mean a partition \( [V]^2 = \langle R \rangle \cup \langle B \rangle \). Equivalently, we ascribe to each edge of the complete graph of order \( p \) a color, either red or blue. This two-coloring defines two edge-induced graphs of order \( p \) and we use \( \langle R \rangle \) and \( \langle B \rangle \) as symbols for these graphs. Let \( F \) and \( G \) be graphs without isolated vertices. The statement \( K_p \rightarrow (F, G) \) means that if \( |V| = p \), then for every possible two-coloring \( \langle R \rangle \cup \langle B \rangle \) of \( [V]^2 \), either \( \langle R \rangle \) contains an isomorphic copy of \( F \) or \( \langle B \rangle \) contains \( G \). The Ramsey number \( r(F, G) \) is the smallest natural number \( p \) such that \( K_p \rightarrow (F, G) \).

**Lemma.** Let \( \langle R \rangle \cup \langle B \rangle \) be a two-coloring of \( [V]^2 \) such that \( \langle R \rangle \) contains no \( K_3 \). Let \( X \) and \( Y \) be nonempty disjoint subsets of \( V \) such that \( |X| \) is odd. The set \( Y(X, R) \), consisting of those vertices of \( Y \) which are adjacent to \( \forall \) vertices of \( X \) in \( \langle R \rangle \), spans a complete graph in \( \langle B \rangle \).

**Proof.** If \( |Y(X, R)| = 1 \), there is nothing to prove. Otherwise, suppose that \( u \) and \( v \) are distinct vertices in \( Y(X, R) \) and note that \( X_R(u) \cap X_R(v) \neq \emptyset \). Thus \( uu \in B \) since \( \langle R \rangle \) contains no \( K_3 \).

The simplest example of this lemma is that for any vertex \( v \), \( N(v) \) spans a complete graph in \( \langle B \rangle \). The following example is a bit more interesting and will be used several times in what follows.

**Corollary.** Let \( \langle R \rangle \cup \langle B \rangle \) be a two-coloring of \( [V]^2 \), where \( |V| \geq 11 \). Suppose that \( \langle R \rangle \) contains no \( K_3 \), but that \( \langle B \rangle \) does contain a \( K_3 \). Then \( \langle B \rangle \) contains \( K_5 - P_3 \).

**Proof.** Let \( X \) span \( K_5 \) in \( \langle B \rangle \) and let \( Y = V - X \). If \( |Y(X, R)| \geq 6 \), then
(B) contains a $K_6$. Otherwise, $Y(X, B) \neq \emptyset$, i.e., there is a vertex of $Y$ which is adjacent in $Y$ to at least three vertices of $X$. We thus have the stated conclusion.

We are now prepared to prove the main result.

**Theorem.** With the graphs $G_6$ through $G_{12}$ as identified (by their complements) in Figure 1, the following table gives $r(K_3, G)$ for every connected graph $G$ of order six.

<table>
<thead>
<tr>
<th>Class</th>
<th>$G$</th>
<th>$r(K_3, G)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>$G \subseteq G_0$</td>
<td>11</td>
</tr>
<tr>
<td>(b)</td>
<td>$G_1, \ldots, G_6$</td>
<td>12</td>
</tr>
<tr>
<td>(c)</td>
<td>$G_7, G_8$</td>
<td>13</td>
</tr>
<tr>
<td>(d)</td>
<td>$G_9, \ldots, G_{12}$</td>
<td>14</td>
</tr>
<tr>
<td>(e)</td>
<td>$K_6-x$</td>
<td>17</td>
</tr>
<tr>
<td>(f)</td>
<td>$K_6$</td>
<td>18</td>
</tr>
</tbody>
</table>

**Proof.** (a) With $|V| = 10$ let $(R, B)$ be the two-coloring of $[V]^2$ in which $\langle R \rangle = K_{2,2}$. This well-known example shows that if $G$ is any connected graph of order six, then $r(K_3, G) \geq 11$. In the other direction, we now argue that $K_{11} \rightarrow (K_3, G_0)$. Let $V = \{v_1, v_2, \ldots, v_{11}\}$ and suppose, to the contrary, that there exists a two-coloring $(R, B)$ of $[V]^2$ such that $\langle R \rangle$ contains no $K_3$ and $\langle B \rangle$ contains no $G_0$. In view of the corollary to the lemma, we may assume that $\langle B \rangle$ contains no $K_5$. However, in [3] Clancy found that $K_{11} \rightarrow (K_5, K_5-x)$. Thus, we may assume that $\{v_1, v_2, \ldots, v_d\}$ induces $K_d-x$ in $\langle B \rangle$, with only the edge $v_1v_2$ being red. Let $X = \{v_3, v_4, v_5\}$, $Y = \{v_6, v_7, \ldots, v_{11}\}$ and apply the lemma. Since $\langle B \rangle$ contains no $K_6$, it follows that $Y(X, B) \neq \emptyset$, i.e., there is a vertex of $Y$ which is adjacent in $\langle B \rangle$ to at least two vertices of $X$. The same vertex must be adjacent in $\langle B \rangle$ to either $v_3$ or $v_4$, otherwise there is a red $K_5$. Consequently, we find a copy of $G_0$ in $\langle B \rangle$, thus completing the proof.

(b) With $V = Z_{11}$ let $(R, B)$ be the two-coloring of $[V]^2$ defined by setting, for $x \neq y$, $xy \in R$ if $x - y = \pm 1$ or $\pm 3 \pmod{11}$. It is easy to see that $\langle R \rangle$ contains no $K_3$. We now wish to demonstrate that $\langle B \rangle$ contains none of the graphs $G_1$ through $G_6$. Since $G_5$ is a subgraph of $G_1$ through $G_6$, it suffices to prove that $\langle B \rangle$ contains neither $G_3$ nor $G_6$. However, if $\langle B \rangle$ did contain $G_5$, it would, in fact, contain $G_3$ since neither of the two triangles in $G_5$ are in $\langle R \rangle$. Thus, we need only demonstrate that $\langle B \rangle$ contains neither $G_3$ nor $G_6$. If $\langle B \rangle$ did contain either $G_3$ or $G_6$, there would be a set of four vertices, $X$, which spans $K_4$ in $\langle B \rangle$ and two vertices $u, v \in X$ which are adjacent in $\langle B \rangle$ and such that $X_6(u) \cup X_6(v) = X$. Note that $\langle B \rangle$ is vertex symmetric and that if $X$ spans $K_4$ in $\langle B \rangle$, then $X$ is of
the form \( \{x, x+2, x+4, x+6\} \) (mod 11). Thus, we may confine our attention to the case \( X = \{0, 2, 4, 6\} \). For each \( v \notin X \) we list \( X_B(v) \) below.

\[
\begin{array}{c|c}
  v & X_B(v) \\
  \hline
  9 & \{0, 2, 4\} \\
  10 & \{4, 6\} \\
  1 & \{6\} \\
  3 & \emptyset \\
  5 & \{0\} \\
  7 & \{0, 2\} \\
  8 & \{2, 4, 6\}
\end{array}
\]

For every pair \( u, v \notin X \) such that \( X_B(u) \cup X_B(v) = X \), we find that \( uv \notin B \). Thus, \( B \) contains neither \( G_3 \) nor \( G_6 \).

We now wish to prove that \( K_{12} \to (K_3, G_1) \). Since \( G_2 \) through \( G_6 \) are subgraphs of \( G_9 \), this will complete the proof of (b). With \( V = \{v_1, v_2, \ldots, v_{12}\} \) let us suppose that there exists a two-coloring \((R, B)\) of \([V]^2\) such that \((R)\) contains no \( K_3 \) and \((B)\) contains no \( G_1 \). In view of the corollary to the lemma, we may assume that \((B)\) contains no \( K_3 \). On the other hand, \((B)\) must contain \( K_5 - x \). Thus, as in the proof of (a), we may assume that \( \{v_1, v_2, \ldots, v_3\} \) induces \( K_4 - x \) in \((B)\), with only the edge \( v_1v_2 \) being red. Let \( X \) denote the set \( \{v_0, v_1, \ldots, v_{12}\} \) and form the partition \( X = (X_{RR}, X_{RB}, X_{BR}, X_{BB}) \) according to whether the pair of edges \( (xv_1, xv_2) \) is in \( R \times R, R \times B, B \times R, \) or \( B \times B \). Of course \( X_{RR} = \emptyset \) since there is no red \( K_5 \). Also, \( |X_{RB}| \) and \( |X_{BR}| \) are both \( \leq 3 \) since \((B)\) contains no \( K_5 \). It follows that \( X_{BB} = \emptyset \). If for some \( x \in X_{BB} \) any one of the edges \( xv_3, xv_4, \) or \( xv_5 \) is in \( B \), then \((B)\) contains \( G_1 \), contrary to our assumption. Therefore, we must assume that all three of the edges belong to \( R \). If, for some \( x \) in \( X_{RB} \) or \( X_{RR} \), all three of the edges \( xv_3, xv_4, \) and \( xv_5 \) belong to \( R \), then \((B)\) contains \( K_5 - P_3 \). Since we must reject this possibility, and in view of the fact that there is no red \( K_5 \), we see that every vertex in \( X_{BB} \) is adjacent in \((B)\) to every other vertex of \( X \). Thus \( X_B(v_1) \) and \( X_B(v_2) \) span complete graphs in \((B)\). Now either \( |X_B(v_1)| \) or \( |X_B(v_2)| \) is at least 4 and so we find a \( K_5 \) in \((B)\), producing a contradiction and so completing the proof.

(c) With \( V = \{v_1, v_2, \ldots, v_{12}\} \) consider the two-coloring of \([V]^2\) in which \((R)\) is the graph shown in Figure 2. It is easily seen that \((R)\) contains no \( K_5 \). We now wish to demonstrate that \((B)\) contains no \( G_6 \), and hence no \( G_5 \). Note that every vertex of \( G_6 \) is of degree four. Thus, if \((B)\) did contain a copy of \( G_6 \) there would be a choice of six vertices from \( V \) such that the graph induced in \((R)\) by these six vertices has no vertex of degree two or more. Consider the sets \( X_1 = \{v_1, v_2, v_3, v_4\} \), \( X_2 = \{v_5, v_6, \ldots, \} \), and \( X_3 = \{v_7, v_8, \ldots, \} \). If \( v_5 \) is adjacent to \( v_1 \) and \( v_5 \) is adjacent to \( v_2 \), then \((B)\) contains \( G_6 \), contradicting our assumption. Thus, there must exist a \( v \in X_{BB} \) such that \( v \) is adjacent to \( v_1 \) and \( v \) is adjacent to \( v_2 \). Since \( v \) is adjacent to \( v_3 \), \( v \) is adjacent to \( v_4 \), and \( v \) is adjacent to \( v_5 \), \((B)\) contains \( G_6 \), producing a contradiction and so completing the proof.
\{v_5, v_6, v_7, v_8\} and \(X_2 = \{v_9, v_{10}, v_{11}, v_{12}\}\) and note that we must choose at most two, and so exactly two, vertices from each of these sets. Suppose that \(v_1\) is chosen from \(X_1\). Then \(v_7\) and \(v_8\) must be chosen from \(X_2\) and \(v_{10}\) and \(v_{12}\) must then be chosen from \(X_3\). This leaves no possible choice for the second vertex from \(X_1\). Now by symmetry the only other case to consider is where \(v_2\) and \(v_4\) are chosen from \(X_1\) and \(v_9\) and \(v_{11}\) are chosen from \(X_3\). This leaves no viable choice for the two vertices from \(X_2\) and thus shows that \((B)\) contains no \(G_6\).

The proof of (c) will be completed by showing that \(K_{13} \rightarrow (K_5, G_7)\). Let \(V = \{v_1, v_2, \ldots, v_5\}\) and suppose that there exists a two-coloring \((R, B)\) of \(|V|^2\) such that \((R)\) contains no \(K_5\) and \((B)\) contains no \(G_7\). It is known that there is a unique two-coloring of \(|V|^2\) which avoids both a \(K_5\) in \((R)\) and a \(K_4\) in \((B)\) [4] and we note that in this particular two-coloring \((B)\) contains \(G_7\). Thus, in the two-coloring whose existence we have assumed, we must admit that \((B)\) contains \(K_5\). Thus, without loss of generality, we may suppose that \(X = \{v_1, v_2, \ldots, v_5\}\) spans \(K_5\) in \((B)\). Let \(Y = V - X\) and recall from the lemma the fact that \(Y(X, R)\) spans a complete graph in \((B)\). It follows that \(|Y(X, R)| \leq 5\) and so \(|Y(X, B)| \geq 3\). Since \((B)\) contains no \(K_5 - x, y\) each vertex in \(Y(X, B)\) is adjacent to precisely three vertices of \(X\) in \((B)\). Observe that if \(u, v \in Y(X, B)\) then \(X_R(u) \cap X_R(v)\) is either 0 or 2, for if \(X_R(u) \cap X_R(v) = \{w\}\), then \(u\) and \(v\) together with \(X - w\) span \(G_7\) in \((B)\). In view of the fact that \(|Y(X, B)| \geq 3\), there is no loss of generality in assuming that \(X_R(v_6) = X_R(v_7) = \{v_1, v_2\}\) and that \(X_R(v_9) = \{v_3, v_4\}\). Observe that \(X' = \{v_3, v_4, \ldots, v_5\}\) also spans \(K_5\) in \((B)\). Let \(Y' = V - X'\) and note that \(v_3\) and \(v_4\) are elements of \(Y'(X', B)\). Let \(v\) be any one of the vertices \(v_3\) through \(v_{13}\). There are three cases to consider.

(i) If \(v \in Y(X, B)\) then \(X_R(v) = X_R(v_9) = \{v_3, v_4\}\). (ii) If \(v \in Y'(X', B)\) then \(X_R(v) = X_R(v_7) = \{v_1, v_2\}\) and so must contain either \(v_3\) or \(v_4\). Also, a second vertex \(w \in Y'(X', B)\), if it exists, must satisfy \(X_R(w) = X_R(v)\). (iii) If \(v \in Y(X, R) \cap Y'(X', R)\) then, for the following reason, \(N_R(v)\) contains \(\{v_3, v_4, v_5\}\). Since \((R)\) contains no \(K_3, N_R(v)\) contains either \(\{v_1, v_2\}\) or \(\{v_6, v_7\}\). In either case, \(N_R(v)\) contains \(\{v_3, v_4, v_5\}\) since \(v\) is in both \(Y(X, R)\) and \(Y'(X', R)\). From (i)–(iii) we see that either \(N_R(v_1)\) or \(N_R(v_2)\) contains \(\{v_3, v_0, \ldots, v_{13}\}\). This gives a \(K_6\) in \((B)\) and so a contradiction.

(d) With \(V = Z_{13}\) let \((R, B)\) be the two-coloring of \(|V|^2\) defined by setting, for \(x \neq y\), \(xy \in R\) if \(x - y\) is a cubic residue \((\text{mod } 13)\). This is the unique two-coloring of \(|V|^2\) which avoids a \(K_5\) in \((R)\) and a \(K_4\) in \((B)\). Now \((B)\) contains none of the graphs \(G_6\) through \(G_{12}\). For \(G_6\) through \(G_{11}\) this is perfectly obvious, since each of these graphs contain \(K_5\). If \((B)\) did contain \(G_{12}\), it would, in fact, have to contain \(G_9\) since the triangle of \(G_{12}\) cannot be in \((R)\).
To see that $K_{14} \rightarrow (K_3, G)$ for $i = 9$ through 12 it is enough to note that in any two-coloring which avoids a $K_3$ in $\langle R \rangle$ there must be a $K_3$ in $\langle B \rangle$ and then apply the corollary to the lemma.

(e) With $V = GF(16)$ let $\langle R, B \rangle$ be the two-coloring of $|V|^2$ defined by setting, for $u \neq v, uv \in R$ if $u - v$ is a cubic residue of the field. This two-coloring may be thought of as being obtained from the famous three-coloring due to Greenwood and Gleason and used in their proof that $r(K_3, K_3, K_3) = 17$ [5]; just combine the last two colors into one. On this basis, it is immediately recognized that $\langle R \rangle$ contains no $K_5$, and $\langle B \rangle$ contains no $K_6$. We now wish to demonstrate that, indeed, $\langle B \rangle$ contains no $K_6 - x$. The graph $\langle R \rangle$ is edge symmetric, and on this basis, a search for $K_6 - x$ in $\langle B \rangle$ may be confined to taking an arbitrary pair of vertices $u, v$ with $uv \in R$ and then checking to see if the subgraph of $\langle B \rangle$ induced by $N_b(u) \cap N_b(v)$ contains $K_4$. As representatives of the nonzero cubic residues we take $x^3, x^3 + x^2, x^3 + x, x^3 + x^2 + x + 1, and 1$. The cubic nonresidues are $x, x^2, x + 1, x^2 + x, x^2 + x + 1, x^2 + x + 1, x^3 + x^2 + x, x^3 + x^2 + 1, and x^3 + 1$. We take $u = 0, v = 1$ and so find that $N_b(u) \cap N_b(v) = \{x, x + 1, x^2, x^2 + 1, x^2 + x, x^2 + x + 1\}$. Observing that the pairs $\{x, x + 1\}, \{x^2, x^2 + 1\}, \{x^2 + x, x^2 + x + 1\}$ are adjacent in $\langle R \rangle$, we see that there is no blue $K_4$ spanned by vertices in $N_b(u) \cap N_b(v)$. Thus $\langle B \rangle$ does not contain $K_6 - x$.

With $|V| = 17$ let $\langle R, B \rangle$ be an arbitrary two-coloring of $|V|^2$. Select an arbitrary vertex $v \in V$. If its degree in $\langle R \rangle$ is at least six, then either $\langle R \rangle$ contains $K_2$ or $\langle B \rangle$ contains $K_6$. Otherwise, the degree of $v$ in $\langle B \rangle$ is at least eleven and in the two-coloring of $[N_b(v)]^2$ there is either a red $K_5$ or a blue $K_6 - x$. Thus $K_{17} \rightarrow (K_3, K_6 - x)$.

(f) This is the well-known result from classical Ramsey theory [4, 6].

Finally, we need to verify that this theorem covers all connected graphs of order six. The first class (a) dealt with in the theorem consists of all those connected graphs of order six whose complements contain $P_4$. Otherwise, $G$ is a connected graph of order six such that the connected components of $\overline{G}$ are drawn from $K_{1,n}$ for $n = 1, 2, 3, 4$ and $K_3$. It is evident that classes (b) through (f) exhaust these possibilities.

References


