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Models for Unsupervised Learning

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MODELS FOR UNSUPERVISED LEARNING

by

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Dedication

To my wonderful family who encouraged me,

and in loving memory of my mother.

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Abstract

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Learning is a feature of living organisms which is crucial in the process of adaptation. It is understood that a stimulus triggers a chain of neurophysiological reactions in an organism, and as a consequence we say that the organism is learning from that initial exposure. Several researchers have dealt with understanding and modeling learning through mathematical systems. Due to the complexity of the brain, scholars reduced the problem to a simpler mechanism consisting of neurons, which are processing additive units, interconnected with pathways, called synapses. A primary goal was to derive a system of equations that captures the changes synaptic parameters undergo in a learning process, and identify stimuli that generates a flow of changes that will converge over time. This would represent a stable reflection of the learning process.

My dissertation explores generalizations of models for unsupervised learning proposed in literature. The first model is due to Oja and Karhunen and reflects the changes of a network connecting weights or synaptic parameters following a Hebbian principle and incorporating a forgetting term to allow convergence. The second model, due to Cox and Adams, generalizes the Oja-Karhunen model by introducing errors in the learning process.

These paradigms are presented as systems of differential equations explored in three settings:

1. The finite dimensional Euclidean space over the reals;
2. The infinite dimensional Hilbert space of square summable sequences equipped with the standard inner product; and

3. The infinite dimensional Banach space of bounded operators on a separable complex Hilbert space.

In each setting the existence and uniqueness of local or global solutions is well established, a form for solutions is derived, and the asymptotic behavior is determined. In the third setting we use the polar factorization of operators to decompose the system into two components where an explicit form for solutions is given.

In the Cox-Adams model we also explore the impact of the error factor in the long-term behavior of the solutions.

Contents

1 Introduction

Learning is a process an organism undergoes in order to acquire knowledge and gain skill for future interaction with the environment. There are two main types of learning listed in literature: "supervised" and "unsupervised" learning. In a supervised learning environment, the organism is guided and tested through the learning process, while in an unsupervised learning no control or testing is available. In the former, the organism is trained with a collection of desirable input-output. Even though in the latter no clear training occurs, but rather an exposure to stimuli eventually will lead the organism to a better adaptation.

In this work we deal with models for unsupervised learning. Such a model is either a continuous or a discrete system of equations, which follows rules that are both biologically motivated and also constrained by a possible implementation in an actual device, known as an "artificial neural network". A network architecture consists of processing units, also designated as neurons, interconnected with pathways (or synapses) through which information can flow. When a stimulus excites the neurons of a network, it generates a series of signals that propagates through the network's pathways as quantifiable information. This process discovers significant patterns of a data set and it entails a variety of changes of the network's internal parameters, as for example, the network connecting weights. The aim is to define a converging algorithm that performs the network's adaptation without any outside control. If convergence occurs, then the network is fully characterized and ready to perform as an educated device.

Several researchers have proposed systems that perform unsupervised learning, see Amari [3], Haykin [30], or Hertz, [31].

We render an abbreviated (simplified) development of the background for the systems we address in this dissertation, as shown in Botelho and Jamison [9].

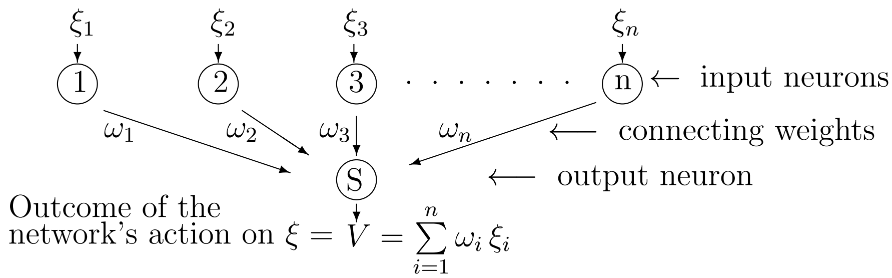

Figure 1

Figure 1 illustrates the structure of a feed forward neural network with n input neurons which are connected to a single output neuron S. The inputs into respective neurons, $\{\xi_i\}_i$, are randomly drawn from a probability distribution. The term ω_i represents the weight (or strength) of the connection between the input neuron i and the output neuron S. Information changes as it travels through each synapse, via multiplication by that synapse's respective connecting weight. All the altered inputs arriving at neuron S are combined defining the output component

$$V = \sum_{i=1}^{n} \omega_i \, \xi_i. \tag{1.1}$$

Hebbian learning [30, see Haykin] sets the rate of change of a weight expected value $\langle \omega_i \rangle$ (denoted by ω_i for simplicity) as proportional to the product of the pre- and post- synaptic activity, $\dot{\omega}_i = \alpha \, V \xi_i$, with some constant α. For convenience we introduce the following time rescaling: Let $\tau = \alpha \, t$ (then $\dot{\tau} = \alpha$) and

$$\overline{\omega_i}(\tau) = \omega_i \left(\frac{\tau}{\alpha} \right) = \omega_i(t).$$

Then

$$\dot{\omega}_i(t) = \frac{d}{d\tau} [\overline{\omega_i}(\tau)] \frac{d\tau}{dt} = \frac{d}{d\tau} [\overline{\omega_i}(\tau)] \, \alpha.$$

Thus

$$\frac{d}{d\tau}[\overline{\omega_i}(\tau)] = \frac{1}{\alpha}\,\dot{\omega}_i(t) = \frac{1}{\alpha}(\alpha V \xi_i) = V\xi_i.$$

So we may adjust the Hebbian learning rule to reflect this time rescaling:

$$\dot{\omega}_i = V\xi_i\,.$$

However, we see that such an equation leads to exponential growth and synaptic saturation, since, for example, if n = 1, then

$$\dot{\omega}_1(t) = V\xi_1 = \omega_1\xi_1\xi_1 = \omega_1\xi_1^2.$$

Thus

$$\frac{\dot{\omega}_1(t)}{\omega_1(t)} = \xi_1^2,$$

and $ln|\omega_1(t)| = \xi_1^2 t + C$. Therefore $\omega_1(t) = e^{\xi_1^2 t + C} = e^{\xi_1^2 t}e^C$, which tends to infinity as t tends to infinity. To alleviate this problem, in 1988 Kohonen [37] introduced a quadratic "forgetting" term, $-V^2\omega_i$, which limits the synaptic weight growth. This forgetting term is incorporated in the learning rule proposed by Oja [41],

$$\dot{\omega}_i = V(\xi_i - V\omega_i) = V\xi_i - V^2\omega_i. \tag{1.2}$$

To illustrate that this forgetting term actually resolves the non-convergence issue, we revisit the one input network case,

$$\dot{\omega}_1 = V\xi_1 - V^2\omega_1 = (\omega_1\xi_1)\xi_1 - (\omega_1\xi_1)^2\omega_1 = \omega_1\xi_1^2 - \omega_1^3\xi_1^2.$$

$$\omega_1^{-3}\dot{\omega}_1 = \omega_1^{-2}\xi_1^2 - \xi_1^2.$$

Set $v = \omega_1^{-2}$, then $\dot{v} = -2\omega_1^{-3}\dot{\omega}_1$, or equivalently, $\frac{1}{-2}\dot{v} = \omega_1^{-3}\dot{\omega}_1$. Thus we solve the equation

$$\frac{\dot{v}}{-2} - v\xi_1^2 = -\xi_1^2.$$

$$\dot{v} + 2v\xi_1^2 = 2\xi_1^2.$$

$$\dot{v}e^{2\,\xi_1^2\,t} + (2\,v\xi_1^2)e^{2\,\xi_1^2\,t} = (2\,\xi_1^2)e^{2\,\xi_1^2\,t}.$$

Integrating, we find

$$v\,e^{2\,\xi_1^2\,t} = e^{2\,\xi_1^2\,t} + C.$$

Hence

$$\omega_1 = \frac{1}{\sqrt{1 + Ce^{-2\,\xi_1^2\,t}}}\,,$$

which converges to one as t tends to infinity, and thus eventually becomes uniformly asymptotically stable.

We then consider a Hebbian learning scheme with a quadratic forgetting term as in (1.2), $\dot{\omega}_i = V\xi_i - V^2\omega_i$. Substituting the input value (1.1), $V = \sum_{i=1}^{n} \omega_i\,\xi_i$, into equation (1.2), we obtain

$$\dot{\omega}_i = \sum_{j=1}^{n} \omega_j\xi_j\xi_i - \left(\sum_{j=1}^{n} \omega_j\xi_j\right)^2 \omega_i = \sum_{j=1}^{n} \omega_j\xi_j\,\xi_i - \sum_{j=1}^{n}\sum_{k=1}^{n} \omega_j\xi_j\,\xi_k\omega_k\,\omega_i. \tag{1.3}$$

We set M to be the symmetric input correlation matrix whose (i j) entry equals the expected value $\langle \xi_i\,\xi_j \rangle$ of the correlation between the inputs into neurons i and j, respectively. Let Ω denote the $1 \times n$ row matrix $[\omega_1\ \omega_2\ \cdots\ \omega_n]$ which represents the connecting weights associated with the network's connections to the output neuron S. Therefore we may convert equation (1.3) to the matrix representation, $\dot{\Omega} = \Omega M - \Omega M \Omega^T \Omega$, or equivalently, $\dot{\Omega}^T = M^T\Omega^T - \Omega^T\Omega M^T \Omega^T$. Since M is symmetric, this becomes $\dot{\Omega}^T = M\Omega^T - \Omega^T\Omega M\Omega^T$. Hence for $Z = \Omega^T$, we have $\dot{Z} = MZ - ZZ^TMZ$, known as the the Oja and Karhunen model of unsupervised learning (see [43]). This model generalizes the standard Hebbian learning rule for a single neuron utilizing principal component analysis, and is represented by the system of differential equations (2.1) (on p. 11):

$$\begin{cases} \dot{Z} = M\,Z \;-\; Z\;Z^T\,M\,Z \\ Z(0) = Z_0, \end{cases}$$

with Z a time dependent column vector in $\mathbb{R}^n$, Z^T a row vector equal to the transpose of Z, and M a symmetric $n \times n$ matrix with real entries. This system traces the evolution of the connecting weights of a network consisting of n input neurons linked to a single output neuron. We refer the reader to Ham [28], Haykin [30], or Hertz [31] for other interpretations of the model.

We also consider system

$$\begin{cases} \dot{Z}_E = E\,M\,Z_E \;-\; Z_E\;Z_E^T\;M\,Z_E \\ Z(0) = Z_0, \end{cases}$$

where the matrix E is a symmetric, positive definite (p. 26), $n \times n$ matrix. In the Cox-Adams model [1, 44], E represents an error matrix. One example of such a matrix is the non-singular tri-diagonal matrix $[e_{ij}]_{ij}$ given by

$$e_{ij} = \begin{cases} 1 - \varepsilon & \text{if } i = j \\ \varepsilon/2 & \text{if } |i - j| = 1 \\ 0 & \text{otherwise}, \end{cases}$$

where ε denotes the probability of the formation of a temporary synapse. We refer the reader to Botelho and Jamison [11] for a detailed interpretation of the Cox-Adams model for a network with n input neurons and a single output neuron. For interesting examples of the tridiagonal self-adjoint operators, see Dombrowski [19, 20] or Duren [22, 23].

In this work, we derive explicit global solutions, and analyze stability, for both the Oja-Karhunen system and the Cox-Adams system of differential equations in several settings. We extend an important result by Oja that qualifies system (2.1) as a statistical principal component analyzer. The actual knowledge of the solutions not only permits a detailed analysis of the long term behavior of the system but also information of the history of the network evolution.

We also develop a technique based on the polar decomposition of operators to solve our system explicitly. This way we construct the decomposition of a solution as the product of a partial isometry and a positive operator (p. 16). Hence we obtain an explicit form for the solution to investigate, and to characterize both the long term, or asymptotic, behavior, as well as information about the previous history.

We summarize the content of each chapter and state our major contributions.

In **Chapter 2** we derive an explicit form for global solutions of system (2.1), ***Theorem 2.1*** *(see p. 11): Let Z_0 be a nonzero vector in $\mathbf{R}^n$ with norm $\|Z_0\| \leq 1$ and M a symmetric $n \times n$ matrix. Then the system*

$$\begin{cases} \dot{Z} = M\,Z \,-\, Z\,Z^T\,M\,Z \\ Z(0) = Z_0 \end{cases}$$

has a unique global solution.

We use the symmetry of M and then the existence of an orthogonal matrix P that diagonalizes M to set a change of variables that reduces the given system to (see equation (2.3) on p. 13)

$$\begin{cases} \dot{\omega}_i = \lambda_i \omega_i - \omega_i \sum_{j=1}^{n} \lambda_j \omega_j^2 \\ \omega_i(0) = \alpha_i, \quad \text{the i-the coordinate of } P^T Z_0, \ \ for\ i = 1, \ldots, n, \end{cases}$$

with λ_i an eigenvalue of M, and W is a column vector in $\mathbb{R}^n$; the i^{th} entry given by (see equation (2.7) on p. 15)

$$\omega_i = \frac{\alpha_i e^{\lambda_i t}}{\sqrt{1 + \sum_{j=1}^{n} \alpha_j^2 \left(e^{2\lambda_j t} - 1\right)}}, \ \forall\, i = 1, \cdots, n,$$

with $\alpha_i = \omega_i(0)$ representing the i^{th} coordinate of the initial condition $P^T Z_0$.

We then characterize the stability behavior and determine that the omega-limit set of a solution $Z(t)$ consists of a single vector in the unit n-ball:

Proposition 2.8 *(p. 17): Let M be a symmetric $n \times n$ matrix and Z_0 a non-trivial vector in R^n such that $\|Z_0\| \leq 1$. If Z(t) is a solution to system (2.2), then Z_∞ consists of a single vector in the unit $n-$ball.*

In **Chapter 3** we consider the Oja-Karhunen model acting on an infinite dimensional space, ℓ_2, the Hilbert space of all square summable sequences equipped with the standard inner product. In this new setting we derive the explicit form for global solutions, and we analyze their convergence properties, see Theorem 3.3, p. 20. We find that for every t,

$$Z(t) = \sum_i \frac{z_i^0\, e^{\lambda_i t}}{\sqrt{1 + \sum_j (z_j^0)^2 (e^{2\lambda_j t} - 1)}}\, u_i,$$

where $\{u_i\}_i$ is an orthonormal countable basis of eigenvectors corresponding to the real eigenvalues $\{\lambda_i\}$ of M. We also show that the $\omega-$limit set of the solution consists of a single vector in ℓ_2.

In **Chapter 4** we propose a class of admissible error matrices that allows a reduction of the Cox-Adams model to the Oja-Karhunen model. This new scheme relies on a partial transfer of randomness from E to the input correlation matrix M. Consequently, standard methods also apply to system (4.1) (see p. 26), and an explicit form for the solutions can be derived. The next theorem establishes the existence of maximal solutions in a particular case:

Theorem 4.5 *(p. 33): Let Z_0 be a vector in $\mathbb{R}^n$ such that $\|Z_0\| \leq 1$. If both E and M are positive definite symmetric, commuting, $n \times n$ matrices, then there exists t_0 such that*

$$\begin{cases} \dot{Z}_E = E\,M\,Z_E \;-\; Z_E\;Z_E^T\;M\,Z_E \\ Z_E(0) = Z_0 \end{cases}$$

has a unique maximal solution $Z_E(t)$, defined for $t > t_0$.

We establish that the solution $Z_E(t) = PW_E(t)$, with $W_E(t) = [\omega_i^E]_{i=1,\dots n}$,

$$\omega_i^E(t) = \frac{\alpha_i\, e^{\mu_i t}}{\sqrt{1 + \sum\limits_j \left(\frac{\lambda_j}{\mu_j}\right) \alpha_j^2\, (e^{2\mu_j\, t} - 1)}},$$

where $\{\lambda_i\}$ represents the eigenvalues of M and $\{\mu_i\}$ represent the eigenvalues of the matrix E.

We then discuss the impact of temporary connections on the overall network evolution and asymptotic behavior.

In order to detect the directional impact of the error matrix on the asymptotic behavior of solutions we study the variation of θ_E, the angle between the vectors W and W_E, which is equal to the angle between Z and Z_E.

$$\lim_{t\to\infty} \cos(\theta_E(t)) = \frac{\sum\limits_{i=max\{i_0,j_0\}}^{n} \alpha_i^2}{\sqrt{\sum\limits_{i=i_0}^{n} \alpha_i^2 \sum\limits_{i=j_0}^{n} \alpha_i^2}}.$$

If $i_0 = j_0$ then

$$\lim_{t\to\infty} \cos(\theta_E(t)) = 1.$$

At this point, we can conclude that whenever the eigenvalues of M and EM are positive, and if the multiplicities of the largest principal eigenvalues of M and EM are equal, then the error factor E has no directional impact in the flow.

In Chapter 5 we consider a generalization of system (2.1) to the infinite dimensional setting, the Banach space $\mathcal{B}(\mathcal{H})$ of bounded operators on a separable Hilbert space. We utilize the polar decomposition of operators that allows us to derive a "scalar" system and a "polar" system associated with the original system. Both systems are solved explicitly. These two solutions combined define the local solution for (2.1), given certain mild constrains on the initial conditions. The explicit form for local solutions is used to derive the existence of global solutions and for the stability analysis:

Theorem 5.34 *(see p. 76): If Z_0 is invertible and commutes with the normal operator M, then there exists $\varepsilon > 0$ and a unique differentiable mapping*

$Z : (-\varepsilon, \varepsilon) \to \mathcal{B}(\mathcal{H})$ such that

$$\dot{Z} = M\,Z \;-\; Z\,Z^*\,M\,Z \text{ and } Z(0) = Z_0$$

if and only if $Z(t) = \sqrt{V(t)}\, P(t)$ *with*

$$V(t) = \left[Id + (V_0^{-1} - Id)e^{-(M+M^*)t}\right]^{-1} \;\; \textit{and } P(t) = e^{\int_0^t -\frac{1}{2}(M-M^*)(V(\xi)-Id)d\xi} P_0.$$

We also consider the Cox-Adams system with E representing an invertible, self-adjoint, positive operator on $\mathcal{H}$ and M a self-adjoint operator on $\mathcal{H}$. The operator valued, time dependent Z_E now represents the continuous change of connecting weights. We present a scheme that explicitly solves this system. First a natural change of variables reduces the Cox-Adams system to a system where no synaptic formation occurs. However the probabilistic effect transfers to the input correlation operator M. This system reduces to an Oja type model. As a result we have the following corollary:

Corollary 5.42 *(p. 85): Let E be an invertible, positive, self-adjoint operator. If* Z_0 *is an invertible operator in* $\mathcal{B}(\mathcal{H})$, M *a self-adjoint operator that commutes with* Z_0*, the elements of the spectrum of* M *are strictly positive, and* $\|\sqrt{E}(Z_0\, Z_0^*)^{-1}\sqrt{E} - I\| < 1$, *then there exists* $\epsilon > 0$ *so that*

$$I + \left[\sqrt{E}(Z_0\, Z_0^*)^{-1}\sqrt{E} - I\right]\, exp\left(-2\sqrt{E}M\sqrt{E})\, t\right)$$

is invertible on the interval $(-\varepsilon, \infty)$ *and*

$$\lim_{t\to\infty} \left[I + \left[\sqrt{E}(Z_0\, Z_0^*)^{-1}\sqrt{E} - I\right]\, exp\left(-2\sqrt{E}M\sqrt{E})\, t\right)\right] = I.$$

Furthermore (p. 85) we observe that

$$\lim_{t\to\infty} Z(t) = P_0.$$

This provides a filtering procedure that selects the polar component of the initial condition.

2 Models for Unsupervised Learning

In this chapter we consider the following system proposed by Oja and Karhunen for the task of unsupervised learning:

$$\begin{cases} \dot{Z} = M\,Z \;-\; Z\;Z^T\,M\,Z \\ Z(0) = Z_0. \end{cases} \tag{2.1}$$

Z is an $n-$column vector with real entries, Z^T is the transpose of Z and M is a symmetric matrix. The time dependent vector Z represents the evolution of the connecting weights of a network from an initial stage Z_0.

A convenient change of variables allows us to decouple the linearized system and find an explicit form for the solution. We then establish conditions for the existence of global solutions. We investigate the long term stability of this system and also its dependence on the initial conditions.

We note that the stability behavior of solutions reflects the learning performed by the system [41, Oja]. We show that, from certain initial conditions, the system evolves to a single vector of connecting weights and, consequently, the learning process stabilizes. The network is then said to emerge as an educated device.

2.1 Oja-Karhunen model: Existence of Global Solutions

We start by establishing the existence and uniqueness of solutions as stated in the following theorem.

Theorem 2.1. *Let Z_0 be a nonzero vector in $\mathbf{R}^n$ with norm $\|Z_0\| \leq 1$ and M a symmetric $n \times n$ matrix. Then the system*

$$\begin{cases} \dot{Z} = M\,Z \;-\; Z\;Z^T\,M\,Z \\ Z(0) = Z_0 \end{cases} \tag{2.2}$$

has a unique global solution.

We first state a well-known result regarding symmetric matrices:

Proposition 2.2. *(Bronson [15, p. 419]) For every $n \times n$ real symmetric matrix A there exists an $n \times n$ real orthogonal matrix P such that $P^T AP = D$, where D is a diagonal matrix.*

Proof of Theorem 2.1: Since the matrix M is symmetric, M is similar to a diagonal matrix J; the diagonal entries are the eigenvalues of M, see Anton [4, p. 277]. Therefore there exists an $n \times n$ orthogonal matrix P (i.e. $P^T P = PP^T = I$) satisfying $P^T MP = J$. Thus system (2.2) becomes

$$\begin{cases} \dot{Z} = (PJP^T)\,Z \;-\; Z\;Z^T\,(PJP^T)\,Z \\ Z(0) = Z_0. \end{cases}$$

We set $W = P^T Z$, and (2.2) is equivalent to

$$\begin{cases} \dot{W} = (I - WW^T)JW, \\ W(0) = PZ_0. \end{cases}$$

Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ denote the eigenvalues of M, and W= $[\omega_i]_{i=1,\ldots,n}$ is a column vector in $\mathbb{R}^n$, where the i^{th} component satisfies

$$\dot{\omega}_i = \sum_{j=1}^{n} (I - WW^T)_{ij}(JW)_j.$$

$$
(I - WW^T)_{ij} = \begin{cases} 1 - \omega_i^2 & \text{if } i = j \\ -\omega_i \omega_j & \text{if } i \neq j, \end{cases}
$$

we have

$$
\begin{cases} \dot{\omega}_i = \lambda_i \omega_i - \omega_i \sum_{j=1}^{n} \lambda_j \omega_j^2 \\ \omega_i(0) = \alpha_i, \text{ the i-the coordinate of } P^T Z_0, \ for \ i = 1, \cdots, n. \end{cases} \tag{2.3}
$$

The existence and uniqueness theorem for differential equations implies that ω_i is the constant function equal to zero, if $\alpha_i = 0$ (see Hartman [29, p. 46]).

We assume $\alpha_i \neq 0$ for every $i = 1, \cdots n$. Thus for t in a small neighborhood of zero, $\omega_i(t) \neq 0$. Therefore (2.3) can be written as

$$
\frac{\dot{\omega}_i}{\omega_i} = \lambda_i - \sum_{j=1}^{n} \lambda_j \omega_j^2, \ \ i = 1, \ldots, n,
$$

then

$$
\frac{\dot{\omega}_i}{\omega_i} - \lambda_i = \frac{\dot{\omega}_1}{\omega_1} - \lambda_1, \ i = 2, \cdots, n.
$$

We employ standard integration techniques to derive that

$$
ln|\omega_i| - \lambda_i \, t = ln|\omega_1| - \ \lambda_1 \, t + C,
$$

which is equivalent to

$$
|\omega_i| \, e^{-\lambda_i \, t} = |\omega_1| \, e^{-\,\lambda_1 \, t} \, e^C, \ \ for \ i = 2, ..., n.
$$

We denote the sign of ω_i by $sgn(\omega_i)$, and set $K = sgn(\omega_i)\, sgn(\omega_1)\, e^C$, then

$$
\omega_i \, e^{-\lambda_i \, t} = K \, e^{-\,\lambda_1 \, t} \, \omega_1. \tag{2.4}
$$

Hence for $t=0$, $K=\frac{\alpha_i}{\alpha_1}$, and (2.4) becomes

$$\omega_i\, e^{-\lambda_i\, t} = \frac{\alpha_i}{\alpha_1}\, e^{-\,\lambda_1\, t}\, \omega_1.$$

Consequently we obtain,

$$\omega_i = \frac{\alpha_i}{\alpha_1} e^{(\lambda_i-\lambda_1)t}\omega_1. \tag{2.5}$$

We substitute (2.5) into (2.3) to get the following Euler equation:

$$\dot{\omega}_1(t) - \lambda_1\omega_1 = -\omega_1^3 \sum_{j=1}^{n} \lambda_j \left(\frac{\alpha_j}{\alpha_1}\right)^2 e^{2(\lambda_j-\lambda_1)\,t}. \tag{2.6}$$

If we set $v=\omega_1^{-2}$, then $\dot{v} = -2\omega_1^{-3}\dot{\omega}_1$, and thus (2.6) becomes

$$\dot{v} + 2\lambda_1 v = 2\sum_{j=1}^{n} \lambda_j \left(\frac{\alpha_j}{\alpha_1}\right)^2 e^{2(\lambda_j-\lambda_1)t}.$$

Therefore

$$v e^{2\lambda_1 t} = \sum_{j=1}^{n} \left(\frac{\alpha_j}{\alpha_1}\right)^2 e^{2\lambda_j t} + K, \text{ with } K \text{ constant,}$$

and hence

$$\omega_1^{-2} e^{2\lambda_1 t} = \sum_{j=1}^{n} \left(\frac{\alpha_j}{\alpha_1}\right)^2 e^{2\lambda_j t} + \frac{1}{\alpha_1^2} - \sum_{j=1}^{n} \left(\frac{\alpha_j}{\alpha_1}\right)^2.$$

This implies

$$\omega_1 = \frac{\alpha_1\, e_1^{\lambda} t}{\sqrt{1 - \sum\limits_{j=1}^{n} \alpha_j^2 + \sum\limits_{j=1}^{n} \alpha_j^2\, e^{2\lambda_j\, t}}}.$$

We substitute the expression above into (2.5) to derive that

$$\omega_i = \frac{\alpha_i}{\alpha_1} e^{(\lambda_i-\lambda_1)t} \frac{\alpha_1 e^{\lambda_1 t}}{\sqrt{1 - \sum\limits_{j=1}^{n} \alpha_j^2 - \sum\limits_{j=1}^{n} \alpha_j^2 e^{2\lambda_j t}}}.$$

We have

$$\omega_i = \frac{\alpha_i e^{\lambda_i t}}{\sqrt{1 + \sum_{j=1}^{n} \alpha_j^2 \left(e^{2\lambda_j t} - 1\right)}}, \quad \text{for } i = 1, \cdots, n. \tag{2.7}$$

We now conclude that if there is a solution, it must be unique and given as in (2.7). We observe that the orthogonality of P implies that $\|W_0\| = \|Z_0\|$. Since $e^{2\lambda_j t} > 0$ and $\sum_{j=1}^{n} \alpha_j^2 \leq 1$, we have that ω_i is well defined for all t. Clearly, at $t = 0$:

$$\omega_i = \frac{\alpha_i}{\sqrt{1 - \sum_{j=1}^{n} \alpha_j^2 (1-1)}} = \frac{\alpha_i}{1} = \alpha_i.$$

We also conclude that $1 + \sum_{j=1}^{n} \alpha_j^2 \left(e^{2\lambda_j t} - 1\right) > 0$, for all t. It is straightforward to check that ω_i, as defined above, satisfies (2.3). Since $W = P^T Z$, with P an orthogonal matrix, $Z(t) = PW(t)$ is a global solution of (2.2) such that $Z(0) = Z_0$. This completes the proof. $\square$

We have shown that for every i, ω_i is well defined for all t, provided that the initial condition Z_0 is in the unit ball, i.e. $\|Z_0\| \leq 1$. We now determine whether there are initial conditions outside the unit ball under which ω_i is defined for $t \geq 0$, equivalently we determine constraints such that

$$1 + \sum_{j=1}^{n} \alpha_j^2 \left(e^{2\lambda_j t} - 1\right) > 0, \text{ for } t \geq 0.$$

Without loss of generality, we assume $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$.

For $t > 0$, we have that $e^{2\lambda_j t} - 1 \geq e^{2\lambda_1 t} - 1$. If

$$\sum_{j=1}^{n} \alpha_j^2 (e^{2\lambda_1 t} - 1) > -1, \text{ then } 1 - \sum_{j=1}^{n} \alpha_j^2 + \sum_{j=1}^{n} \alpha_j^2 e^{2\lambda_j t} > 0.$$

Remark 2.3. *We find that $\omega_i(t)$ is well-defined under the following conditions:*

(a) If $\sum_{j=1}^{n}\alpha_j^2 \leq 1$, *then* $\omega_i(t)$ *is defined for every* $t \geq 0$.

(b) If $\sum_{j=1}^{n}\alpha_j^2 > 1$ *and* $\lambda_1 > 0$, *then* $\omega_i(t)$ *is defined for every*

$t \geq \frac{1}{2\lambda_1} ln \left(1 - \frac{1}{\sum_j \alpha_j^2}\right)$, *in particular for* $t \geq 0$.

We are interested in maximal solutions defined for $t \geq 0$ *since for those we investigate their asymptotic behavior in a forthcoming section.*

Now we recall the definition positive semi-definite matrix.

Definition 2.4. *Matrix M is* **positive, or positive semi-definite**, *if* $x^T M x \geq 0$ *for every x in* $\mathbb{R}^n$.

Corollary 2.5. *If M is a positive semi-definite* $n \times n$ *matrix and* Z_0 *a vector in* $\mathbb{R}^n$, *then there exists* $t_0 < 0$ *such that system (2.2) has a unique maximal solution defined for* $t > t_0$.

Proof: Since M is positive semi-definite, $x^T M x \geq 0$ for every x in $\mathbb{R}^n$. Thus $x^T \lambda x = \lambda \|x\|^2 \geq 0$, where λ is the eigenvalue associated with the eigenvector x. Since x is not the zero vector, we must have that $\lambda \geq 0$. Hence all the eigenvalues of M are nonnegative. The result follows from the Remark 2.3. □

Remark 2.6. *There is a broad spectrum of important matrices known to be positive semi-definite. It is of interest to mention two important classes of such matrices.*

1. *Let* $B = [b_{ij}]_{i,j}$ *be defined from a list of positive numbers* $p_1, p_2, \cdots p_n$, *as follows:*

   $$b_{i,j} = min\{p_i, p_j\}.$$

   The matrix B is positive semi-definite. Matrices of this form are covariance matrices arising in the Theory of Brownian motion, see [8].

2. Given $a > 0$ and $r \in [-1, 1]$, we define two positive semi-definite $n \times n$ matrices, $E = [e_{ij}]_{i,j=1,\cdots n}$ and $F = [f_{ij}]_{i,j=1,\cdots n}$, as follows:

$$e_{ij} = \frac{1}{(p_i + p_j)^a} \quad \textit{and} \quad f_{ij} = \frac{p_i^r + p_j^r}{p_1 + p_j}.$$

We refer the reader to references [50] and [51] for some history on the applications that led to the study of these type of matrices.

These matrices satisfy the hypotheses of Corollary 2.5.

2.2 Stability Analysis

We now study the asymptotic behavior of solutions of system (2.2). More precisely, we give a characterization of their $\omega-$limit sets. First we recall the definition of $\omega-$ limit set of $\{Z(t)\}_{t\in R}$, (for additional information see Arrowsmith [5]).

Definition 2.7. *A vector $u \in \mathbf{R}^n$ is in the ω-limit set of $Z(t)$ if there exists a sequence of times (t_n) converging to $+\infty$ such that $\|u - Z(t_n)\|$ converges to zero. We denote by Z_∞ the $\omega-$limit set of $Z(t)$.*

Oja followed a qualitative analysis of system (2.2) to conclude in [41, 42] that if Z(t) is not a constant solution but converges as $t \to \infty$, then its limit is a norm one eigenvector of M corresponding to the largest eigenvalue. We now use the form for the solutions encountered in the previous section (Theorem 2.1) to give an explicit description of the system's long term dynamical behavior.

Proposition 2.8. *Let M be a symmetric $n \times n$ matrix and Z_0 a non-trivial vector in R^n such that $\|Z_0\| \leq 1$. If Z(t) is a solution to system (2.2), then Z_∞ consists of a single vector in the unit $n-$ball.*

Proof: We follow the notation used in the proof of Theorem 2.1 and assume that $\lambda_1 \leq \cdots \leq \lambda_k < \lambda_{k+1} = \cdots = \lambda_n$, where $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of M. If $\alpha_i = 0$ then $\omega_i(t) \equiv 0$ for all t. We assume $\alpha_i \neq 0$ for $j = 1, \cdots, n$. From (2.7),

$$\omega_i(t) = \frac{\alpha_i \, e^{\lambda_i t}}{\sqrt{1 - \sum\limits_{j=1}^{n} \alpha_j^2 (1 - \, e^{2\lambda_j t})}}.$$

Therefore we have:

For $\|Z_0\| < 1$: If $i \leq k$, or if $\lambda_n < 0$, then $\lim\limits_{t\to\infty} \omega_i(t) = 0$.

If $i > k$ and $\lambda_n > 0$, then $\lim\limits_{t\to\infty} \omega_i(t) = \frac{\alpha_i}{\sqrt{\sum\limits_{j=k+1}^{n} \alpha_j^2}}$.

If $i > k$ and $\lambda_n = 0$, then $\lim\limits_{t\to\infty} \omega_i(t) = \frac{\alpha_i}{\sqrt{1 - \sum\limits_{j=1}^{k} \alpha_j^2}}$.

For $\|Z_0\| = 1$: If $i \leq k$, then $\lim\limits_{t\to\infty} \omega_i(t) = 0$.

If $i > k$ then $\lim\limits_{t\to\infty} \omega_i(t) = \frac{\alpha_i}{\sqrt{\sum\limits_{j=k+1}^{n} \alpha_j^2}}$.

Since the i^{th} coordinate of W_∞ is equal to $\underset{t\to\infty}{lim} \omega_i(t)$, W_∞ is a single vector, and since P is orthogonal, $Z_\infty = PW_\infty$ is a single vector in the unit n-ball. $\square$

Remark 2.9. *The stationary solutions of system (2.2) are either vectors in the kernel of M or norm 1 eigenvectors of M, see Botelho and Jamison [9]. The previous corollary then implies that the $\omega-$limit set of $Z(t)$ consists of a single stationary solution, which is either an eigenvector of M associated with the largest eigenvalue, or a vector in the kernel of M.*

3 Learning Systems on an Infinite Dimensional Space (ℓ_2)

In this section we investigate a theoretical question, which is a natural extension of the system considered in the previous sections. We investigate the convergence properties of the Oja-Karhunen model acting on an infinite dimensional space, ℓ_2, the Hilbert space of all square summable sequences equipped with the standard inner product. We assume that the input set of values, now a sequence in ℓ_2, determines a compact and self-adjoint operator M on ℓ_2.

The Spectral Theorem for compact operators reveals a finite dimensional eigenspace associated with each nontrivial eigenvalue, and hence this new setting allows a representation of unsupervised learning through finite dimensional templates. The techniques employed previously extend to this more general setting. We start by recalling the definition of a compact operator.

Definition 3.1. *(Ringrose [47, p. 9]) Suppose X is a Banach space and T is a bounded linear operator on X. The operator T is said to be a compact operator, if given any bounded sequence (x_n) (i.e., $sup\{\|x_n\| : n = 1, 2, ...\} < \infty$), in X, there exists a subsequence $(x_{n(q)})$ in X such that the sequence $T(x_{n(q)})$ converges in X.*

Theorem 3.2. Spectral Theorem for compact and self-adjoint operators *(Zimmer [52, p. 57] or Ringrose [47, p. 55]): Suppose $\mathcal{H}$ is a separable Hilbert space and $T \in \mathcal{B}(\mathcal{H})$ (i.e., $\mathcal{B}(\mathcal{H})$ is the continuous linear maps from $\mathcal{H}$ to $\mathcal{H}$) is compact and self-adjoint. Then $\mathcal{H}$ has an orthonormal basis consisting of eigenvectors for T, $\{u_i\}$, relative to which T has the representation*

$$Tv = \sum_i \lambda_i \, u_i \otimes u_i(v), \;\; \forall \; v \in \mathcal{H},$$

λ_i is a real eigenvalue of T and $u_i \otimes u_i(v) = \langle v, \, u_i \rangle \, u_i$.

The compactness of M (Ringrose [47, Theorem 1.8.7, p. 58]) also implies that the eigenspace, $\{v : (M - \lambda I)\, v = 0\}$, associated with each eigenvalue λ is finite dimensional and whenever M has an infinite number of distinct eigenvalues $\{\lambda_i\}$,

$$\lim_{i\to\infty} \lambda_i = 0.$$

For completeness we give a proof [52, Zimmer, p. 52] of the above statement. We suppose $\underset{i\to\infty}{lim}\lambda_i \neq 0$. Then there exists $\varepsilon > 0$, so that $S = \{i : |\lambda_i| > \varepsilon\}$ is infinite. Then for $i, j \in S$, consider the sequence $\{Mu_i\}_{i\in S}$, with u_i a unit eigenvector with eigenvalue λ_i, then

$$\begin{aligned}
\|Mu_i - Mu_j\|^2 &= \langle Mu_i - Mu_j, Mu_i - Mu_j\rangle \\
&= \|Mu_i\|^2 + \|Mu_j\|^2 - 2\langle Mu_i, Mu_j\rangle \\
&= \|\lambda_i u_i\|^2 + \|\lambda_j u_j\|^2 - 0 \\
&= |\lambda_i|^2 + |\lambda_j|^2 \\
&> 2\varepsilon^2.
\end{aligned}$$

Therefore $\{Mu_i\}_{i\in S}$ obviously has no convergent subsequence. Consequently M is not compact, which is a contradiction. Thus $\underset{i\to\infty}{lim}\, \lambda_i = 0$ as claimed.

We now use the spectral representation of compact operators to establish the following existence theorem.

Theorem 3.3. *If M is a self-adjoint compact operator on ℓ_2, and Z_0, a sequence in ℓ_2, is such that $\|Z_0\| \leq 1$, then*

$$\begin{cases} \dot{Z} = M\,Z \;-\; Z\;Z^T\,M\,Z \\ Z(0) = Z_0, \end{cases} \tag{3.1}$$

has a unique global solution.

Proof: The Spectral Theorem for Compact and Self-adjoint Operators asserts the existence of an orthonormal countable basis of eigenvectors $\{u_i\}$ of M, with corresponding (real) eigenvalues $\{\lambda_i\}$, so that M has the representation

$$M = \sum_i \lambda_i \, u_i \otimes \, u_i,$$

where $u_i \otimes u_j$ denotes the rank one operator on ℓ_2 defined by $u_i \otimes u_j(v) = \langle v, u_j \rangle \, u_i$, for every $v \in \ell_2$. Thus

$$MZ = \sum_i \lambda_i \langle Z, \, u_i \rangle \, u_i. \tag{3.2}$$

Since $\{u_i\}_i$ is an orthonormal basis, we have that $Z(t) = \sum\limits_i z_i(t) \, u_i$ and $Z_0 = \sum\limits_i z_i^0 \, u_i$, with the Fourier coefficients $z_i(t) = \langle Z(t), \, u_i \rangle$, and $z_i^0 = \langle Z_0, \, u_i \rangle$. Thus (3.2) becomes

$$MZ = \sum_i \lambda_i z_i \, u_i.$$

Incorporating this into system (3.1), we obtain the following system which the i^{th} Fourier coefficient must satisfy

$$\begin{cases} \dot{z}_i = \lambda_i \, z_i \; - \; z_i \, \sum\limits_j \lambda_j z_j^2 \\ \\ z_i(0) = z_i^0. \end{cases}$$

We follow a strategy similar to the one presented in the proof of Theorem 2.1. If a solution exists, it must be of the form

$$Z(t) = \sum_i \frac{z_i^0 \, e^{\lambda_i t}}{\sqrt{1 + \sum\limits_j (z_j^0)^2 (e^{2\lambda_j t} - 1)}} \, u_i.$$

We now show that, for every t, $Z(t)$ is differentiable, and that both $Z(t)$ and $\dot{Z}(t)$ are in ℓ_2.

Fix t, and let

$$\lambda_m = min\{\lambda_j\}_j,$$

$$\lambda_M = max\{\lambda_j\}_j.$$

We first show that $Z(t) \in \ell_2$: Since $\|Z_0\|^2 = \sum_j (z_j^0)^2 \leq 1$, in the denominator

$$\begin{aligned} 1 + \sum_j (z_j^0)^2 (e^{2\lambda_j t} - 1) &\geq 1 + \sum_j (z_j^0)^2 (e^{2\lambda_m t} - 1) \\ &\geq e^{2\lambda_m t} \sum_j (z_j^0)^2 > 0. \end{aligned} \tag{3.3}$$

Thus

$$0 \leq \frac{1}{1 + \sum_j (z_j^0)^2 (e^{2\lambda_j t} - 1)} \leq \frac{1}{e^{2\lambda_m t} \sum_j (z_j^0)^2},$$

and

$$\left| \frac{z_i^0 e^{\lambda_i t}}{1 + \sum_j (z_j^0)^2 (e^{2\lambda_j t} - 1)} \right|^2 \leq K (z_i^0)^2, \quad \text{with } K = \frac{e^{2(\lambda_M - \lambda_m)t}}{\|Z_0\|^2}.$$

Since $Z_0 \in \ell_2$ and the series $\sum_j |z_i^0|^2$ is convergent we conclude that $Z(t) \in \ell_2$, for every t. First we observe that if Z is differentiable, then

$$\dot{Z}(t) = \sum_i \frac{\left(\lambda_i \left[1 + \sum_j (z_j^0)^2 \left(e^{2\lambda_j t} - 1 \right) \right] - \sum_j (z_j^0)^2 \, \lambda_j e^{2\lambda_j t} \right) e^{\lambda_i t} z_i^0}{\sqrt{\left[1 + \sum_j (z_i^0)^2 (e^{2\lambda_j t} - 1) \right]^3}} \, u_i.$$

We show that the sequence

$$\left\{ \frac{\left(\lambda_i \left[1+\sum_j (z_j^0)^2\,(e^{2\lambda_j t}-1)\right] - \sum_j (z_j^0)^2\,\lambda_j e^{2\lambda_j t}\right) e^{\lambda_i t} z_i^0}{\sqrt{\left[1+\sum_j (z_i^0)^2 (e^{2\lambda_j t}-1)\right]^3}} \right\}_i \in \ell_2.$$

For a fixed t,

$$\left(\lambda_i \left[1+\sum_j (z_j^0)^2\,(e^{2\lambda_j t}-1)\right] - \sum_j (z_j^0)^2\,\lambda_j e^{2\lambda_j t}\right)\; e^{\lambda_i t}$$

$$\leq \left(\lambda_M \left[1+\sum_j (z_j^0)^2\,(e^{2\lambda_M t}-1)\right] - \sum_j (z_j^0)^2\,\lambda_m e^{2\lambda_m t}\right)\; e^{\lambda_M t}$$

$$\leq \left(\lambda_M \left[1+\sum_j (z_j^0)^2\,(e^{2\lambda_M t})\right] - \sum_j (z_j^0)^2\,\lambda_m e^{2\lambda_m t}\right)\; e^{\lambda_M t}$$

$$\leq \left(\lambda_M + \left[\lambda_M e^{2\lambda_M t} - \lambda_m e^{2\lambda_m t}\right] \sum_j (z_j^0)^2\right) e^{\lambda_M t},$$

which is a constant, denoted by K. As shown before, the denominator

$$\left[1+\sum_j (z_j^0)^2 (e^{2\lambda_j t}-1)\right]^{\frac{3}{2}} \geq \left(e^{2\lambda_m t} \sum_j (z_j^0)^2\right)^{3/2},$$

which implies that

$$\left| \frac{\left(\lambda_i \left[1+\sum_j (z_j^0)^2\,(e^{2\lambda_j t}-1)\right] - \sum_j (z_j^0)^2\,\lambda_j e^{2\lambda_j t}\right) e^{\lambda_i t} z_i^0}{\sqrt{\left[1+\sum_j (z_i^0)^2 (e^{2\lambda_j t}-1)\right]^3}} \right|^2$$

$$\leq \frac{K^2(z_i^0)^2}{\left(e^{2\lambda_m t}\sum_j (z_j^0)^2\right)^{3/2}}.$$

Therefore the series

$$\sum_i \left| \frac{\left(\lambda_i \left[1+\sum_j (z_j^0)^2\,(e^{2\lambda_j t}-1)\right] - \sum_j (z_j^0)^2\,\lambda_j e^{2\lambda_j t}\right) e^{\lambda_i\,t} z_i^0}{\sqrt{\left[1+\sum_j (z_i^0)^2 (e^{2\lambda_j t}-1)\right]^3}} \right|^2$$

converges since $Z_0 \in \ell_2$. Thus $\dot{Z}(t) \in \ell_2$, and is well-defined. It is straightforward to check that $Z(t)$ satisfies system (3.1). This concludes the proof. □

Next, we give the definitions of weak, strong, and uniform ω-limit set of a solution $Z(t)$.

Definition 3.4. *(Arrowsmith [5]). An operator $L \in \mathcal{B}(\mathcal{H})$ is in the*

(a) ***weak*** *ω-limit set of $Z(t)$, $L \in \omega_*(Z(t))$, if there exists a sequence of times (t_n) converging to $+\infty$ such that $\langle (L-Z)(t_n)(u),\, v\rangle$ converges to zero, for every u and $v \in \mathcal{H}$.*

(b) ***strong*** *ω-limit set of $Z(t)$, $L \in \omega_s(Z(t))$, if there exists a sequence of times (t_n) converging to $+\infty$ such that $\|(L-Z)(t_n)(v)\|$ converges to zero, for every $v \in \mathcal{H}$.*

(c) ***uniform*** *ω-limit set of $Z(t)$, $L \in \omega_u(Z(t))$, if there exists a sequence of times*

(t_n) converging to $+\infty$ such that $\|(L-Z)(t_n)\|$ converges to zero.

We denote by Z_∞^w, Z_∞^s, and Z_∞^u the weak, strong, and the uniform limit sets, respectively.

Corollary 3.5. *Let M be a self-adjoint compact operator on ℓ_2 and $\|Z_0\| \leq 1$. If $Z(t)$ is the solution of*

$$\begin{cases} \dot{Z} = M\,Z \;-\; Z\,Z^T\,M\,Z \\ Z(0) = Z_0, \end{cases}$$

then Z^s_∞ consists of a single vector in ℓ_2.

Proof: Since M is a compact operator, the sequence $\{\lambda_i\}_i$ of eigenvalues of M converges to zero. We denote by λ_s the supremum of $\{\lambda_i\}_i$, and set $\Lambda_s = \{k : \lambda_k = \lambda_s\}$, possibly an empty set. Proposition 2.8 implies that if Z^s_∞ is nonempty, it consists of a single vector, $Z_\infty = \sum\limits_i z_i^\infty u_i$, with

$$z_i^\infty = \begin{cases} 0 & \text{if} \quad i \notin \Lambda_s, \\ 0 & \text{if} \quad i \in \Lambda_s,\ \lambda_s < 0,\ \text{and}\ \|Z_0\| < 1, \\ \frac{z_i^0}{\sqrt{\sum\limits_{k\in\Lambda_s} (z_k^0)^2}} & \text{if} \quad i \in \Lambda_s,\ \lambda_s < 0,\ \text{and}\ \|Z_0\| = 1, \\ \frac{z_i^0}{\sqrt{\sum\limits_{k\in\Lambda_s} (z_k^0)^2}} & \text{if} \quad i \in \Lambda_s \text{ and } \lambda_s > 0, \\ \frac{z_i^0}{\sqrt{1-\sum\limits_{k\notin\Lambda_s} (z_k^0)^2}} & \text{if} \quad i \in \Lambda_s \text{ and } \lambda_s = 0, \end{cases}$$

since both $Z(t_n, Z_0) \to Z_\infty$ and $\|Z(t_n, Z_0)\| \to \|Z_\infty\|$. □

Remark 3.6. *It remains open to establish whether the strong limit set determined in the Corollary 3.5 is in fact the uniform limit set. It follows from the corollary above that the weak limit set is equal to the strong limit set.*

4 Synaptic Formations in a Learning Process

In this chapter we consider a generalization of Oja-Karhunen model proposed by Cox and Adams in [18]. This new model incorporates a probabilistic component allowing the creation of temporary connections during a learning process. This model is given by the system:

$$\begin{cases} \dot{Z}_E = E\, M\, Z_E \,-\, Z_E\, Z_E^T\, M\, Z_E \\ Z(0) = Z_0, \end{cases} \tag{4.1}$$

with Z_E a vector of connecting weights in $\mathbb{R}^n$, and M a symmetric $n \times n$ input correlation matrix. Since E represents a small perturbation of the identity, we deem E to be positive definite. Hence it is invertible. Here we address a more general situation than the one considered in Botelho and Jamison [9, 11].

4.1 Existence Theorem for the Cox-Adams Model

In this section we use a convenient change of coordinates to reduce the system to an Oja-Karhunen system. The positivity assumption on E allows us to transfer the probabilistic component to the input correlation matrix and then previous methods apply to solve the system explicitly.

Definition 4.1. *(Kolman, [38]) An $n \times n$ symmetric matrix C with the property that $X^T\, C\, X > 0$, for every nonzero vector X in $\mathbb{R}^n$ is called positive definite.*

We state our first result on the existence of global solutions.

Theorem 4.2. *Let M and E be symmetric $n \times n$ matrices, E positive definite, and $Z_0 \in \mathbf{R}^n$. Then there exists an $r > 0$ such that the system*

$$\begin{cases} \dot{Z}_E = E\,M\,Z_E \,-\, Z_E\; Z_E^T\,M\,Z_E \\ Z(0) = Z_0, \;\; \|Z_0\| \leq r \end{cases} \tag{4.2}$$

has a unique global solution.

Proof: Since E is positive definite, we denote by $E^{1/2}$ the positive square root of E. Also, since E is invertible, $E^{1/2}$ is also invertible. Therefore we write (4.2) as

$$\begin{cases} \dot{Z}_E = E^{1/2}E^{1/2}\,M\,E^{1/2}E^{-1/2}Z_E \,-\, Z_E\; Z_E^T\,E^{-1/2}E^{1/2}\,M\,E^{1/2}E^{-1/2}Z_E \\ Z(0) = Z_0. \end{cases}$$

Multiplying on the left by $E^{-1/2}$,

$$\begin{cases} E^{-\frac{1}{2}}\dot{Z}_E = (E^{\frac{1}{2}}ME^{\frac{1}{2}})E^{-\frac{1}{2}}Z_E - (E^{-\frac{1}{2}}Z_E)(Z_E^TE^{-\frac{1}{2}})(E^{\frac{1}{2}}ME^{\frac{1}{2}})(E^{-\frac{1}{2}}Z_E) \\ E^{-\frac{1}{2}}Z(0) = E^{-\frac{1}{2}}Z_0. \end{cases}$$

We set $W = E^{-1/2}\,Z_E$, and $M_1 = E^{1/2}ME^{1/2}$. We observe that M_1 is a also symmetric matrix. Thus system (4.2) is equivalent to

$$\begin{cases} \dot{W} = M_1\,W \,-\, W\;W^T\,M_1\,W, \\ W(0) = E^{-1/2}Z_0. \end{cases}$$

If $r = \frac{1}{\|E^{-1/2}\|}$ and $\|Z_0\| \leq r$, we have that $\|W(0)\| \leq 1$, and the statement now follows from Theorem 2.1, with $Z(t) = E^{1/2}W$. □

We now state a Proposition established by Hladnik and Olmadič:

Proposition 4.3. *(Hladnik-Olmadič [32]) Let A and B be operators on a Hilbert space H, let B be positive and denote by P the positive square root of the operator B. Then, the spectrum $\sigma(AB) = \sigma(BA) = \sigma(PAP)$.*

We denote by $\sigma(EM)$ the set of all eigenvalues of the matrix EM. Since $E^{1/2}$ is the positive square root of E, by Proposition 4.3,

$$\sigma(EM) = \sigma(ME) = \sigma(E^{1/2}ME^{1/2}) = \sigma(M_1).$$

Moreover, if E and M are both positive definite symmetric matrices, then Hu-yun [34, p. 147] establishes upper and lower bounds for the eigenvalues of EM in terms of the eigenvalues of E and M:

$$\frac{2}{n}\frac{(min\,\nu_j)^2\,(min\,\lambda_j)^2}{(min\,\nu_j)^2\,+(min\,\lambda_j)^2} < \mu_i < \frac{n}{2}\left[(max\,\nu_j)^2 + (max\,\lambda_j)^2\right], \qquad 1 \leq j \leq n, \tag{4.3}$$

where $\nu_1, \cdots, \nu_n$; $\lambda_1, \cdots, \lambda_n$; and $\mu_1, \cdots, \mu_n$ are the eigenvalues of E, M, and $M_1 = ME = EM$, respectively. These considerations imply the following corollary.

Corollary 4.4. *Let M and E be positive definite, symmetric $n \times n$ matrices. Then for every Z_0, a nonzero vector in $\mathbf{R}^n$, there exists $t_0 < 0$ such that*

$$\begin{cases} \dot{Z}_E = E\,M\,Z_E \,-\, Z_E\,Z_E^T\,M\,Z_E \\ Z(0) = Z_0 \end{cases}$$

has a unique maximal solution $Z(t)$, defined for $t > t_0$, and the $\omega-$limit set of $Z_E^s(t)$ consists of a single vector.

As observed in Remark 2.9, the nontrivial omega limit sets are the unit eigenvectors of EM or vectors in the kernel of EM. The long term evolution of solutions may retrieve unit eigenvectors associated with the largest eigenvalue of EM.

4.2 A Particular Case

In this section we consider error matrices that are not necessarily invertible; however, we impose a more restrictive assumption on the initial stimulus. Specifically, we assume the error matrix E and the input correlation matrix M are commuting symmetric matrices. This implies that that E and M are simultaneously diagonalizable via an orthogonal matrix P, see Bellman [7]. Examples of error matrices satisfying the conditions just described are polynomial matrices on M, or matrices which are uniformly approximated by polynomials on M. The system now considered is

$$\begin{cases} \dot{Z}_E = E\, M\, Z_E \,-\, Z_E\; Z_E^T\; M\, Z_E \\ Z_E(0) = Z_0, \end{cases} \tag{4.4}$$

with Z_E a column of connecting weights, M a symmetric input correlation matrix, and E a symmetric matrix that commutes with M. We set $M_1 = EM = ME$ and P an orthogonal matrix such $P^T MP = D$ and $P^T M_1 P = D_1$, with D and D_1 diagonal matrices. If $W_E = P^T Z_E$ and $W_0 = P^T Z_0$, then (4.4) becomes

$$\begin{cases} \dot{W}_E = D_1\, W_E \,-\, W_E\; W_E^T\, D\, W_E \\ W_E(0) = W_0. \end{cases}$$

If there is no error then $D = D_1$, and the results are as before. We assume that the eigenvalues of D_1 are $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$, and the eigenvalues of D are $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. We represent by ω_i^E and $\omega_i^E(0)$, the i^{th} component of W_E and W_0, respectively. If $\omega_i^E(0) = 0$, then $\omega_i^E \equiv 0$. Without loss of generality, we assume $\omega_i^E(0) \neq 0$ for $i = 1, 2, ..., n$. For simplicity of notation we set $\alpha_i = \omega_i^E(0)$. For t in a small neighborhood of zero, $\omega_i^E(t) \neq 0$. Utilizing techniques as in Section 2.1, we have that

$$\dot{\omega}_i = \mu_i \omega_i \, - \, \omega_i \, \sum_{j=1}^{n} \lambda_j \omega_j^2. \tag{4.5}$$

Thus

$$\frac{\dot{\omega_i^E}}{\omega_i^E} = \mu_i \, - \, \sum_{j=1}^{n} \lambda_j (\omega_j^E)^2,$$

which holds for i = 1, ... n. Thus

$$\frac{\dot{\omega_i^E}}{\omega_i^E} - \mu_i = \frac{\dot{\omega_1^E}}{\omega_1^E} - \mu_1.$$

Employing standard integration techniques, we have

$$\omega_i^E(t) = \frac{\alpha_i}{\alpha_1} e^{(\mu_i - \mu_1)t} \omega_1^E, \quad i = 1, 2, ..., n. \tag{4.6}$$

The following appropriate computations establish equation (4.6):

$$ln|\omega_i^E| - \mu_i \, t = ln|\omega_1^E| - \mu_1 \, t + K.$$

$$e^{ln|\omega_i^E|} e^{-\mu_i \, t} = e^{ln|\omega_1^E|} e^{-\mu_1 \, t} e^K.$$

$$|\omega_i^E| e^{-\mu_i \, t} = |\omega_1^E| \, e^{-\mu_1 \, t} e^K = e^K |\omega_1^E| \, e^{-\mu_1 \, t}.$$

Denote the sign of ω_i^E by $sgn(\omega_i^E)$, then

$$sgn(\omega_i^E) \, \omega_i^E = sgn(\omega_i^E) \, e^K \, \omega_1^E e^{(\mu_i - \mu_1) \, t}, \quad i = 1, 2, ..., n.$$

$$\omega_i^E = sgn(\omega_i^E) \, sgn(\omega_i^E) \, e^K \, \omega_1^E e^{(\mu_i - \mu_1) \, t}, \quad i = 1, 2, ..., n.$$

$$\omega_i^E = C_1 \, \omega_1^E e^{(\mu_i - \mu_1) \, t}, \quad i = 1, 2, ..., n, \text{ where } C_1 \text{ is a constant.}$$

Solving for C_1, we set $t = 0$, $\omega_i^E(0) = C_1 \omega_1^E(0) e^0 = \alpha_i$. Thus $C_1 \alpha_1 = \alpha_i$. And $C_1 = \frac{\alpha_i}{\alpha_1}$. Hence

$$\omega_i^E(t) = \frac{\alpha_i}{\alpha_1} \omega_1^E e^{(\mu_i - \mu_1)t},$$

which verifies equation (4.6).

Substituting (4.6) into (4.5), we have

$$\dot{\omega_1^E} = \mu_1 \omega_1^E - \omega_1^E \sum_{j=1}^{n} \lambda_j \left(\frac{\alpha_j}{\alpha_1} e^{(\mu_j - \mu_1)\, t} \omega_1^E \right)^2 .$$

$$\dot{\omega_1^E} - \mu_1 \omega_1^E = -(\omega_1^E)^3 \sum_{j=1}^{n} \lambda_j \left(\frac{\alpha_j}{\alpha_1} \right)^2 e^{2(\mu_j - \mu_1)\, t}.$$

Thus

$$\dot{\omega_1^E} (\omega_1^E)^{-3} - \mu_1 (\omega_1^E)^{-2} = - \sum_{j=1}^{n} \lambda_i \left(\frac{\alpha_j}{\alpha_1} \right)^2 e^{2(\mu_j - \mu_1)\, t}. \tag{4.7}$$

We order the eigenvalues by $\mu_1 \leq \cdots < \mu_k = 0 = \cdots = \mu_{k+p} < \mu_{k+p+1} \leq \cdots \leq \mu_n$.

Setting

$$v = (\omega_1^E)^{-2},$$

then

$$\dot{v} = -2 (\omega_1^E)^{-3} \, \dot{\omega}_1^E.$$

Therefore (4.7) becomes

$$-\frac{1}{2} \dot{v} - \mu_1 v = - \sum_{j=1}^{n} \lambda_j \left(\frac{\alpha_j}{\alpha_1} \right)^2 e^{2(\mu_j - \mu_1)\, t}.$$

$$\dot{v} + 2 \mu_1 v = 2 \sum_{j=1}^{n} \lambda_j \left(\frac{\alpha_j}{\alpha_1} \right)^2 e^{2(\mu_j - \mu_1)\, t}.$$

$$e^{2\mu_1 t}\,\dot{v} + 2\mu_1 e^{2\mu_1 t}\,v = 2\sum_{j=1}^{n} \lambda_j \left(\frac{\alpha_j}{\alpha_1}\right)^2 e^{2\mu_j\,t}.$$

Set Λ to be a subset of $\{1,\cdots,n\}$ consisting of those values j for which $\mu_j = 0$. Then

$$e^{2\mu_1 t}\,\dot{v} + 2\mu_1 e^{2\mu_1 t}\,v = 2\sum_{j\in\Lambda} \lambda_j \left(\frac{\alpha_j}{\alpha_1}\right)^2 + 2\sum_{j\notin\Lambda} \lambda_j \left(\frac{\alpha_j}{\alpha_1}\right)^2 e^{2\mu_j\,t}.$$

Equivalently,

$$e^{2\mu_1 t}\,(\omega_1^E)^{-2} = 2\sum_{j\in\Lambda} \lambda_j \left(\frac{\alpha_j}{\alpha_1}\right)^2 t + \sum_{j\notin\Lambda} \frac{\lambda_j}{\mu_j} \left(\frac{\alpha_j}{\alpha_1}\right)^2 e^{2\mu_j\,t} + K_1, \tag{4.8}$$

with

$$K_1 = \frac{1}{\alpha_1^2} - \sum_{j\notin\Lambda} \frac{\lambda_j}{\mu_j} \left(\frac{\alpha_j}{\alpha_1}\right)^2.$$

Therefore (4.8) yields

$$e^{2\mu_1 t}\,(\omega_1^E)^{-2} = 2\sum_{j\in\Lambda} \lambda_j \left(\frac{\alpha_j}{\alpha_1}\right)^2 t + \sum_{j\notin\Lambda} \frac{\lambda_j}{\mu_j} \left(\frac{\alpha_j}{\alpha_1}\right)^2 e^{2\mu_j\,t} + \frac{1}{\alpha_1^2} - \sum_{j\notin\Lambda} \frac{\lambda_j}{\mu_j} \left(\frac{\alpha_j}{\alpha_1}\right)^2.$$

$$(\omega_1^E)^{-2} = e^{-2\mu_1 t}\left[2\sum_{j\in\Lambda} \lambda_j \left(\frac{\alpha_j}{\alpha_1}\right)^2 t + \sum_{j\notin\Lambda} \frac{\lambda_j}{\mu_j} \left(\frac{\alpha_j}{\alpha_1}\right)^2 e^{2\mu_j\,t} + \frac{1}{\alpha_1^2} - \sum_{j\notin\Lambda} \frac{\lambda_j}{\mu_j} \left(\frac{\alpha_j}{\alpha_1}\right)^2\right].$$

$$\omega_1^E = \frac{e^{2\mu_1 t} sgn(\alpha_1)}{\sqrt{2\sum_{j\in\Lambda} \lambda_j \left(\frac{\alpha_j}{\alpha_1}\right)^2 t + \sum_{j\notin\Lambda} \frac{\lambda_j}{\mu_j} \left(\frac{\alpha_j}{\alpha_1}\right)^2 e^{2\mu_j\,t} + \frac{1}{\alpha_1^2} - \sum_{j\notin\Lambda} \frac{\lambda_j}{\mu_j} \left(\frac{\alpha_j}{\alpha_1}\right)^2}}.$$

Equivalently,

$$\omega_i^E(t) = \frac{\alpha_i\, e^{\mu_i t}}{\sqrt{1 + 2\sum_{j\in\Lambda} (\lambda_1 \alpha_1^2)\, t + \sum_{j\notin\Lambda} \left(\frac{\lambda_j}{\mu_j}\right) \alpha_j^2\, (e^{2\mu_j\,t} - 1)}}.$$

The next theorem establishes the existence of maximal solutions:

Theorem 4.5. *Let Z_0 be a vector in $\mathbb{R}^n$ such that $\|Z_0\| \leq 1$. If both E and M are positive definite, symmetric, commuting, $n \times n$ matrices, then there exists t_0 such that*

$$\begin{cases} \dot{Z}_E = E\, M\, Z_E \;-\; Z_E\; Z_E^T\; M\, Z_E \\ Z_E(0) = Z_0 \end{cases}$$

has a unique maximal solution $Z_E(t)$, defined for $t > t_0$.

Proof: The left inequality in (4.3) asserts that EM is also positive definite, see Hu-yun [34]. Previous considerations imply that $Z_E(t) = PW_E(t)$, with $W_E(t) = [\omega_i^E]_{i=1,\ldots n}$ and

$$\omega_i^E(t) = \frac{\alpha_i\, e^{\mu_i t}}{\sqrt{1 + \sum_j \left(\frac{\lambda_j}{\mu_j}\right) \alpha_j^2\, (e^{2\mu_j\, t} - 1)}}.$$

Therefore ω_i^E is well defined for $t > t_0$, for some $t_0 \in \mathbb{R}$. $\square$

4.3 Stability of Solutions of Cox-Adams Model

First we find that the $\omega-$limit set for the Cox-Adams model is also a singleton. Then we investigate the impact of the error factor in the learning process. Now for every i, $\omega_i^E(t)$ is well defined for $t \geq t_0$, and we claim that the ω-limit set is given by

$$\lim_{t\to\infty} w_i^E(t) = \begin{cases} \dfrac{\alpha_i}{\sqrt{\sum\limits_{\{j:\mu_j=\mu_n\}} \left(\frac{\lambda_j}{\mu_j}\right)\alpha_j^2}}, & \text{for } \mu_i = \mu_n \\ \\ 0 & \text{for } \mu_i \neq \mu_n. \end{cases}$$

$$\omega_i^E(t) = \frac{\alpha_i\, e^{\mu_i t}}{\sqrt{1 + \sum\limits_j \left(\frac{\lambda_j}{\mu_j}\right) \alpha_j^2\, (e^{2\mu_j\, t} - 1)}}$$

$$= \frac{\alpha_i\, e^{\mu_i t}}{\sqrt{1 - \sum\limits_j \left(\frac{\lambda_j}{\mu_j}\right) \alpha_j^2 + \sum\limits_j \left(\frac{\lambda_j}{\mu_j}\right) \alpha_j^2 e^{2\mu_j\, t}}}$$

$$= \frac{\alpha_i}{\sqrt{e^{-2\mu_i t}\left(1 - \sum\limits_j \left(\frac{\lambda_j}{\mu_j}\right) \alpha_j^2\right) + \sum\limits_j \left(\frac{\lambda_j}{\mu_j}\right) \alpha_j^2 e^{2(\mu_j - \mu_i)\, t}}}.$$

Recall that M and M_1 are positive definite, $0 < \lambda_1 \leq \lambda_2 \cdots \leq \lambda_n$, and $0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n$. If $\mu_i = \mu_n$, then $\lim\limits_{t\to\infty} \sum\limits_{\{j:\mu_j \neq \mu_n\}} \left(\frac{\lambda_j}{\mu_j}\right) \alpha_j^2 e^{2(\mu_j - \mu_i)\, t} = 0$, and

$$\lim_{t\to\infty} \omega_i^E = \frac{\alpha_i}{\sqrt{\sum\limits_{\{j:\mu_j = \mu_n\}} \left(\frac{\lambda_j}{\mu_j}\right) \alpha_j^2}}.$$

If $\mu_i \neq \mu_n$, then $\mu_i < \mu_n$, and $\lim\limits_{t\to\infty} \sum\limits_{\{j:\mu_j = \mu_n\}} \left(\frac{\lambda_j}{\mu_j}\right) \alpha_j^2 e^{2(\mu_j - \mu_i)\, t} = \infty$. Thus $\lim\limits_{t\to\infty} \omega_i^E = 0$. This verifies our claim. We also notice that the single vector in the limit set might not be in the unit ball when synaptic creation is possible,

$$\|W_\infty^E\| = \sqrt{\sum_{i=1}^n \frac{\alpha_i^2}{\left(\sqrt{\sum\limits_{\{j:\mu_j = \mu_n\}} \left(\frac{\lambda_j}{\mu_j}\right) \alpha_j^2}\right)^2}} = \sqrt{\frac{\sum\limits_{\{i:\mu_i = \mu_n\}} \mu_i \alpha_i^2}{\sum\limits_{\{j:\mu_j = \mu_n\}} \lambda_j \alpha_j^2}}.$$

In order to detect the directional impact of the error matrix on the asymptotic behavior of solutions we study the variation of θ_E, the angle between the vectors

W and W_E. We start by observing that the angle between W and W_E is equal to the angle between Z and Z_E. Also

$$\|W\| = \sqrt{\sum_{i=1}^{n} \omega_i^2} = \frac{\sqrt{\sum\limits_{i=1}^{n} \alpha_i^2 e^{2\lambda_i t}}}{\sqrt{1 + \sum\limits_{j=1}^{n} \alpha_j^2 \left(e^{2\lambda_j t} - 1\right)}},$$

and

$$\|W_E\| = \sqrt{\sum_{i=1}^{n} (\omega_i^E)^2} = \frac{\sqrt{\sum\limits_{i=1}^{n} \alpha_i^2 e^{2\lambda_i t}}}{\sqrt{1 + \sum\limits_{j=1}^{n} \frac{\lambda_j}{\mu_j} \alpha_j^2 \left(e^{2\lambda_j t} - 1\right)}}.$$

Now

$$\cos(\theta_E(t)) = \frac{\langle W, W_E \rangle}{\|W\| \, \|W_E\|}$$

$$= \frac{\sum\limits_{i=1}^{n} \alpha_i^2 e^{(\lambda_i + \mu_i)\, t}}{\sqrt{1+\sum\limits_{j=1}^{n} \alpha_j^2 \left(e^{2\lambda_j t} - 1\right)} \sqrt{1+\sum\limits_{j=1}^{n} \frac{\lambda_j}{\mu_j} \alpha_j^2 \left(e^{2\lambda_j t} - 1\right)}} \cdot \frac{\sqrt{1+\sum\limits_{j=1}^{n} \alpha_j^2 \left(e^{2\lambda_j t} - 1\right)}}{\sqrt{\sum\limits_{i=1}^{n} \alpha_i^2 e^{2\lambda_i t}}} \frac{\sqrt{1+\sum\limits_{j=1}^{n} \frac{\lambda_j}{\mu_j} \alpha_j^2 \left(e^{2\lambda_j t} - 1\right)}}{\sqrt{\sum\limits_{i=1}^{n} \alpha_i^2 e^{2\mu_i t}}}$$

$$= \frac{\sum\limits_{i=1}^{n} \alpha_i^2 e^{(\lambda_i + \mu_i)\, t}}{\sqrt{\sum\limits_{i=1}^{n} \alpha_i^2 e^{2\lambda_i t} \sum\limits_{i=1}^{n} \alpha_i^2 e^{2\mu_i t}}}.$$

We order the eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots < \lambda_{i_0} = \cdots = \lambda_n,\ \mu_1 \leq \mu_2 \leq \cdots < \mu_{i_0} = \cdots = \mu_n,$ of M and M_1, respectively, with $i_0 = min\{i : \lambda_i = \lambda_n\}$, and $j_0 = min\{j : \mu_j = \mu_n\}$. Let $k = max\{i_0, j_0\}$. Then

$$\lim_{t \to \infty} \cos(\theta_E(t)) = \lim_{t \to \infty} \frac{\sum\limits_{i=1}^{n} \alpha_i^2 e^{(\lambda_i - \lambda_k)\, t} e^{(\mu_i - \mu_k)\, t}}{\sqrt{\sum\limits_{i=1}^{n} \alpha_i^2 e^{2(\lambda_i - \lambda_k) t} \sum\limits_{i=1}^{n} \alpha_i^2 e^{2(\mu_i - \mu_k) t}}}$$

$$= \lim_{t\to\infty} \frac{\sum_{i=1}^{k-1} \alpha_i^2 e^{(\lambda_i-\lambda_k)\,t} e^{(\mu_i-\mu_k)\,t} + \sum_{i=k}^{n} \alpha_i^2 e^{(\lambda_i-\lambda_k)\,t} e^{(\mu_i-\mu_k)\,t}}{\sqrt{\left(\sum_{i=1}^{i_0-1} \alpha_i^2 e^{2(\lambda_i-\lambda_k)t} + \sum_{i=i_0}^{n} \alpha_i^2 e^0\right) \left(\sum_{i=1}^{j_0-1} \alpha_i^2 e^{2(\mu_i-\mu_k)t} + \sum_{i=j_0}^{n} \alpha_i^2 e^{(\mu_i-\mu_k)t}\right)}}$$

$$= \frac{\sum_{i=max\{i_0,\,j_0\}}^{n} \alpha_i^2}{\sqrt{\left(\sum_{i=i_0}^{n} \alpha_i^2\right) \sum_{i=j_0}^{n} \alpha_i^2}}.$$

Therefore

$$\lim_{t\to\infty} \cos(\theta_E(t)) = \frac{\sum_{i=max\{i_0,j_0\}}^{n} \alpha_i^2}{\sqrt{\sum_{i=i_0}^{n} \alpha_i^2 \sum_{i=j_0}^{n} \alpha_i^2}}.$$

If $i_0 = j_0$ then

$$\lim_{t\to\infty} \cos(\theta_E(t)) = 1.$$

At this point we conclude that whenever the eigenvalues of M and EM are positive, and if the multiplicities of the largest principal eigenvalues of M and EM are equal, then the error factor E has no directional impact in the asymptotic behavior of the solution.

5 Learning Systems on Spaces of Bounded Linear Operators

In this chapter we consider the following generalization of the Oja and Karhunen model:

$$\begin{cases} \dot{Z} = M\,Z \,-\, Z\,Z^*\,M\,Z \\ Z(0) = Z_0, \end{cases} \tag{5.1}$$

in which the time dependent variables are bounded linear operators on $\mathcal{B}(\mathcal{H})$, the Banach space of bounded liner operators on a separable complex Hilbert space $\mathcal{H}$. The operator Z^* is the adjoint of Z and M is a normal operator (i.e. $M^*M = MM^*$) on $\mathcal{H}$. Some of the results in this chapter may be found in [13]. We apply the polar decomposition of operators to solve explicitly system (5.1), provided Z_0 is invertible and commutes with M. This method allows a decomposition of (5.1) into a "scalar" system and a "polar" system. The "scalar" system is an Euler type equation, for which well-known techniques can be extended to this new setting in order to derive an explicit form for solutions. The "polar" system is a first order non autonomous linear differential equation, that can also be solved explicitly. These two components of the solution are combined to define a representation for the solution of (5.1). This representation allows us to study the long-term and the stability behavior of the flow under some mild initial conditions. The main result established in this chapter is given by the following theorem:

Theorem 5.34, (see p. 76): If Z_0 is invertible and commutes with the normal operator M, then there exists $\varepsilon > 0$ and a unique differentiable mapping $Z : (-\varepsilon, \varepsilon) \to \mathcal{B}(\mathcal{H})$ such that

$$\dot{Z} = M\,Z \,-\, Z\,Z^*\,M\,Z \text{ and } Z(0) = Z_0$$

if and only if for $Z(t) = \sqrt{V(t)}\,P(t)$ with

$$V(t) = \left[I + (V_0^{-1} - I)e^{-(M+M^*)t}\right]^{-1} \;\; and\; P(t) = e^{\int_0^t -\frac{1}{2}(M-M^*)(V(\xi)-I)d\xi} P_0.$$

This form for the solutions allows us to analyze the asymptotic behavior of the system following techniques developed in section 5.3. As a consequence we state the following Corollary:

Corollary 5.38, (p. 82): If Z_0 is an invertible operator in $\mathcal{B}(\mathcal{H})$, M is a self-adjoint operator that commutes with Z_0, $\|(Z_0 Z_0^*)^{-1} - I\| < 1$, and the spectrum of M is strictly positive, then $\lim_{t\to\infty} Z(t) = P_0$, the "polar" factor of the decomposition of the initial condition.

5.1 Local Existence and Uniqueness of Solutions

The main theorem of this section establishes the local existence and uniqueness of solutions of system (5.1). We first set some notation and state results to be used in our proof. Given $\rho > 0$, we set $B_\rho(Z_0) = \{Z \in \mathcal{B}(\mathcal{H}) : \|Z - Z_0\| \leq \rho\}$, and denote by $\mathcal{C}\left([-\varepsilon, \varepsilon], B_\rho(Z_0)\right)$ the space of all continuous functions defined on the interval $[-\varepsilon,\, \varepsilon]$ with values in $B_\rho(Z_0)$. The space, $\mathcal{C}\left([-\varepsilon, \varepsilon], \mathcal{B}(\mathcal{H})\right)$, equipped with the norm $\| Z \|_\infty = sup\{\|Z(t)\| : t \in [-\varepsilon,\, \varepsilon\,]\}$, is a Banach space, with

$$\|Z - Z_0\| = \|Z - Z_0\|_\infty = sup\{\|Z(t) - Z_0(t)\| : |t| \leq \varepsilon\};$$

where $\|Z(t) - Z_0(t)\| = sup\left\{||(Z(t) - Z_0(t))(v)||_{\mathcal{H}} : v \in \mathcal{H}, ||v|| = 1\right\}$.

We use a version of the classical fixed point theorem due to Tychonov to prove the existence of a positive number ε and a unique differentiable path

$Z : (-\varepsilon, \varepsilon) \to \mathcal{B}(\mathcal{H})$ so that $\dot{Z(t)} = MZ(t) - Z(t)Z(t)^*MZ(t)$, with $Z(0) = Z_0$, [12]. We start by recalling a variation of Tychonov's Fixed Point Theorem as stated in Hartman.

Theorem 5.1. *(Hartman [29, Theorem 0.1, p. 404]) Let $\mathfrak{D}$ be a Banach space of elements x, y, ... with norms $|x|, |y|, ...$, let T_0 be a map from the ball $|x| \leq \rho$ in $\mathfrak{D}$ into $\mathfrak{D}$ satisfying $|T_0[x] - T_0[y]| \leq \theta|x - y|$ for some θ, $0 < \theta < 1$. Let $m = |T_0[0]|$ and $m \leq \rho(1 - \theta)$. Then there exists a unique fixed point x_0 of T_0, i.e., $T_0[x_0] = x_0$.*

We also need the following well known results from Operator Theory.

Lemma 5.2. *(Furuta [24, Theorem 1(i), p. 35], or Zimmer [52, Lemma 1.2.21, p. 28]) Let T be a bounded linear operator on a Hilbert space $\mathcal{H}$. Then T^* is a bounded linear operator on $\mathcal{H}$, and $\|T^*\| = \|T\|$.*

Lemma 5.3. *(Furuta [24, Corollary 2(i), p. 36], or Zimmer [52, p. 75]) Let Z be a bounded linear operator on $\mathcal{H}$. Then $\|Z^*Z\| = \|ZZ^*\| = \|Z\|^2$.*

Theorem 5.4. *If M and Z_0 are bounded operators on a complex Hilbert space $\mathcal{H}$, then there exist positive numbers ε and ρ, and a unique differentiable map $Z : (-\varepsilon, \varepsilon) \to B_\rho(Z_0)$ such that $\dot{Z}(t) = MZ(t) - Z(t)Z^*(t)MZ(t)$, and $Z(0) = Z_0$.*

Proof of Theorem: The map $T : B_\rho(Z_0) \to \mathcal{B}(\mathcal{H})$, given by

$$T(Z) = MZ - ZZ^*MZ$$

satisfies a Lipschitz condition, since for $Z_1, Z_2 \in B_\rho(Z_0)$,

$$\begin{aligned}
\|TZ_1 - TZ_2\| &= \|MZ_1 - Z_1Z_1^*MZ_1 - (MZ_2 - Z_2Z_2^*MZ_2)\| \\
&\leq \|MZ_1 - MZ_2\| + \|Z_1Z_1^*MZ_1 - Z_2Z_2^*MZ_2\| \\
&\leq \|M\| \; \|Z_1 - Z_2\| + \|Z_1Z_1^*MZ_1 - Z_2Z_2^*MZ_2\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \|M\|\,\|Z_1 - Z_2\| + \|Z_1 Z_1^* M Z_1 \overbrace{- Z_1 Z_1^* M Z_2 + Z_1 Z_1^* M Z_2} - Z_2 Z_2^* M Z_2\| \\
&\leq \|M\|\,\|Z_1 - Z_2\| + \|\underline{Z_1 Z_1^* M} Z_1 - \underline{Z_1 Z_1^* M} Z_2\| + \|Z_1 Z_1^* \underline{M Z_2} - Z_2 Z_2^* \underline{M Z_2}\| \\
&\leq \|M\|\,\|Z_1 - Z_2\| + \|Z_1 Z_1^* M\|\,\|Z_1 - Z_2\| + \|Z_1 Z_1^* - Z_2 Z_2^*\|\,\|M Z_2\| \\
&\leq \underline{\|M\|}\,\|Z_1 - Z_2\| + \|Z_1\|\,\|Z_1^*\|\,\underline{\|M\|}\,\|Z_1 - Z_2\| + \|Z_1 Z_1^* - Z_2 Z_2^*\|\,\underline{\|M\|}\,\|Z_2\| \\
&\leq \|M\|\,(\,\|Z_1 - Z_2\| + \|Z_1\|\,\|Z_1\|\,\|Z_1 - Z_2\| + \|Z_1 Z_1^* - Z_2 Z_2^*\|\,\|Z_2\|) \\
&\leq \|M\| \left(\|Z_1 - Z_2\| + \|Z_1\|^2 \|Z_1 - Z_2\| + \|Z_1 Z_1^* \overbrace{- Z_2 Z_1^* + Z_2 Z_1^*} - Z_2 Z_2^*\|\,\|Z_2\| \right) \\
&\leq \|M\| \left[\underline{\|Z_1 - Z_2\|} + \|Z_1\|^2 \underline{\|Z_1 - Z_2\|} + (\|Z_1 \underline{Z_1^*} - Z_2 \underline{Z_1^*}\| + \|\underline{Z_2} Z_1^* - \underline{Z_2} Z_2^*\|)\,\|Z_2\| \right] \\
&\leq \|M\| \left[\left(1 + \|Z_1\|^2\right) \underline{\|Z_1 - Z_2\|} + (\underline{\|Z_1 - Z_2\|}\|Z_1^*\| + \|Z_2\|\,\|Z_1^* - Z_2^*\|)\,\|Z_2\| \right] \\
&\leq \|M\| \left[1 + \|Z_1\|^2 + \|Z_1\|\,\|Z_2\| + \|Z_2\|^2\right] \|Z_1 - Z_2\| \\
&\leq \|M\| \ \left[1 + 3(\|Z_0\| + \rho)^2\right] \ \|Z_1 - Z_2\|.
\end{aligned}
$$

Set $\rho = 2\|Z_0\|$ (this constant is conveniently chosen for forthcoming estimates) then

$$
\begin{aligned}
\|TZ_1 - TZ_2\| &\leq \|M\| \ \left[1 + 3(\|Z_0\| + 2\|Z_0\|)^2\right] \ \|Z_1 - Z_2\| \\
&\leq \|M\| \ \left[1 + 3(3\|Z_0\|)^2\right] \ \|Z_1 - Z_2\| \\
&\leq \|M\| \left(1 + 27\|Z_0\|^2\right) \|Z_1 - Z_2\|. \qquad (5.2)
\end{aligned}
$$

Hence T satisfies the Lipschitz condition.

We now choose $\varepsilon > 0$ so that

$$
\theta = \|M\| \ (1 + 27\|Z_0\|^2)\ \varepsilon < \frac{1}{2}. \qquad (5.3)
$$

We define $\widetilde{Z}_0 : [-\varepsilon, \varepsilon] \to \mathcal{B}(\mathcal{H})$ as the constant operator function $\widetilde{Z}_0(t) = Z_0$.

Clearly $\widetilde{Z}_0 \in \ \mathcal{C}([-\varepsilon, \varepsilon], B_\rho(Z_0))$.

Now we define a function F on $\mathcal{C}([-\varepsilon,\varepsilon], B_\rho(Z_0)) \in \mathcal{C}([-\varepsilon,\varepsilon], \mathcal{B}(\mathcal{H}))$, by $F(Z)(t) = \widetilde{Z}_0 + \int\limits_0^t T(Z(\xi))d\xi = Z_0 + \int\limits_0^t T(Z(\xi))d\xi$.

$$
\begin{array}{rcl}
F : \mathcal{C}([-\epsilon,\, \epsilon],\, \mathcal{B}_\rho(Z_0)) & \longrightarrow & \mathcal{C}([-\epsilon,\, \epsilon],\, \mathcal{B}_\rho(Z_0)) \\
Z : [\varepsilon, \varepsilon] \rightsquigarrow B_\rho(Z_0) & \rightsquigarrow & F(Z) : [-\varepsilon, \varepsilon] \to B_\rho(Z_0) \\
t \rightsquigarrow Z(t) & \rightsquigarrow & t \rightsquigarrow \widetilde{Z}_0(t) + \int_0^t T(Z(\xi))\, d\xi.
\end{array}
$$

Z is an operator-valued function from $[-\varepsilon,\varepsilon]$ into the "ball" $B_\rho(Z_0)$ inside the space of bounded operators $\mathcal{B}(\mathcal{H})$.

We show F is well defined, that is, we show that $\|F(Z)(t) - Z_0\|_\infty \le \rho$, and thus, $F(Z)(t) \in B_\rho(Z_0)$.

$$
\begin{aligned}
\|F(Z)(t) - Z_0\|_\infty &= \left\| \widetilde{Z}_0(t) + \int_0^t [MZ(\xi) - Z(\xi)Z(\xi)^* MZ(\xi)]\, d\xi - Z_0 \right\|_\infty \\
&\le \left\| \int_0^t [MZ(\xi) - Z(\xi)Z(\xi)^* MZ(\xi)]\, d\xi \right\|_\infty \\
&\le \varepsilon\, \|MZ - ZZ^* MZ\|_\infty \\
&\le \varepsilon\, \|I - ZZ^*\|_\infty \|M\|\, \|Z\|_\infty \\
&\le \varepsilon \|M\|\, (1 + \|ZZ^*\|_\infty)\, \|Z\|_\infty \\
&= \varepsilon\, \|M\|\, \left(1 + \|Z\|_\infty^2\right)\, \|Z\|_\infty.
\end{aligned}
$$

Since $Z \in B_\rho(Z_0)$, then $\|Z\| - \|Z_0\| \le \|Z - Z_0\| \le \rho = 2\|Z_0\|$,

and hence, $\|Z\| \le 3\|Z_0\|$; so

$$
\begin{aligned}
\|F(Z)(t) - Z_0\|_\infty &\le \varepsilon \|M\|\, \left(1 + (3\|Z_0\|)^2\right)\, (3\|Z_0\|) \\
&= 3\varepsilon \|M\|\, \left(1 + 9\|Z_0\|^2\right)\, \|Z_0\| \\
&\le 3\varepsilon \|M\|\, \left(1 + 27\|Z_0\|^2\right)\, \|Z_0\| \\
&\le 3\left(\frac{1}{2}\right) \|Z_0\| \le 2\|Z_0\| = \rho.
\end{aligned}
$$

Therefore we have shown that F(Z)(t) $\in \mathcal{B}_\rho(Z_0)$, and thus F is well-defined.

F is also a contraction, since

$$\begin{aligned}
\|F(Z_1)-F(Z_2))\|_\infty &= \left\| Z_0 + \int_0^t T(Z_1(\xi))d\xi - \left(Z_0 + \int_0^t T(Z_2(\xi))d\xi \right) \right\|_\infty \\
&= \left\| \int_0^t \left[T(Z_1(\xi)) - T(Z_2(\xi)) \right] d\xi \right\|_\infty \\
&= sup \left\{ \left\| \int_0^t \left[T(Z_1(\xi)) - T(Z_2(\xi)) \right] d\xi \right\|_\infty : |t| \le \varepsilon \right\} \\
&\le \varepsilon \|T(Z_1) - T(Z_2)\|_\infty \\
&\le \varepsilon \|M\| \left(1 + 27\|Z_0\|^2\right) \|Z_1 - Z_2\| \text{ , by equation (5.2);} \\
&= \theta \|Z_1 - Z_2\| < \frac{1}{2}\|Z_1 - Z_2\|, \quad \text{by equation (5.3).}
\end{aligned}$$

Since this inequality holds for every $Z_1, Z_2 \in B_\rho(Z_0)$, Theorem 5.1, on page 39, asserts that F has a unique fixed point, i.e. there exists $Z \in \mathcal{C}\left([-\varepsilon,\varepsilon], B_\rho(Z_0)\right)$ such that

$$F(Z)(t) = Z(t) = Z_0 + \int_0^t T(Z(\xi))d\xi = Z_0 + \int_0^t \left[MZ(\xi) - Z(\xi)Z^*(\xi)MZ(\xi)\right] d\xi.$$

Therefore, taking the derivative, $\dot{Z}(t) = MZ(t) - Z(t)Z(t)^*MZ(t)$ and $Z(0) = Z_0$.

We now show that $\dot{Z}(t) = \lim\limits_{h\to 0} \frac{Z(t+h)-Z(t)}{h} = T(Z(t)) = MZ(t) - Z(t)Z(t)^*MZ(t)$ uniformly:

Since we concluded above that T satisfies the Lipschitz condition, then

$$\|TZ_1 - TZ_2\| \le K\|Z_1 - Z_2\|, \text{ for some } K > 0.$$

The map

$$Z : [-\varepsilon,\varepsilon] \to B_\rho(Z_0) \;\; (t \rightsquigarrow Z(t) \in B_\rho(Z_0) \subseteq \mathcal{B}(\mathcal{H}))$$

is continuous and $Z(t)$ is a continuous bounded operator. Thus given $\varepsilon_1 > 0$, there exists a $\delta > 0$, so that if $|h| < \delta$, then $K\|Z(t+h) - Z(t)\| < \varepsilon_1$. Without loss of generality we assume $h > 0$, a similar reasoning applies for $h < 0$.

$$\left\| \frac{Z(t+h) - Z(t)}{h} - T(Z(t)) \right\| = \frac{1}{h} \|Z(t+h) - Z(t) - hT(Z(t))\|$$

$$
\begin{aligned}
&= \frac{1}{h}\left\| \int_0^{t+h} T(Z(\xi))d\xi - \int_0^{t} T(Z(\xi))d\xi - h\left(\int_0^1 d\xi\right) T(Z(t))\right\| \\
&= \frac{1}{h}\left\| \int_t^{t+h} T(Z(\xi))d\xi - (t+h-t)\int_0^1 T(Z(t))d\xi\right\| \\
&= \frac{1}{h}\left\| \int_t^{t+h} T(Z(\xi))d\xi - \int_t^{t+h} T(Z(t))d\xi\right\| \\
&\leq \frac{1}{h}\int_t^{t+h} \|T(Z(\xi)) - T(Z(t))\|\, d\xi \\
&\leq \frac{1}{h}\int_t^{t+h} K\,\|Z(\xi) - Z(t)\|\, d\xi \\
&< \frac{1}{h}\int_t^{t+h} K\frac{\varepsilon_1}{K}d\xi = \frac{\varepsilon_1}{h}\int_t^{t+h} d\xi \\
&\leq \varepsilon_1. \qquad \square
\end{aligned}
$$

We observe that each such solution is a differentiable path of bounded operators.

5.2 The Polar Representation of Operators

Every complex number can be written as the product of a nonnegative number and a complex number of modulus one, i.e., $z = re^{i\theta}$. A polar form for an operator on $\mathbb{C}^n$ is represented as a product of a positive operator and a unitary operator. For operators on an infinite-dimensional Hilbert space, a similar result is valid and the representation obtained is, under suitable hypotheses, unique.

We use the polar decomposition [47, Ringrose, p. 48] of operators to separate our system into two unique systems: the "scalar" system and the "polar" system associated with (5.1). Before proving this result we need to introduce the notion of a partial isometry and the polar decomposition of an operator. We now collect some definitions and results to be used in the forthcoming proofs.

Definition 5.5. *(Furuta [24, p. 52]) An operator U on a Hilbert space $\mathcal{H}$ is a partial isometry if there exists a closed subspace D of $\mathcal{H}$ such that*

$$\|Ux\| = \|x\| \text{ for any } x \in D,$$
$$\text{and } Ux = 0 \text{ for any } x \in D^{\perp} = \{x \in \mathcal{H} : \langle x, y\rangle = 0 \;\; \forall\, y \in D\}.$$

Remark 5.6. *It is shown in Furuta [24, Theorem 3, p. 55] that an operator U is a partial isometry if and only if $UU^*U = U$.*

Definition 5.7. *(Furuta [24] p 38) T is a positive operator, denoted by $T \geq 0$, if the inner product $(Tx, x) \geq 0$ for all $x \in H$.*

Lemma 5.8. *(Furuta [24, pp. 57-58]) Let T be a bounded operator on a Hilbert space $\mathcal{H}$, equipped with an inner product denoted by $\langle\ ,\ \rangle$, then*

1. *There exists a unique positive bounded operator S so that $S^2 = T^*T$. The operator S is self-adjoint and is denoted by $|T|$ (or $\sqrt{T^*T}$).*
2. *$T = P\,|T|$, with P a partial isometry.*

Theorem 5.9. ***Square root of a positive operator.*** *(Furuta [24, p. 46], or Retherford [46, p. 72]). For any positive operator A, there exists the unique positive operator S such that $S^2 = A$ (denoted by $S = A^{\frac{1}{2}}$).*

By definition of adjoint operator, $\langle x, Z_0 y\rangle = \langle Z_0^* x, y\rangle$ holds $\forall$ x, y $\in \mathcal{H}$.

Lemma 5.10. *(Christensen [16, p. 40]) Every bounded and positive operator $U : \mathcal{H} \to \mathcal{H}$ has a unique bounded and positive square root W. The operator W has the following properties:*

(i) If U is self-adjoint, then W is self-adjoint.

(ii) If U is invertible, then W is also invertible.

(iii) W can be expressed as a limit (in the strong operator topology) of a sequence of polynomials in U, and commutes with U.

The decomposition of T stated in Lemma 5.8.2 is unique and is called the **polar decomposition** of the operator T, whenever the kernel of P = the kernel of $|T|$. In particular, Lemma 5.8.2 also implies that for the operator $Z_0^* \in \mathcal{H}$, $Z_0^* = Q_0\sqrt{(Z_0^*)^* Z_0^*} = Q_0\sqrt{Z_0 Z_0^*}$, where Q_0 is a partial isometry. Therefore we have that

$$Z_0 = (Z_0^*)^* = (Q_0\sqrt{Z_0 Z_0^*})^* = \sqrt{(Z_0 Z_0^*)^*}\, Q_0^* = \sqrt{Z_0 Z_0^*} P_0, \text{ with } P_0 = Q_0^*.$$

Hence,

$$Z_0 = \sqrt{Z_0 Z_0^*} P_0. \tag{5.4}$$

For $t \in (-\varepsilon, \varepsilon)$, let $Z(t)$ denote a local solution of (5.1) and set

$$V = ZZ^*.$$

$$\begin{aligned}
\dot{V} &= \frac{d}{dt}[ZZ^*] \\
&= \lim_{h\to 0} \frac{Z(t+h)Z^*(t+h) - Z(t)Z^*(t)}{h} \\
&= \lim_{h\to 0} \frac{Z(t+h)Z^*(t+h) - Z(t)Z^*(t+h) + Z(t)Z^*(t+h) - Z(t)Z^*(t)}{h} \\
&= \lim_{h\to 0} \left[\frac{Z(t+h) - Z(t)}{h} Z^*(t+h) + Z(t)\frac{Z^*(t+h) - Z^*(t)}{h}\right] \\
&= \dot{Z}Z^* + Z\dot{Z}^*.
\end{aligned}$$

It is a routine calculation to verify that $V(t)$ is a local solution of the system

$$\begin{cases} \dot{V} = MV + VM^* - VMV - VM^*V \\ V(0) = V_0 = Z_0 Z_0^*. \end{cases} \tag{5.5}$$

In fact, if Z(t), $t \in (-\varepsilon, \varepsilon)$, is a solution of the initial valued problem (5.1), then

$$\dot{Z} = MZ - ZZ^*MZ. \tag{5.6}$$

Multiplying (5.6) on the right by Z^*, yields

$$\dot{Z}Z^* = MZZ^* - ZZ^*MZZ^* = MV - VMV. \tag{5.7}$$

Taking the adjoint of (5.6),

$$\dot{Z}^* = Z^*M^* - Z^*M^*ZZ^*. \tag{5.8}$$

Multiplying (5.8) on the left by Z produces

$$Z\dot{Z}^* = ZZ^*M^* - ZZ^*M^*ZZ^* = VM^* - VM^*V. \tag{5.9}$$

Adding (5.7)and (5.9) together, we have

$$(\dot{ZZ^*}) = \dot{Z}Z^* + Z\dot{Z}^* = MV + VM^* - VMV - VM^*V.$$

Therefore,

$$\dot{V} = \frac{d}{dt}V = \frac{d}{dt}ZZ^* = MV + VM^* - V(M + M^*)V. \tag{5.10}$$

In the next section we consider for $\varepsilon > 0$, the map $V : (-\varepsilon, \varepsilon) \to \mathcal{B}(\mathcal{H})$, defined by $t \rightsquigarrow Z(t)Z(t)^*$.

5.2.1 The "Scalar" System

In this section we first establish the local existence of a unique differentiable map $V(t)$, which satisfies system (5.5). By standard local existence and uniqueness of solutions theorems, this solution must locally be given by

$$V(t) = Z(t)Z(t)^*.$$

Next we use Fuglede-Putnam Theorem 5.14, and employ Picard's iterative method to establish commutativity properties of $\{V(t)\}_{t\in(\varepsilon,\,\varepsilon)}$. We then establish the local existence of the operator V^{-1} and derive several properties which allow us to construct the "scalar" system in the polar decomposition of the solution to system (5.1).

The following theorem is a consequence of Theorem 5.1.

Theorem 5.11. *If M and Z_0 are bounded operators on a complex Hilbert space $\mathcal{H}$, then there exist a positive number ε and a unique differentiable map $V : (-\varepsilon,\, \varepsilon) \to \mathcal{B}(\mathcal{H})$ such that $\dot{V}(t) = MV(t) + V(t)M^* - V(t)MV(t) - V(t)M^*V(t)$ and $V(0) = V_0 = Z_0Z_0^*$.*

Proof: Since the proof follows similar arguments to those given for the Local Existence Theorem 5.4, it is omitted. It establishes that $T : B_\rho(V_0) \to \mathcal{B}(\mathcal{H})$, given by $V \to MV + VM^* - VMV - VM^*V$, satisfies a Lipschitz condition. □

The local solution described in Theorem 5.11 is a differentiable path of bounded operators. By Tychonov Fixed Point Theorem V(t) is unique; consequently we must have that $V(t) \equiv Z(t)Z(t)^*$.

Definition 5.12. *(Furuta [24, p. 52]) An operator U on a Hilbert space $\mathcal{H}$ is said to be an **isometry operator** if*

$$\|Ux\| = \|x\| \text{ for any } x \in H,$$

$$\langle Ux, Uy\rangle = \langle x, y\rangle \text{ for any } x, y \in \mathcal{H}.$$

*An operator U on a Hilbert space $\mathcal{H}$ is said to be a **unitary operator** if if U is an isometry operator from $\mathcal{H}$ onto $\mathcal{H}$.*

Theorem 5.13. *(Furuta [24])*

(i) An operator U on a Hilbert space $\mathcal{H}$ is an isometry operator if and only if $U^*U = I$.

(ii) An operator U on a Hilbert space $\mathcal{H}$ is a unitary operator if and only if $U^*U = UU^* = I$.

Fuglede-Putnam Theorem stated next will be used to establish commutativity properties of the family $\{V(t)\}_{t\in(-\varepsilon,\varepsilon)}$.

Theorem 5.14. *(Furuta [24, Theorem F-P (Fuglede-Putnam), p. 67]) Let A and B be normal operators (i.e. $A^*A = AA^*$). If $AX = XB$ holds for some operator X, then $A^*X = XB^*$.*

Recall that for $t \in (-\varepsilon, \varepsilon)$, $V(t) = Z(t)Z(t)^*$ satisfies system (5.5) on page 46. Under certain commutativity assumptions, we derive an explicit form for the local solution of system (5.5). If, in addition to the normality of M, we assume M and Z_0 commute, and Z_0 is invertible, we use Picard's Iterative method to show that V(t) is a family of commuting and invertible operators that also commute with M. This allows us to give a different form to (5.5) and reduce the system to an Euler type equation.

Proposition 5.15. *Let M and Z_0 be bounded operators on a complex Hilbert space $\mathcal{H}$, For $k = 1, ...,$ let*

$$\begin{aligned} T(t, V_1) &= M\,V_0 + V_0 M^* - V_0 M V_0 - V_0 M^* V_0, \\ T(t, V_k) &= M V_{k-1}(t, V_0) + V_{k-1}(t, V_0) M^* \\ &\qquad - V_{k-1}(t, V_0) M V_{k-1}(t, V_0) - V_{k-1}(t, V_0) M^* V_{k-1}(t, V_0), \end{aligned}$$

Then there exists $\varepsilon > 0$ such that the sequence

$$V_0(t, V_0) \equiv V_0,$$

$$V_1(t, V_0) = V_0 + \int_0^t T(\xi, V_0) d\xi,$$

$$\vdots$$

$$V_n(t, V_0) = V_0 + \int_0^t T(\xi, V_{n-1}) d\xi,$$

$$\vdots$$

converges uniformly to $V(t) = Z(t)Z^*(t)$ for $t \in (-\varepsilon,\ \varepsilon)$.

Proof: We choose ϵ such that $2\|M\|\,(1 + 4\|V_0\|)\,\varepsilon \,<\, \frac{1}{4}$. We recall that $V \in B_\rho(V_0)$ satisfies system (5.5), and also satisfies the inequality

$$\|V - V_0\| \le \rho, \quad \texttt{for}\ |t| \le \varepsilon.$$

$T(t, V) = MV(t) + V(t)M^* - V(t)MV(t) - V(t)M^*V(t)$ is a continuous operator-valued function of t and V, with values in $\mathcal{B}(\mathcal{H})$. It is easy to see that there exists N such that $\|T(t, V)\| \le N$ for $|t| \le \epsilon$. We set $\delta = min\{\varepsilon,\ \rho/N\}$. Also, T(t,V) satisfies the Lipschitz condition, i.e. for $|t| \leq \delta$, and any $V_1, V_2 \in B_\rho(V_0)$,

$$\|T(t,V_1) - T(t,V_2)\| < K\|V_1 - V_2\|, \text{ for some } K > 0. \tag{5.11}$$

By employing Picard's Iterative Method, or The Method of Successive Approximations (Ince [35, pp. 62-66], Hartman [29, pp. 8-10], or Rainville [45, pp. 266-267]), we show that for all values of $t \in (-\delta, \delta)$,

$$V(t) = V_0 + \int_0^t T(\xi, V(\xi))d\xi$$

may be defined as the uniform limit of the sequence of operators $V_n(t, V_0)\}_n$:

For t in the interval $(-\delta, \delta)$, we consider the sequence $\{V_n(t, V_0)\}_n$ given by $V_0(t, V_0) \equiv V_0$, and

$$V_n(t, V_0) = V_0 + \int_0^t T(\xi, V_{n-1})d\xi. \tag{5.12}$$

We follow an induction procedure to show that

$$\|V_n(t, V_0) - V_0\| \leq \rho, \text{ for } |t| < \delta.$$

Suppose that $\|V_{n-1}(t, V_0) - V_0\| \leq \rho$.

Since $\|T(t, V_{n-1})\| \leq N$, we have

$$\begin{aligned} \|V_n(t, V_0) - V_0\| &= \left\| \int_0^t T(\xi, V_{n-1})d\xi \right\| \\ &\leq \int_0^{|t|} \|T(\xi, V_{n-1})\| d\xi \\ &\leq N\,|t| \leq N\delta \leq \rho. \end{aligned}$$

We now prove that

$$\|V_n(t, V_0) - V_{n-1}(t, V_0)\| \leq \frac{NK^{n-1}}{n!} \, |t|^n. \tag{5.13}$$

Clearly, n = 1 holds:

$$\begin{aligned}
\|V_1(t, V_0) - V_0(t)\| &= \left\| V_0 + \int_0^t T(\xi, V_0) d\xi - V_0(t)) \right\| \\
&= \left\| \int_0^t T(\xi, V_0) d\xi \right\| \\
&\leq \int_0^{|t|} \|T(\xi, V_0)\| \, d\xi \\
&\leq N \, |t| \\
&= \frac{NK^{1-1}}{(1)!} \, |t|^1.
\end{aligned}$$

Now suppose that, for $|t| \leq \delta$, then

$$\|V_{n-1}(t, V_0) - V_{n-2}(t, V_0)\| \leq \frac{NK^{n-2}}{(n-1)!} \, |t|^{n-1}.$$

Then by (5.12),

$$\begin{aligned}
\|V_n(t, V_0) - V_{n-1}(t, V_0)\| &\leq \int_0^{|t|} \|T(\xi, V_{n-1}) - T(\xi, V_{n-2})\| d\xi \\
&\leq \int_0^{|t|} K\|V_{n-1}(\xi, V_0) - V_{n-2}(\xi, V_0)\| d\xi, \text{ by Lipschitz condition (5.11),} \\
&\leq K \int_0^{|t|} \frac{NK^{n-2}}{(n-1)!} \xi^{n-1} d\xi \\
&= \frac{NK^{n-1}}{(n-1)!} \int_0^{|t|} \xi^{n-1} d\xi \\
&= \frac{NK^{n-1}}{n!} \, |t|^n.
\end{aligned}$$

Therefore the inequality (5.13) holds for all values of n. Hence the series

$$V_0 + \sum_{j=1}^{\infty}(V_j(t, V_0) - V_{j-1}(t, V_0))$$

is uniformly convergent when $|t| \leq \delta$.

We show that each term is continuous: Given $\varepsilon > 0$, we find $\delta > 0$, so that if $|t_1 - t_2| < \delta$, then

$\|V_j(t_1, V_0) - V_{j-1}(t_1, V_0) - [V_j(t_2, V_0) - V_{j-1}(t_2, V_0)]\,\| < \varepsilon.$

$\|V_j(t_1, V_0) - V_j(t_2, V_0) + V_{j-1}(t_1, V_0) - V_{j-1}(t_2, V_0)\|$

$\leq \left\|V_0 + \int_0^{t_1} T(\xi, V_{j-1})d\xi - \left(V_0 + \int_0^{t_2} T(\xi, V_{j-1})d\xi\right)\right\| + \left\|\int_{t_2}^{t_1} T(\xi, V_{j-2})d\xi\right\|$

$= \left\|\int_{t_2}^{t_1} T(\xi, V_{j-1})d\xi\right\| + \left\|\int_{t_2}^{t_1} T(\xi, V_{j-2})d\xi\right\|$

$\leq |t_1 - t_2|\,\|T\| + |t_1 - t_2|\,\|T\|$

$= 2|t_1 - t_2|\,\|T\| < \varepsilon$, if $|t_1 - t_2| < \frac{\varepsilon}{2\|T\|}$.

Hence we see that each term is a continuous operator-valued function of t.

But

$$V_n(t, V_0) = V_0 + \sum_{j=1}^{n}(V_j(t, V_0) - V_{j-1}(t, V_0)).$$

Consequently the limit function

$$V(t) = \lim_{n\to\infty} V_n(t, V_0) \tag{5.14}$$

exists and is a continuous operator-valued function of t in the interval $(-\delta, \delta)$. So for each n = 1, 2,....., there exists an ε_n, satisfying the inequality

$$\|V(\xi, V_0) - V_{n-1}(\xi, V_0)\| \leq \varepsilon_n.$$

Thus

$$\left\| \int_0^t [T(\xi, V(\xi)) - T(\xi, V_{n-1})] d\xi \right\| \leq \int_0^{|t|} K \|V(\xi, V_0) - V_{n-1}(\xi, V_0))\| d\xi \quad \text{by (5.11)}$$
$$\leq K\, \varepsilon_n |t|$$
$$\leq K\, \varepsilon_n\, \delta, \tag{5.15}$$

where ε_n is independent of t and tends to zero as n tends to infinity.

Therefore

$$\begin{aligned} \lim_{n\to\infty} V_n(t, V_0) &= V_0 + \lim_{n\to\infty} \int_0^t T(\xi, V_{n-1}) d\xi \\ &= V_0 + \int_0^t \lim_{n\to\infty} T(\xi, V_{n-1}) d\xi \quad \text{using (5.15)} \\ &= V_0 + \int_0^t T(\xi, V(\xi)) d\xi. \end{aligned}$$

It follows that V(t) is a solution of the integral equation

$$V(t) = V_0 + \int_0^t T(\xi, V(\xi)) d\xi.$$

The operator-valued function $T(\xi, V(\xi))$ is continuous in the interval $(-\delta, \delta)$.

Consequently

$$\dot{V}(t) = \frac{d}{dt} \int_0^t T(\xi, V(\xi)) d\xi = T(t, V(t)).$$

Thus the limit-function V(t) satisfies the differential equation; and $V(0) = V_0$. □

Proposition 5.16. *If M is a normal operator (i.e. $M^*M = MM^*$) that commutes with Z_0, then for $t \in (-\varepsilon, \varepsilon)$, $V(t)$ commutes with both M and M^*.*

Proof: We use induction to establish the commutativity of both M and M^* with $V_n(t, V_0)$, for each n. Then the result follows from Proposition 5.15. Fuglede-Putnam Theorem 5.14 asserts that if a normal operator M commutes with an operator Z_0, then M^* also commutes with Z_0. This implies that Z_0 and Z_0^* commute with both M and M^*. Therefore $V_0 = Z_0 Z_0^*$ also commutes with both M and M^*. Since $T(\xi, V_0) = MV_0 + V_0M^* - V_0MV_0 - V_0M^*V_0$,

$$V_1(t, V_0)M = \left(V_0 + \int_0^t T(\xi, V_0) d\xi\right) M$$

$$= \left(V_0 + \int_0^t \left(MV_0 + V_0M^* - V_0MV_0 - V_0M^*V_0\right) d\xi\right) M.$$

And since M is normal and commutes with V_0,

$$= M\left(V_0 + \int_0^t \left(MV_0 + V_0M^* - V_0MV_0 - V_0M^*V_0\right) d\xi\right).$$

Therefore

$$V_1(t, V_0)M = MV_1(t, V_0).$$

Similarly, $V_1(t, V_0)$ also commutes with M^*.

We now assume V_{n-1} commutes with both M and M^*. Then each term of

$T(\xi, V_{n-1}) =$

$MV_{n-1}(\xi, V_0)+V_{n-1}(\xi, V_0)M^*-V_{n-1}(\xi, V_0)MV_{n-1}(\xi, V_0)-V_{n-1}(\xi, V_0)M^*V_{n-1}(\xi, V_0)$

commutes with both M and M^*. Thus

$V_n(t, V_0)M = \left(V_0 + \int_0^t T(\xi, V_{n-1})d\xi\right) M = M\left(V_0 + \int_0^t T(\xi, V_{n-1})d\xi\right) = MV_n(t, V_0).$

Hence $V_n(t, V_0)M = MV_n(t, V_0)$ for every n.

Similarly, $V_n(t, V_0)$ commutes with M^* for every n. Therefore by (5.14),

$$V(t)M = \lim_{n\to\infty} V_n(t, V_0)M = \lim_{n\to\infty} MV_n(t, V_0) = MV(t).$$

and

$$V(t)M^* = \lim_{n\to\infty} V_n(t, V_0)M = \lim_{n\to\infty} MV_n(t, V_0) = M^*V(t).$$

□

Now we show that $\{V(t)\}_{t\in(-\varepsilon,\varepsilon)}$, is a family of commuting operators.

Proposition 5.17. *If M is a normal operator that commutes with Z_0, then $\{V(t)\}_{t\in(-\varepsilon,\varepsilon)}$ is a family of commuting operators for $t \in (-\varepsilon, \varepsilon)$.*

Proof: If W is an operator that commutes with M, then since M is normal, it follows from Fuglede-Putman Theorem 5.14, that W commutes with both M and M^*. Thus $MW + WM^* - WMW - WM^*W$ also commutes with both M and M^*. Since $V_n(t, V_0)$ is a polynomial in t with commuting operators as coefficients, it follows from an induction argument that

$$V_n(t_1, V_0)V_n(t_2, V_0) = V_n(t_2, V_0)V_n(t_1, V_0)$$

for t_1 and t_2 in the interval $(-\varepsilon, \varepsilon)$. Therefore $V(t_1)$ and $V(t_2)$ commute. □

Lemma 5.18. *If $Z_0 \in \mathcal{B}(\mathcal{H})$ is invertible, then Z_0^* and $Z_0Z_0^*$ are invertible.*

Proof: Assume Z_0 is invertible. Then

$$Z_0^*(Z_0^{-1})^* = (Z_0^{-1}Z_0)^* = I^* = (Z_0Z_0^{-1})^* = (Z_0^{-1})^*Z_0.$$

This implies that Z_0^* is invertible and $(Z_0^*)^{-1} = (Z_0^{-1})^*$.

We also have that

$$(Z_0 Z_0^*)\ [(Z_0^*)^{-1} Z_0^{-1}] \ = \ Z_0[Z_0^*(Z_0^*)^{-1}]Z_0^{-1} \ = \ Z_0 I Z_0^{-1} = Z_0 Z_0^{-1} = I,$$

and

$$[(Z_0^*)^{-1} Z_0^{-1}]\ (Z_0 Z_0^*) = (Z_0^*)^{-1}(Z_0^{-1} Z_0) Z_0^* = (Z^*)^{-1} I Z_0^* = I.$$

Therefore, $V_0 = Z_0 Z_0^*$ is invertible. □

We define an algebra, a Banach algebra $\mathfrak{B}$, and establish invertibility of an element in $\mathfrak{B}$:

Definition 5.19. *(Royden, [48, p. 210]) A linear space A of functions in $\mathcal{C}(\mathcal{X})$, the set of all continuous real-valued functions on X, is called an **algebra** if the product of any two elements in A is again in A.*

Definition 5.20. *(Douglas, [21, p. 31]) A **Banach Algebra** $\mathfrak{B}$ is an algebra over $\mathbb{C}$ with identity 1 which has a norm making it into a Banach space and satisfying $\|1\| = 1$ and the inequality $\|fg\| \leq \|f\|\,\|g\|$, for f and g in $\mathfrak{B}$.*

Proposition 5.21. *(Douglas [21, Proposition 2.5, p. 32]). If f is in the Banach algebra $\mathfrak{B}$ and $\|I - f\| < 1$, then f is invertible.*

Proposition 5.22. *(Douglas [21, Proposition 2.7, p. 32]). The set of invertible operators is open in $\mathcal{B}(\mathcal{H})$.*

Thus by Proposition 5.22 since V_0 is an invertible operator, there exists an $\varepsilon > 0$ such that every $V(t)$ in the open ball $\overset{o}{B}_\varepsilon(V_0) = \{V \in \mathcal{B}(\mathcal{H}) : \|V_0 - V\| < \varepsilon\}$ is invertible.

For completeness we provide the following details. Since the map $V : [\varepsilon, \varepsilon] \to B_\rho(V_0) \subseteq \mathcal{B}(\mathcal{H})$, given by $t \to V(t) \in B_\rho(V_0)$ is continuous, and V(t) is a continuous bounded operator, given $\varepsilon = \frac{1}{\|V_0^{-1}\|}$ there exists $\delta > 0$ so that if $|t| < \delta$, then $\|V(0) - V(t)\| < \frac{1}{\|V_0^{-1}\|}$. Then $1 > \|V_0^{-1}\| \, \|V_0 - V(t)\| \geq \|I - V_0^{-1}V(t)\|$.

Thus Proposition 5.21 implies that the operator $V_0^{-1}V(t)$ is invertible, and thus $V(t) = V_0(V_0^{-1}V(t))$ is invertible for $|t| < \delta$.

Now we assume that $V(t)$, for $t \in (-\varepsilon, \varepsilon)$, is an invertible operator that commutes with M.

Lemma 5.23. *(Furuta [24, Corollary 2, p. 36]) Let T be an operator on $\mathcal{H}$. Then $\|T^*T\| = \|TT^*\| = \|T\|^2$.*

Since $\langle ZZ^*x, x\rangle = \langle Z^*x, Z^*x\rangle = \|Z^*x\|^2 \geq 0$, the operator Z^*Z is positive, thus we denote the unique positive square root of $V = ZZ^*$ by $V^{\frac{1}{2}}$ (Theorem 5.9, p. 44).

We now list additional properties of the local family of operators $\{V(t)\}_{t\in(-\varepsilon,\varepsilon)}$.

Lemma 5.24. *If Z_0 is invertible, M is normal, $MZ_0 = Z_0M$, and $\{V(t)\}_{t\in(-\varepsilon,\varepsilon)}$ is a local solution of (5.5), then*

1. *Z_0^* and $V_0 = Z_0Z_0^*$ are invertible, with $(Z_0^*)^{-1} = (Z_0^{-1})^*$.*
2. *$(V^{-1})^* = V^{-1}$.*
3. *$V^{1/2} = \sqrt{V}$ is invertible, and $\left(\sqrt{V}\right)^{-1} = \sqrt{V^{-1}}$; also denote by $V^{-1/2}$.*
4. *$\left((\sqrt{V})^{-1}\right)^* = \left(\sqrt{V}\right)^{-1}$.*
5. *$MV^{-1} = V^{-1}M$.*
6. *$V^{-1}M^* = M^*V^{-1}$.*
7. *$\dot{V}V^{-1} = V^{-1}\dot{V}$.*
8. *$\dot{V}V^{-\frac{1}{2}} = V^{-\frac{1}{2}}\dot{V}$, or equivalently, $\dot{V}\left(\sqrt{V}\right)^{-1} = \left(\sqrt{V}\right)^{-1}\dot{V}$.*

Proof:

1. Prove Z_0^* is invertible:

$$(Z_0^*)\left(Z_0^{-1}\right)^* = \left(Z_0^{-1}\, Z_0\right)^* = I^* = \left(Z_0\, Z_0^{-1}\right)^* = \left(Z_0^{-1}\right)^* (Z_0^*) .$$

   Thus, $(Z_0^*)^{-1} = (Z_0^{-1})^*$.

   Now we prove that $V_0 = Z_0 Z_0^*$ is invertible:

   $$\begin{aligned}\left[(Z_0^*)^{-1}\, Z_0^{-1}\right](Z_0\, Z_0^*) &= (Z_0^*)^{-1}\,\left(Z_0^{-1} Z_0\right)\, Z_0^* \\ &= (Z_0^*)^{-1}\; I\, Z_0^* = (Z_0^*)^{-1}\; Z_0^* = I.\end{aligned}$$

   Similarly, $(Z_0\, Z_0^*)\left[(Z_0^*)^{-1}\, Z_0^{-1}\right] = I$. And V_0 is invertible.

2. Prove $(V^{-1})^* = V^{-1} : (V^{-1}V)^* = (VV^{-1})^* = I$.

   Thus $V^*(V^{-1})^* = (V^{-1})^* V^* = I$.

   Since V is self-adjoint $V(V^{-1})^* = (V^{-1})^* V = I$. Consequently $(V^{-1})^* = V^{-1}$.

3. Prove $\sqrt{V}$ is invertible:

   Let x be an arbitrary element in $\mathcal{H}$. Then there exists $y \in \mathcal{H}$, such that $Vx = y$. Therefore, the inner product

$$\langle V^{-1}y, y\rangle = \langle V^{-1}(Vx), Vx\rangle = \langle x, Vx\rangle = \overline{\langle Vx, x\rangle} \geq 0,$$

   since V is a positive operator. Hence V^{-1} is a positive operator. Thus by Theorem 5.9 (p. 44) there exists a unique positive operator A such that $A^2 = V^{-1}$, and $A = \sqrt{V^{-1}}$. Thus $A^2 = \sqrt{V^{-1}}\sqrt{V^{-1}} = V^{-1}$.

   But we also have that for the operator $B = \left(\sqrt{V}\right)^{-1}$,

$$B^2 = \left[\left(\sqrt{V}\right)^{-1}\right]^2 = \left(\sqrt{V}\right)^{-1}\left(\sqrt{V}\right)^{-1} = \left(\sqrt{V}\,\sqrt{V}\right)^{-1} = V^{-1}.$$

In other words, the operator B also satisfies $B^2 = V^{-1}$. Since the operator A is unique, we must have that $\left(\sqrt{V}\right)^{-1} = \sqrt{V^{-1}}$, and $\sqrt{V}$ is invertible. For simplicity we also denote the inverse by $V^{-1/2}$.

4. Prove $\left(V^{-\frac{1}{2}}\right)^* = \left(V^{\frac{1}{2}}\right)^{-1}$:
   $\sqrt{V}\left(\sqrt{V}\right)^{-1} = I = \left(\sqrt{V}\right)^{-1}\sqrt{V}$
   $\left(\left(\sqrt{V}\right)^{-1}\right)^*\left(\sqrt{V}\right)^* = I = \left(\sqrt{V}\right)^*\left(\left(\sqrt{V}\right)^{-1}\right)^*$
   Thus $\left(\left(\sqrt{V}\right)^{-1}\right)^* = \left(\left(\sqrt{V}\right)^*\right)^{-1} = \left(\sqrt{V}\right)^{-1}$, by Lemma 5.10, p. 44.

5. Prove $MV^{-1} = V^{-1}M$: Since V and M commute, and V is invertible, we have
   $V^{-1}(VM)V^{-1} = V^{-1}(MV)V^{-1}$.
   Hence $MV^{-1} = V^{-1}M$.

6. The equation $V^{-1}M^* = M^*V^{-1}$ follows from (2) and (5) above.

7. Prove $\dot{V}V^{-1} = V^{-1}\dot{V}$: Now $\dot{V} = MV + VM^* - VMV - VM^*V$. (see (5.10)). Since V and V^{-1} commute with both M and M^*, then

$$\dot{V} = (M + M^*)V - (M + M^*)V^2 = (M + M^*)V - V(M + M^*)V.$$

Thus

$$\dot{V}V^{-2} = (M + M^*)V^{-1} - (M + M^*),\ \textit{and} \tag{5.16}$$

$$V^{-1}\dot{V}V^{-1} = V^{-1}(M + M^*) - (M + M^*) = (M + M^*)V^{-1} - (M + M^*).$$

Therefore

$$V^{-1}\dot{V}V^{-1} = \dot{V}V^{-2} \quad \textit{and thus} \quad V^{-1}\dot{V} = \dot{V}V^{-1}.$$

8. Prove $\dot{V}\left(\sqrt{V}\right)^{-1} = \left(\sqrt{V}\right)^{-1}\dot{V}$: We shall compute the derivative of equation

$$VV^{-\frac{1}{2}} = V^{\frac{1}{2}}. \tag{5.17}$$

The derivative of the left hand side of equation (5.17):

$$\frac{d}{dt}\left[VV^{-\frac{1}{2}}\right] = \lim_{h\to 0}\frac{V(t+h)V^{-\frac{1}{2}}(t+h) - V(t)V^{-\frac{1}{2}}(t)}{h}$$

$$= \lim_{h\to 0}\frac{V(t+h)V^{-\frac{1}{2}}(t+h)\overbrace{-V(t+h)V^{-\frac{1}{2}}(t) + V(t+h)V^{-\frac{1}{2}}(t)}-V(t)V^{-\frac{1}{2}}(t)}{h}$$

$$= \lim_{h\to 0}V(t+h)\frac{V^{-\frac{1}{2}}(t+h) - V^{-\frac{1}{2}}(t)}{h} + \lim_{h\to 0}\frac{V(t+h) - V(t)}{h}V^{-\frac{1}{2}}(t)$$

$$= \lim_{h\to 0}V^{\frac{1}{2}}(t+h)\frac{V^{\frac{1}{2}}(t) - V^{\frac{1}{2}}(t+h)}{h}V^{-\frac{1}{2}}(t) + \lim_{h\to 0}\frac{V(t+h) - V(t)}{h}V^{-\frac{1}{2}}(t)$$

$$= \lim_{h\to 0}V^{\frac{1}{2}}(t+h)\frac{\left(V^{\frac{1}{2}}(t)-V^{-\frac{1}{2}}(t+h)\right)\left[\left(V^{\frac{1}{2}}(t)+V^{\frac{1}{2}}(t+h)\right)\left(V^{\frac{1}{2}}(t)+V^{\frac{1}{2}}(t+h)\right)^{-1}\right]}{h}V^{-\frac{1}{2}}(t)$$

$$+ \lim_{h\to 0}\frac{V(t+h) - V(t)}{h}V^{-\frac{1}{2}}(t)$$

$$= \lim_{h\to 0}V^{\frac{1}{2}}(t+h)\frac{V(t) - V(t+h)}{h}\left(V^{\frac{1}{2}}(t) + V^{\frac{1}{2}}(t+h)\right)^{-1}V^{-\frac{1}{2}}(t)$$

$$+ \lim_{h\to 0}\frac{V(t+h) - V(t)}{h}V^{-\frac{1}{2}}(t)$$

$$= -V^{\frac{1}{2}}(t)\dot{V}(t)\left(V^{\frac{1}{2}}(t) + V^{\frac{1}{2}}(t)\right)^{-1}V^{-\frac{1}{2}}(t) + \dot{V}(t)V^{-\frac{1}{2}}(t)$$

$$= -V^{\frac{1}{2}}\dot{V}\left(2V^{\frac{1}{2}}\right)^{-1}V^{-\frac{1}{2}} + \dot{V}V^{-\frac{1}{2}}$$

$$= -V^{\frac{1}{2}}\dot{V}\left(\frac{1}{2}V^{-\frac{1}{2}}\right)V^{-\frac{1}{2}} + \dot{V}V^{-\frac{1}{2}}$$

$$= -\frac{1}{2}V^{\frac{1}{2}}\dot{V}V^{-1} + \dot{V}V^{-\frac{1}{2}} \tag{5.18}$$

The derivative of the right hand side of equation (5.17):

$\frac{d}{dt}\left[V^{\frac{1}{2}}\right] =$

$$= \lim_{h\to 0} \frac{V^{\frac{1}{2}}(t+h) - V^{\frac{1}{2}}(t)}{h} \left[\left(V^{\frac{1}{2}}(t+h) + V^{\frac{1}{2}}(t)\right)\left(V^{\frac{1}{2}}(t+h) + V^{\frac{1}{2}}(t)\right)^{-1}\right]$$

$$= \lim_{h\to 0} \frac{V(t+h) - V(t)}{h} \left(V^{\frac{1}{2}}(t+h) + V^{\frac{1}{2}}(t)\right)^{-1}$$

$$= \dot{V}(t) \left(V^{\frac{1}{2}}(t) + V^{\frac{1}{2}}(t)\right)^{-1}$$

$$= \dot{V}(t) \left(2V^{\frac{1}{2}}(t)\right)^{-1}$$

$$= \dot{V}(t)\, \frac{1}{2}V^{-\frac{1}{2}}(t) = \frac{1}{2}\dot{V}(t)\, V^{-\frac{1}{2}}(t)$$

Combining this result with (5.18), we have

$$-\frac{1}{2}V^{\frac{1}{2}}\dot{V}V^{-1} + \dot{V}V^{-\frac{1}{2}} = \frac{1}{2}\dot{V}\, V^{-\frac{1}{2}}$$

$$\dot{V}V^{-\frac{1}{2}} - \frac{1}{2}\dot{V}\, V^{-\frac{1}{2}} = \frac{1}{2}V^{\frac{1}{2}}\dot{V}V^{-1}$$

$$\frac{1}{2}\dot{V}\, V^{-\frac{1}{2}} = \frac{1}{2}V^{\frac{1}{2}}\dot{V}V^{-1}$$

$$\dot{V}\, V^{-\frac{1}{2}} = V^{\frac{1}{2}}\dot{V}V^{-1}$$

$$V^{-\frac{1}{2}}\left(\dot{V}\, V^{-\frac{1}{2}}\right)V^{\frac{1}{2}} = V^{-\frac{1}{2}}\left(V^{\frac{1}{2}}\dot{V}V^{-1}\right)V^{\frac{1}{2}}$$

$$V^{-\frac{1}{2}}\dot{V} = \dot{V}V^{-\frac{1}{2}}$$

□

We use the next Theorem in the proof of the following Proposition.

Theorem 5.25. *(Rudin, [49, Theorem 10.12, p. 235]) If A is a Banach algebra, then $G(A)$, the set of all invertible elements of A, is an open subset of A, and the mapping $x \to x^{-1}$ is a homeomorphism of $G(A)$ onto $G(A)$.*

Proposition 5.26. *Let Z_0 be an invertible operator in $\mathcal{B}(\mathcal{H})$, M normal, and $MZ_0 = Z_0 M$. If, for $t \in (-\varepsilon, \varepsilon)$, $V(t)$ is a solution of the system*

$$\begin{cases} \dot{V} = MV + VM^* - VMV - VM^*V \\ V(0) = V_0 = Z_0 Z_0^*, \end{cases}$$

then

$$V(t) = (I + Ce^{-(M+M^*)\,t})^{-1},$$

with $C = V_0^{-1} - I$.

Proof: We select $\varepsilon > 0$ so that $V(t)$ is invertible for every $t \in (-\varepsilon, \varepsilon)$, and let $(\dot{V^{-1}})$ denote the derivative of V^{-1} with respect to t. Then $(\dot{V^{-1}}) = -V^{-2}\dot{V}$:

$$\begin{aligned} (\dot{V^{-1}}) &= \lim_{h\to 0} \frac{V^{-1}(t+h) - V^{-1}(t)}{h} \\ &= \lim_{h\to 0} V^{-1}(t+h)V^{-1}(t)\frac{V(t) - V(t+h)}{h} \\ &= -V^{-2}\dot{V}. \end{aligned}$$

We claim the convergence is uniform for $t \in (-\varepsilon, \varepsilon)$, We want to show that for fixed $t \in (-\varepsilon, \varepsilon)$, given $\varepsilon_1 > 0$, there exists $\delta > 0$, so that if $|h| < \delta$,

$$\left\| (\dot{V^{-1}}) - \left(-V^{-2}\dot{V}\right) \right\| < \varepsilon_1.$$

Since V is invertible, $0 < \|V^{-1}\|$ is bounded (see Douglas [21, p. 76]). And since the map $t \to V(t)$ is continuous, and $V(t)$ is a continuous bounded operator, the map $t \to V^{-1}(t)$ is continuous since it is a composition of continuous functions $(t \to V(t) \to V^{-1}(t))$, see Rudin [49, Theorem 10.12, p. 235]). Therefore there exists $\delta_1 > 0$ so that if $|h| < \delta_1$, then $\|V(t+h) - V(t)\| < \frac{|h|}{\|V^{-1}(t)\|}$; and there exists

$\delta_2 < 0$ so that if $|h| < \delta_2$, then $\|V^{-1}(t+h) - V^{-1}(t)\| < \epsilon_1$. If $|h| < \delta = min\{\delta_1, \delta_2\}$, then

$$\left\| \frac{V^{-1}(t+h) - V^{-1}(t)}{h} + V^{-2}(t) \frac{V(t+h) - V(t)}{h} \right\|$$

$$= \left\| V^{-1}(t+h)V^{-1}(t) \left[\frac{V(t) - V(t+h)}{h} \right] + V^{-2}(t) \frac{V(t+h) - V(t)}{h} \right\|$$

$$\leq \frac{1}{|h|} \left\| -V^{-1}(t+h)V^{-1}(t) \left[V(t+h) - V(t) \right] + V^{-2}(t) \left[V(t+h) - V(t) \right] \right\|$$

$$= \frac{1}{|h|} \left\| \left[-V^{-1}(t+h)V^{-1}(t) + V^{-2}(t) \right] \left[V(t+h) - V(t) \right] \right\|$$

$$= \frac{1}{|h|} \left\| \left[V^{-1}(t+h) - V^{-1}(t) \right] \left(-V^{-1}(t) \right) \left[V(t+h) - V(t) \right] \right\|$$

$$\leq \frac{1}{|h|} \left\| V^{-1}(t+h) - V^{-1}(t) \right\| \left\| V^{-1}(t) \right\| \left\| V(t+h) - V(t) \right\|$$

$$< \frac{1}{|h|} (\varepsilon_1) \left\| V^{-1}(t) \right\| \left(\frac{|h|}{\|V^{-1}(t)\|} \right) = \varepsilon_1.$$

Therefore

$$(\dot{V^{-1}}) = \frac{d}{dt}(V^{-1}) = -V^{-2}\dot{V}.$$

Equation (5.16) implies that

$$(\dot{V^{-1}}) = (M + M^*) - (M + M^*)V^{-1}.$$

Thus,

$$(\dot{V^{-1}}) + (M + M^*)V^{-1} = M + M^*.$$

So

$$e^{(M+M^*)\,t} \left[(\dot{V^{-1}}) + (M + M^*)V^{-1} \right] = e^{(M+M^*)\,t}(M + M^*).$$

$$e^{(M+M^*)\,t}(\dot{V^{-1}}) + e^{(M+M^*)\,t}(M + M^*)V^{-1} = e^{(M+M^*)\,t}(M + M^*).$$

Thus,

$$e^{(M+M^*)t}V^{-1}(t) = e^{(M+M^*)t} + C,$$

with C $\in \mathcal{B}(\mathcal{H})$.

Multiplying this equation by $e^{-(M+M^*)\,t}$, we obtain, $V^{-1}(t) = I + Ce^{-(M+M^*)\,t}$. Also, from this equation we have that

$$C = e^{(M+M^*)\,t}V^{-1}(t) - e^{(M+M^*)t} = e^{(M+M^*)\,t}[V^{-1}(t) - I].$$

Now since C is a constant operator which holds $\forall\ t \in (-\varepsilon, \varepsilon)$, $\quad C = e^0[V^{-1}(0) - I] = V^{-1}(0) - I$. Thus

$$V(t) = \left(I + Ce^{-(M+M^*)\,t}\right)^{-1}, \ \text{ with } \quad C = V_0^{-1} - I.$$

proving our claim. □

It is a straightforward calculation to verify that $V(t) = (I + Ce^{-(M+M^*)t})^{-1}$ satisfies (5.5).

For completeness we show that

$$\frac{d}{dt}(I + Ce^{-(M+M^*)t})^{-1} = \left[(I + Ce^{-(M+M^*)t})^{-2}\right] Ce^{-(M+M^*)t}(M + M^*).$$

$\dot{V}(t) = \frac{d}{dt}[(I + Ce^{-(M+M^*)t})^{-1}] = \underset{h\to 0}{lim}\frac{[I+Cexp(-(M+M^*)(t+h))]^{-1}-[I+Cexp(-(M+M^*)t)]^{-1}}{h}$.

Factoring out $\left[I + Ce^{(-(M+M^*)(t+h))}\right]^{-1}$ on the left, and $\left[I + Ce^{(-(M+M^*)t)}\right]^{-1}$ on the right yields

$$= \underset{h\to 0}{lim}\frac{\left[I+Ce^{-(M+M^*)(t+h)}\right]^{-1}\left[(I+Ce^{-(M+M^*)t})-(I+Ce^{-(M+M^*)(t+h)})\right]\left[I+Ce^{-(M+M^*)t}\right]^{-1}}{h}.$$

Consider the middle term.

$$\frac{(I + Ce^{-(M+M^*)t}) - (I + Ce^{-(M+M^*)(t+h)}}{h}$$

$$= \frac{1}{h}\left[Ce^{-(M+M^*)t} - Ce^{-(M+M^*)(t+h)}\right]$$

$$= \frac{Ce^{-(M+M^*)t}}{h}\left[I - e^{-(M+M^*)h}\right]$$

$$= \frac{Ce^{-(M+M^*)t}}{h}\left[I - \sum_{n=0}^{\infty}\frac{[-(M+M^*)h]^n}{n!}\right]$$

$$= \frac{Ce^{-(M+M^*)t}}{h}\left[I - I + (M+M^*)h - \frac{(M+M^*)^2h^2}{2!} + - \cdots\right]$$

$$= Ce^{-(M+M^*)t}\left[(M+M^*) - \frac{(M+M^*)^2 h}{2!} + - \cdots\right]$$

which tends to $Ce^{-(M+M^*)t}(M+M^*)$ as $h \to 0$.

Therefore the derivative is equal to

$\left[I + Ce^{-(M+M^*)t}\right]^{-1} Ce^{-(M+M^*)t}(M+M^*)\left[I + Ce^{-(M+M^*)t}\right]^{-1}$

$= \left[I + Ce^{-(M+M^*)t}\right]^{-2} Ce^{-(M+M^*)t}(M+M^*)$, since all terms commute.

Consequently

$$\dot{V}(t) = \frac{d}{dt}[(I + Ce^{-(M+M^*)\,t})^{-1}] = V^2(t)C(M+M^*)e^{-(M+M^*)t}. \tag{5.19}$$

It is a forthright calculation to verify that the family $\{V(t)\}_{t\in(m,M)}$ is a maximal solution of system (5.5), provided that $M = sup\{t : I + Ce^{-(M+M^*)t}$ is invertible$\}$ and $m = inf\{t : I + Ce^{-(M+M^*)t}$ is invertible$\}$.

5.2.2 The "Polar" System

In this section we derive the "polar" system associated with (5.1) and find the explicit form for solutions. For every $t \in (-\varepsilon, \varepsilon)$ we have that

$$Z(t) = \sqrt{V(t)}P(t) = V^{\frac{1}{2}}(t)P(t),$$

with $V^{\frac{1}{2}}$ representing the unique positive operator, so that $V = V^{\frac{1}{2}}V^{\frac{1}{2}} = ZZ^*$ and P is a partial isometry. (Since $V^{\frac{1}{2}}$ is self-adjoint, $(V^{\frac{1}{2}})^* = V^{\frac{1}{2}}$; and since P is a partial isometry, $P = PP^*P$).

Differentiating the equation $Z(t) = V^{\frac{1}{2}}(t)P(t)$:

$$\begin{aligned}
\frac{d}{dt}\left[V^{\frac{1}{2}}(t)P(t)\right] &= \lim_{h\to 0}\frac{V^{\frac{1}{2}}(t+h)P(t+h) - V^{\frac{1}{2}}(t)P(t)}{h} \\
&= \lim_{h\to 0}\frac{V^{\frac{1}{2}}(t+h)P(t+h) \overbrace{-V^{\frac{1}{2}}(t+h)P(t) + V^{\frac{1}{2}}(t+h)P(t)} -V^{\frac{1}{2}}(t)P(t)}{h} \\
&= \lim_{h\to 0} V^{\frac{1}{2}}(t+h)\frac{P(t+h) - P(t)}{h} + \lim_{h\to 0}\frac{V^{\frac{1}{2}}(t+h) - V^{\frac{1}{2}}(t)}{h}P(t) \\
&= \lim_{h\to 0} V^{\frac{1}{2}}(t+h)\frac{P(t+h) - P(t)}{h} + \\
&\quad + \lim_{h\to 0}\frac{\left(\left[V^{\frac{1}{2}}(t+h) + V^{\frac{1}{2}}(t)\right]^{-1}\left[V^{\frac{1}{2}}(t+h) + V^{\frac{1}{2}}(t)\right]\right)\left[V^{\frac{1}{2}}(t+h) - V^{\frac{1}{2}}(t)\right]}{h}P(t) \\
&= \lim_{h\to 0} V^{\frac{1}{2}}(t+h)\frac{P(t+h) - P(t)}{h} \\
&\quad + \lim_{h\to 0}\left[V^{\frac{1}{2}}(t+h) + V^{\frac{1}{2}}(t)\right]^{-1}\frac{[V(t+h) - V(t)]}{h}P(t) \\
&= V^{\frac{1}{2}}(t)\dot{P}(t) + \left[2V^{\frac{1}{2}}(t)\right]^{-1}\dot{V}(t)P(t).
\end{aligned}$$

Thus

$$\frac{d}{dt}\left[V^{\frac{1}{2}}(t)P(t)\right] = V^{\frac{1}{2}}(t)\dot{P}(t) + \frac{1}{2}V^{-\frac{1}{2}}(t)\dot{V}(t)P(t). \tag{5.20}$$

Therefore we have that

$$\begin{aligned}
\dot{Z} = V^{\frac{1}{2}}\dot{P} + \frac{1}{2}V^{-\frac{1}{2}}\dot{V}P &= MZ - ZZ^*MZ, \ by\ (5.1); \\
&= (I - V)MZ, \text{ since } V = ZZ^*.
\end{aligned}$$

Thus

$$V^{\frac{1}{2}}\dot{P} = (I-V)MZ - \frac{1}{2}V^{-\frac{1}{2}}\dot{V}P.$$

Therefore

$$\dot{P} = V^{-\frac{1}{2}}(I-V)MZ - \frac{1}{2}V^{-1}\dot{V}P,$$

and since, $Z = V^{\frac{1}{2}}P$,

$$\dot{P} = V^{-\frac{1}{2}}(I-V)MV^{\frac{1}{2}}P - \frac{1}{2}V^{-1}\dot{V}P.$$

Hence

$$\dot{P} = V^{-\frac{1}{2}}MV^{\frac{1}{2}}P - V^{-\frac{1}{2}}VMV^{\frac{1}{2}}P - \frac{1}{2}V^{-1}\dot{V}P.$$

The commutativity of V and M implies that

$$\dot{P} = MP - VMP - \frac{1}{2}V^{-1}\dot{V}P.$$

Consequently we have that

$$\dot{P} = \left[M - VM - \frac{1}{2}V^{-1}\dot{V}\right]P. \tag{5.21}$$

We set $A(t) = M - V(t)M - \frac{1}{2}V^{-1}(t)\dot{V}(t)$. Equation (5.21) becomes

$$\dot{P} = A(t)P.$$

Now $V(t) = \left(I + Ce^{-(M+M^*)\,t}\right)^{-1}$, and by equation (5.19) we have that $\dot{V} = V^2C(M+M^*)e^{-(M+M^*)t}$. Therefore

$$\begin{aligned} A(t) &= M - VM - \frac{1}{2}V^{-1}V^2Ce^{-(M+M^*)t}(M+M^*) \\ &= M - VM - \frac{1}{2}VCe^{-(M+M^*)t}(M+M^*). \end{aligned}$$

We now show that $A(t) = -\frac{1}{2}(M - M^*)(V(t) - I)$.

$$\begin{aligned}A(t) &= M - V\left[M + \frac{1}{2}C(M + M^*)e^{-(M+M^*)t}\right]\\ &= M - \left[M + \frac{1}{2}C(M + M^*)e^{-(M+M^*)t}\right]V\end{aligned}$$

Since $\|Ce^{-(M+M^*)t}\| < 1$, $V(t) = (I + Ce^{-(M+M^*)t})^{-1}$ may be written as the convergent series [26, Griffel, p. 219], $\sum_{n=0}^{\infty}(-Ce^{-(M+M^*)\,t})^n$. Thus

$$\begin{aligned}A(t) &= M - \left[M + \tfrac{1}{2}C(M + M^*)e^{-(M+M^*)t}\right]\left[\sum_{n=0}^{\infty}(-1)^n C^n e^{-n(M+M^*)\,t}\right]\\ &= M - \left[M + \tfrac{1}{2}C(M + M^*)e^{-(M+M)^*t}\right]\left[I - Ce^{-(M+M^*)t} + C^2e^{-2(M+M^*)t} - + \cdots\right]\\ &= M - \big[M - MCe^{-(M+M^*)t} + MC^2e^{-2(M+M^*)t} - + \cdots\\ &\qquad \cdots + \tfrac{1}{2}C(M + M^*)e^{-(M+M^*)\,t} - \tfrac{1}{2}C^2(M + M^*)e^{-2(M+M^*)\,t} + - \cdots\big]\\ &= \tfrac{1}{2}(M - M^*)\left[Ce^{-(M+M^*)t} - C^2e^{-2(M+M^*)t} + C^3e^{-3(M+M^*)t} - +......\right]\\ &= \tfrac{1}{2}(M - M^*)\left[I - I + Ce^{-(M+M^*)t} - C^2e^{-2(M+M^*)t} + C^3e^{-3(M+M^*)t} - +.....\right]\\ &= \tfrac{1}{2}(M - M^*)\left[I - (I - Ce^{-(M+M^*)t} + C^2e^{-2(M+M^*)t} - C^3e^{-3(M+M^*)t} + -......)\right]\\ &= \tfrac{1}{2}(M - M^*)\left[I - \sum_{n=0}^{\infty}(-1)^n C^n e^{-n(M+M^*)t}\right]\\ &= \tfrac{1}{2}(M - M^*)\left[I - (I + Ce^{-(M+M^*)t})^{-1}\right]\\ &= \tfrac{1}{2}(M - M^*)(I - V(t)).\end{aligned}$$

Therefore

$$A(t) = -\frac{1}{2}(M - M^*)(V(t) - I).$$

Using this result, equation (5.21) reduces to

$$\dot{P} = -\frac{1}{2}(M - M^*)(V(t) - I)P.$$

It follows from Lemma 5.16, that for every t_1 and t_2 in the interval $(-\varepsilon, \varepsilon)$, we have $A(t_1)A(t_2) = A(t_2)A(t_1)$.

In the next Lemma, we use the following notable Spectral Theorem for Hermitian (self-adjoint) Operators.

The Spectral Theorem for Hermitian Operators 5.27. *[27, Halmos, p. 69] If A is a Hermitian operator, then there exists a (necessarily real and necessarily unique) compact, complex spectral measure E, called the spectral measure of A, such that $A = \int \lambda dE(\lambda)$.*

Lemma 5.28. *Let M be a bounded operator acting on a Hilbert space $\mathcal{H}$, with the spectrum $\sigma(M + M^*)$ strictly positive. Then the operator $M + M^*$ is invertible.*

Proof: Since $M + M^*$ is self-adjoint, by the Spectral Theorem for Hermitian Operators, there exists a real, unique, compact spectral measure E, such that $M + M^* = \int \lambda dE(\lambda)$. Since $\sigma(M + M^*)$ is strictly positive, any eigenvalue λ of $M + M^*$ is strictly greater than zero. Thus λ^{-1} exists, and $\int \lambda^{-1} dE(\lambda) = (M + M^*)^{-1}$. Therefore the operator $M + M^*$ is invertible. $\square$

Lemma 5.29. *Let $\varepsilon > 0$ be given, and let M be a bounded operator acting on a Hilbert space $\mathcal{H}$, $V(t) = (I + Ce^{-(M+M^*)t})^{-1}$, with the spectrum $\sigma(M + M^*)$ strictly positive, and $\|C\| < 1$. Then $\forall\, t \in (-\varepsilon, \varepsilon)$, $A(t) = -\frac{1}{2}(M - M^*)(V(t) - I)$ is continuous.*

Proof: Let F:$(-\varepsilon, \varepsilon) \to \mathcal{B}(\mathcal{H})$, be given by F(t) = A(t), an operator-valued curve. Without loss of generality assume $t_1 \leq t_2$.

Then $\| -\frac{1}{2}(M - M^*)[V(t_1) - V(t_2)]\| \leq \frac{1}{2}\|M - M^*\| \|V(t_1) - V(t_2)\|$.

$$\begin{aligned}\|V(t_1) - V(t_2)\| &= \left\|(I + Ce^{-(M+M^*)t_1})^{-1} - (I + Ce^{-(M+M^*)t_2})^{-1}\right\|, \\ &= \Big\|(I + Ce^{-(M+M^*)\,t_1})^{-1} \left[(I + Ce^{-(+M^*)\,t_2})\right.\end{aligned}$$

$$-(I + Ce^{(M+M^*)\,t_1})\big]\,(I + Ce^{-(M+M^*)\,t_2})^{-1}\big\|$$

$$\leq \big\|(I + Ce^{-(M+M^*)t_1})^{-1}\big\|\;\big\|Ce^{-(M+M^*)t_1}$$

$$-Ce^{-(M+M^*)t_2}\big\|\;\big\|(I + Ce^{-(M+M^*)t_2})^{-1}\big\|.$$

The first norm:

$$\begin{aligned}
\big\|(I + Ce^{-(M+M^*)t_1})^{-1}\big\| &= \left\|\sum_{n=0}^{\infty}(-1)^n C^n e^{-n(M+M^*)t_1}\right\| \\
&= \big\|I - Ce^{-(M+M^*)t_1} + C^2 e^{-2(M+M^*)t_1} - +...\big\| \\
&\leq 1 + \|I\| + \|C\|e^{2\|M\|\varepsilon} + \frac{\|C\|^2 e^{4\|M\|\epsilon}}{2!} + \frac{\|C\|^3 e^{6\|M\|\epsilon}}{3!} + ... \\
&= 1 + \sum_{n=0}^{\infty}\frac{\left(\|C\|e^{2\|M\|\epsilon}\right)^n}{n!} \\
&= 1 + e^{\|C\|e^{2\|M\|\varepsilon}} = L.
\end{aligned}$$

Similarly, we find that $\|(I + Ce^{-(M+M^*)t_2})^{-1}\| \leq L$.

Now we have $\|Ce^{-(M+M^*)t_1} - Ce^{-(M+M^*)t_2}\|$

$= \left\|C\left[\sum_{n=0}^{\infty}\frac{(-1)^n(M+M^*)^n t_1^n}{n!} - \sum_{n=0}^{\infty}\frac{(-1)^n(M+M^*)^n t_2^n}{n!}\right]\right\|$

$= \left\|C\sum_{n=0}^{\infty}\frac{(-1)^n(M+M^*)^n}{n!}\left(t_1^n - t_2^n\right)\right\|$

$= \left\|C\left[-(M+M^*)(t_1 - t_2) + \frac{(M+M^*)^2}{2!}(t_1^2 - t_2^2) - \frac{(M+M^*)^3}{3!}(t_1^3 - t_2^3) + -...\right]\right\|$

$\leq \|C\|\,|t_1 - t_2|\left\|\left[-(M+M^*) + \frac{(M+M^*)^2}{2!}(t_1 + t_2) - \frac{(M+M^*)^3}{3!}(t_1^2 + t_1 t_2 - t_2^2) + -...\right]\right\|$

$\leq \|C\|\,|t_1 - t_2|\,\left(2\|M\| + 2\|M\|\varepsilon + \frac{2^2}{2!}\|M\|^2 2\varepsilon + \frac{2^3}{3!}\|M\|^3 3\varepsilon + ... + \frac{2^n}{n!}\|M\|^n n\varepsilon + ...\right)$

$= |C\|\,|t_1 - t_2|\,\left(2\|M\| + \sum_{n=0}^{\infty}\frac{(2\|M\|)^n n\varepsilon}{n!}\right)$

$= |C\|\,|t_1 - t_2|\,(2\|M\| + K)$, for some constant $K > 0$; since, by the ratio test,

$$\lim_{n\to\infty}\frac{2^{n+1}\|M\|^{n+1}(n+1)\varepsilon}{(n+1)!}\;\cdot\frac{n!}{2^n\|M\|^n n\varepsilon} = \lim_{n\to\infty}\frac{2\|M\|}{n} = 0.$$

So $\exists\,\varepsilon_1 > 0$, satisfying

$$\left\|\sum_{n=0}^{\infty}\frac{(2\|M\|)^n n\varepsilon}{n!}-K\right\| < \varepsilon_1.$$

Thus

$$\|V(t_1)-V(t_2)\| \;\leq\; L^2\;\|C\|(2\|M\|+K)\;|t_1-t_2| < \varepsilon_2,$$

for $|t_1 - t_2| < \frac{\varepsilon_2}{L^2\;\|C\|(2\|M\|+K)}$, for some $\varepsilon_2 \;\leq\; \varepsilon_1$. Hence we conclude that A(t) is bounded and continuous. $\square$

Now we show that the integral of A exists:

Lemma 5.30. *Let* $\varepsilon \;>\; 0$, *and let* M *be a bounded operator acting on a Hilbert space* $\mathcal{H}$, *with the spectrum* $\sigma(M+M^*)$ *strictly positive, and* $\|C\| < 1$. *For every* $t \in (-\varepsilon, \varepsilon)$ *define* $V(t) = (I + Ce^{-(M+M^*)t})^{-1}$, *and* $A(t) = -\frac{1}{2}(M-M^*)(V(t)-I)$. *Then* $F(t) = \int\limits_0^t A(\xi)\,d\xi$ *exists.*

Proof: Let Δ be the set of all partitions of the closed interval [0, t]. Then a partition $\sigma_n \;\in\; \Delta$ if and only if $\sigma_n \;:\; t_0 \;=\; 0 \;<\; t_1 \;<\; .... \;<\; t_n \;=\; t$. For partitions $\sigma, \pi \in \Delta$, if σ is a refinement of π then every subinterval of σ is contained in some subinterval of π, and every partition point in π is also a partition point in σ. Let

$$S(A,\sigma) = \sum_{j=1}^{n} A(t_{j-1})(t_j - t_{j-1}) \;\in\; \mathcal{B}(\mathcal{H}).$$

$\|\sigma_n\| \;=\; max\{|\,t_j - t_{j-1}|$, j = 1, ... n$\}$. Let $\varepsilon \;=\; \frac{1}{n}$. Then $\exists\, \sigma_n$ such that for every refinement σ of σ_n, with $\|\sigma_n\| < \delta$ (for some $\delta > 0$),

$$\|S(A,\sigma_n) - S(A,\sigma_n)\| < \varepsilon.$$

Let $F_n = S(A, \sigma_n)$. For the sequence $(F_n)_{n=1,2,...}$, $\|F_n - F_m\| < \frac{1}{min\{n,m\}} \to 0$.

Thus F_n is a Cauchy sequence in $\mathcal{B}(\mathcal{H})$ and $\forall\, \upsilon \in \mathcal{H}$,

$$\|F_n\upsilon - F_m\upsilon\| \;\leq\; \|F_n - F_m\|\;\|\upsilon\| \underset{n\,\to\,\infty}{\longrightarrow} 0. \tag{5.22}$$

So $(F_n v)$ is a Cauchy sequence in $\mathcal{H}$, hence converges. We can therefore define an operator $F : v \to \underset{n\to\infty}{lim}(F_n v)$.

We must show that $F \in \mathcal{B}(\mathcal{H})$ and $F_n \to F$ uniformly in the operator norm.

Clearly F is linear; we now show that F is bounded.

Because (F_n) is a Cauchy sequence, it follows from (5.22) that for any $\varepsilon > 0$ there is an n_0 such that

$$\|F_n v - F_m v\| < \|v\|\varepsilon \quad \text{for all} \quad n,\, m > n_0,\ v \in \mathcal{H}.$$

Taking the limit as $m \to \infty$ gives

$$\|F_n v - Fv\| \leq \|v\|\varepsilon \quad \text{for all} \quad n > n_0,\ v \in \mathcal{H}. \tag{5.23}$$

This shows that $F_n - F$ is a bounded operator for $n > n_0$, and it follows that

$$F = F_n + (F - F_n) \in \mathcal{B}(\mathcal{H}).$$

Alternately, since F_n is bounded, there exists some $K > 0$ such that

$$\|Fv\| \leq \|Fv - F_n v\| + \|F_n v\| \leq \varepsilon\|v\| + \|F_n\|\,\|v\| \leq (1 + \|F_n\|)\|v\| \leq K\|v\|,$$

and F is bounded, and thus in $\mathcal{B}(\mathcal{H})$.

We now show that F_n converges to F uniformly in the operator norm. From (5.23) it follows that for any $\varepsilon > 0$,

$$\|F_n - F\| = sup\{\|F_n v - Fv\|_{\mathcal{H}} : \|v\| = 1\} \leq \varepsilon$$

for $n > n_0$, which shows that $\|F_n - F\| \to 0$, that is $F_n \to F$.

Hence $F(t) = \int\limits_0^t A(\xi)\, d\xi$ exists. $\square$

Next we show that $e^B A = Ae^B$:

Lemma 5.31. *Let $\varepsilon > 0$ be given, A be a bounded operator on a Hilbert space $\mathcal{H}$, and $B(t) = \int_0^t A(\xi)\,d\xi$. Then $e^{B(t)}A(t) = A(t)e^{B(t)}$ $\forall\ t \in (-\varepsilon, \varepsilon)$.*

Proof: Using the properties of the exponential operator-valued function $e^T = \sum_{n=0}^{\infty} \frac{T^n}{n!}$, we have that

$$
\begin{aligned}
\frac{d}{dt}[e^{B(t)}] &= \lim_{h\to 0} \frac{e^{B(t+h)} - e^{B(t)}}{h} = \lim_{h\to 0} \frac{\left[e^{B(t+h)-B(t)} - I\right] e^{B(t)}}{h} \\
&= \lim_{h\to 0} \frac{\left[\sum_{n=0}^{\infty} \frac{[B(t+h)-B(t)]^n}{n!} - I\right] e^{B(t)}}{h} \\
&= \lim_{h\to 0} \frac{\left[I + B(t+h) - B(t) + \frac{[B(t+h)-B(t)]^2}{2!} + \cdots - I\right] e^{B(t)}}{h} \\
&= \lim_{h\to 0} \left[\frac{B(t+h) - B(t) + \frac{(B(t+h)-B(t))^2}{2!} + \cdots}{h}\right] e^{B(t)}
\end{aligned}
$$

The Fréchet derivative [26] gives us

$$B(t+h) - B(t) = B'(t)h + h\mathcal{O}(h) = [B'(t) + \mathcal{O}(h)]h,$$

where $\mathcal{O}(h) \to 0$ as $h \to 0$. Squaring this equation we find that

$$[B(t+h) - B(t)]^2 = [B'(t) + \mathcal{O}(h)]^2 h^2 = \mathcal{O}(h^2) = h\mathcal{O}(h),$$

which also converges to 0 as h tends to 0.

So for each $n \geq 2$,

$$\frac{[B(t+h) - B(t)]^n}{n!} = \frac{[B'(t) + \mathcal{O}(h)]^n h^n}{n!} = \mathcal{O}(h^2) = h\mathcal{O}(h) \longrightarrow 0 \text{ as } h \to 0.$$

Therefore $\sum_{n=1}^{\infty} \frac{[B(t+h)-B(t)]^n}{n!} = B'(t)h + \mathcal{O}(h^2)$.

Thus $\lim\limits_{h\to 0}\frac{\sum_{n=1}^{\infty}\frac{[B(t+h)-B(t)]^n}{n!}}{h} = \lim\limits_{h\to 0}\frac{B'(t)h+\mathcal{O}(h^2)}{h} = \lim\limits_{h\to 0}[B'(t)+\mathcal{O}(h)] = \dot{B}(t).$

Thus $\frac{d}{dt}[e^{B(t)}] = \dot{B}(t)e^{B(t)}$.

Similarly,

$$\frac{d}{dt}[e^{B(t)}] = \lim_{h\to 0}\frac{e^{B(t+h)} - e^{B(t)}}{h}$$

$$= \lim_{h\to 0}\frac{e^{B(t)}\left(e^{B(t+h)-B(t)} - I\right)}{h}$$

$$= e^{B(t)}\dot{B}(t)).$$

Therefore

$$\frac{d}{dt}[e^{B(t)}] = e^{B(t)}\dot{B}(t)) = \dot{B}(t)e^{B(t)}. \tag{5.24}$$

Hence $e^{B(t)}A(t) = A(t)e^{B(t)}$. □

We verify that $P(t) = e^{\int_0^t A(\xi)d\xi}P_0$ $(|t| < \varepsilon)$ is a solution of the "polar" system

$$\begin{cases} \dot{P} = -\frac{1}{2}(M - M^*)(V(t) - I)\,P \\ P(0) = P_0. \end{cases} \tag{5.25}$$

Proof:

$$\begin{aligned} \frac{d}{dt}\left[e^{\int_0^t A(\xi)d\xi}P_0\right] &= e^{\int_0^t A(\xi)d\xi}P_0A(t) \\ &= A(t)e^{\int_0^t A(\xi)d\xi}P_0 \\ &= A(t)P(t) \\ &= -\frac{1}{2}(M - M^*)(V(t) - I)\,P(t) \end{aligned}$$

$$\text{And}\quad P(0) = e^{\int_0^0 A(\xi)d\xi}P_0 = P_0.$$

□

Show that P_0 is a partial isometry:

$Z_0 = \sqrt{Z_0 Z_0^*}\, P_0$, by equation (5.4) p. 45.

$\left(\sqrt{Z_0 Z_0^*}\right)^{-1} Z_0 = P_0$,

$Z_0^* \left(\left(\sqrt{Z_0 Z_0^*}\right)^{-1}\right)^* = P_0^*$.

$Z_0^* \left(\sqrt{Z_0 Z_0^*}\right)^{-1} = P_0^*$, by Lemma 5.24.4, p. 57. Thus

$$\begin{aligned}
P_0 P_0^* P_0 &= \left(\sqrt{Z_0 Z_0^*}\right)^{-1} Z_0\, Z_0^* \left(\sqrt{Z_0 Z_0^*}\right)^{-1} \left(\sqrt{Z_0 Z_0^*}\right)^{-1} Z_0 \\
&= \left(\sqrt{Z_0 Z_0^*}\right)^{-1} \left(\sqrt{Z_0\, Z_0^*}\sqrt{Z_0\, Z_0^*}\,\right) \left(\sqrt{Z_0 Z_0^*}\right)^{-1} \left(\sqrt{Z_0 Z_0^*}\right)^{-1} Z_0 \\
&= \left(\left(\sqrt{Z_0 Z_0^*}\right)^{-1} \sqrt{Z_0\, Z_0^*}\right) \left(\sqrt{Z_0\, Z_0^*}\, \left(\sqrt{Z_0 Z_0^*}\right)^{-1}\right) \left(\sqrt{Z_0 Z_0^*}\right)^{-1} Z_0 \\
&= (I)\,(I)\, \left(Z_0 Z_0^*\right)^{-1} Z_0 \\
&= \left(\sqrt{Z_0 Z_0^*}\right)^{-1} Z_0 \\
&= P_0.
\end{aligned}$$

Remark 5.32. *For every t, $P(t) = e^{\int_0^t A(\xi)d\xi} P_0$ is a partial isometry. This follows from Remark 5.6 (p. 44), since we have*

$$P(t)\, P(t)^*\, P(t) = e^{\int_0^t A(\xi)d\xi} P_0\, P_0^* e^{-\int_0^t A(\xi)d\xi}\, e^{\int_0^t A(\xi)d\xi}\, P_0 = P(t).$$

The previous considerations prove the following theorem.

Theorem 5.33. *If Z_0 is an invertible operator in $\mathcal{B}(\mathcal{H})$, M is normal, $MZ_0 = Z_0 M$, then $P(t)$ (for $|t| < \varepsilon$) is a solution of the system (5.25) if and only if*

$$P(t) = e^{\int_0^t A(\xi)d\xi} P_0,$$

where $A(t) = -\frac{1}{2}(M - M^*)(V(t) - I)$.

Now we are ready to restate and prove the Main Theorem of this chapter.

Theorem 5.34. *If Z_0 is invertible and commutes with the normal operator M, then there exist $\varepsilon > 0$ and a unique differentiable mapping $Z : (-\varepsilon, \varepsilon) \to \mathcal{B}(\mathcal{H})$ such that*

$$\dot{Z} = M\,Z \;-\; Z\,Z^*\,M\,Z \text{ and } Z(0) = Z_0$$

if and only if $\quad Z(t) = \sqrt{V(t)}\;P(t)$ *with*

$$V(t) = \left[I + (V_0^{-1} - I)e^{-(M+M^*)t}\right]^{-1} \text{ and } P(t) = e^{\int_0^t -\frac{1}{2}(M-M^*)(V(\xi)-I)d\xi} P_0.$$

Proof: Using the commutativity of $V^{-1/2}$ and $\dot{V}$ established in Lemma 5.24.8 (p. 57), equation (5.20) (p. 66) may be written as

$$\frac{d}{dt}\left[\sqrt{V}P\right] = \frac{1}{2}\dot{V}\,V^{-\frac{1}{2}}\,P + V^{\frac{1}{2}}\dot{P}.$$

Thus invoking Proposition 5.26 (p. 62), Theorem 5.33 (p. 75), and Remark 5.6 (p. 44), we find

$$\begin{aligned}
Z &= V^{\frac{1}{2}}P\\
\dot{Z} &= \frac{1}{2}\dot{V}V^{-\frac{1}{2}}P + V^{\frac{1}{2}}\dot{P}\\
&= \frac{1}{2}\left[(M+M^*)V - (M+M^*)V^2\right]V^{-\frac{1}{2}}P + V^{\frac{1}{2}}\left[-\frac{1}{2}(M-M^*)\right](V-I)P\\
&= \frac{1}{2}\left[(M+M^*)V^{\frac{1}{2}} - (M+M^*)V^{\frac{3}{2}} - (M-M^*)\left(V^{\frac{3}{2}} - V^{\frac{1}{2}}\right)\right]P\\
&= \frac{1}{2}\left[MV^{\frac{1}{2}} + M^*V^{\frac{1}{2}} - MV^{\frac{3}{2}} - M^*V^{\frac{3}{2}} - M\left(V^{\frac{3}{2}} - V^{\frac{1}{2}}\right) + M^*\left(V^{\frac{3}{2}} - V^{\frac{1}{2}}\right)\right]P\\
&= \frac{1}{2}\left[M\left(V^{\frac{1}{2}} - V^{\frac{3}{2}} - V^{\frac{3}{2}} + V^{\frac{1}{2}}\right) + M^*\left(V^{\frac{1}{2}} - V^{\frac{3}{2}} + V^{\frac{3}{2}} - V^{\frac{1}{2}}\right)\right]P\\
&= \frac{1}{2}\left[M\left(2V^{\frac{1}{2}} - 2V^{\frac{3}{2}}\right)\right]P\\
&= M\left(V^{\frac{1}{2}} - V^{\frac{3}{2}}\right)P\\
&= MV^{\frac{1}{2}}P - MV^{\frac{3}{2}}P
\end{aligned}$$

$$= MV^{\frac{1}{2}}P - MV^{\frac{3}{2}}PP^*P \quad \text{(Since } P \text{ is a partial isometry, } P = PP^*P.\text{)}$$
$$= M\left(V^{\frac{1}{2}}P\right) - \left(V^{\frac{1}{2}}PP^*V^{\frac{1}{2}}\right)M\left(V^{\frac{1}{2}}P\right)$$
$$= MZ - ZZ^*MZ \quad \text{(Since} \quad Z = V^{\frac{1}{2}}P\text{)}.$$

□

5.3 Stability Analysis

In this section the long term behavior of solutions is established. Theorem 5.34 states that

$$Z(t) = \left[I + (V_0^{-1} - I)e^{-(M+M^*)t}\right]^{-\frac{1}{2}} e^{\int_0^t -\frac{1}{2}(M-M^*)(V(\xi)-I)d\xi} P_0$$

is a solution of (5.1) p. 37, provided that $I + (V_0^{-1} - I)e^{-(M+M^*)t}$ is invertible. We consider additional assumptions on M that assure the existence of solution for every $t \in (-\varepsilon, \infty)$.

Lemma 5.35. *If Z_0 is an invertible operator in $\mathcal{B}(\mathcal{H})$, M a normal operator that commutes with Z_0, $\|(Z_0\, Z_0^*)^{-1} - I\| < 1$, and the spectrum of $M + M^*$ is strictly positive, then there exists $\epsilon > 0$ so that $I + ((Z_0\, Z_0^*)^{-1} - I)e^{-(M+M^*)t}$ is invertible on the interval $(-\varepsilon,\, \infty)$ and*

$$\lim_{t\to\infty} \left[I + ((Z_0\, Z_0^*)^{-1} - I)e^{-(M+M^*)t}\right] = I.$$

Proof: Proposition 5.26 (p. 62) implies that $I + ((Z_0\, Z_0^*)^{-1} - I)e^{-(M+M^*)t}$ is invertible on the interval $(-\varepsilon,\, \varepsilon)$. Since the spectrum of $M+M^*$ is strictly positive, i.e. $\varrho \in \sigma(M + M^*)$ then $\varrho \geq \lambda > 0$, then

$$\|((Z_0\, Z_0^*)^{-1} - I)e^{-(M+M^*)t}\| \leq \|(Z_0\, Z_0^*)^{-1} - I\|\, e^{-\lambda t} < 1.$$

Therefore $I + ((Z_0\, Z_0^*)^{-1} - I)e^{-(M+M^*)t}$ is invertible for $t \in (-\varepsilon, \infty)$, and

$$\lim_{t \to \infty} \left[I + ((Z_0\, Z_0^*)^{-1} - I)e^{-(M+M^*)t}\right] = \left[I + ((Z_0\, Z_0^*)^{-1} - I) \cdot 0\right] = I.$$

$\square$

In the following proposition we use the logarithmic function of an operator. We refer the reader to Kato [36, p. 524], Conway [17, p. 178], or Douglas [21, Lemma 2.13, p. 34] for more details. According to Conway [17, p. 177-178], every invertible normal operator T has a logarithm A, and T commutes with A. Therefore if $T \in B(H)$, then T commutes with $log(T) = A$; A is called a logarithm of T. According to Douglas [21, p. 34], if $\|I - T\| < 1$, then

$$log(T) = \sum_{n=1}^{\infty} \frac{-1\,(I - T)^n}{n}.$$

Equivalently, we may write

$$log(T) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\,(T - I)^n}{n}, \ \ \texttt{and}$$

$$-log(T) = -\sum_{n=1}^{\infty} \frac{-1\,(T - I)^n}{n} = -\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\,(T - I)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n\,(T - I)^n}{n}.$$

Remark 5.36. *If $T = exp(A)$, then $log(T) = A$.*

So, if $\|I - T\| < 1$, then

$$exp\,(log(T)) = T.$$

$$\texttt{If}\ \ \|exp(A) - I\| < 1,\ \ \texttt{then}\ \ log(exp(A)) = A.$$

Proposition 5.37. *Let Z_0 be an invertible operator in $\mathcal{B}(\mathcal{H})$, M a normal operator that commutes with Z_0, $\|(Z_0\, Z_0^*)^{-1} - I\| < 1$, and the spectrum of $M+M^*$ is strictly positive. If $Z(t)$ is the maximal solution of system (5.1) with $Z(0) = Z_0$, then*

$$\lim_{t\to\infty} Z(t) = exp\left[\frac{1}{2}(M - M^*)(M + M^*)^{-1} log(Z_0 Z_0^*)^{-1}\right] P_0,$$

where P_0 is the partial isometry in the polar decomposition of Z_0.

Proof: Theorem 5.34 (p. 76) states that $Z(t) = V(t)^{\frac{1}{2}} P(t)$ with

$$V(t) = \left[I + (V_0^{-1} - I)exp(-(M + M^*)\, t)\right]^{-1}, \text{ and}$$

$$P(t) = exp\left(\int_0^t -\frac{1}{2}(M - M^*)(V(\xi) - I)d\xi\right) P_0.$$

Therefore, with $C = V_0^{-1} - I$, we have that

$$\int_0^t -\frac{1}{2}(M - M^*)(V(\xi) - I)d\xi = \int_0^t \frac{1}{2}(M - M^*)(I - V(\xi))d\xi$$

$$= \frac{1}{2}(M - M^*) \int_0^t \left[I - \left(I + Ce^{-(M+M)\,\xi}\right)^{-1}\right] d\xi$$

$$= \frac{1}{2}(M - M^*) \int_0^t \left[I - \sum_{n=0}^{\infty} (-1)^n C^n e^{-n(M+M^*)\xi}\right] d\xi$$

$$= \frac{1}{2}(M - M^*) \int_0^t \left[I - (I - Ce^{-(M+M^*)\xi} + C^2 e^{-2(M+M^*)\xi} - + \cdots\right] d\xi$$

$$= \frac{1}{2}(M - M^*) \int_0^t \left[Ce^{-(M+M^*)\xi} - C^2 e^{-2(M+M^*)\xi} + - \cdots\right] d\xi$$

$$= \frac{1}{2}(M - M^*) \int_0^t \sum_{n=1}^{\infty} (-1)^{n+1} C^n e^{-n(M+M^*)\xi} d\xi$$

$$= -\frac{1}{2}(M - M^*) \int_0^t \sum_{n=1}^{\infty} (-1)^n C^n e^{-n(M+M^*)\xi} d\xi$$

$$= -\frac{1}{2}(M - M^*) \sum_{n=1}^{\infty} \int_0^t (-1)^n C^n e^{-n(M+M^*)\xi} d\xi$$

By (5.24) p. 74,

$$= -\frac{1}{2}(M - M^*) \sum_{n=1}^{\infty} \left[(-1)^n C^n e^{-n(M+M^*)\xi} [-n(M + M^*)]^{-1}\right]_{\xi=0}^{t}$$

$$= \frac{1}{2}(M - M^*)(M + M^*)^{-1} \left[\sum_{n=1}^{\infty} \frac{\left(-Ce^{-(M+M^*)t}\right)^n}{n} - \sum_{n=1}^{\infty} \frac{(-C)^n}{n} \right]$$

Since $\|Cexp(-(M + M^*)t)\| \leq \|C\| < 1$, for $t \geq 0$, we have that

$$\sum_{n=1}^{\infty} \frac{(-Cexp[-(M + M^*)t])^n}{n} = -log\left(Cexp[-(M + M^*)t] + I\right),$$

$$\text{and } \sum_{n=1}^{\infty} \frac{(-C)^n}{n} = -log(C + I).$$

Therefore, we have that

$$\begin{aligned}
&\int_0^t -\frac{1}{2}(M - M^*)(V(\xi) - I) d\xi \\
&\quad = \frac{1}{2}(M - M^*)(M + M^*)^{-1} \left[- log\left(Cexp[-(M + M^*)\,t] + I\right) + log(C + I)\right] \\
&\quad = \frac{1}{2}(M - M^*)(M + M^*)^{-1} \left[log(C + I) - log\left(Cexp[-(M + M^*)\,t] + I\right)\right]
\end{aligned} \tag{5.26}$$

We also observe that

$$\begin{aligned}
&exp[log(C + I) - log(C\,exp[-(M + M^*)\,t\,] + I)] \\
&\qquad = exp\,[log(C + I)] \cdot \left(exp\,[log\,(C\,exp\,[-(M + M^*)\,t\,] + I\,)\,]\,\right)^{-1}
\end{aligned}$$

$$= (C+I)\,(C\,exp\,[-(M+M^*)\,t\,]+I\,)^{-1}\,,$$

hence

$$\begin{aligned}
&\lim_{t\to\infty} exp\Big[log(C+I) - log\,(Cexp\,[-(M+M^*)\,t\,]+I\,)\ \Big] \\
&\qquad\qquad = \lim_{t\to\infty}(C+I)\,[C\,exp\,[-(M+M^*)\,t\,]+I\,]^{-1} \\
&\qquad\qquad = (C+I)(C\cdot 0+I)^{-1} \\
&\qquad\qquad = (C+I)I^{-1} \\
&\qquad\qquad = C+I \\
&\qquad\qquad = V_0^{-1}.
\end{aligned}$$

For large values of t we have that

$$\Big\|exp\,[log(C+I) - log\,(C\,exp\,[-(M+M^*)\,t\,]+I\,)\,] - I\Big\| < 1,$$

and thus

$$\lim_{t\to\infty} log[exp[log(C+I) - log(C\,exp(-(M+M^*)t)+I)]\,] = log(V_0^{-1}).$$

This implies that

$$\begin{aligned}
\lim_{t\to\infty} P(t) &= \lim_{t\to\infty} exp\left(\int_0^t -\frac{1}{2}(M-M^*)(V(\xi)-I)d\xi\right)P_0; \quad \text{by (5.26)} \\
&= \lim_{t\to\infty} exp\Bigg(\frac{1}{2}(M-M^*)(M+M^*)^{-1} \\
&\qquad\qquad [log(C+I) - log(Cexp(-(M+M^*)t)+I)]\Bigg)P_0 \\
&= \lim_{t\to\infty} exp\Bigg(\frac{1}{2}(M-M^*)(M+M^*)^{-1} \\
&\qquad\qquad log\Big(exp\Big[log(C+I) - log(Cexp(-(M+M^*)t)+I)\Big]\Big)\Bigg)P_0
\end{aligned}$$

$$= exp\left(\frac{1}{2}(M - M^*)(M + M^*)^{-1}log\left(V_0^{-1}\right)\right)P_0,$$

and with $V(t) = \left[I + (V_0^{-1} - I)exp(-(M + M^*)\,t)\right]^{-1}$, using Lemma 5.35 p. 77,

$$\lim_{t\to\infty} Z(t) = \lim_{t\to\infty} \sqrt{V(t)}\,P(t) = exp\left(\frac{1}{2}(M - M^*)(M + M^*)^{-1}log\left(V_0^{-1}\right)\right)P_0.$$

□

Corollary 5.38. *If Z_0 is an invertible operator in $\mathcal{B}(\mathcal{H})$, M a self-adjoint operator that commutes with Z_0, $\|(Z_0\,Z_0^*)^{-1} - I\| < 1$, and the spectrum of M is strictly positive, then the limit of $Z(t)$ as t goes to infinity is equal to P_0, the "polar" factor of the decomposition of the initial condition.*

5.4 General Solution for the Cox-Adams Learning Model

In this section we consider the system

$$\dot{Z} = EMZ_E - Z_E Z_E^* M Z_E, \tag{5.27}$$

with E representing an invertible, positive, self-adjoint operator on $\mathcal{H}$ and M a self-adjoint operator on $\mathcal{H}$. Some of the results in this section may be found in [14]. The operator valued, time dependent Z now represents the continuous change of connecting weights according to the rule described in equation (5.27).

We present a scheme that explicitly solves system (5.27). First a natural change of variables reduces (5.27) to a static system where no synaptic formation occurs. However, the probabilistic effect transfers to the input correlation operator M. System (5.27) reduces to an Oja type model. We follow a strategy applied before. Theorem 5.4 implies the local existence and uniqueness of solutions.

Since E is positive and invertible, we rewrite equation (5.27) as follows:

$$\dot{Z}_E = \left(\sqrt{E}\sqrt{E}\right) M \left(\sqrt{E}(\sqrt{E})^{-1}\right) Z_E - Z_E Z_E^* \left((\sqrt{E})^{-1}\sqrt{E}\right) M \left(\sqrt{E}(\sqrt{E})^{-1}\right) Z_E,$$

equivalently,

$$\begin{aligned}(\sqrt{E})^{-1}\dot{Z}_E = &\left(\sqrt{E}M\sqrt{E}\right)\left((\sqrt{E})^{-1}Z_E\right) \\ &- \left((\sqrt{E})^{-1}Z_E\right)\left(Z_E^*(\sqrt{E})^{-1}\right)\left(\sqrt{E}M\sqrt{E}\right)\left((\sqrt{E})^{-1}Z_E\right).\end{aligned}$$

We set $W = (\sqrt{E})^{-1}Z_E$ and $S = \sqrt{E}\, M\, \sqrt{E}$.

We observe that S is a hermitian operator. Then system (5.27) reduces to

$$\begin{cases} \dot{W} = SW - WW^*SW \\ W(0) = W_0, \end{cases}$$

where $W_0 = (\sqrt{T})^{-1}Z_0$.

Proposition 5.39. *If W_0 is invertible and commutes with the hermitian operator S, then there exist $\varepsilon > 0$ and a unique differentiable mapping $W : (-\varepsilon, \varepsilon) \to \mathcal{B}(\mathcal{H})$ such that*

$$\dot{W} = S\,W \;-\; W\;W^*\,S\,W \;\textit{and}\; W(0) = W_0,$$

$$\textit{if and only if}\;\; W(t) = V(t)^{1/2}\,P(t),\;\textit{with}$$

$$V(t) = \left[I + (V_0^{-1} - I)\,exp(-2St)\,\right]^{-1}$$

$$\textit{and}\quad P(t) = P_0.$$

Proof: Since the operator S is hermitian ($S = S^*$), the statement follows from Theorem 5.34 p. 76. □

Theorem 5.40. *Let E be an invertible, positive, self-adjoint operator and M be a hermitian operator. If Z_0 is invertible and commutes with M, then there exist $\varepsilon > 0$ and a unique differentiable mapping $Z_E : (-\varepsilon, \varepsilon) \to \mathcal{B}(\mathcal{H})$ such that*

$$\begin{cases} \dot{Z}_E = EM\, Z_E \,-\, Z_E\, Z_E^*\, M\, Z_E \\ Z(0) = Z_0, \end{cases}$$

if and only if

$$Z_E(t) = (EV(t))^{1/2}\, P(t),$$

$$V(t) = \left[I + \left(\sqrt{E}(Z_0 Z_0^*)^{-1}\sqrt{E} - I\right) exp\left(-2\sqrt{E}\, M\, \sqrt{E}\, t\right)\right]^{-1}$$

$$and\ \ P(t) = P_0.$$

Proof: This follows from Proposition 5.39. □

The following lemma is used in the stability analysis of the Cox-Adams model.

Lemma 5.41. *If Z_0 is an invertible operator in $\mathcal{B}(\mathcal{H})$, M a normal operator that commutes with Z_0, $\|(Z_0\, Z_0^*)^{-1} - I\| < 1$, and the elements of the spectrum of M are strictly positive, then there exists $\epsilon > 0$ so that*

$$I + \left[(Z_0\, Z_0^*)^{-1} - I\right]\, exp\,(-(M + M^*)\, t\,)$$

is invertible on the interval $(-\varepsilon, \infty)$ and

$$\lim_{t \to \infty} \left[I + ((Z_0\, Z_0^*)^{-1} - I)\, exp\,(-(M + M^*)\, t)\,\right] = I.$$

As a result we have the following corollary.

Corollary 5.42. *Let E be an invertible, positive, and self-adjoint operator. If Z_0 is an invertible operator in $\mathcal{B}(\mathcal{H})$, M a self-adjoint operator that commutes with Z_0, the elements of the spectrum of M are strictly positive, and $\|\sqrt{E}(Z_0\, Z_0^*)^{-1}\sqrt{E} - I\| < 1$, then there exists $\epsilon > 0$ so that*

$$I + \left[\sqrt{E}(Z_0\, Z_0^*)^{-1}\sqrt{E} - I\right]\, exp\left(-2\sqrt{E}M\sqrt{E})\, t\right)$$

is invertible on the interval $(-\varepsilon, \infty)$ and

$$\lim_{t\to\infty}\left[I + \left[\sqrt{E}(Z_0\, Z_0^*)^{-1}\sqrt{E} - I\right]\, exp\left(-2\sqrt{E}M\sqrt{E})\, t\right)\right] = I.$$

Remark 5.43. *We observe that, under the assumptions listed in Corollary 5.42, we have*

$$\lim_{t\to\infty} Z(t) = P_0.$$

This provides a filtering procedure that selects the polar component of the initial condition.

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