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The Dissertation Committee for Malgorzata Marta Czerwinska certifies that this is the final approved version of the following electronic dissertation: “Geometric Properties of Symmetric Spaces of Measurable Operators.”

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GEOMETRIC PROPERTIES OF SYMMETRIC SPACES OF MEASURABLE OPERATORS

A Dissertation

Presented for the

Doctor of Philosophy

Degree

The University of Memphis

Małgorzata M. Czerwińska,

May, 2011

I would like to dedicate this doctoral dissertation to my mother, Hanna Czerwińska. She instilled in me the value of hard work and gave me strength to pursue my goals. There is no doubt in my mind that without her support and love I could not have completed this process.

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I wish to express my gratitude to Peter Dodds, Ben De Pagter and Fedor Sukochev, who allowed us to use their unpublished manuscript of the monograph *Theory of Noncommutative Integration* [31]. This well organized material discusses in details various aspects of noncommutative theory and it was a tremendous help in my research. I am especially greateful to Dr. Sukochev, for all his support and guidance during my visit in Sydney.

ABSTRACT

Czerwińska, Małgorzata Marta. Ph.D. The University of Memphis. May, 2011. Geometric properties in symmetric spaces of measurable operators. Major Professor: Anna Kamińska.

Symmetric spaces of measurable operators $E(\mathcal{M}, \tau)$, known also as *noncommutative symmetric spaces*, were introduced first by Ovčinnikov in 1970 [58, 59]. They emerged as a generalization of the theory of unitary matrix spaces introduced by Schatten in sixties [71], as well as the theory of noncommutative L_p spaces introduced by Segal and Dixmier in the early fifties [20, 66]. Their study provides a unified approach to the theory of ideals of compact operators in Hilbert space due to Schatten [71], and to the classical theory of rearrangement invariant Banach function spaces [4, 51]. With the development of noncommutative theory, it was natural to expect the space $E(\mathcal{M}, \tau)$ to reflect the properties of the corresponding symmetric function space E. Establishing those lifting-type results from the space E to $E(\mathcal{M}, \tau)$ effectively reduces the study on geometric structures in noncommutative settings, to the corresponding questions in symmetric spaces of measurable functions.

In this dissertation we explore strongly extreme points, complex extreme points, points of complex local uniform rotundity, smooth points, strongly smooth points of the unit ball in $E(\mathcal{M}, \tau)$ and their global counterparts, midpoint local uniform rotundity, complex rotundity, complex local uniform rotundity, smoothness, Fréchet smoothness, respectively. Moreover, we investigate exposed and strongly exposed points in $E(\mathcal{M}, \tau)$.

Contents

1 Introduction and Definitions

1.1 Introduction

In 1937, von Neumann [57] showed that if $\|\cdot\|_E$ is a symmetric norm on $\mathbb{R}^n$ then one can define a norm on the space of $n \times n$ matrices by

$$\|x\|_E = \|s_1(x), \dots, s_n(x)\|_E,$$

where $s_n(x)$ are singular numbers of the matrix x (i.e. the eigenvalues of $|x| = \sqrt{x^*x}$) in decreasing order. The next step in infinite dimensional spaces was done in 1960 by Schatten [71]. He defined the unitary matrix space C_E, the ideal of compact operators on a separable Hilbert space affiliated to a symmetric sequence space E. One of the points of interest in the theory of unitary matrix spaces was to investigate what properties of symmetric sequence space E are inherited by the unitary matrix space C_E [37]. It was shown by Arazy in [3] that E is isometrically embedded in C_E, and that the isometry V can be chosen with respect to any compact operator x in such a way that $V(s(x)) = x$, where $s(x) = \{s_n(x)\}_{n=1}^{\infty}$ is the sequence of singular numbers of x. Therefore many geometrical properties of $x \in C_E$ are also satisfied by $s(x) \in E$. In the same paper Arazy showed that $x \in C_E$ is an extreme point (resp. smooth, resp. exposed, resp. strongly exposed) of the unit ball in C_E if and only if $s(x)$ is an extreme point of the unit ball in E (resp. smooth, resp. exposed, resp. strongly exposed). Lifting the uniform rotundity from E to C_E was considered by Tomczak-Jaegermann in [75].

The beginning of the theory of symmetric spaces of measurable operators can be traced back to the early fifties. It was then when Segal and Dixmier [66, 20] laid out the foundation for noncommutative L_p spaces, by introducing the concept of noncommutative integration in the more restricted settings of semifinite von Neumann algebras. The notion of the singular value function of the measurable operator, the generalization of usual singular numbers for compact operators on a Hilbert space, was introduced in a Bourbaki seminar note by Grothendieck [39].

In 1970 Ovčinnikov initiated the theory of symmetric spaces of measurable operators [58, 59]. In his work the central role is played by rearrangement invariant structure induced by singular value functions. Similar ideas were presented in the subsequent work of Yeadon [79, 80]. Later in 1989, a method of constructing symmetric spaces of measurable operators was developed by P. G. Dodds, T. K. Dodds and B. de Pagter [22, 23]. The authors adapted the notion of measurablility introduced by Nelson [56], which is considerably more general than the one permitted by methods of [58, 81]. In fact, for closed operators affiliated with a semifinite von Neumann algebra with a normal, faithful, semifinite trace τ, the notion of τ-measurability in the sense of Nelson is equivalent to requiring the existence of an everywhere finite decreasing rearrangement. It is P. G. Dodds', T. K. Dodds' and B. de Pagter's efficient construction of $E(\mathcal{M},\tau)$ spaces that laid the foundation for the theory of symmetric spaces of measurable operators and started its fast development.

The construction introduced in [22, 23] leads to the definition of the symmetric spaces $E(\mathcal{M},\tau)$ of τ-measurable operators associated to a symmetric Banach function space E and semifinite von Neumann algebra $(\mathcal{M},\tau)$, used in this dissertation. The space $E(\mathcal{M},\tau)$ consists of all τ-measurable operators x for which the singular value function $\mu(x)$ belongs to E and it is equipped with the norm $\|x\|_{E(\mathcal{M},\tau)} = \|\mu(x)\|_E$.

Since the noncommutative theory emerged, it has attracted the attention of the several specialists in functional analysis and operator theory such as G. Pisier [64], N. Kalton [46, 47], M. Junge [43], U. Haagerup [40], Q. Xu [78, 77], P. Dodds [21], T. Dodds [21], F. Sukochev [69], B. de Pagter [60], and many others.

There are two closely related directions in studying symmetric spaces of measurable operators. One focuses on lifting topological or geometrical properties from the commutative setting to the noncommutative one. The other concentrates on reducing problems in the noncommutative case to those in the commutative one.

Until now it has been shown that many geometric properties such as rotundity [8], (local) uniform rotundity [9], (uniform) Kadec-Klee [28, 29], Banach-Saks [30] and several others are lifted from E to $E(\mathcal{M},\tau)$. In particular, Chilin, Krygin and Sukochev [8] characterized extreme points of the unit ball in $E(\mathcal{M},\tau)$ in terms of its singular value function $\mu(x)$ in the unit ball of E. Later on in [9] they showed that $E(\mathcal{M},\tau)$ inherits from E its local uniform rotundity and uniform rotundity. Good examples to illustrate the second trend in noncommutitive research are works [24, 32], allowing to reduce weakly compact subsets of $E(\mathcal{M},\tau)$ to those in E.

Despite strong analogy between the commutative and noncommutative settings, in some aspects noncommutative spaces behave very differently than their commutative counterparts. The first difficulty that one can encounter when studying noncommutative spaces is that the usual triangle inequality for the modulus of complex numbers is no longer valid for the modulus of operators. In general, we do not have that $|x+y| \leq |x|+|y|$, for measurable operators x, y. One of the most spectacular divergence of the noncommutative spaces from their commutative versions appears in the "unconditional structure". It was shown by Gordon and Lewis [38] that the Schatten class C_p fails to have any unconditional basis when $p \neq 2$, contrary to ℓ_p spaces. A very interesting survey by Pisier and Xu [64] classifies the similarities and differences between the usual L_p spaces and noncommutative L_p spaces.

It was shown by Ovčinnikov in [58, 59] that the noncommutative L_p spaces associated with a semifinite von Neumann algebra form an interpolation scale with respect to both the real and complex interpolation methods. In [25] it was observed that some interpolation theorems for noncommutative symmetric spaces can be deduced from their commutative counterparts. Certain methods of interpolation has proven to be crucial in establishing a theory of Köthe duality for symmetric spaces of measurable operators in [26].

This dissertation compliments the existing body of results on geometric properties in $E(\mathcal{M},\tau)$ by considering complex extreme points ($\mathbb{C}$-extreme points), complex local uniform extreme points ($\mathbb{C}-LUR$ points), strongly extreme points ($MLUR$-points), smooth and strongly smooth points of the unit ball of $E(\mathcal{M},\tau)$ and associated to them complex rotundity ($\mathbb{C}$-rotundity), complex local uniform rotundity ($\mathbb{C}-LUR$), midpoint local uniform rotundity ($MLUR$), smoothness and Fréchet smoothness of $E(\mathcal{M},\tau)$. We will also characterize exposed and strongly exposed points of the unit ball in $E(\mathcal{M},\tau)$ in terms of corresponding properties of their singular value functions.

The knowledge on smooth, exposed and extreme points of the unit ball of a space has important consequences in studying its isometric structure. As applications of those geometric properties, let us mention the Krein-Milman theorem [65], the existence and uniqueness of smooth points with applications in best approximation [67], or the fact that isometries preserve extreme points [35].

Complex rotundity properties followed real convexity notions as their complex analogies. The concepts of $\mathbb{C}$-extreme points and $\mathbb{C}$-rotund spaces have been introduced by Thorp and Whitley in [74] in connection with the strong maximum modulus theorem of vector-valued analytic functions. Its liaison to holomorphic spaces has been further confirmed by Globevnik's work in [36] who investigated complex uniformly rotund spaces and showed among others that peak points of the ball algebra over a Banach space X are complex extreme points of its unit ball B_X. Along the same line, for instance, are the recent results in [1]. The complex geometric properties also found other applications, for instance, in studying (local) geometry of Banach spaces [19]. Moreover, as it has been recently observed [42], $\mathbb{C}$-extreme points and $\mathbb{C}$-rotundity of a complex Banach lattice E are equivalent to upper monotone points and strict monotonicity of its real part E_r, respectively. That observation could be very useful in studying complex properties in Banach lattices, and we will apply it later in the dissertation.

Smooth and strongly smooth points are related to Gâteaux differentiability and Fréchet differentiability of the norm, respectively [17, 18]. Unlike other forms of derivatives, the Gâteaux and Fréchet differential of functions may be nonlinear and therefore they find important applications in nonlinear analysis. It is a classical fact, due to Day [15], that every separable Banach space admits an equivalent Gâteaux smooth renorming. Klee and Kadec [44, 49] showed independently that any Banach space X with separable dual X^*, admits a Fréchet differentiable norm. Differentiability properties of norms on Banach spaces are characterized by the convexity properties of its dual. In fact for the reflexive spaces, there is full duality between strict convexity and smoothness.

Exposed points were first defined by Straszewicz in 1935 in the case of finite-dimensional spaces [72]. The concept of strongly exposed points was introduced by Lindenstrauss in 1963 [54]. There is a connection between strongly exposed points and the Radon-Nikodym property. Phelps showed that a Banach space X has a Radon-Nikodym property if every non-empty closed, bounded convex subset is contained in a closed convex hull of its strongly exposed points [62]. In a strictly convex Banach space all points of its unit sphere are exposed, while in a locally uniformly convex Banach space all points of its unit sphere are strongly exposed.

For all the facts on geometric properties mentioned here and for more historical details, we refer to [18, 53, 17, 63].

In the next section we provide terminology, all necessary facts especially in non-commutative spaces, and we recall some useful results needed further in the dissertation.

In chapter two, we prove that if x is order continuous and E is fully symmetric function space then x is a strongly extreme $(MLUR)$ point of the unit ball in $E_0(\mathcal{M}, \tau)$; whenever, its singular value function $\mu(x)$ is a $MLUR$ point of the unit ball in E_0, where E_0 is a closed subspace of E containing all functions with finite distributions. Furthermore, under the assumption that E is a symmetric function space and the von Neumann algebra $\mathcal{M}$ has σ-finite trace, we obtain that if x is a $MLUR$ point then so is $\mu(x)$. Consequently, we obtain that the $MLUR$ property of a symmetric space E is lifted to $E(\mathcal{M},\tau)$, and vice versa that E inherits this property from $E(\mathcal{M},\tau)$.

Chapter three is devoted to $\mathbb{C}$-extreme points and $\mathbb{C}$-rotundity. We provide a characterization of $\mathbb{C}$-extreme points of the unit ball in $E(\mathcal{M},\tau)$, where E is a symmetric function space, analogous to that of extreme points obtained by Chilin, Krygin and Sukochev in [8, 31]. As a consequence we also get that $E\left(\mathcal{M},\tau\right)$ is $\mathbb{C}$-rotund if and only if E is $\mathbb{C}$-rotund. We also obtain the result relating rotundity properties to monotonicity properties in noncommutative spaces. It is an analogy to the known correlations between rotundity and monotonicity for the Banach function spaces [42]. More precisely, we show that complex rotundity of $E(\mathcal{M},\tau)$ is equivalent to strict monotonicity of the norm $\|\cdot\|_{E(\mathcal{M},\tau)}$ on $E(\mathcal{M},\tau)$.

In the first section of chapter four, we prove analogous results as in the case of strongly extreme points for $\mathbb{C}-LUR$ points and $\mathbb{C}-LUR$ property. We show that if x is order continuous, E is strongly symmetric function space and $\mu(x)$ is a $\mathbb{C}-LUR$ point of the unit ball in E_0, then x is a $\mathbb{C}-LUR$ point of the unit ball in $E_0\left(\mathcal{M},\tau\right)$. Therefore if E is order continuous then $\mathbb{C}-LUR$ property is lifted from E to $E(\mathcal{M},\tau)$, and vice versa E inherits this property from the space $E(\mathcal{M},\tau)$. Finally, we conclude that the norm $\|\cdot\|_{E(\mathcal{M},\tau)}$ on $E(\mathcal{M},\tau)$ is upper locally uniformly monotone, whenever $E(\mathcal{M},\tau)$ is complex locally uniformly rotund.

In the second section of chapter four, we discuss the definitions of $\mathbb{C}-MLUR$ points and $\mathbb{C}-MLUR$ [6] spaces that are formulated analogically as $MLUR$ points and $MLUR$ spaces in [34] for real Banach spaces. We present several equivalent conditions, and in particular, we show that in any complex Banach space these notions are equivalent to $\mathbb{C}-LUR$ points and $\mathbb{C}-LUR$ spaces, respectively. Therefore

$\mathbb{C}-MLUR$ and $\mathbb{C}-LUR$ are not distinguishable contrary to their corresponding real notions $MLUR$ and LUR.

In chapter five, we investigate the relationships between smooth and strongly smooth points of the unit ball of an order continuous symmetric function space E, and of the unit ball of the space of τ-measurable operators $E(\mathcal{M},\tau)$. We prove that x is a smooth point of the unit ball in $E(\mathcal{M},\tau)$ if and only if the decreasing rearrangement $\mu(x)$ of the operator x is a smooth point of the unit ball in E, and either $\mu(\infty; f) = 0$ for the function $f \in S_{E^\times}$ supporting $\mu(x)$, or $s(x^*) = \mathbf{1}$. Under the assumption that the trace τ on $\mathcal{M}$ is σ-finite, we show that x is strongly smooth point of the unit ball in $E(\mathcal{M},\tau)$ if and only if its decreasing rearrangement $\mu(x)$ is a strongly smooth point of the unit ball in E. Consequently, for a symmetric function space E, we obtain corresponding relations between smoothness or strong smoothness of the function f and its decreasing rearrangement $\mu(f)$. Finally, under suitable assumptions, we state results relating the global properties such as smoothness and Fréchet smoothness of the spaces E and $E(\mathcal{M},\tau)$.

The last chapter discusses the correlation between exposed or strongly exposed points of the unit ball of an order continuous symmetric function space E, and of the unit ball of the space of τ-measurable operators $E(\mathcal{M},\tau)$. We prove that an operator x is an exposed or strongly exposed point of the unit ball in $E(\mathcal{M},\tau)$ if and only if its singular value function $\mu(x)$ is an exposed or strongly exposed point of the unit ball in E, respectively.

1.2 Preliminaries

As usual by $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ denote the set of complex, real, non-negative real and natural numbers, respectively. Given $z \in \mathbb{C}$, by $\overline{z}$, $\operatorname{Re} z$ and $\operatorname{Im} z$ denote the conjugate of z, real and imaginary part of z, respectively. For a subset A of a space X, the symbol A^c will mean the complement of A, that is $X \setminus A$.

Given a non-empty subset D of a partially ordered set $(X, \leq)$, notations $\sup D$ or $\bigvee D$ will stand for the supremum of D, whenever it exists. Similarly, $\inf D$ or $\bigwedge D$ will denote the infimum of D. If $\{x_\alpha\} \subset X$ is increasing net and $x = \sup x_\alpha$ exists, then we write $x_\alpha \uparrow x$. Analogously $x_\alpha \downarrow x$ means, that decreasing net $\{x_\alpha\} \subset X$ has infimum x.

Let I stand for either the set of natural numbers $\mathbb{N}$ equipped with the counting measure or for the interval $[0, \alpha)$, $0 < \alpha \leq \infty$, with the Lebesgue measure m. By $L^0 = L^0(I)$ we denote the set of all complex valued measurable functions on I. Given $f \in L^0(I)$, the *support* of f, that is the set of all $t \in I$ for which $f(t) \neq 0$, will be denoted by $\mathrm{supp}(f)$.

The *distribution function* $d(f)$ of a function $f \in L^0$ is given by (see [4, 51])

$$d(\lambda; f) = m\{t \in I : |f(t)| > \lambda\}, \quad \text{for all } \lambda \geq 0.$$

For $f \in L^0$ we define its *decreasing rearrangement* as

$$\mu(t; f) = \inf\{s > 0 : d(s; f) \leq t\}, \qquad t \geq 0.$$

Observe that if $I = [0, \alpha)$, where $\alpha < \infty$, then $\mu(t; f) = 0$ for all $t \geqslant \alpha$. So in this case we will consider $\mu(t; f)$ as a function on the interval $[0, \alpha)$ and interpret $\mu(\infty; f) = 0$. In the case of discrete measure, the elements of L^0 coincide with complex-valued sequences $x = \{x_n\}$, and then $\mu(x) = \mu(n; x)$ is a decreasing rearrangement of x defined equivalently as $\mu(n; x) = \inf\{s > 0 : d(s; x) \leq n - 1\}$, for $n \in \mathbb{N}$.

A Banach space $(E, \|\cdot\|_E)$, where $E \subset L^0(I)$ is called a *symmetric space* if it follows from $f \in E, g \in L^0(I)$ and from the inequality for the decreasing rearrangements $\mu(g) \leq \mu(f)$ that $g \in E$ and $\|g\|_E \leq \|f\|_E$. We will say that E is a *symmetric function* (resp. *sequence*) *space*, whenever E is a symmetric space on $I = [0, \alpha)$ (resp. on $I = \mathbb{N}$).

Given the symmetric space $E \subset L^0[0,\alpha)$, $0 < \alpha \leq \infty$, the symbol E_0 will stand for the set of all elements $f \in E$ for which $\mu(\infty; f) = \lim_{t\to\infty} \mu(t; f) = 0$, that is the distribution function $d(\lambda; f)$ is finite for all $\lambda \in [0, \alpha)$.

Let us recall the definition of L_p spaces, which are classical examples of symmetric spaces. Given $1 \leq p < \infty$, the space L_p consists of all elements $f \in L^0 = L^0(I)$, for which

$$\|f\|_{L_p} = \left(\int_I |f|^p \, dt \right)^{1/p} < \infty.$$

If $p = \infty$, then L_∞ is defined to be a space of all functions $f \in L^0$, with

$$\|f\|_{L_\infty} = \operatorname{esssup} |f| < \infty.$$

If $I = \mathbb{N}$ with a counting measure, it is customary to denote the corresponding L_p space by ℓ_p.

The space $L_1 + L_\infty$ consists of all functions $f \in L^0$ that are representable as a sum $f = g + h$ of functions $g \in L_1$ and $g \in L_\infty$. For each $f \in L_1 + L_\infty$, let

$$\|f\|_{L_1+L_\infty} = \inf\{\|g\|_{L_1} + \|h\|_{L_\infty}\},$$

where the infimum is taken over all representations $f = g + h$, where $g \in L_1$ and $h \in L_\infty$.

The symbol $L_1 \cap L_\infty$ will denote the space of all functions f in the intersection of L_1 and L_∞, equipped with the norm

$$\|f\|_{L_1 \cap L_\infty} = \max\{\|f\|_{L_1}, \|f\|_{L_\infty}\}.$$

It is known that every symmetric function space is *intermediate* between spaces L_1 and L_∞, i.e. $L_1 \cap L_\infty \subset E \subset L_1 + L_\infty$ with continuous embeddings [51]. If E is a symmetric sequence space, then $\ell_1 \subset E \subset \ell_\infty$ [4].

We say that g is *submajorized* by f and write $g \prec f$, whenever

$$\int_0^s \mu(t;g)dt \leq \int_0^s \mu(t;f)dt \quad \text{for every } s > 0.$$

Recall that a symmetric space E is *strongly symmetric* if for $f, g \in E$, if $g \prec f$ then $\|g\|_E \leq \|f\|_E$. We also say that E is *fully symmetric*, whenever for $f \in E$ and $g \in L^0(I)$, if $g \prec f$ then $g \in E$ and $\|g\|_E \leq \|f\|_E$.

An element $f \in E$ is called *order continuous*, if for every $0 \leq f_n \leq |f|$ such that $f_n \downarrow 0$ a.e. it holds $\|f_n\|_E \downarrow 0$. The space E is *order continuous* if every element in E is order continuous. If E is order continuous then it is fully symmetric [51, Chapter II, Theorem 4.10] and $E = E_0$.

The *Köthe dual* $E^\times$ of a symmetric functions space E is defined to be the space of all functions $g \in L^0$, for which $fg \in L_1$ for all $f \in E$. The space $E^\times$ is a symmetric function space, endowed with the norm

$$\|g\|_{E^\times} = \sup\left\{\int_0^\infty |fg|(t)dt : \|f\|_E \leq 1\right\}, \quad g \in E^\times.$$

If E is order continuous, then its dual E^* is isometrically isomorphic to the Köthe dual $E^\times$. We have then that each functional $F \in E^*$ is of the form

$$F(h) = \int_0^\infty h(t)f(t)dt, \text{ for some } f \in E^\times, h \in E.$$

Given a Banach space X, by $B(X)$ we will denote the space of all linear, bounded operators from X to X. The notation $\|\cdot\|_{B(X)}$ will stand for the operator norm on $B(X)$.

We say that a linear operator T on $L_1 + L_\infty$ is *admissible* with respect to the couple (L_1, L_∞) if the restriction of T to L_1 is a bounded operator from L_1 into L_1, and the restriction of T to L_∞ is a bounded operator from L_∞ into L_∞. If an admissible operator T for the couple (L_1, L_∞) is a positive contraction on L_1 and on L_∞, that is $\|T\|_{B(L_1)}, \|T\|_{B(L_\infty)} \leq 1$ and $Tf \geq 0$ for all $f \in L_1 + L_\infty$, then T is said to be a *substochastic* operator. It is known that a fully symmetric function space is an *interpolation space* between L_1 and L_∞, that is every admissible operator T is a bounded operator from E into E.

We then have $\|T\|_{B(E)} \leq \max\left(\|T\|_{B(L_1)}, \|T\|_{B(L_\infty)}\right)$. We state below important Calderon-Ryff's theorem [4, Chapter 2, Theorem 2.10].

Theorem 1.1. *Let $f, g \in L^0$ be nonnegative functions, $f \in L_1 + L_\infty$ and $g \prec f$. Then there exists a substochastic operator T such that $Tf = g$ a.e.*

For this and all other facts and definitions in symmetric Banach function spaces we refer to monographs [4, 51].

Let $\mathcal{M}$ be a semifinite von Neumann algebra with the unit element $\mathbf{1}$ on the Hilbert space H. If $x \in \mathcal{M}$ then by $\|x\|_{\mathcal{M}}$ denote the operator norm in $B(H)$.

A *linear operator* on Hilbert space H is a linear map $x : D(x) \to H$, where the domain $D(x)$ is a linear subspace of H. If $D(x)$ is dense in H, then we say that x is *densely defined*. The operator x is called *closed* whenever its graph is a closed subspace of $H \times H$. Any closed and densely defined linear operator has a closed and densely defined *adjoint* $x^* : D(x^*) \to H$, which is uniquely determined by the relation $\langle x\xi, \eta\rangle = \langle \xi, x^*\eta\rangle$, $\xi \in D(x), \eta \in D(x^*)$. Note that $x^{**} = x$. A closed densely defined linear operator $x : D(x) \to H$ is called *self-adjoint* if $x^* = x$ (meaning that also the domains coincide). If in addition $\langle x\xi, \xi\rangle \geq 0$ for all $\xi \in D(x)$ then x is said to be a *positive operator*. The range and kernel of a linear operator x are denoted by $\operatorname{Range} x$ and $\operatorname{Ker} x$, respectively. The projection onto $\operatorname{Ker} x$ is called a *null projection* of x, denoted by $n(x)$. The projection $s(x) = \mathbf{1} - n(x)$, which is the projection onto $\operatorname{Ker}^\perp x$, is called a *support projection*.

If $u \in B(H)$ satisfies $u^*u = uu^* = \mathbf{1}$, then u is called a *unitary operator*. Moreover, an operator $v \in B(H)$ is a *partial isometry* if the restriction of v to the orthogonal complement of its kernel is an isometry, that is $\|v\xi\|_H = \|\xi\|_H$ for all $\xi \in \mathrm{Ker}^{\perp} v$ [45, 73].

We will see below that every closed and densely defined linear operator has a decomposition analogous to the factorization of a complex number in terms of its modulus and argument.

Theorem 1.2. *(Polar Decomposition) Let $x : D(x) \to H$ be a closed and densely defined operator. There exists a partial isometry u, with $u^*u = s(x)$ and $uu^* = s(x^*)$, such that $x = u\,|x|$. Moreover, if $x = wa$, where a is positive and w is a partial isometry with $w^*w = s(a)$, then $a = |x|$ and $w = v$.*

The last statement in this theorem is usually referred to as the *uniqueness of the polar decomposition.*

Let $\mathcal{M}^+$ be the space of all positive operators in $\mathcal{M}$. The *trace* τ on $\mathcal{M}$ is a map $\tau : \mathcal{M}^+ \to [0, \infty]$, which satisfies the following properties:

(i) $\tau(a + b) = \tau(a) + \tau(b)$ for all $a, b \in \mathcal{M}^+$;

(ii) $\tau(\lambda a) = \lambda\tau(a)$ for all $a \in \mathcal{M}^+$ and $\lambda \in \mathbb{R}^+$;

(iii) $\tau(u^*au) = \tau(a)$ whenever $a \in \mathcal{M}^+$ and u is a unitary operator.

Moreover, the trace $\tau : \mathcal{M}^+ \to [0, \infty]$ is called:

(i) *faithful* if $a \in \mathcal{M}^+$ and $\tau(a) = 0$ imply that $a = 0$;

(ii) *semi-finite* if for every $a \in \mathcal{M}^+$ with $\tau(a) > 0$ there exists $0 \le b \le a$ such that $0 < \tau(b) < \infty$;

(iii) *normal* if $\tau(a_\beta) \uparrow \tau(a)$ whenever $a_\beta \uparrow a$ in $\mathcal{M}^+$.

A von Neumann algebra equipped with a semi-finite, faithful, normal trace is called a *semi-finite von Neumann algebra* [73].

The symbol $P(\mathcal{M})$ will stand for the set of all orthogonal projections in $\mathcal{M}$. The projections p and q are said to be *equivalent* (relative to the von Neumann algebra $\mathcal{M}$) denoted by $p \sim q$, if there exists a partial isometry $v \in \mathcal{M}$ such that $p = v^*v$ and $q = vv^*$. Non-zero projection $p \in \mathcal{P}(\mathcal{M})$ is called *minimal* if $q \in \mathcal{P}(\mathcal{M})$ and $q \le p$

imply that $q = p$ or $q = 0$. The von Neumann algebra $\mathcal{M}$ is called *non-atomic* if it has no minimal orthogonal projections. A projection $p \in P(\mathcal{M})$ is called *σ-finite* (with respect to the trace τ) if there exists a sequence $\{p_n\}$ in $P(\mathcal{M})$ such that $p_n \uparrow p$ and $\tau(p_n) < \infty$. If unit element $\mathbf{1}$ in $\mathcal{M}$ is σ-finite, then we say that the trace τ on $\mathcal{M}$ is σ-finite [45, 73, 31].

By $e^x(\cdot)$ we will denote the spectral measure of a self-adjoint (possibly unbounded) operator x on H. We say that a closed and densely defined operator x is *affiliated* with the von Neumann algebra $\mathcal{M}$ whenever $ux = xu$ for all unitary operators in the commutant $\mathcal{M}'$ of $\mathcal{M}$. Moreover, if $x = u\,|x|$ is the polar decomposition of a closed and densely defined operator x, then x is affiliated with $\mathcal{M}$ if and only if $u \in \mathcal{M}$ and $|x|$ is affiliated with $\mathcal{M}$ [73]. We have then that $s(x) = u^*u = e^{|x|}(0, \tau(\mathbf{1}))$ and $n(x) = \mathbf{1} - s(x) = e^{|x|}\{0\}$.

A closed, densely defined operator x, affiliated with a semi-finite von Neumann algebra $\mathcal{M}$, is called *τ-measurable* if there exists $\lambda > 0$ such that $\tau\left(e^{|x|}(\lambda, \infty)\right) < \infty$. The collection of all τ-measurable operators will be denoted by $S\left(\mathcal{M}, \tau\right)$. The set $S\left(\mathcal{M}, \tau\right)$ is a $*$-algebra with respect to the sum and product defined as the closure of the algebraic sum and product, respectively. The set of all positive, τ-measurable operators will be denoted by $S^+\left(\mathcal{M}, \tau\right)$. Note that the set of all self-adjoint operators in $S\left(\mathcal{M}, \tau\right)$, denoted by $S_h\left(\mathcal{M}, \tau\right)$, is equipped now with the partial order $\geq$, that is for the self-adjoint operators $x, y \in S\left(\mathcal{M}, \tau\right)$, $y \geq x$ whenever $y - x \geq 0$.

For $x \in S\left(\mathcal{M}, \tau\right)$ we define the *singular value function* (or *decreasing rearrangement*) $\mu(x)$ by setting

$$\mu(t; x) = \inf\{\lambda \geq 0 : \ \tau(e^{|x|}(\lambda, \infty)) \leq t\}, \quad t \in [0, \infty).$$

Observe that if $\tau(\mathbf{1}) < \infty$, then $\mu(t; x) = 0$ for all $t \geqslant \tau(\mathbf{1})$. Similarly as in commutative case, $\mu(x)$ is considered as a function on interval $[0, \tau(\mathbf{1}))$, $\mu(\infty; x) =$

$\lim_{t\to\infty} \mu(t;x)$ if $\tau(\mathbf{1}) = \infty$, and $\mu(\infty;x) = 0$ if $\tau(\mathbf{1}) < \infty$. For $x \in S(\mathcal{M},\tau)$,

$$S_0(\mathcal{M},\tau) = \{x \in S(\mathcal{M},\tau) : \ \mu(\infty;x) = 0\}$$

is a $*$-subalgebra in $S(\mathcal{M},\tau)$.

We shall frequently use the following submajorization inequalities: if $x, y \in S(\mathcal{M},\tau)$ then $\mu(\mu(x) - \mu(y)) \prec \mu(x-y)$ and $\mu(x+y) \prec \mu(x) + \mu(y)$ [21, 22]. For the basic properties of the singular value function we refer the reader to [21, 33].

The trace τ on $\mathcal{M}^+$ extends uniquely to an additive, positively homogeneous, unitarily invariant and normal functional $\tilde{\tau} : S(\mathcal{M},\tau)^+ \to [0,\infty]$, which is given by $\tilde{\tau}(x) = \int_0^\infty \mu(t;x)\,dt$ for all $x \in S(\mathcal{M},\tau)^+$ [26]. For convenience, we denote this extension $\tilde{\tau}$ by τ.

Let $\epsilon, \delta > 0$ and $U(\epsilon,\delta) = \{x \in S(\mathcal{M},\tau) : \ \mu(\delta;x) < \epsilon\}$. The collection of all sets $\{U(\epsilon,\delta) : \ \epsilon,\delta > 0\}$ forms a base at zero for a metrizable Hausdorff topology in $S(\mathcal{M},\tau)$ [21, 33, 56]. The convergence of the sequence $\{x_n\}$ to zero in this topology will be referred to as a *convergence in measure* and will be denoted by $x_n \xrightarrow{\tau} 0$.

Given a semifinite von Neumann algebra $(\mathcal{M},\tau)$ and a symmetric Banach function space E on $[0,\tau(\mathbf{1}))$, we define the corresponding *noncommutative space* $E(\mathcal{M},\tau)$ by setting

$$E(\mathcal{M},\tau) = \{x \in S(\mathcal{M},\tau) : \ \mu(x) \in E\}.$$

The space $E(\mathcal{M},\tau)$ equipped with the norm $\|x\|_{E(\mathcal{M},\tau)} := \|\mu(x)\|_E$ is a Banach space [47], and it is called the (*noncommutative*) *symmetric space of operators* associated with $(\mathcal{M},\tau)$ and corresponding to E.

In particular if $E = L_p$ [64] on $[0,\tau(\mathbf{1}))$, $1 \leqslant p < \infty$, then for $x \in L_p(\mathcal{M},\tau)$ we have $\|x\|_{L_p(\mathcal{M},\tau)} = \|\mu(x)\|_{L_p} = \left(\int_0^{\tau(\mathbf{1})} \mu(t;|x|^p)dt\right)^{1/p} = (\tau(|x|^p))^{1/p}$.

If $x \in S(\mathcal{M},\tau)$, then $x \in \mathcal{M}$ if and only if $\mu(x) \in L_\infty[0,\tau(\mathbf{1}))$. We have then that $\|x\|_{\mathcal{M}} = \mu(0;x) = \|\mu(x)\|_{L_\infty}$. Therefore in analogy to $L_p(\mathcal{M},\tau)$ spaces, $1 \leq p < \infty$, the von Neumann algebra $\mathcal{M}$ is also denoted by $L_\infty(\mathcal{M},\tau)$.

The space $L_1(\mathcal{M},\tau)+\mathcal{M}$ consists of all operators $x \in S(\mathcal{M},\tau)$, such that $x = y+z$ where $y \in L_1(\mathcal{M},\tau)$ and $z \in M$. The norm on the space $L_1(\mathcal{M},\tau)+\mathcal{M}$ is defined by setting

$$\|x\|_{L_1(\mathcal{M},\tau)+\mathcal{M}} = \inf\{\|y\|_{L_1(\mathcal{M},\tau)} + \|z\|_{\mathcal{M}} : x = y+z, y \in L_1(\mathcal{M},\tau), z \in \mathcal{M}\},$$

for all $x \in L_1(\mathcal{M},\tau)+\mathcal{M}$. The the space $L_1(\mathcal{M},\tau)\cap\mathcal{M}$ is equipped with the norm

$$\|x\|_{L_1(\mathcal{M},\tau)\cap\mathcal{M}} = \max(\|x\|_{L_1(\mathcal{M},\tau)}, \|x\|_{\mathcal{M}}), \quad x \in L_1(\mathcal{M},\tau)\cap\mathcal{M}.$$

If $\tau(\mathbf{1}) < \infty$ then $\mathcal{M} \subset E(\mathcal{M},\tau) \subset L_1(\mathcal{M},\tau)$, and if $\tau(\mathbf{1}) = \infty$ then $L_1(\mathcal{M},\tau)\cap\mathcal{M} \subset E(\mathcal{M},\tau) \subset L_1(\mathcal{M},\tau)+\mathcal{M}$, with continuous embeddings [31, 26].

Let us explain how noncommutative spaces give rise to symmetric function spaces or Schatten classes.

Example 1.3. Let $0 < \alpha \leq \infty$ and $H = L_2[0,\alpha)$ be a Hilbert space of square integrable functions on $[0,\alpha)$. For $f \in L_\infty[0,\alpha)$ define multiplication operator

$$M_f : L_2[0,\alpha) \to L_2[0,\alpha), \qquad M_f\xi = f\xi, \quad \xi \in L_2[0,\alpha).$$

We have that $\|M_f\|_{B(L_2)} = \|f\|_{L_\infty}$ and the mapping $f \mapsto M_f$ is an algebraic isomorphism and isometry from $L_\infty[0,\alpha)$ into $B(L_2[0,\alpha))$. Moreover, $M_f^* = M_{\overline{f}}$, where $\overline{f}$ is a complex conjugate of f. Defining

$$\mathcal{M} = \{M_f : \quad f \in L_\infty[0,\alpha)\},$$

we have that $\mathcal{M}$ is a commutative von Neumann algebra on the Hilbert space $H = L_2[0,\alpha)$. It is frequently identified with algebra $L_\infty[0,\alpha)$.

The mapping $\tau : \mathcal{M}^+ \to [0,\alpha)$, defined by

$$\tau(M_f) = \int_0^\alpha f(t)dt, \quad 0 \le f \in L_\infty[0,\alpha),$$

is a faithful, normal, semifinite trace on $\mathcal{M}$.

For $f \in L^0[0,\alpha)$, the operator M_f is defined by setting

$$D(M_f) = \{\xi \in L_2[0,\alpha) : \quad f\xi \in L_2[0,\alpha)\},$$

and

$$M_f\xi = f\xi, \quad \xi \in D(M_f).$$

The algebra $S(\mathcal{M},\tau)$ of all τ-measurable operators consists of all operators M_f,$f \in L^0[0,\alpha)$, for which there exists a measurable set $A \subset [0,\alpha)$ such that $m(X \setminus A) < \infty$ and $f\chi_A \in L_\infty[0,\alpha)$. The algebra $S(\mathcal{M},\tau)$ is often identified with the algebra

$$S([0,\alpha),m) = \left\{f \in L^0[0,\alpha) : \ m([0,\alpha) \setminus A) < \infty \text{ and } f\chi_A \in L_\infty[0,\alpha), \text{ for some } A\right\}.$$

For $M_f \in S(\mathcal{M},\tau)$, the singular value function coincides with the decreasing rearrangement of $f \in S([0,\alpha),m)$. Therefore, it is justified to use the same notation for decreasing rearrangement of functions as for singular value function of operators.

Given a symmetric function space E, the noncommutative space $E(\mathcal{M},\tau)$ can be in this case identified with E. Indeed,

$$E(\mathcal{M},\tau) = \{M_f \in S(\mathcal{M},\tau) : \quad \mu(M_f) \in E\} \simeq \{f \in S([0,\alpha),m) : \quad \mu(f) \in E\} = E.$$

Example 1.4. Let H be a Hilbert space and $\mathcal{M} = B(H)$. Given a maximal orthonormal system $\{e_\alpha\}$ in H, we define

$$\operatorname{tr}(a) = \sum_\alpha \langle a e_\alpha, e_\alpha \rangle, \quad a \in B(H)^+.$$

The value of $\operatorname{tr}(a)$ does not depend on the particular choice of the maximal orthonormal system in H. The trace tr is a semi-finite, faithful, normal trace on $B(H)$, called a canonical trace on $B(H)$.

We have that $S(B(H), \operatorname{tr}) = B(H)$. Moreover, if x is a compact operator on H, then the singular value function of x is identified with its sequence of singular numbers, that is

$$\mu(x) = \sum_{n=1}^\infty s_n(x) \chi_{[n-1,n)}.$$

Let E be any symmetric function space. Since $E(B(H), \operatorname{tr}) = \{x \in B(H) : \mu(x) \in E\}$ is two-sided $*$-ideal of $B(H)$ it must be contained in $K(H)$ or equal to $B(H)$. In particular if $E = L_p$, $1 \le p < \infty$, then

$$L_p(B(H), \operatorname{tr}) = \{x \in K(H) : \mu(x) \in L_p\} = \{x \in K(H) : \{s_n(x)\} \in \ell_p\},$$

and

$$\|x\|_{L_p(B(H),\operatorname{tr})} = \|\mu(x)\|_{L_p[0,\infty)} = \|\{s_n(x)\}\|_{\ell_p}.$$

Therefore, symmetric space of measurable operators $L_p(B(H), \operatorname{tr})$ becomes Schatten class of compact operators C_p.

Analogously as for function spaces, an operator $x \in E(\mathcal{M}, \tau)$ is said to be *order continuous* in $E(\mathcal{M}, \tau)$ if for all sequences $\{x_n\} \subset S(\mathcal{M}, \tau)$, whenever $|x| \ge x_n \downarrow 0$ then $\|x_n\|_{E(\mathcal{M},\tau)} \to 0$. If all operators in $E(\mathcal{M}, \tau)$ are order continuous, then we say that $E(\mathcal{M}, \tau)$ is an order continuous space. We will show in chapter 2 that x is an order continuous element of $E(\mathcal{M}, \tau)$ if and only if $\mu(x)$ is an order continuous element of E. Also note that every order continuous symmetric space on $(\mathcal{M}, \tau)$ is embedded in $S_0(\mathcal{M},\tau)$ [7]. Furthermore, x is order continuous if and only if $\|xp_n\|_{E(\mathcal{M},\tau)} \to 0$ for every sequence $\{p_n\} \subset P(\mathcal{M})$, satisfying $p_n \downarrow_n 0$. For more details we refer the reader to [31].

If E is order continuous, then the dual $E\left(\mathcal{M},\tau\right)^*$ can be identified with the Köthe dual $E\left(\mathcal{M},\tau\right)^\times$ [26], where

$$E(\mathcal{M},\tau)^\times = \{x \in S(\mathcal{M},\tau) : xy \in L_1\left(\mathcal{M},\tau\right) \text{ for all } y \in E(\mathcal{M},\tau)\},$$

and it is equipped with the norm

$$\|x\|_{E(\mathcal{M},\tau)^\times} = \sup\{\tau(|xy|) : y \in E(\mathcal{M},\tau), \|y\|_{E(\mathcal{M},\tau)} \leq 1\}, \quad x \in E(\mathcal{M},\tau)^\times.$$

Therefore, if E is order continuous then every functional from $E\left(\mathcal{M},\tau\right)^*$ is of the form

$$\Phi_y(x) = \tau(xy), \quad x \in E(\mathcal{M},\tau),$$

for some $y \in E\left(\mathcal{M},\tau\right)^\times$. It is known that if E a fully symmetric Banach function space on $[0,\tau(\mathbf{1}))$ then $E\left(\mathcal{M},\tau\right)^\times = E^\times\left(\mathcal{M},\tau\right)$ and $E^\times$ is also a fully symmetric Banach function space [26, Propositions 5.4, 5.6]. Therefore, if E is an order continuous symmetric function space, and hence it is a fully symmetric function space, then $E\left(\mathcal{M},\tau\right)^*$ can be identified with a fully symmetric Köthe dual $E^\times\left(\mathcal{M},\tau\right)$.

For the theory of operator algebras, we refer to [45, 73], for noncommutative Banach function spaces to [60, 31, 25, 22, 21], and for unitary matrix spaces to [37]. For more information about the Köthe duality see [26, 31].

Recall the following properties of singular value function. Although, the first two properties are certainly well known, it appears there are no references to them in the literature. We sketch their proofs for the sake of completeness. The condition (6) in Proposition below follows by the same argument as in the proofs of [31, Chapter VII, Proposition 6.2] or of Lemma 1.9.

Proposition 1.5. (1) *For* $x \in S(\mathcal{M},\tau)$, $\mu(|x|+\mu(\infty;x)n(x)) = \mu(x)$.

(2) *If* $x \in S(\mathcal{M},\tau)$ *and* $|x| \geq \mu(\infty;x)s(x)$ *then* $\mu(|x|-\mu(\infty;x)s(x)) = \mu(x) - \mu(\infty;x)$.

(3)*[8, Proposition 2.2] If* $x,y \in S^{+}(\mathcal{M},\tau)$, $y \neq 0$ *and* $x \geq \mu(\infty;x)\mathbf{1}$, *then there exists* $s>0$ *such that* $\mu(s;x) < \mu(s;x+y)$.

(4) *[70, Proposition 3] If* $x,y \in S(\mathcal{M},\tau)$, $y=y^*$, $x \geq 0$, *then* $\mu(t;x) \leq \mu(t;x+iy)$ *for all* $t>0$.

(5) *[8, Proposition 3.5] If* $x,y \in S(\mathcal{M},\tau)$, $y=y^*$, $x \geq \mu(\infty;x)\mathbf{1}$ *and* $\mu(x+iy) = \mu(x)$, *then* $y=0$.

(6) *If* $x,y \in (L_1(\mathcal{M},\tau)+\mathcal{M}) \cap S_0(\mathcal{M},\tau)$ *and* $\mu((x+y)/2) = \mu(x) = \mu(y)$, *then* $x=y$.

Proof. (1) For $x \in S(\mathcal{M},\tau)$, consider the real valued function $f(t) = t+\mu(\infty;x)\chi_{\{0\}}(t)$, $t \geq 0$. Then, by functional calculus, $f(|x|) = |x|+\mu(\infty;x)n(x) \geq 0$ and for $\lambda>0$,

$$e^{|x|+\mu(\infty;x)n(x)}(\lambda,\infty) = e^{|x|}(f^{-1}(\lambda,\infty)) = \begin{cases} e^{|x|}(\lambda,\infty) & \text{if } \lambda \geq \mu(\infty;x) \\ e^{|x|}((\lambda,\infty)\cup\{0\}) & \text{if } \lambda < \mu(\infty;x). \end{cases}$$

Since $\mu(\infty;x) = \inf\{\lambda : \tau(e^{|x|}(\lambda,\infty)) < \infty\}$, $\tau\left(e^{|x|}(\lambda,\infty)\right) = \infty$ for all $\lambda < \mu(\infty;x)$. Hence also $\tau\left(e^{|x|+\mu(\infty;x)n(x)}(\lambda,\infty)\right) = \infty$ for $\lambda < \mu(\infty;x)$, and so $\mu(t;|x|+\mu(\infty;x)n(x)) = \mu(t;x)$ for $t>0$.

(2) Let $x \in S(\mathcal{M},\tau)$ and $|x| \geq \mu(\infty;x)s(x)$. Consider the function $f(t) = t - \mu(\infty;x)\chi_{(0,\infty)}$. Then $f(|x|) = |x|-\mu(\infty;x)s(x) \geq 0$ and for all $\lambda>0$,

$$e^{|x|-\mu(\infty;x)s(x)}(\lambda,\infty) = e^{f(|x|)}(\lambda,\infty) = e^{|x|}\left(f^{-1}(\lambda,\infty)\right) = e^{|x|}(\lambda+\mu(\infty;x),\infty).$$

Thus $||x| - \mu(\infty;x)s(x)| = |x| - \mu(\infty;x)s(x)$ and so,

$$\begin{aligned}\mu(t;|x| - \mu(\infty;x)s(x)) &= \inf\left\{\lambda \geq 0: \quad \tau\left(e^{|x|-\mu(\infty;x)s(x)}(\lambda,\infty)\right) \leq t\right\}\\ &= \inf\left\{\lambda \geq 0: \quad \tau\left(e^{|x|}(\lambda + \mu(\infty;x),\infty)\right) \leq t\right\}, \quad t > 0.\end{aligned}$$

Therefore for all $t > 0$, $\mu(t;|x| - \mu(\infty;x)s(x)) = \mu(t;x) - \mu(\infty;x)$. □

Let us explain how the unitary matrix space C_E can be in fact identified with the symmetric operator space. Using this identification, many lifting-type results from the symmetric sequence space E into the space C_E can be deduced from the corresponding results for the symmetric function space E and the corresponding symmetric operator space $E(\mathcal{M},\tau)$. Recall that given a symmetric sequence space $E \neq \ell_\infty$, the *unitary matrix space* C_E is a subspace of a Banach space of compact operators $K(H) \subset B(H)$ for which the sequence of singular numbers $s(x) = \{s_n(x)\} \in E$, and it is equipped with the norm $\|x\|_{C_E} = \|s(x)\|_E$. Let G be the set of all real functions $f \in L_1[0,\infty) + L_\infty[0,\infty)$ such that

$$\pi(f) = \{\pi_n(f)\} = \left\{\int_{n-1}^{n} \mu(t;f)dt\right\} \in E,$$

and set $\|f\|_G = \|\pi(f)\|_E$. If E is order continuous then $(G, \|\cdot\|_G)$ is an order continuous symmetric function space on $[0,\infty)$ [7, Proposition 6.1]. It is well known that $S\left(B(H),\mathrm{tr}\right) = B(H)$, where tr is the canonical trace on $B(H)$, and the convergence $x_n \xrightarrow{\mathrm{tr}} x$ is equivalent to the norm convergence $\|x - x_n\|_{B(H)} \to 0$, for $x, x_n \in B(H)$. Since $E \neq \ell_\infty$, the symmetric space of measurable operators $G\left(B(H),\mathrm{tr}\right)$ is a proper two-sided $*$-ideal in $B(H)$ and therefore it is contained in $K(H)$. Thus for any $x \in G\left(B(H),\mathrm{tr}\right)$ the singular value function $\mu(x)$ is of the form $\mu(t;x) = \sum_{n=1}^{\infty} s_n(x)\chi_{[n-1,n)}(t)$, $t \geq 0$. Therefore, the spaces C_E and $G\left(B(H),\mathrm{tr}\right)$ coincide as sets and they have identical norms. Let us summarize it below.

Proposition 1.6. *Let $E \neq \ell_\infty$ be a symmetric sequence space.*

Then $C_E = G(B(H), \mathrm{tr})$, where G is the set of all real functions $f \in L_1[0,\infty) + L_\infty[0,\infty)$ such that

$$\pi(f) = \{\pi_n(f)\} = \left\{ \int_{n-1}^{n} \mu(t; f) dt \right\} \in E.$$

Using this identification, the following can be easily observed.

Lemma 1.7. *Let E be a symmetric sequence space and $G = \{f \in L^0[0, \tau(\mathbf{1})) : \pi(f) = \{\int_{n-1}^{n} \mu(t; f)dt\} \in E\}$. If $g \in G^\times$ then $\{\int_{n-1}^{n} g(t)dt\} \in E^\times$ and $\|\{\int_{n-1}^{n} g(t)dt\}\|_{E^\times} \leq \|g\|_{G^\times}$. Moreover, if $b = \{b_n\} \in E^\times$ then $g = \sum_{n=1}^{\infty} b_n \chi_{[n-1,n)} \in G^\times$ and $\|g\|_{G^\times} \leq \|b\|_{E^\times}$.*

Proof. Let $g \in G^\times$ and denote by $\tilde{\pi}_n(g) = \int_{n-1}^{n} g(t)dt$, $n \in \mathbb{N}$. Then for $\tilde{\pi}(g) = \{\tilde{\pi}_n(g)\}$,

$$\begin{aligned} \|\tilde{\pi}(g)\|_{E^\times} &= \sup\left\{ \sum_{n=1}^{\infty} a_n \tilde{\pi}_n(g) : \ \|\{a_n\}\|_E \leq 1 \right\} \\ &= \sup\left\{ \int_0^\infty f(t)g(t)dt : \ f(t) = \sum_{n=1}^{\infty} a_n \chi_{[n-1,n)}, \|f\|_G = \|\{a_n\}\|_E \leq 1 \right\} \leq \|g\|_{G^\times}. \end{aligned}$$

Furthermore, for $g = \sum_{n=1}^{\infty} b_n \chi_{[n-1,n)}$, where $b = \{b_n\} \in E^\times$ and $\mu(b) = \{\mu(n; b)\}$ is its decreasing rearrangement, we have that

$$\begin{aligned} \|g\|_{G^\times} &\leq \sup\left\{ \int_0^\infty \mu(t; f)\mu(t; g)dt : \ \|f\|_G \leq 1 \right\} \\ &= \sup\left\{ \sum_{n=1}^{\infty} \pi_n(f)\mu(n; b) : \ \|\pi(f)\|_E = \|f\|_G \leq 1 \right\} \leq \|b\|_{E^\times}. \end{aligned}$$

□

We will describe next a technique [31] that allows to replace the von Neumann algebra $\mathcal{M}$ by a non-atomic von Neumann algebra $\mathbb{C}\mathbb{1} \otimes \mathcal{M}$.

Let $\mathcal{N} = \{N_f : L_2[0,1) \to L_2[0,1) : \quad f \in L_\infty[0,1)\}$ be a commutative von Neumann algebra with the trace $\eta(N_f) = \int_0^1 f dm$, where m is the Lebesgue measure on $[0,1]$. Let $\mathcal{A} = \mathcal{N}\overline{\otimes}\mathcal{M}$ be a tensor product of von Neumann algebras $\mathcal{N}$ and $\mathcal{M}$, and $\kappa = \eta \otimes \tau$ be a tensor product of the traces η and τ, that is $\kappa(N_f \otimes x) = \eta(N_f)\tau(x)$, [45, 73]. It is well known that $\mathcal{A}$ has no atoms.

Let $\mathbb{1}$ be the identity operator on $L^2(0,1)$ and denote by $\mathbb{C}\mathbb{1} = \{\lambda\mathbb{1} : \lambda \in \mathbb{C}\}$. Let $x \in S(\mathcal{M}, \tau)$ and consider a linear subspace D in $L_2[0,1) \otimes H$ generated by the vectors of the form $\zeta \otimes \xi$, where $\zeta \in L_2[0,1)$ and $\xi \in D(x) \subset H$. For every $\xi = \sum_{i=1}^{n} \zeta_i \otimes \xi_i \in D$ define $(\mathbb{1} \otimes x)(\xi) = \sum_{i=1}^{n} \zeta_i \otimes x\xi_i$. The linear operator $\mathbb{1} \otimes x : D \to L_2[0,1) \otimes H$ with domain D is preclosed and its closure $\mathbb{1}\overline{\otimes}x$ is contained in $S(\mathbb{C}\mathbb{1} \otimes \mathcal{M}, \kappa)$ [31, 61, 68].

The map $\pi : x \to \mathbb{1} \otimes x$, $x \in \mathcal{M}$, is a unital trace preserving $*$-isomorphism from $\mathcal{M}$ onto the von Neumann subalgebra $\mathbb{C}\mathbb{1} \otimes \mathcal{M}$. Consequently, π extends uniquely to a $*$-isomorphism $\tilde{\pi}$ from $S(\mathcal{M}, \tau)$ onto $S(\mathbb{C}\mathbb{1} \otimes \mathcal{M}, \kappa)$. In fact one can show that $\tilde{\pi}(x) = \mathbb{1}\overline{\otimes}x$.

Moreover $\tilde{\pi}$ preserves the singular value function in the sense that $\tilde{\mu}(\tilde{\pi}(x)) = \mu(x)$, where $\tilde{\mu}(\tilde{\pi}(x))$ is the singular value function of $\tilde{\pi}(x)$ computed with respect to the von Neumann algebra $\mathbb{C}\mathbb{1} \otimes \mathcal{M}$ and the trace κ.

Thus,

$$
\begin{aligned}
E(\mathbb{C}\mathbb{1} \otimes \mathcal{M}, \kappa) &= \{y \in S(\mathbb{C}\mathbb{1} \otimes \mathcal{M}, \kappa) : \tilde{\mu}(y) \in E\} \\
&= \{\tilde{\pi}(x) : x \in S(\mathcal{M}, \tau) \text{ and } \mu(x) \in E\},
\end{aligned}
$$

where

$$\|\tilde{\pi}(x)\|_{E(\mathbb{C}\mathbb{1}\otimes\mathcal{M},\kappa)} = \|\tilde{\mu}(\tilde{\pi}(x))\|_E = \|\mu(x)\|_E = \|x\|_{E(\mathcal{M},\tau)}.$$

Hence $\tilde{\pi}$ is a $*$-isomorphism which is also an isometry from $E(\mathcal{M}, \tau)$ onto $E(\mathbb{C}\mathbb{1} \otimes \mathcal{M}, \kappa)$. We refer reader to [31] for details.

Let us summarize the above discussion.

Proposition 1.8. *For any semi-finite von Neumann algebra $(\mathcal{M}, \tau)$ there exists a non-atomic, semi-finite von Neumann algebra $(\mathcal{A}, \kappa)$, such that $E(\mathcal{M}, \tau)$ is isometrically isomorphic to $E(\mathcal{A}, \kappa)$, for any symmetric function space E.*

With this preparation we can prove the next result for any von Neumann algebra. For non-atomic von Neumann algebras it was shown implicitly in the proof of Theorem 2.1 in [9]. Moreover, we will use the identification of the space C_E with $G(B(H), \mathrm{tr})$ to obtain its version for unitary matrix spaces.

Lemma 1.9. *For any x, $x_n \in E_0(\mathcal{M}, \tau)$, $n \in \mathbb{N}$, we have that $x_n \xrightarrow{\tau} x$ whenever $\|\mu(x) - \mu(x_n)\|_E \to 0$ and $\|\mu(x) - \mu((x + x_n)/2)\|_E \to 0$.*

Moreover, if $E \neq \ell_\infty$ is a symmetric sequence space, and $x, x_n \in C_E$, $n \in \mathbb{N}$, then from $\|s(x) - s(x_n)\|_E \to 0$ and $\|s(x) - s((x + x_n)/2)\|_E \to 0$, it follows that $x_n \xrightarrow{\mathrm{tr}} x$.

Proof. Suppose that $x_n \in E_0(\mathcal{M}, \tau)$, $n \in \mathbb{N}$, and let $\|\mu(x) - \mu(x_n)\|_E \to 0$ and $\|\mu(x) - \mu((x + x_n)/2)\|_E \to 0$. Thus in view of the assumptions and the remarks preceding the lemma,

$$\|\tilde{\mu}(\tilde{\pi}(x)) - \tilde{\mu}(\tilde{\pi}(x_n))\|_E \to 0,$$

and

$$\|\tilde{\mu}(\tilde{\pi}(x)) - \tilde{\mu}((\tilde{\pi}(x) + \tilde{\pi}(x_n))/2)\|_E = \|\tilde{\mu}(\tilde{\pi}(x)) - \tilde{\mu}(\tilde{\pi}((x + x_n)/2))\|_E \to 0.$$

Since $\mathbb{C}\mathbb{1} \otimes \mathcal{M}$ is non-atomic, it follows that $\tilde{\pi}(x) - \tilde{\pi}(x_n) \xrightarrow{\kappa} 0$. The latter is equivalent to the fact that for a.e. $t > 0$, $\tilde{\mu}(t; \tilde{\pi}(x) - \tilde{\pi}(x_n)) \to 0$. Thus for a.e. $t > 0$,

$$\mu(t; x - x_n) = \tilde{\mu}(t; \tilde{\pi}(x - x_n)) = \tilde{\mu}(t; \tilde{\pi}(x) - \tilde{\pi}(x_n)) \to 0$$

and so $x - x_n \xrightarrow{\tau} 0$.

To show the result for C_E, it is enough to observe that for any $x, y \in C_E = G(B(H), \mathrm{tr})$, we have that

$$\begin{aligned}\|\mu(x)-\mu(y)\|_G &= \Big\| \sum_{n=1}^{\infty} (s_n(x)-s_n(y))\chi_{[n-1,n)} \Big\|_G \\ &= \Big\| \pi\Big(\sum_{n=1}^{\infty} (s_n(x)-s_n(y))\chi_{[n-1,n)} \Big) \Big\|_E = \|s(x)-s(y)\|_E .\end{aligned}$$

□

An immediate conclusion from [9, Lemma 1.1], stating that for $x, x_n \in S(\mathcal{M}, \tau)$, $x_n \xrightarrow{\tau} x$ whenever $|x_n| \xrightarrow{\tau} |x|$ and $|x + x_n|/2 \xrightarrow{\tau} |x|$, is the next useful result.

Corollary 1.10. *If $x, y \in S(\mathcal{M}, \tau)$ and $|x| = |y| = 2^{-1}|x+y|$, then $x = y$.*

The following two embedding results will be very useful to show that certain properties of x in $E(\mathcal{M}, \tau)$ are inherited by $\mu(x)$.

Proposition 1.11. *[31, 25] Suppose that $\mathcal{M}$ is a non-atomic von Neumann algebra with a faithful, normal, σ-finite trace τ. Let $x \in L_1(\mathcal{M}, \tau) + \mathcal{M}$ and $x \in S_0^+(\mathcal{M}, \tau)$. Then there exist a non-atomic commutative von Neumann subalgebra $\mathcal{N} \subset \mathcal{M} \subset B(H)$ and a $*$-isomorphism V acting from the $*$-algebra $S([0, \tau(\mathbf{1})), m)$ into the $*$-algebra $S(\mathcal{N}, \tau)$, such that*

$$V(\mu(x)) = x \quad \textit{and} \quad \mu(V(f)) = \mu(f) \quad \textit{for all } f \in S([0, \tau(\mathbf{1})), m).$$

Consequently, the Banach function lattice $E = E[0, \tau(\mathbf{1}))$ is isometrically embedded into $E(\mathcal{M}, \tau)$.

Given any linear operator $x : H \to H$ and a subspace $H_0 \subset H$, by $x|_{H_0}$ we denote the restriction of x to H_0. For a von Neumann algebra $\mathcal{M}$ and the projection $p \in P(\mathcal{M})$ define

$$\mathcal{M}_p = \left\{ x_p = px|_{p(H)} : \quad x \in \mathcal{M} \right\}.$$

The set $\mathcal{M}_p$ is a von Neumann algebra of $B(p(H))$, with the unit element p. We define $\tau_p : \mathcal{M}_p^+ \to [0, \infty]$ by setting

$$\tau_p(x_p) = \tau(pxp),$$

where $x \in \mathcal{M}^+$. It is well known that τ_p is a semi-finite, normal, faithful trace on $\mathcal{M}_p^+$, and it is finite if and only if $\tau(p) < \infty$.

The following can be easily observed.

Lemma 1.12. *If $x \in E(\mathcal{M}, \tau)$ and $\mu(\infty; x) = 0$, then for $p = s(x^*) \vee s(x)$ (resp. $p = s(x)$), the trace τ_p is σ-finite on $\mathcal{M}_p$.*

Proof. Setting $p_n = e^{|x|}(\frac{1}{n}, \infty) \vee e^{|x^*|}(\frac{1}{n}, \infty)$ (resp. $p_n = e^{|x|}(\frac{1}{n}, \infty)$), we have that $p_n \uparrow p$ and $\tau_p(p_n) = \tau(p_n) < \infty$, $n \in \mathbb{N}$. □

Next lemma together with Proposition 1.11 will ensure that E is isometrically embedded into $E(\mathcal{M}_p, \tau_p)$, for some $p \in P(\mathcal{M})$, even if the trace τ on the non-atomic von Neumann algebra $\mathcal{M}$ is not σ-finite.

Lemma 1.13. *If $\mathcal{M}$ be a non-atomic, semi-finite von Neumann algebra, there exists a σ-finite projection $p \in P(\mathcal{M})$ such that $\tau(p) = \tau(\mathbf{1})$.*

Proof. If $\tau(\mathbf{1}) < \infty$, then $p = \mathbf{1}$. Suppose next that $\tau(\mathbf{1}) = \infty$. Since $\mathcal{M}$ is non-atomic, we can find a projection $p_1 \in P(\mathcal{M})$ such that $1 < \tau(p_1) < \infty$. Choose a projection $p_2 \in P(\mathcal{M})$, satisfying $p_2 \leq p_1^\perp$ and $1 < \tau(p_2) < \infty$. Next, take a projection $p_3 \in P(\mathcal{M})$ such that $p_3 \leq (p_1 + p_2)^\perp$ and $1 < \tau(p_3) < \infty$. Let $\{p_n\} \subset P(\mathcal{M})$ be a sequence of projections obtained by repeating the process described above. Set $p = \bigvee p_n$. Since $\{p_n\}$ is a sequence of mutually orthogonal projections, we have in fact that $p = \sum_n p_n$ and $\tau(p) = \sum_n \tau(p_n) = \infty$. Setting now $q_n = \sum_{i=1}^n p_i$, it follows that $q_n \uparrow p$, $\tau(q_n) < \infty$, $n \in \mathbb{N}$, and therefore p is a σ-finite projection. □

For any $x \in S(\mathcal{M}, \tau)$ we have that $e^{|x_p|}(s, \infty) = e^{|pxp|}(s, \infty)$, $s \geq 0$, and consequently $\mu^{\tau_p}(x_p) = \mu(pxp)$, where $\mu^{\tau_p}(x_p)$ is a singular value function of x_p computed with respect to the reduced von Neumann algebra $\mathcal{M}_p$ and the trace τ_p [31]. Furthermore, if $\mathcal{M}$ is non-atomic, then $\mathcal{M}_p$ is also non-atomic. Letting

$$pS(\mathcal{M}, \tau)p = \{pxp : \quad x \in S(\mathcal{M}, \tau)\},$$

the set $pS(\mathcal{M}, \tau)p$ is a $*$-subalgebra of $S(\mathcal{M}, \tau)$ with the unit element p. It is also well known [31, Chapter 3, Section 7] that if $x \in S(\mathcal{M}, \tau)$ then $x_p \in S(\mathcal{M}_p, \tau_p)$, and the map $\Phi_p : x \to x_p$, $x \in pS(\mathcal{M}, \tau)p$, is a unital $*$-isomorphism from $pS(\mathcal{M}, \tau)p$ onto $S(\mathcal{M}_p, \tau_p)$.

Proposition 1.14. *Suppose that $\mathcal{M}$ is a non-atomic von Neumann algebra with a faithful, normal, σ-finite trace τ. Let $x \in S_0^+(\mathcal{M}, \tau)$, $p \in P(\mathcal{M})$ be such that $\tau(p) = \tau(\mathbf{1})$ and $xp = px = x$. Then there exist a $*$-isomorphism W acting from the $*$-algebra $S([0, \tau(\mathbf{1})), m)$ into the $*$-algebra $pS(\mathcal{M}, \tau)p$, such that*

$$W(\mu(x)) = x \quad \text{and} \quad \mu(W(f)) = \mu(f) \quad \text{for all } f \in S([0, \tau(\mathbf{1})), m).$$

Proof. Let $x \in S_0^+(\mathcal{M}, \tau)$ and $xp = px = x$. Then $x_p \in S_0^+(\mathcal{M}_p, \tau_p)$. In fact by the remarks before, $x_p \in S^+(\mathcal{M}_p, \tau_p)$ and $\mu^{\tau_p}(x_p) = \mu(pxp) = \mu(x)$. Hence, $\mu^{\tau_p}(\infty; x_p) = \mu(\infty; x) = 0$. Applying Proposition 1.11 to the element $x_p \in S_0^+(\mathcal{M}_p, \tau_p)$ and in view of $\tau_p(\mathbf{1}_p) = \tau(p) = \tau(\mathbf{1})$, there exists a $*$-isomorphism V from $S([0, \tau(\mathbf{1})), m)$ into $S(\mathcal{M}_p, \tau_p)$ such that $V(\mu^{\tau_p}(x_p)) = x_p$ and $\mu^{\tau_p}(V(f)) = \mu(f)$, for all $f \in S([0, \tau(\mathbf{1})), m)$. The function $\Psi_p(y_p) = \Phi_p^{-1}(y_p) = pyp$ for $y_p \in S(\mathcal{M}_p, \tau_p)$ is a $*$-isomorphism from $S(\mathcal{M}_p, \tau_p)$ onto $pS(\mathcal{M}, \tau)p$.

Letting $W = \Psi_p \circ V$, W is a $*$-isomorphism from $S([0, \tau(\mathbf{1})), m)$ into $pS(\mathcal{M}, \tau)p$.

Since $pxp = x$, $\mu(x) = \mu^{\tau_p}(x_p)$ and so,

$$W(\mu(x)) = \Psi_p(V(\mu(x))) = \Psi_p(V(\mu^{\tau_p}(x_p))) = \Psi_p(x_p) = pxp = x.$$

Note that for any $y \in S(\mathcal{M}_p, \tau_p)$, $py = y$ and $y = yp|_{p(H)} = (yp)_p$. Finally by $V(f) \in S(\mathcal{M}_p, \tau_p)$,

$$\begin{aligned}\mu(W(f)) &= \mu(\Psi_p(V(f))) = \mu(\Psi_p((V(f)p)_p)) = \mu(pV(f)p)\\ &= \mu(p\,|V(f)|\,p) = \mu(p\,|V(f)|\,pp) = \mu^{\tau_p}\left((|V(f)|\,p)_p\right)\\ &= \mu^{\tau_p}(|V(f)|) = \mu^{\tau_p}(V(f)) = \mu(f),\end{aligned}$$

which finishes the proof. □

Observe that any $*$-homomorphism V is positive, that is for any $f \in S([0, \tau(\mathbf{1})), m)$, if $f \geqslant 0$ then $V(f) \geqslant 0$. Indeed, since $V(\sqrt{f}) = V(\overline{\sqrt{f}}) = \left(V(\sqrt{f})\right)^*$, if follows that

$$V(f) = V(\sqrt{f}\sqrt{f}) = V(\sqrt{f})V(\sqrt{f}) = V(\sqrt{f})\left(V(\sqrt{f})\right)^* = \left|V(\sqrt{f})\right|^2 \geqslant 0.$$

Since the next four facts will be applied several times in this dissertation, we state them below for further reference. For the proof of the first two propositions, we refer to [26, Propositin 3.4] and [31, Chapter III, Proposition 4.30, Proposition 4.32].

Proposition 1.15. *If $x, y \in S(\mathcal{M}, \tau)$ and $xy, yx \in L^1(\mathcal{M}, \tau)$, then $\tau(xy) = \tau(yx)$.*

It follows from the above proposition that if $x \in E(\mathcal{M}, \tau)$ and $y \in E^\times(\mathcal{M}, \tau)$, therefore $xy, yx \in L^1(\mathcal{M}, \tau)$ then $\tau(xy) = \tau(yx)$.

Proposition 1.16. *If $x, y \in S^+(\mathcal{M}, \tau)$ and $xy \in L^1(\mathcal{M}, \tau)$, then $x^{\frac{1}{2}}yx^{\frac{1}{2}}, y^{\frac{1}{2}}xy^{\frac{1}{2}} \in L^1(\mathcal{M}, \tau)$ and*

$$\tau(xy) = \tau(x^{\frac{1}{2}}yx^{\frac{1}{2}}) = \tau(y^{\frac{1}{2}}xy^{\frac{1}{2}}).$$

As a consequence, we have that if $x, y \in S^{+}(\mathcal{M},\tau)$ and $x \in E(\mathcal{M},\tau)$, $y \in E^{\times}(\mathcal{M},\tau)$ then $\tau(xy) \geq 0$.

The following proposition is a generalization of result from [31] proved only for $E = L_1$.

Proposition 1.17. *Let E be a symmetric function space on $[0,\tau(\mathbf{1}))$. If $x \in E(\mathcal{M},\tau)$ and $y \in E^{\times}(\mathcal{M},\tau)$ then*

$$|\tau(xy)| \leq \tau(|x^*|\,|y|)^{\frac{1}{2}}\tau(|x|\,|y^*|)^{\frac{1}{2}}.$$

Proof. Let $x = u\,|x|$ and $y = v\,|y|$ be the polar decompositions of x and y, respectively. Clearly, $xv\,|y|^{\frac{1}{2}}$ and $|y|^{\frac{1}{2}}$ belong to $S(\mathcal{M},\tau)$. Moreover,

$$\tau\left(\left|xv\,|y|^{\frac{1}{2}}\,|y|^{\frac{1}{2}}\right|\right) = \tau(|xy|) \leq \|x\|_{E(\mathcal{M},\tau)}\|y\|_{E^{\times}(\mathcal{M},\tau)} < \infty,$$

and

$$\begin{aligned}\tau\left(\left||y|^{\frac{1}{2}}\,xv\,|y|^{\frac{1}{2}}\right|\right) &= \int \mu(t;|y|^{\frac{1}{2}}\,xv\,|y|^{\frac{1}{2}})dt \leq \int \mu^{\frac{1}{2}}(t,y)\mu(t;x)\mu^{\frac{1}{2}}(t,y)dt \\ &= \int \mu(t;x)\mu(t;y)dt \leq \|x\|_{E(\mathcal{M},\tau)}\|y\|_{E^{\times}(\mathcal{M},\tau)} < \infty.\end{aligned}$$

Thus $xv\,|y|^{\frac{1}{2}}\,|y|^{\frac{1}{2}}, |y|^{\frac{1}{2}}\,xv\,|y|^{\frac{1}{2}} \in L^1(\mathcal{M},\tau)$, and so by Proposition 1.15

$$\tau(xy) = \tau(xv\,|y|^{\frac{1}{2}}\,|y|^{\frac{1}{2}}) = \tau(|y|^{\frac{1}{2}}\,xv\,|y|^{\frac{1}{2}}) = \tau(|y|^{\frac{1}{2}}\,u\,|x|\,v\,|y|^{\frac{1}{2}}) = \tau(|y|^{\frac{1}{2}}\,u\,|x|^{\frac{1}{2}}\,|x|^{\frac{1}{2}}\,v\,|y|^{\frac{1}{2}}).$$

Denote by $a = |y|^{\frac{1}{2}}\,u\,|x|^{\frac{1}{2}}$ and by $b = |x|^{\frac{1}{2}}\,v\,|y|^{\frac{1}{2}}$. Then

$$\|a\|^2_{L^2(\mathcal{M},\tau)} = \tau(a^*a) = \tau(|x|^{\frac{1}{2}}\,u^*\,|y|\,u\,|x|^{\frac{1}{2}}) \leq \int \mu(t;x)\mu(t;y)dt < \infty,$$

and $a \in L^2(\mathcal{M},\tau)$. Similarly, one can show that $b \in L^2(\mathcal{M},\tau)$. Recall that the inner product on $L^2(\mathcal{M},\tau)$ may be defined by setting $\langle a,b\rangle = \tau(b^*a)$, $a,b \in L^2(\mathcal{M},\tau)$. Applying the Cauchy-Schwarz inequality, it follows that

$$|\tau(xy)| = |\tau(ab)| = |\langle b,a^*\rangle| \le \langle a^*,a^*\rangle^{\frac{1}{2}}\langle b,b\rangle^{\frac{1}{2}} = \tau(aa^*)^{\frac{1}{2}}\tau(b^*b)^{\frac{1}{2}}$$
$$= \tau\left(|y|^{\frac{1}{2}}u\,|x|\,u^*\,|y|^{\frac{1}{2}}\right)^{\frac{1}{2}}\tau\left(|y|^{\frac{1}{2}}v^*\,|x|\,v\,|y|^{\frac{1}{2}}\right)^{\frac{1}{2}}.$$

Again using the fact that $x \in E(\mathcal{M},\tau)$ and $y \in E^{\times}(\mathcal{M},\tau)$, it is not difficult to see that

$$|y|^{\frac{1}{2}}u\,|x|\,u^*\,|y|^{\frac{1}{2}}, u\,|x|\,u^*\,|y|^{\frac{1}{2}}\,|y|^{\frac{1}{2}} \in L^1(\mathcal{M},\tau),$$

and

$$|y|^{\frac{1}{2}}v^*\,|x|\,v\,|y|^{\frac{1}{2}}, |x|\,v\,|y|^{\frac{1}{2}}\,|y|^{\frac{1}{2}}v^* \in L^1(\mathcal{M},\tau).$$

Hence by Proposition 1.15,

$$|\tau(xy)| \le \tau\left(u\,|x|\,u^*\,|y|\right)^{\frac{1}{2}}\tau\left(|x|\,v\,|y|\,v^*\right)^{\frac{1}{2}} = \tau(|x^*|\,|y|)^{\frac{1}{2}}\tau(|x^*|\,|y|)^{\frac{1}{2}},$$

using for the last equality the fact that $u\,|x|\,u^* = |x^*|$ and $v\,|y|\,v^* = |y^*|$. □

Lemma 1.18. *Let $x \in B_{E(\mathcal{M},\tau)}$ and $y \in B_{E^{\times}(\mathcal{M},\tau)}$. If $\tau(xy) = 1$ and $s(y) = s(x^*)$ then $|y| \ge \mu(\infty;y)s(y)$.*

Proof. Let $x \in E(\mathcal{M},\tau)$ and $y \in E^{\times}(\mathcal{M},\tau)$. By Proposition 1.17, we have that

$$1 = \tau(xy) \le \tau(|x^*|\,|y|)^{\frac{1}{2}}\tau(|x|\,|y^*|)^{\frac{1}{2}} \le 1.$$

Moreover, $\tau(|x^*|\,|y|) \ge 0$ and $\tau(|x|\,|y^*|) \ge 0$ by Proposition 1.16. Consequently, $\tau(|x^*|\,|y|) = 1$. If $\mu(\infty;y) = 0$ the claim follows instantly. Suppose that $\mu(\infty;y) > 0$ and let $0 < \epsilon < 1$. Consider an operator $|y| + \epsilon\,|y|\,e^{|y|}[0,\beta]$, where $\beta = \frac{1}{1+\epsilon}\mu(\infty;y)$.

We will show first that $\mu(|y| + \epsilon\,|y|\,e^{|y|}[0,\beta]) = \mu(y) \in B_{E^{\times}}$. Indeed, set $a = |y| + \epsilon\,|y|\,e^{|y|}[0,\beta] = |y|\,e^{|y|}(\beta,\infty) + (1+\epsilon)\,|y|\,e^{|y|}[0,\beta]$, and consider the real valued function $f(t) = (1+\epsilon)t\chi_{[0,\beta]} + t\chi_{(\beta,\infty)}$, $t \geq 0$. We have $a = f(|y|)$, and it follows that

$$e^a(\lambda,\infty) = e^{|y|}(f^{-1}(\lambda,\infty)) = \begin{cases} e^{|y|}\left(\frac{1}{1+\epsilon}\lambda,\infty\right) & \text{if } \lambda < \mu(\infty;x) \\ e^{|y|}(\lambda,\infty) & \text{if } \lambda \geq \mu(\infty;x) \end{cases},$$

for all $\lambda \geq 0$. Since $\mu(\infty;y) = \inf\{s \geq 0 : \ \tau(e^{|y|}(s,\infty) < \infty\}$, $\tau(e^a(\lambda,\infty)) = \infty$ for $\lambda < \mu(\infty;y)$. Thus for any $t > 0$,

$$\mu(t;a) = \inf\{\lambda : \ \tau\left(e^{a_+}(\lambda,\infty)\right) \leq t\} = \inf\{\lambda \geq \mu(\infty;x) : \ \tau\left(e^{|y|}(\lambda,\infty)\right) \leq t\} = \mu(t;y),$$

which yields that $\mu(a) = \mu(y)$.

Thus $\mu(|y| + \epsilon\,|y|\,e^{|y|}[0,\beta]) = \mu(y) \in B_{E^\times}$, and so

$$\begin{aligned} 1 + \epsilon\tau(|x^*|\,|y|\,e^{|y|}[0,\beta]) &= \tau(|x^*|\,|y|) + \epsilon\tau(|x^*|\,|y|\,e^{|y|}[0,\beta]) \\ &= \tau\left(|x^*|\,(|y| + \epsilon\,|y|\,e^{|y|}[0,\beta])\right) \leq 1. \end{aligned}$$

Consequently by Proposition 1.16, $\tau(|x^*|^{\frac{1}{2}}\,|y|\,e^{|y|}[0,\beta]\,|x^*|^{\frac{1}{2}}) = \tau(|x^*|\,|y|\,e^{|y|}[0,\beta]) = 0$, and since the operator $|x^*|^{\frac{1}{2}}\,|y|\,e^{|y|}[0,\beta]\,|x|^{\frac{1}{2}} \geq 0$, it follows that $|x^*|^{\frac{1}{2}}\,|y|\,e^{|y|}[0,\beta]\,|x^*|^{\frac{1}{2}} = 0$. Hence taking $\xi \in \text{Range }|x^*|^{\frac{1}{2}}$, that is $\xi = |x^*|^{\frac{1}{2}}\,\xi_0$, for some $\xi_0 \in D(|x^*|^{\frac{1}{2}})$, we get that

$$\langle |y|\,e^{|y|}[0,\beta]\xi,\xi\rangle = \langle |y|\,e^{|y|}[0,\beta]\,|x^*|^{\frac{1}{2}}\,\xi_0, |x^*|^{\frac{1}{2}}\,\xi_0\rangle = \langle |x^*|^{\frac{1}{2}}\,|y|\,e^{|y|}[0,\beta]\,|x^*|^{\frac{1}{2}}\,\xi_0,\xi_0\rangle = 0.$$

Therefore $|y|\,e^{|y|}[0,\beta] = 0$ on $\overline{\text{Range }|x^*|^{\frac{1}{2}}}$ and thus $\text{Ker}^\perp(|y|\,e^{|y|}[0,\beta]) \subset \text{Ker }|x^*|^{\frac{1}{2}} = \text{Ker}\,x^*$. On the other hand, in view of $s(y) = s(x^*)$, $\text{Ker}^\perp(|y|\,e^{|y|}[0,\beta]) \subset \text{Ker}^\perp y = \text{Ker}^\perp x^*$, which implies that $\text{Ker}^\perp(|y|\,e^{|y|}[0,\beta]) = \{0\}$ and $|y|\,e^{|y|}[0,\beta] = 0$. Since ϵ was arbitrarily small, $|y|\,e^{|y|}[0,\mu(\infty;y)) = 0$. Then $e^{|y|}(0,\mu(\infty;y)) = 0$, and so

$n(y) = e^{|y|}[0, \mu(\infty; y))$ and $s(y) = e^{|y|}[\mu(\infty; y), \infty)$. Finally,

$$|y| = \int_{[\mu(\infty;y),\infty)} \lambda de^{|y|}(\lambda) \geq \mu(\infty; y)e^{|y|}[\mu(\infty; y), \infty) = \mu(\infty; y)s(y).$$

□

Let $(X, \|\cdot\|)$ be a Banach space over the field of complex numbers. By B_X and S_X we denote the unit ball and the unit sphere in X, respectively. Below there are given definitions of several geometric properties of X which will be subjects of this dissertation.

We say that $x \in S_X$ is a *strongly extreme point* of the unit ball B_X, or $MLUR$ point of B_X [53], if for any $\{y_n\}$, $\{z_n\} \subset B_X$, $\|2x - y_n - z_n\| \to 0$ implies that $\|y_n - z_n\| \to 0$. Equivalently, $x \in S_X$ is a strongly extreme point if for any $\{y_n\} \subset X$, $\|x \pm y_n\| \to 1$ implies $\|y_n\| \to 0$. A Banach space X is called *midpoint locally uniformly rotund* ($MLUR$) space, if every element from the unit sphere S_X is a strongly extreme point. Midpoint local uniform rotundity was defined by Anderson [2]. He showed that a strictly convex reflexive Banach space with the Kadec-Klee property is midpoint locally uniformly convex. For more on the $MLUR$-property in real Banach spaces, on its role and relations to other geometric properties we refer to [53].

A point x of S_X is said to be a *complex extreme* point ($\mathbb{C}$-*extreme* point) of the unit ball B_X [74] if for every $\lambda \in \mathbb{C}$, $|\lambda| \leq 1$ and y in X, whenever $x + \lambda y \in B_X$ then $y = 0$. Equivalently, x is a complex extreme point for B_X whenever $x \pm y, x \pm iy \in B_X$, $y \in B_X$, then $y = 0$. The space X is said to be *complex rotund* ($\mathbb{C}$-*rotund*) space, if every element from the unit sphere S_X is a $\mathbb{C}$-extreme point. By $\mathbb{C} - \text{ext}\,(B_X)$ denote the set of all complex extreme points of B_X. Complex extreme points were introduced by Thorp and Whitley in [74], where they showed that L_1 is a complex rotund space. One of the most important properties of the space of complex analytic functions is the validity of the maximum modulus principle. As it was shown in [74], the maximum modulus principle holds true for analytic functions with values in complex rotund Banach spaces.

A point $x \in S_X$ is a *point of complex local uniform rotundity* ($\mathbb{C}-LUR$ point) [76] if for every $\epsilon > 0$ there exists $\delta(x, \epsilon) > 0$ such that

$$\sup_{\lambda=\pm 1,\pm i} \|x + \lambda y\| \geq 1 + \delta(x, \epsilon)$$

for every $y \in X$ satisfying $\|y\| \geq \epsilon$. Equivalently, x is a $\mathbb{C}-LUR$ point whenever from $\|x+\lambda y_n\| \to 1$, $\{y_n\} \subset X$, $\lambda = \pm 1, \pm i$ it follows that $\|y_n\| \to 0$. If every point of the unit sphere of X is a $\mathbb{C}-LUR$ point, then X is called a *complex locally uniformly rotund* ($\mathbb{C}-LUR$) space. In 2000, Wang and Teng [76] defined $\mathbb{C}-LUR$ points and $\mathbb{C}-LUR$ spaces and obtained criteria for this property in the class of Musielak-Orlicz spaces of vector-valued functions.

It is clear that the real geometric properties such as uniform rotundity, local uniform rotundity and rotundity imply their complex analogies, that is, complex uniform rotundity, complex local uniform rotundity and complex rotundity, respectively.

There exist also the notions of complex strongly extreme points ($\mathbb{C}-MLUR$ points) and $\mathbb{C}-MLUR$ spaces. They are defined analogously as $MLUR$ points and $MLUR$ spaces [6] following the idea contained in [34]. However as we will show in the second section of the fourth chapter the notions of $\mathbb{C}-LUR$ points and $\mathbb{C}-MLUR$ points coincide in any complex Banach space, and consequently the properties $\mathbb{C}-LUR$ and $\mathbb{C}-MLUR$ are equivalent. We wish to point out that there exist $\mathbb{C}-LUR$ spaces that are not complex uniformly rotund, and complex rotund spaces that are not $\mathbb{C}-LUR$. We will discuss briefly some examples in the last section of chapter 4.

An element $x \in S_X$ is a *smooth point* of B_X if there exists a unique normalized functional $F \in X^*$ which supports B_X at x; i.e. $F(x) = 1$. We will say then that the functional F *supports* x. A Banach space X is said to be *smooth* (or *Gâteaux* *smooth*) if every x from the unit sphere is a smooth point [16, 17, 18, 53].

It is worth to observe that the unique functional $F \in X^*$ supporting a smooth point x is an extreme point of B_{X^*}. Indeed, letting $F = (F_1 + F_2)/2$, where $F_1, F_2 \in B_{X^*}$, we have $2 = 2F(x) = F_1(x) + F_2(x)$. Thus since $|F_1(x)|, |F_2(x)| \leqslant 1$ it follows that $F_1(x) = F_2(x) = 1$. Using now the fact that F is a unique functional supporting x, we get that $F_1 = F_2 = F$.

Let $x \in S_X$ be a smooth point of B_X and F be its supporting functional. If for any sequence $\{F_n\} \subset X^*$, satisfying $\|F_n\|_{X^*} \to 1$, the condition $F_n(x) \to 1$ implies $\|F_n - F\|_{X^*} \to 0$ then x is called a *strongly smooth point* of B_X, and we say that F *strongly supports* x. It is standard to check that equivalent definition of strongly smooth points arises when condition $\|F_n\|_{X^*} \to 1$ in the above statement is replaced by $\{F_n\} \subset B_{X^*}$. A Banach space X is said to be *Fréchet smooth* if every x from the unit sphere is a strongly smooth point.

It is easy to observe that the functional $F \in S_{X^*}$ which strongly supports $x \in S_X$, is a strongly extreme point of B_{X^*}. Indeed, let $\|F \pm F_n\|_{X^*} \to 1$, for the sequence $\{F_n\} \subset X^*$. Then in view of the inequality $|1 \pm F_n(x)| = |(F \pm F_n)(x)| \leqslant \|F \pm F_n\|_{X^*}\|x\|_X$, it follows that $\overline{\lim}_n |1 \pm F_n(x)| \leqslant 1$, and so $\lim_n |1 \pm F_n(x)| = 1$ by Lemma 1.20. Therefore $\lim_n F_n(x) = 0$ and $(F - F_n)(x) \to 1$. By the assumption that F strongly supports x, $\|F_n\|_{X^*} = \|F - (F - F_n)\|_{X^*} \to 0$, proving that F is a strongly extreme point of B_{X^*}.

There is a certain level of duality between smoothness and convexity properties. Klee [48] showed that if dual X^* of a Banach space X is smooth, then X is strictly convex, and if X^* is strictly convex, then X is smooth. Therefore for the reflexive Banach spaces there is a complete duality between smoothness and strict convexity. It is also known that if X^* is locally uniformly rotund, then X is Fréchet smooth [18]. For the applications of smoothness and Fréchet smoothness, we refer to [17, 18].

An element $x \in S_X$ is an *exposed point* of B_X if there exists a normalized functional

$F \in X^*$ which supports B_X exactly at x, i.e. $F(x) = 1$ and $F(y) \neq 1$ for every $y \in B_X \setminus \{x\}$. We say that F *exposes* B_X at x.

It is not difficult to see that every exposed point of B_X is an extreme point of B_X.

Let $x \in S_X$ be an exposed point of B_X and suppose that the functional F exposes B_X at x. If $F(x_n) \to 1$ implies $\|x - x_n\| \to 0$ for all sequences $\{x_n\} \subset B_X$, then x is a *strongly exposed point* of B_X and F *strongly exposes* B_X at x. It is well known that every strongly exposed point of B_X is strongly extreme.

Straszewicz [72] and later Mil'man [55] started a discussion on exposed points (called by them accessible points). It is known that in a strictly convex (resp. locally uniformly convex) space all boundary points of the closed unit ball are exposed (resp. strongly exposed). In fact, we have that in every normed space a convex weakly compact set is the closed convex hull of its set of strongly exposed points [16].

Monotone properties of Banach lattices are closely related to their complex rotundity properties [42, 52]. The interplay between those properties is an important factor in investigating complex properties in Banach lattices. Let us recall some monotonicity notions employed further in this dissertation.

A point f in a partially ordered normed linear space $(F, \|\cdot\|_F)$ is called *upper monotone* point, or UM point, if for any $g \in F$ the condition $g \geq f$ and $g \neq f$ implies that $\|g\|_F > \|f\|_F$.

We say that the norm $\|\cdot\|_F$ on F is *strictly monotone* (SM for short) is for every $f, g \geq 0$ we have that $\|f\|_F < \|g\|_F$, whenever $f \leq g$ and $f \neq g$.

An element $0 \leq f \in S_F$ is called *upper locally uniform monotone point*, or $ULUM$ point, if for any sequence $\{f_n\}$ such that $f \leq f_n$, if $\|f_n\|_F \to 1$ then $\|f_n - f\|_F \to 0$. If every point in S_{F^+} is a $ULUM$ point, then we say that the norm on F is *upper locally uniformly monotone*.

For a Banach lattice E over the field of complex numbers, we define its *real part* as

$$E_r = \{f \in E : \quad \mathrm{Im}\,(f) = 0\}\,,$$

with the norm induced from E.

The following results of H. Hudzik and A. Narloch [42] relating monotonicity and complex rotundity properties for the Banach lattice E appear very useful in our investigations.

Theorem 1.19. (1) *An element f of a complex Banach lattice E is a $\mathbb{C}$-extreme point if and only if $|f|$ is an UM point in its real part E_r.*

(2) *If f is a $\mathbb{C}-LUR$ point of a complex Banach lattice E, then $|f|$ is an $ULUM$ point in its real part E_r.*

(3) *Complex Banach lattice E is complex rotund if and only if its norm $\|\cdot\|$ on E is strictly monotone.*

(4) *If a complex Banach lattice E is complex locally uniformly rotund then its norm $\|\cdot\|$ on E is upper locally uniformly monotone.*

We finish this section with an elementary result in Banach spaces.

Lemma 1.20. *Let $(X, \|\cdot\|)$ be a normed space. If $\overline{\lim}_n \|x \pm y_n\| \le 1$, $\|x\| = 1$ and $\{y_n\} \subset X$ for $n \in \mathbb{N}$, then $\lim\limits_n \|x \pm y_n\| = 1$.*

Proof. Let $\|x\| = 1$ and $\overline{\lim}_n \|x \pm y_n\| \le 1$. From the inequality

$$2 = 2\|x\| \le \|x + y_n\| + \|x - y_n\|$$

it follows that $\overline{\lim\limits_n} \|x \pm y_n\| = 1$. Note that $1 - \|x - y_n\|/2 \le \|x + y_n\|/2$ and therefore

$$1 - \overline{\lim_n} \|x - y_n\|/2 \le \underline{\lim}_n \,(1 - \|x - y_n\|/2) \le \underline{\lim}_n \|x + y_n\|/2.$$

Since $\overline{\lim\limits_n} \|x \pm y_n\| = 1$, we get that $1 \le \underline{\lim}_n \|x + y_n\| \le \overline{\lim}_n \|x + y_n\| = 1$, and consequently $\lim_n \|x + y_n\| = 1$. Similarly one can show that $\lim_n \|x - y_n\| = 1$. $\square$

1.3 Convention

Convention. Throughout this dissertation the terms decreasing or increasing mean non-increasing or non-decreasing, respectively. Unless stated otherwise, the equality or inequality between functions indicate that they are satisfied almost everywhere. For any positive, decreasing function $g \in L^0[0,\infty)$, a set $B = \{t \geqslant 0 : g(t) = b\}$ for some $b > 0$, is called an interval of constancy of g if $mB > 0$. Semifinite von Neumann algebra $\mathcal{M}$ is always fixed and has faithful, semifinite, normal trace τ. By E we will always denote a symmetric function space on $[0,\alpha)$, where $0 < \alpha = \tau(\mathbf{1}) \leqslant \infty$, and by m the Lebesgue measure on $\mathbb{R}$. If f is a Lebesgue integrable function on $[0,\alpha)$ then we write $\int f(t)\,dt := \int_0^\alpha f(t)\,dt$, unless stated otherwise. We interpret $\infty \cdot 0 = 0$ if necessary, where 0 is a zero projection.

2 Strongly Extreme Points and Midpoint Local Uniform Rotundity

In this chapter we will discuss strongly extreme points and midpoint local uniform rotundity of $E(\mathcal{M},\tau)$. The results of this chapter have been published in [11]. We will first show that if $\mu(x)$ is an order continuous, strongly extreme point of the unit ball in E_0, then x is a strongly extreme point of the unit ball in $E_0(\mathcal{M},\tau)$.

We shall need the following two lemmas to prove the claim. To ensure that E is an interpolation space between L^1 and L^∞ we need to assume that E is a fully symmetric space [4, 51].

Lemma 2.1. *Suppose that E is fully symmetric and $\mu(x)$ is a $MLUR$ point of the unit ball in E. Then for any $\{y_n\}$, $\{z_n\} \subset \mathrm{B}_{E(\mathcal{M},\tau)}$ the convergence $\|2x-y_n-z_n\| \to 0$ implies that $\|\mu(x)-\mu(y_n)\| \to 0$ and $\|\mu(x)-\mu(z_n)\| \to 0$.*

Proof. Suppose that $\mu(x) \in S_E$ is a $MLUR$ point of B_E, and $\|2x-y_n-z_n\|_{E(\mathcal{M},\tau)} \to 0$, $\{y_n\},\{z_n\} \subset B_{E(\mathcal{M},\tau)}$. Note that $\mu(2x) \prec \mu(2x-y_n-z_n)+\mu(y_n)+\mu(z_n)$. By Theorem 1.1, for any $n \in \mathbb{N}$, there exists a substochastic linear operator $T_n : E \to E$ such that

$$T_n\mu(2x-y_n-z_n)+T_n\mu(y_n)+T_n\mu(z_n) = \mu(2x).$$

Therefore

$$\|\mu(2x)-T_n\mu(y_n)-T_n\mu(z_n)\|_E = \|T_n\mu(2x-y_n-z_n)\|_E \leq \|2x-y_n-z_n\|_{E(\mathcal{M},\tau)},$$

which implies that $\|2\mu(x)-T_n\mu(y_n)-T_n\mu(z_n)\|_E \to 0$. Applying the fact that $\mu(x)$ is a $MLUR$ point of B_E, we get

$$\|\mu(x)-T_n\mu(y_n)\|_E \to 0 \quad \text{and} \quad \|\mu(x)-T_n\mu(z_n)\|_E \to 0.$$

Therefore it remains to show that $\|T_n\mu(y_n)-\mu(y_n)\|_E \to 0$ and $\|T_n\mu(z_n)-\mu(z_n)\|_E \to$

0. Consider the function sequence $f_n = T_n\mu(y_n)/2 - \mu(y_n)/2$. For every $n \in \mathbb{N}$,

$$\begin{aligned}
\|\mu(x) - f_n\|_E &= \|\mu(x) - T_n\mu(y_n)/2 + \mu(y_n)/2\|_E \\
&\leq \|\mu(x) - T_n\mu(y_n)\|_E + \|T_n\mu(y_n)\|_E/2 + \|\mu(y_n)\|_E/2 \\
&\leq \|\mu(x) - T_n\mu(y_n)\|_E + 1,
\end{aligned}$$

and in view of $\mu(x) - \frac{1}{2}\mu(y) \prec \mu(x - \frac{1}{2}y)$,

$$\begin{aligned}
\|\mu(x) + f_n\|_E &= \|\mu(x) + T_n\mu(y_n)/2 - \mu(y_n)/2\|_E \\
&\leq \|\mu(x) - \mu(y_n/2)\|_E + \|T_n\mu(y_n)\|_E/2 \\
&\leq \|x - y_n/2\|_{E(\mathcal{M},\tau)} + \|y_n\|_{E(\mathcal{M},\tau)}/2 \\
&\leq \|x - y_n/2 - z_n/2\|_{E(\mathcal{M},\tau)} + \|y_n\|_{E(\mathcal{M},\tau)}/2 + \|z_n\|_{E(\mathcal{M},\tau)}/2 \\
&\leq \|x - y_n/2 - z_n/2\|_{E(\mathcal{M},\tau)} + 1.
\end{aligned}$$

Consequently,

$$\overline{\lim_n}\|\mu(x) \pm f_n\|_E \leq 1.$$

Now by Lemma 1.20 we have that $\lim_n \|\mu(x) \pm f_n\|_E = 1$. Applying the assumption that $\mu(x)$ is a $MLUR$ point, we deduce that $\|f_n\|_E = \frac{1}{2}\|T_n\mu(y_n) - \mu(y_n)\|_E \to 0$. Similarly, one can show that $\|T_n\mu(z_n) - \mu(z_n)\|_E \to 0$. □

Before we state the first main theorem of this chapter, we need few facts about order continuous elements of $E(\mathcal{M},\tau)$. Next proposition relates order continuity of the operator x with order continuity of its singular value function.

Proposition 2.2. *An operator $x \in E(\mathcal{M},\tau)$ is order continuous element of $E(\mathcal{M},\tau)$ if and only if $\mu(x)$ is order continuous element of E.*

Proof. Suppose that $\mu(x)$ is order continuous element of E and $0 \downarrow_n x_n \leqslant |x|$. If $\mu(x)$

is order continuous, then $\mu(\infty; x) = 0$ and by [26, Lemma 3.5], $\mu(t; x_n) \downarrow_n 0$ for all $t > 0$. Therefore $\|x_n\|_{E(\mathcal{M},\tau)} = \|\mu(x_n)\|_E \to 0$ and x is an order continuous element of $E(\mathcal{M}, \tau)$.

Let x be an order continuous element of $E(\mathcal{M}, \tau)$. Hence $\mu(\infty; x) = 0$ and by Lemma 1.12 for $p = s(x) \vee s(x^*)$ the trace τ_p on $\mathcal{M}_p$ is σ-finite. It is easy to show that x_p is an order continuous element of $E(\mathcal{M}_p, \tau_p)$. Indeed, let $\{y_n\} \subset E(\mathcal{M}_p, \tau_p)$ be such that $0 \downarrow_n y_n \leqslant |x_p| = |x|_p$. Then for all $n \in \mathbb{N}$, $y_n = (x_n)_p$ for some positive $x_n \in E(\mathcal{M}, \tau)$. Moreover, from $0 \downarrow_n (x_n)_p \leqslant |x|_p$ it follows that $0 \downarrow_n px_np \leqslant p\,|x|\,p \leqslant |x|$. Using the assumption that x is order continuous, $\|y_n\|_{E(\mathcal{M}_p,\tau_p)} = \|px_np\|_{E(\mathcal{M},\tau)} \to 0$, proving that x_p is order continuous. Moreover $\mu^{\tau_p}(x_p) = \mu(pxp) = \mu(x)$, where μ^{τ_p} is a singular value function computed with respect to the trace τ_p and the von Neumann algebra $\mathcal{M}_p$. Therefore we can assume that the trace τ is σ-finite. Consider a $*$-isomorphism $\tilde{\pi}$ from $E(\mathcal{M}, \tau)$ onto $E(\mathbb{C}\mathbb{1} \otimes \mathcal{M}, \kappa)$, where $\mathbb{C}\mathbb{1} \otimes \mathcal{M}$ is a non-atomic von Neumann algebra (see the discussion preceding Proposition 1.8). It is not difficult to see that x is order continuous in $E(\mathcal{M}, \tau)$ if and only if $\tilde{\pi}(x)$ is order continuous in $E(\mathbb{C}\mathbb{1} \otimes \mathcal{M}, \kappa)$. Since for the singular value function $\tilde{\mu}(\tilde{\pi}(x))$ of $\tilde{\pi}(x)$ computed with respect to the von Neumann algebra $\mathbb{C}\mathbb{1} \otimes \mathcal{M}$ and the trace κ, we have that $\tilde{\mu}(\tilde{\pi}(x)) = \mu(x)$, it can be assumed that the von Neumann algebra $\mathcal{M}$ is non-atomic.

Suppose that a.e $0 \downarrow_n f_n \leqslant \mu(x)$. By Proposition 1.11, there exists a $*$-isomorphism V from $S([0, \tau(\mathbf{1})), m)$ into $S(\mathcal{M}, \tau)$ such that $V(\mu(x)) = |x|$ and $\mu(V(f)) = \mu(f)$ for all $f \in S([0, \tau(\mathbf{1})), m)$. Therefore $0 \downarrow_n V(f_n) \leqslant V(\mu(x)) = |x|$ and using the fact that x is order continuous, $\|f_n\|_E = \|V(f_n)\|_{E(\mathcal{M},\tau)} \to 0$. □

The following convergence result is known under the stronger assumption that the whole space E is order continuous [9, Proposition 1.1]. Its analogy for the symmetric sequence space $E \neq \ell_\infty$ and the unitary matrix space C_E follows instantly, by the same argument as in the proof of Lemma 1.9.

Proposition 2.3. *Let E be strongly symmetric. For an order continuous element $x \in E(\mathcal{M},\tau)$ and $\{x_n\} \subset E(\mathcal{M},\tau)$, the following conditions are equivalent.*

(i) $\|x - x_n\|_{E(\mathcal{M},\tau)} \to 0$,

(ii) $\|\mu(x) - \mu(x_n)\|_E \to 0$ *and* $x_n \xrightarrow{\tau} x$.

Moreover, if $E \neq \ell_\infty$ is a strongly symmetric sequence space, $x \in C_E$ is order continuous in C_E and $\{x_n\} \subset C_E$, then the following conditions are equivalent.

(i') $\|x - x_n\|_{C_E} \to 0$,

(ii') $\|s(x) - s(x_n)\|_E \to 0$ *and* $x_n \xrightarrow{\text{tr}} x$.

Proof. Since $\mu(x) - \mu(x_n) \prec \mu(x - x_n)$, $n \in \mathbb{N}$, and the embedding of $E(\mathcal{M},\tau)$ in $S(\mathcal{M},\tau)$ is continuous, (i) implies (ii).

Suppose now that (ii) holds. Note that it is enough to show that there exists a subsequence that satisfies (i), since this implies that every subsequence has a subsequence which satisfies (i). If $x_n - x \xrightarrow{\tau} 0$, by [31, Chapter II, Lemma 5.15], passing to subsequence of $\{x_n\}$, there exists a sequence $\{p_j\} \subset P(\mathcal{M})$ satisfying $p_j \uparrow \mathbf{1}$, $\tau(p_j^\perp) \to 0$ as $j \to \infty$ and $\|(x - x_n)p_j\|_{\mathcal{M}} \to 0$ as $n \to \infty$ for all $j \in \mathbb{N}$.

For all projections p in $E(\mathcal{M},\tau)$ and all $n, j \in \mathbb{N}$ we have the following

$$
\begin{aligned}
\|(x - x_n)p\|_{E(\mathcal{M},\tau)} &\leq \|(x - x_n)p_j p\|_{E(\mathcal{M},\tau)} + \|(x - x_n)p_j^\perp p\|_{E(\mathcal{M},\tau)} \qquad (2.1)\\
&\leq \|(x - x_n)p_j\|_{\mathcal{M}} \|p\|_{E(\mathcal{M},\tau)} + \|\mu(x)\chi_{[0,\tau(p_j^\perp))} + \mu(x_n)\chi_{[0,\tau(p_j^\perp))}\|_E\\
&\leq \|(x - x_n)p_j\|_{\mathcal{M}} \|p\|_{E(\mathcal{M},\tau)} + \|\mu(x_n)\chi_{[0,\tau(p_j^\perp))} - \mu(x)\chi_{[0,\tau(p_j^\perp))}\|_E\\
&+ 2\|\mu(x)\chi_{[0,\tau(p_j^\perp))}\|_E \leq \|(x - x_n)p_j\|_{\mathcal{M}} \|p\|_{E(\mathcal{M},\tau)}\\
&+ \|\mu(x_n) - \mu(x)\|_E + 2\|\mu(x)\chi_{[0,\tau(p_j^\perp))}\|_E.
\end{aligned}
$$

Consider first the case when $x_n \in S_0(\mathcal{M},\tau)$ for all $n \in \mathbb{N}$. Since x is order continuous, also $x \in S_0(\mathcal{M},\tau)$. Setting $q(x) = s(x) \vee s(x^*)$ and $q(x_n) = s(x_n) \vee s(x_n^*)$, $n \in \mathbb{N}$, and $p = \vee_{n=1}^\infty q(x_n) \vee q(x)$, the trace τ_p is σ-finite on the von Neumann algebra $\mathcal{M}_p$ by Lemma 1.12. Moreover, for all $n \in \mathbb{N}$, $\|x_p - (x_n)_p\|_{E(\mathcal{M}_p,\tau_p)} = \|p(x -$

$x_n)p\|_{E(\mathcal{M},\tau)} = \|x - x_n\|_{E(\mathcal{M},\tau)}$, $\mu^{\tau_p}((x_n)_p) = \mu(x_n)$, $\mu^{\tau_p}(x_p) = \mu(x)$ and $\mu^{\tau_p}(x_p - (x_n)_p) = \mu(x - x_n)$, where μ^{τ_p} is the singular value function computed with respect to the trace τ_p and $\mathcal{M}_p$. Therefore, we can assume that the trace τ is σ-finite. Consider the non-atomic von Neumann algebra $\mathbb{C}\mathbb{1} \otimes \mathcal{M}$ and a $*$-isomorphism $\tilde{\pi} : x \to \mathbb{1}\overline{\otimes}x$ from $S(\mathcal{M},\tau)$ onto $S(\mathbb{C}\mathbb{1} \otimes \mathcal{M}, \kappa)$. Since $\tilde{\pi}$ preserves the singular value function, it can be also assumed that the von Neumann algebra $\mathcal{M}$ is non-atomic.

Let $\{e_k\} \subset P(\mathcal{M})$ be such that $e_k \uparrow_k \mathbf{1}$ and $\tau(e_k) < \infty$, for all $k \in \mathbb{N}$. By Proposition 1.11, for each $n \in \mathbb{N}$ there exists a $*$-isomorphism V_n from $S([0,\tau(\mathbf{1})), m)$ into $S(\mathcal{M},\tau)$, such that $V_n(\mu(x_n)) = |x_n|$ and $\mu(V_n(f)) = \mu(f)$, for all $f \in S([0,\tau(\mathbf{1})), m)$. Let $q_{nk} = V_n(\chi_{[0,\tau(e_k))})$, $n,k \in \mathbb{N}$. Clearly, for $n,k \in \mathbb{N}$, q_{nk} is a projection, where $\tau(q_{nk}) = \tau(e_k) < \infty$, and therefore $q_{nk} \in E(\mathcal{M},\tau)$. By the fact that $|x_n q_{nk}^{\perp}| = \left||x_n|\, q_{nk}^{\perp}\right|$ and $q_{nk}^{\perp} = \mathbf{1} - q_{nk} = V_n(\chi_{[0,\tau(\mathbf{1}))}) - V_n(\chi_{[0,\tau(e_k))}) = V_n(\chi_{[\tau(e_k),\tau(\mathbf{1}))})$, for all $n,k \in \mathbb{N}$ we have that

$$\begin{aligned}\|x_n q_{nk}^{\perp}\|_{E(\mathcal{M},\tau)} &= \|\,|x_n|\, q_{nk}^{\perp}\|_{E(\mathcal{M},\tau)} \\ &= \|V_n(\mu(x_n))V_n(\chi_{[\tau(e_k),\tau(\mathbf{1}))})\|_{E(\mathcal{M},\tau)} \\ &= \|\mu(x_n)\chi_{[\tau(e_k),\tau(\mathbf{1}))}\|_E \\ &\leq \|\mu(x_n)\chi_{[\tau(e_k),\tau(\mathbf{1}))} - \mu(x)\chi_{[\tau(e_k),\tau(\mathbf{1}))}\|_E \\ &+ \|\mu(x)\chi_{[\tau(e_k),\tau(\mathbf{1}))}\|_E \leq \|\mu(x_n) - \mu(x)\|_E \\ &+ \|\mu(x)\chi_{[\tau(e_k),\tau(\mathbf{1}))}\|_E.\end{aligned} \tag{2.2}$$

Let $\epsilon > 0$. Since x and $\mu(x)$ are order continuous by Proposition 2.2, $\|xe_k^{\perp}\|_{E(\mathcal{M},\tau)} \to 0$, $\|\mu(x)\chi_{[\tau(e_k),\tau(\mathbf{1}))}\|_E \to 0$ and $\|\mu(x)\chi_{[0,\tau(p_j{}^{\perp}))}\|_E \to 0$ as $k,j \to \infty$. Therefore, there exist $k_0, j_0 \in \mathbb{N}$ such that $\|xe_{k_0}^{\perp}\|_{E(\mathcal{M},\tau)} \leq \epsilon/5$, $\|\mu(x)\chi_{[\tau(e_{k_0}),\tau(\mathbf{1}))}\|_E \leq \epsilon/5$ and $\|\mu(x)\chi_{[0,\tau(p_{j_0}^{\perp}))}\|_E \leq \epsilon/20$. Observe that $\|q_{nk}\|_{E(\mathcal{M},\tau)} = \|V_n\left(\chi_{[0,\tau(e_k))}\right)\|_{E(\mathcal{M},\tau)} = \|\chi_{[0,\tau(e_k))}\|_E = \|e_k\|_{E(\mathcal{M},\tau)}$ and $\mathbf{1} = q_{nk} + q_{nk}^{\perp} = q_{nk} + q_{nk}^{\perp}(e_k + e_k^{\perp}) = q_{nk} + q_{nk}^{\perp}e_k + q_{nk}^{\perp}e_k^{\perp}$,

$n, k \in \mathbb{N}$. This, combined with inequalities 2.1 and 2.2, implies that for all $n \in \mathbb{N}$

$$\begin{aligned}\|x - x_n\|_{E(\mathcal{M},\tau)} &\leq \|(x - x_n)q_{nk_0}\|_{E(\mathcal{M},\tau)} + \|(x - x_n)q_{nk_0}^{\perp}e_{k_0}\|_{E(\mathcal{M},\tau)} \\ &+ \|xq_{nk_0}^{\perp}e_{k_0}^{\perp}\|_{E(\mathcal{M},\tau)} + \|x_n q_{nk_0}^{\perp}e_{k_0}^{\perp}\|_{E(\mathcal{M},\tau)} \\ &\leq \|(x - x_n)q_{nk_0}\|_{E(\mathcal{M},\tau)} + \|(x - x_n)e_{k_0}\|_{E(\mathcal{M},\tau)} \\ &+ \|xe_{k_0}^{\perp}\|_{E(\mathcal{M},\tau)} + \|x_n q_{nk_0}^{\perp}\|_{E(\mathcal{M},\tau)} \\ &\leq 2\|(x - x_n)p_{j_0}\|_{\mathcal{M}}\|e_{k_0}\|_{E(\mathcal{M},\tau)} + 3\|\mu(x_n) - \mu(x)\|_E + 3/5\epsilon.\end{aligned}$$

Since $\|(x - x_n)p_{j_0}\|_{\mathcal{M}} \to 0$ and $\|\mu(x_n) - \mu(x)\|_E \to 0$ as $n \to \infty$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\|(x - x_n)p_{j_0}\|_{\mathcal{M}} \leq \epsilon/(10\|e_{k_0}\|_{E(\mathcal{M},\tau)})$ and $\|\mu(x_n) - \mu(x)\|_E \leq \epsilon/15$. Consequently, for all $n \geq N$ $\|x - x_n\|_{E(\mathcal{M},\tau)} \leq \epsilon$.

Suppose now that for some $n \in \mathbb{N}$, $\mu(\infty; x_n) > 0$. Then $\mu(x_n) > c\chi_{[0,\tau(\mathbf{1}))}$ for some constant $c > 0$ and therefore $\mu(\mathbf{1}) = \chi_{[0,\tau(\mathbf{1}))} \in E$, which implies that $\mathbf{1} \in E(\mathcal{M}, \tau)$. By the relation 2.1, it follows that

$$\begin{aligned}\|x - x_n\|_{E(\mathcal{M},\tau)} \leq \|(x - x_n)p_j\|_{\mathcal{M}}\|\mathbf{1}\|_{E(\mathcal{M},\tau)} + \|\mu(x_n) - \mu(x)\|_E \\ + 2\|\mu(x)\chi_{[0,\tau(p_j^{\perp}))}\|_E,\end{aligned}$$

for all $n, j \in \mathbb{N}$. Therefore, similarly as in the previous case, one can show that $\|x - x_n\|_{E(\mathcal{M},\tau)} \to 0$ as $n \to \infty$, and the claim follows. □

Our next result relies essentially on Lemma 1.9, and therefore we have to assume that the operators are elements of $E_0(\mathcal{M}, \tau)$.

Theorem 2.4. *Let E be fully symmetric and x be an order continuous element of $E(\mathcal{M}, \tau)$. If the singular value function $\mu(x)$ is a MLUR point in B_{E_0} then x is a MLUR point in $B_{E_0(\mathcal{M},\tau)}$.*

Proof. Let $\mu(x) \in S_{E_0}$ be a $MLUR$ point in B_{E_0}, and suppose that $\|2x - y_n -$

$z_n\|_{E(\mathcal{M},\tau)} \to 0$, where $\{y_n\}, \{z_n\} \subset B_{E_0(\mathcal{M},\tau)}$. By Lemma 2.1, we have

$$\|\mu(x) - \mu(y_n)\|_E \to 0 \quad \text{and} \quad \|\mu(x) - \mu(z_n)\|_E \to 0.$$

Also

$$\|2x - (y_n + x)/2 - (z_n + x)/2\|_{E(\mathcal{M},\tau)} = \|x - y_n/2 - z_n/2\|_{E(\mathcal{M},\tau)} \to 0,$$

with $(y_n + x)/2$, $(z_n + x)/2$ from the unit ball in $E_0(\mathcal{M},\tau)$. Again, referring to Lemma 2.1 we get

$$\|\mu(x) - \mu\left((y_n + x)/2\right)\|_E \to 0 \quad \text{and} \quad \|\mu(x) - \mu\left((z_n + x)/2\right)\|_E \to 0.$$

Now, applying Lemma 1.9 it follows that $y_n \xrightarrow{\tau} x$ and $z_n \xrightarrow{\tau} x$. Thus $\|x - y_n\|_{E(\mathcal{M},\tau)} \to 0$ and $\|x - z_n\|_{E(\mathcal{M},\tau)} \to 0$ by Proposition 2.3, and x is a $MLUR$ point of $B_{E_0(\mathcal{M},\tau)}$. $\square$

Next, we want to establish that if $x \in S_{E(\mathcal{M},\tau)}$ is strongly extreme, then $\mu(x)$ is a strongly extreme point of B_E. We will need first the following elementary lemma.

Lemma 2.5. *Let $x \in S(\mathcal{M},\tau)$. If $n(x) \sim n(x^*)$, then there exists an isometry w such that $x = w|x|$.*

Proof. Suppose that $n(x) \sim n(x^*)$, that is $n(x) = v^*v$ and $n(x^*) = vv^*$, where v is a partial isometry. Let $x = u|x|$ be the polar decomposition of x, that is u is a partial isometry with $\operatorname{Ker} u = \operatorname{Ker} x$. Set $w = u + v$. We claim that w is an isometry, that is $w^*w = \mathbf{1}$. To see it, note first that since $\operatorname{Ker}(u^*) = \operatorname{Ker}(x^*)$, $u^*vv^* = u^*n(x^*) = 0$. Thus $|v^*u|^2 = u^*vv^*u = 0$ and $v^*u = 0$. Hence

$$\begin{aligned} w^*w &= (u^* + v^*)(u + v) = u^*u + u^*v + v^*u + v^*v \\ &= u^*u + (v^*u)^* + v^*u + v^*v = s(x) + n(x) = \mathbf{1}, \end{aligned}$$

proving that w is an isometry.

Since v^*v is a projection on the $\mathrm{Ker}^{\perp} v$ and $0 = (|x|\, n(x))^* = n(x)\,|x| = v^*v\,|x|$, it follows that Range $|x| \subset \mathrm{Ker}\, v$. Consequently $v\,|x| = 0$. Therefore

$$w\,|x| = (u + v)\,|x| = u\,|x| + v\,|x| = u\,|x| = x,$$

and the proof is complete. □

Theorem 2.6. *Suppose that the trace τ on $\mathcal{M}$ is σ-finite. If x is a $MLUR$ point in $B_{E(\mathcal{M},\tau)}$ then $\mu(x)$ is a $MLUR$-point in B_E and either*

(i) $\mu(\infty; x) = 0$, *or*

(ii) $n(x)\mathcal{M}n(x^*) = 0$ *and* $|x| \geq \mu(\infty; x)s(x)$.

Proof. Since strongly extreme points are preserved by the linear isometry, it can be assumed that the von Neumann algebra $\mathcal{M}$ is non-atomic (see Proposition 1.8).

Suppose that x is a $MLUR$ point of the unit ball in $E(\mathcal{M}, \tau)$. Since every strongly extreme point is extreme, conditions (i) and (ii) are satisfied by the well known criterion on extreme points in $B_{E(\mathcal{M},\tau)}$ [8]. It remains to show that $\mu(x)$ is a $MLUR$ point in B_E.

Assume first that $\tau(s(x)) = \infty$. By Proposition 1.5(2) and by (ii), $\mu(|x| - \mu(\infty; x)s(x)) = \mu(x) - \mu(\infty; x)$. Consequently $\mu(\infty; |x| - \mu(\infty; x)s(x)) = 0$ and $|x| - \mu(\infty; x)s(x) \in S_0^+(\mathcal{M}, \tau)$. Also, in view of $s(x)\,|x| = |x|\, s(x)$ we have

$$s(x)\,(|x| - \mu(\infty; x)s(x)) = (|x| - \mu(\infty; x)s(x))\, s(x) = |x| - \mu(\infty; x)s(x).$$

Hence applying Proposition 1.14 to the element $|x| - \mu(\infty; x)s(x)$ with $p = s(x)$, there is a $*$-isomorphism W from $S\,([0, \infty), m)$ into $s(x)S(\mathcal{M}, \tau)s(x)$, such that

$$W(\mu(|x| - \mu(\infty; x)s(x))) = |x| - \mu(\infty; x)s(x) \quad \text{and} \quad \mu(W(f)) = \mu(f),$$

for all $f \in S([0,\infty), m)$. Since $W(1) = s(x)$, where $1 = \chi_{[0,\infty)}$, it follows that

$$\begin{aligned}|x| - \mu(\infty; x)s(x) &= W(\mu(|x| - \mu(\infty; x)s(x))) = W(\mu(x) - \mu(\infty; x)) \\ &= W(\mu(x)) - \mu(\infty; x)W(1) = W(\mu(x)) - \mu(\infty; x)s(x)\end{aligned}$$

and consequently $W(\mu(x)) = |x|$.

Let $\|\mu(x) + \lambda f_n\|_E \to 1$ for $\lambda = \pm 1$, where $f_n \in B_E$, $n \in \mathbb{N}$. Clearly,

$$\begin{aligned}\lim_n \| \, |x| + \lambda W(f_n)\|_{E(\mathcal{M},\tau)} &= \lim_n \|W(\mu(x)) + \lambda W(f_n)\|_{E(\mathcal{M},\tau)} \\ &= \lim_n \|\mu(x) + \lambda f_n\|_E = 1.\end{aligned}$$

Let $x = u\,|x|$ be a polar decomposition of x. Since for $\lambda = \pm 1$,

$$\begin{aligned}\overline{\lim_n}\|x + \lambda u W(f_n)\|_{E(\mathcal{M},\tau)} &= \overline{\lim_n}\|u\,|x| + \lambda u W(f_n)\|_{E(\mathcal{M},\tau)} \\ &\le \lim_n \| \, |x| + \lambda W(f_n)\|_{E(\mathcal{M},\tau)} = 1\end{aligned}$$

and $\|x\|_{E(\mathcal{M},\tau)} = 1$, by Lemma 1.20 we have that

$$\lim_n \|x + \lambda u W(f_n)\|_{E(\mathcal{M},\tau)} = 1, \quad \lambda = \pm 1.$$

Using the assumption that x is a strongly extreme point, we get the convergence $\|uW(f_n)\|_{E(\mathcal{M},\tau)} \to 0$. Recall that $u^*u = s(x)$. Hence, also $\|s(x)W(f_n)\|_{E(\mathcal{M},\tau)} \to 0$. But the image of the isomorphism W is contained in $s(x)S(\mathcal{M},\tau)s(x)$, where the unit element is $s(x)$. Therefore $s(x)W(f_n) = W(f_n)$ and consequently

$$\|f_n\|_E = \|\mu(f_n)\|_E = \|\mu(W(f_n))\|_E = \|W(f_n)\|_{E(\mathcal{M},\tau)} \to 0.$$

This concludes the proof in the case when $\tau(s(x)) = \infty$.

Suppose now that $\tau(s(x)) < \infty$. Thus $\mu(\infty; x) = 0$. Let $x = u\,|x|$ be a polar decomposition of x. Since $s(x) = u^*u$ and $s(x^*) = uu^*$, $s(x) \sim s(x^*)$ and $\tau(s(x^*)) = \tau(s(x)) < \infty$. Hence $s(x)$ and $s(x^*)$ are finite, equivalent projections in $\mathcal{M}$ and by [73, Chapter 5, Proposition 1.38], $n(x) \sim n(x^*)$. Therefore by Lemma 2.5 there exists an isometry w, such that $x = w\,|x|$.

Let $\|\mu(x) + \lambda f_n\|_E \to 1$ for $\lambda = \pm 1$, where $f_n \in B_E$, $n \in \mathbb{N}$. Proposition 1.11, applied to the operator $|x|$, implies the existence of an $*$-isomorphism V from $S([0, \tau(\mathbf{1})), m)$ into $S(\mathcal{M}, \tau)$, such that $V(\mu(x)) = |x|$ and $\mu(V(f)) = \mu(f)$, for all $f \in S([0, \tau(\mathbf{1})), m)$. Note that

$$\begin{aligned}\lim_n \|\,|x| \pm V(f_n)\|_{E(\mathcal{M},\tau)} &= \lim_n \|V(\mu(x)) \pm V(f_n)\|_{E(\mathcal{M},\tau)} \\ &= \lim_n \|\mu(x) \pm f_n\|_E = 1.\end{aligned}$$

Since w is an isometry, for $\lambda = \pm 1$ we have

$$\begin{aligned}\lim_n \|x + \lambda w V(f_n)\|_{E(\mathcal{M},\tau)} &= \lim_n \|w\,|x| + \lambda w V(f_n)\|_{E(\mathcal{M},\tau)} \\ &= \lim_n \|\,|x| + \lambda V(f_n)\|_{E(\mathcal{M},\tau)} = 1.\end{aligned}$$

Now in view of x being a $MLUR$ point of $B_{E(\mathcal{M},\tau)}$, we get $\|wV(f_n)\|_{E(\mathcal{M},\tau)} \to 0$. Hence

$$\lim_n \|f_n\|_E = \lim_n \|V(f_n)\|_{E(\mathcal{M},\tau)} = \lim_n \|wV(f_n)\|_{E(\mathcal{M},\tau)} = 0,$$

which proves that $\mu(x)$ is a $MLUR$ point of B_E, and ends the proof. □

The next corollary combines the results of Theorems 2.4 and 2.6.

Corollary 2.7. *Let E be a symmetric function space on $[0, \tau(\mathbf{1}))$ and $\mathcal{M}$ be a von Neumann algebra with a faithful, normal, semi-finite trace τ.*

(1) *Let x be an order continuous element of $E(\mathcal{M}, \tau)$, where E is fully symmetric.*

Then $\mu(x)$ is a $MLUR$ point of B_{E_0} if and only if x is a $MLUR$ point of $B_{E_0(\mathcal{M},\tau)}$.

(2) *The space E is $MLUR$ if and only if $E(\mathcal{M},\tau)$ is a $MLUR$ space.*

Proof. If $\mu(x)$ is a $MLUR$ point of B_{E_0} then x is a $MLUR$ point in $B_{E_0(\mathcal{M},\tau)}$ by Theorem 2.4. The implication in the other direction is proved in Theorem 2.6, under the additional assumption that the trace τ on $\mathcal{M}$ is σ-finite. Suppose now that x is a $MLUR$ point of $B_{E_0(\mathcal{M},\tau)}$ and $\|x \pm x_n\|_{E(\mathcal{M},\tau)} \to 1$, for $\{x_n\} \subset E_0(\mathcal{M},\tau)$. Set $q_n = \vee_{n=1}^{\infty} s(x_n) \vee s(x_n^*)$, and $q = s(x) \vee s(x^*)$. Since $x, x_n \in S_0(\mathcal{M},\tau)$, q, q_n are σ-finite projections on $\mathcal{M}$ (see Lemma 1.12). Hence for $p = \vee_{n=1}^{\infty}(q_n \vee q)$, the trace τ_p is σ-finite in $\mathcal{M}_p$. Moreover $pxp = x$ and $px_np = x_n$, for all $n \in \mathbb{N}$. Therefore without loss of generality we can assume that the trace τ is σ-finite. By Theorem 2.6 it follows now that $\mu(x)$ is a $MLUR$ point of B_{E_0}.

The second claim follows immediately from the well known fact that any $MLUR$ space E must be order continuous [53]. □

To relate $MLUR$ property of order continuous function f and its decreasing rearrangement $\mu(f)$, we apply Corollary 2.7(1) to the commutative von Neumann algebra $\mathcal{M} = L_\infty[0,\tau(\mathbf{1}))$.

Corollary 2.8. *Let f be an order continuous function in a fully symmetric space E. Then f is a $MLUR$ point of B_{E_0} if and only if $\mu(f)$ is a $MLUR$ point of B_{E_0}.*

Before proving the next theorem, we shall need a version of Lemma 2.1 for a symmetric sequence space E and the unitary matrix space C_E. The proof of the atomic variant of Lemma 2.1 can be conducted in the same way as for symmetric function space E, replacing singular value functions with sequences of singular numbers.

Theorem 2.9. *Let $E \subset c_0$ be a fully symmetric sequence space. Then C_E is a $MLUR$ space if and only if E is a $MLUR$ space.*

Proof. Since E is isometrically embedded in C_E [3, Proposition 1.1], if C_E is a $MLUR$ space then so is E. By Proposition 1.6, $C_E = G(B(H), \mathrm{tr})$ and $\|x\|_{C_E} = \|x\|_{G(B(H),\mathrm{tr})}$

for any compact operator x. Therefore proceeding analogously as in the proof of Theorem 2.4, we can show that if E is a $MLUR$ space then so is C_E. Indeed, let $s(x) = \{s_n(x)\}$ be a $MLUR$ point of B_E, and suppose that $\|2x - y_n - z_n\|_{C_E} \to 0$, where $\{y_n\}, \{z_n\} \subset B_{C_E}$. By Lemma 2.1, proven for the symmetric sequence space E, it follows that

$$\|s(x) - s(y_n)\|_E \to 0, \quad \|s(x) - s\left((y_n + x)/2\right)\|_E \to 0,$$

and

$$\|s(x) - s(z_n)\|_E \to 0, \quad \|s(x) - s\left((z_n + x)/2\right)\|_E \to 0.$$

Then, since $\mu(x) = \sum_{i=1}^{\infty} s_i(x)\chi_{[i-1,i)}$ and $\mu(y_n) = \sum_{i=1}^{\infty} s_i(y_n)\chi_{[i-1,i)}$ for all $n \in \mathbb{N}$, we have that

$$\|\mu(x) - \mu(y_n)\|_G = \left\| \sum_{i=1}^{\infty} (s_i(x) - s_i(y_n))\chi_{[i-1,i)} \right\|_G = \|s(x) - s(y_n)\|_E \to 0.$$

Similarly, it can be shown that $\|\mu(x) - \mu\left((y_n + x)/2\right)\|_G \to 0$, $\|\mu(x) - \mu(z_n)\|_G \to 0$ and $\|\mu(x) - \mu\left((z_n + x)/2\right)\|_G \to 0$. Now, by Lemma 1.9 it follows that $y_n \stackrel{\text{tr}}{\to} x$ and $z_n \stackrel{\text{tr}}{\to} x$. Consequently, by Proposition 2.3 applied to the symmetric space $G\left(B(H), \text{tr}\right)$ it follows that

$$\|x - y_n\|_{C_E} = \|x - y_n\|_{G(B(H),\text{tr})} \to 0 \text{ and } \|x - z_n\|_{C_E} = \|x - z_n\|_{G(B(H),\text{tr})} \to 0,$$

proving that x is a $MLUR$ point of B_{C_E}. Therefore if E is a $MLUR$ space, then so is the space C_E. □

3 Complex Extreme Points and Complex Rotundity

The main result of this chapter, Theorem 3.11, states a criterion for an operator x to be a complex extreme point of $B_{E(\mathcal{M},\tau)}$. This criterion is analogous to the characterization of extreme points obtained in [8, 31]. The results included in this chapter can be found in [11].

We will need several auxiliary results. The first two lemmas describe elementary characteristics of extreme points.

Lemma 3.1. *Let $x, y \in B_{E(\mathcal{M},\tau)}$ and let $\mu(t;x) \leq \mu(t;y)$ for all $t \in [0,\infty)$. If there exists $t_0 > 0$ such that $\mu(t_0;x) < \mu(t_0;y)$ then $\mu(x)$ is not $\mathbb{C}$-extreme point of* B_E.

Proof. By the assumption $\mu(x), \mu(y) \in B_E$ and for all $t > 0$, $\mu(t;x) \leq \mu(t;y)$. Suppose that $\mu(x)$ is $\mathbb{C}$-extreme. Then by Theorem 1.19(1), $\mu(x)$ is an UM-point in its real part E_r. If $\mu(t_0;x) < \mu(t_0;y)$ for some $t_0 > 0$, by the right continuity of the singular value function, there exists a set A of positive measure such that $\mu(t;x) < \mu(t;y)$ for every $t \in A$. By the upper monotonicity of $\mu(x)$ we get that

$$1 = \|\mu(x)\|_E < \|\mu(y)\|_E,$$

contradicting the fact that $\mu(y) \in B_E$. □

Lemma 3.2. *Let $x \in S(\mathcal{M},\tau)$ be a self-adjoint operator. Then $x \in \mathbb{C}-\mathrm{ext}\left(B_{E_h(\mathcal{M},\tau)}\right)$ if and only if $x \in \mathbb{C} - \mathrm{ext}\left(B_{E(\mathcal{M},\tau)}\right)$.*

Proof. It is enough to show the implication only in one direction. Suppose that $x \in S_{E(\mathcal{M},\tau)}$, $x = x^*$ and $x \in \mathbb{C} - ext(B_{E_h(\mathcal{M},\tau)})$. Let $x + \lambda y \in B_{E(\mathcal{M},\tau)}$, $\lambda = \pm 1, \pm i$, where $y \in B_{E(\mathcal{M},\tau)}$. Denoting by $y_1 = (y^* + y)/2$ and $y_2 = (y - y^*)/(2i)$, we get

$y = y_1 + iy_2$, where both y_1 and y_2 are self-adjoint. Note that for all $\lambda = \pm 1, \pm i$,

$$\begin{aligned}\|x + \lambda y_1\|_{E(\mathcal{M},\tau)} &= \|x + \lambda(y + y^*)/2\|_{E(\mathcal{M},\tau)} \leq \|x + \lambda y\|_{E(\mathcal{M},\tau)}/2 \\ &+ \|x + \lambda y^*\|_{E(\mathcal{M},\tau)}/2 = \|x + \lambda y\|_{E(\mathcal{M},\tau)}/2 \\ &+ \|x + \overline{\lambda} y\|_{E(\mathcal{M},\tau)}/2 \leq 1,\end{aligned}$$

and

$$\begin{aligned}\|x + \lambda y_2\|_{E(\mathcal{M},\tau)} &= \|x + \lambda(y - y^*)/(2i)\|_{E(\mathcal{M},\tau)} \leq \|x - \lambda i y\|_{E(\mathcal{M},\tau)}/2 \\ &+ \|x + \lambda i y^*\|_{E(\mathcal{M},\tau)}/2 = \|x - \lambda i y\|_{E(\mathcal{M},\tau)}/2 \\ &+ \|x - \overline{\lambda} i y\|_{E(\mathcal{M},\tau)}/2 \leq 1.\end{aligned}$$

By the assumption that x is a $\mathbb{C}$-extreme point it follows that $y_1 = y_2 = 0$ and consequently $y = 0$. □

Lemma 3.3. *For any $x \in S(\mathcal{M},\tau)$, $n(x)S(\mathcal{M},\tau)n(x^*) = 0$ if $n(x)\mathcal{M}n(x^*) = 0$.*

Proof. Suppose that $n(x)\mathcal{M}n(x^*) = 0$ and let $y \in S(\mathcal{M},\tau)$. Recall that if a is a closed, linear operator with the domain $\mathcal{D}(a) = H$, then by the Closed Graph Theorem, $a \in B(H)$. Furthermore, if a is a bounded, linear operator affiliated with $\mathcal{M}$, that is $ba = ab$ for all $b \in \mathcal{M}'$, then $a \in (\mathcal{M}')' = \mathcal{M}$. Since $e^{|y|}[0,n](H) \subset \mathcal{D}(|y|) = \mathcal{D}(y)$, $ye^{|y|}[0,n] \in B(H)$ and by the τ-measurability of y, $ye^{|y|}[0,n]$ is affiliated with $\mathcal{M}$. Therefore for all $n \in \mathbb{N}$, $ye^{|y|}[0,n] \in \mathcal{M}$ and by the assumption we have $n(x)ye^{|y|}[0,n]n(x^*) = 0$. We will show now that

$$n(x)ye^{|y|}[0,n]n(x^*) \xrightarrow{\tau} n(x)yn(x^*), \text{ as } n \to \infty.$$

By τ-measurability of $|y|$ there exists $n_1 \in \mathbb{N}$ for which $\tau\left(e^{|y|}(n_1,\infty)\right) < \infty$. Since

$$e^{|y|}(n_1,\infty) - e^{|y|}(n,\infty) \uparrow e^{|y|}(n_1,\infty),$$

by the normality of trace τ it follows that

$$\tau\left(e^{|y|}(n_1,\infty) - e^{|y|}(n,\infty)\right) \to \tau\left(e^{|y|}(n_1,\infty)\right),$$

where for $n \geq n_1$,

$$\tau\left(e^{|y|}(n_1,\infty) - e^{|y|}(n,\infty)\right) = \tau\left(e^{|y|}(n_1,\infty)\right) - \tau\left(e^{|y|}(n,\infty)\right).$$

Thus $\tau\left(e^{|y|}(n,\infty)\right) \to 0$ as $n \to \infty$. Consequently, for all $t \in \mathbb{R}^+$,

$$\begin{aligned} &\mu\left(t; n(x)ye^{|y|}[0,n]n(x^*) - n(x)yn(x^*)\right) = \mu\left(t; n(x)ye^{|y|}(n,\infty)n(x^*)\right) \\ &\leq \mu(t; ye^{|y|}(n,\infty)) = \mu(t;y)\chi_{\left[0,\tau\left(e^{|y|}(n,\infty)\right)\right)}(t) \to 0. \end{aligned}$$

Thus $n(x)ye^{|y|}[0,n]n(x^*) \xrightarrow{\tau} n(x)yn(x^*)$ and since $n(x)ye^{|y|}[0,n]n(x^*) = 0$ for all $n \in \mathbb{N}$, the claim follows. $\square$

Lemma 3.4. *Let $x \in S(\mathcal{M},\tau)$. Then $|x| \geq \mu(\infty;x)s(x)$ if and only if $|x^*| \geq \mu(\infty;x)s(x^*)$.*

Proof. Suppose that $|x| \geq \mu(\infty;x)s(x)$. Let $x = u\,|x|$ be the polar decomposition of x. Then $u\,|x|\,u^* = |x^*|$ and $|x^*| \geq \mu(\infty;x)us(x)u^*$. Indeed,

$$\begin{aligned} \langle |x^*|\,\xi,\xi\rangle &= \langle u\,|x|\,u^*\xi,\xi\rangle = \langle |x|\,u^*\xi,u^*\xi\rangle \geq \mu(\infty;x)\,\langle s(x)u^*\xi,u^*\xi\rangle \\ &= \mu(\infty;x)\,\langle us(x)u^*\xi,\xi\rangle, \end{aligned}$$

for any ξ in the domain of $|x^*|$. Applying now the well known equalities, $s(x) = u^*u$

and $s(x^*) = uu^*$, we get $us(x)u^* = uu^*uu^* = s(x^*)$, and so $|x^*| \geq \mu(\infty; x)s(x^*)$.

If $|x^*| \geq \mu(\infty; x)s(x^*)$, then by the above argument $|x| = |(x^*)^*| \geq \mu(\infty; x)s(x)$.

$\square$

We shall need the following results, in particular Corollary 3.6, to prove that x is a complex extreme point whenever $\mu(x)$ is a complex extreme point.

Lemma 3.5. *Let $x \in S(\mathcal{M}, \tau)$ and $x \geq \mu(\infty; x)\mathbf{1}$. If $\mu(x) \in \mathbb{C} - \text{ext}\,(B_E)$ then $x \in \mathbb{C} - \text{ext}\left(B_{E(\mathcal{M},\tau)}\right)$.*

Proof. Let $x \in S(\mathcal{M}, \tau)$, $x \geq \mu(\infty; x)\mathbf{1}$ and $\mu(x) \in \mathbb{C} - \text{ext}\,(B_E)$. Suppose that $x \pm y$, $x \pm iy$ belong to $B_{E(\mathcal{M},\tau)}$, for some $y \in B_{E(\mathcal{M},\tau)}$. In view of Lemma 3.2, we can assume without lost of generality that y is a self-adjoint operator. Now by Proposition 1.5(4), for all $t > 0$,

$$\mu(t; x) \leq \mu(t; x + iy).$$

Since $\mu(x) \in \mathbb{C} - \text{ext}\,(B_E)$ and $\mu(x + iy) \in B_E$, by Lemma 3.1 it follows that for all $t > 0$,

$$\mu(t; x) = \mu(t; x + iy).$$

Then Proposition 1.5(5) implies that $y = 0$, and the claim follows.

$\square$

Corollary 3.6. *If $\mu(x) \in \mathbb{C} - \text{ext}\,(\mathrm{B}_E)$ and $|x| \geq \mu(\infty; x)s(x)$, then $|x| + \mu(\infty; x)n(x) \in \mathbb{C} - \text{ext}\left(\mathrm{B}_{E(\mathcal{M},\tau)}\right)$. Consequently, if $\mu(x) \in \mathbb{C} - \text{ext}\,(\mathrm{B}_E)$ and $\mu(\infty; x) = 0$, then $|x| \in \mathbb{C} - \text{ext}\left(\mathrm{B}_{E(\mathcal{M},\tau)}\right)$.*

Proof. It follows immediately from Lemma 3.5, since $|x| \geq \mu(\infty; x)s(x)$ implies that $|x| + \mu(\infty; x)n(x) \geq \mu(\infty; x)\mathbf{1}$ and by Proposition 1.5(1), $\mu(|x| + \mu(\infty; x)n(x)) = \mu(x)$. $\square$

After all of these preliminary results, we are ready for our first main claim in this chapter.

Theorem 3.7. *An element $x \in S_{E(\mathcal{M},\tau)}$ is a $\mathbb{C}$-extreme point of $\mathrm{B}_{E(\mathcal{M},\tau)}$ whenever $\mu(x)$ is a $\mathbb{C}$-extreme point of B_E and one of the following conditions holds:*

(i) $\mu(\infty; x) = 0$,

(ii) $n(x)\mathcal{M}n(x^*) = 0$ *and* $|x| \geq \mu(\infty; x)s(x)$.

Proof. Suppose that $\mu(x)$ is a $\mathbb{C}$-extreme point and $x \pm y$, $x \pm iy$ belong to $B_{E(\mathcal{M},\tau)}$, for $y \in B_{E(\mathcal{M},\tau)}$. Let $x = u\,|x|$ and $x^* = u^*\,|x^*|$ be the polar decomposition of x and x^*, respectively. Since $\mathrm{Ker}\, u = \mathrm{Ker}\, x$ and $\mathrm{Ker}\,(u^*) = \mathrm{Ker}\,(x^*)$, $un(x) = u^*n(x^*) = 0$. Hence $x = u(|x| + \mu(\infty; x)n(x))$ and $x^* = u^*(|x^*| + \mu(\infty; x)n(x^*))$. Thus $|x| + \mu(\infty; x)n(x) = u^*x$ and $|x^*| + \mu(\infty; x)n(x^*) = ux^*$, and so

$$|x| + \mu(\infty; x)n(x) + \lambda u^*y, \quad |x^*| + \mu(\infty; x)n(x^*) + \lambda uy^* \in B_{E(\mathcal{M},\tau)}$$

for all $\lambda = \pm 1, \pm i$.

In view of the assumptions (i) or (ii) and Lemma 3.4, $|x| \geq \mu(\infty; x)s(x)$ and $|x^*| \geq \mu(\infty; x)s(x^*)$. Since $\mu(x) = \mu(x^*)$ is a $\mathbb{C}$-extreme point, Corollary 3.6 implies that $|x|+\mu(\infty; x)n(x)$ and $|x^*|+\mu(\infty; x)n(x^*)$ are complex extreme points of $B_{E(\mathcal{M},\tau)}$. Therefore $u^*y = uy^* = 0$. Hence $s(x^*)y = uu^*y = 0$ and $ys(x) = yu^*u = 0$, since $(yu^*u)^* = u^*uy^* = 0$. Therefore $y = (s(x^*) + n(x^*))y(s(x) + n(x)) = n(x^*)yn(x)$.

If (ii) is satisfied, then by Lemma 3.3, $n(x^*)S\,(\mathcal{M}, \tau)\,n(x) = 0$ and consequently $y = n(x^*)yn(x) = 0$.

Consider now the case when (i) holds true, that is $\mu(\infty;x)=0$. Then we have

$$\begin{aligned}|x+\lambda y|^2 &= |x+\lambda n(x^*)yn(x)|^2 = (x+\lambda n(x^*)yn(x))^*\,(x+\lambda n(x^*)yn(x))\\ &= \left(x^*+\overline{\lambda}n(x)y^*n(x^*)\right)(x+\lambda n(x^*)yn(x))\\ &= x^*x+\lambda x^*n(x^*)yn(x)+\overline{\lambda}n(x)y^*n(x^*)x\\ &+ n(x)y^*n(x^*)n(x^*)yn(x) = |x|^2+|n(x^*)yn(x)|^2\,.\end{aligned}$$

Also

$$\begin{aligned}(|x|+|y|)^2 &= (|x|+|n(x^*)yn(x)|)^2 = |x|^2+|x|\,|n(x^*)yn(x)|\\ &+ |n(x^*)yn(x)|\,|x|+|n(x^*)yn(x)|^2\,.\end{aligned}$$

Let $n(x^*)yn(x) = v\,|n(x^*)yn(x)|$ be the polar decomposition of $n(x^*)yn(x)$. Then $|n(x^*)yn(x)| = v^*n(x^*)yn(x) = n(x)y^*n(x^*)v$, and so $|n(x^*)yn(x)|\,|x|=0$. Hence,

$$|x+\lambda y| = |x|+|y| \text{ for } \lambda=\pm 1,\pm i,$$

and so $|x|+|y| \in B_{E(\mathcal{M},\tau)}$. Since $\mu(|x|+|y|)\geq\mu(x)$, Lemma 3.1 implies that $\mu(|x|+|y|)=\mu(x)$. By Proposition 1.5(3), if $|y|\neq 0$ then $\mu(s;|x|+|y|)>\mu(s;|x|)$ for some $s>0$. Thus $|y|=0$, and consequently $y=0$. This concludes the proof in case when $\mu(\infty;x)=0$. □

In order to show the reverse statement, we will need the next two lemmas.

Lemma 3.8. *If x is a $\mathbb{C}$-extreme point of $\mathrm{B}_{E(\mathcal{M},\tau)}$ then $|x|\geq\mu(\infty;x)s(x)$.*

Proof. Suppose that $\mu(\infty;x)>0$. Let $x=u\,|x|$ be the polar decomposition of x. Fix $0<\epsilon<1$ and consider the following operators

$$a_{\pm} = |x| \pm \epsilon\,|x|\,e^{|x|}[0,\beta],$$

$$b_{\pm} = |x| \pm i\epsilon\, |x|\, e^{|x|}[0,\beta],$$

where $\beta = \frac{1}{1+\epsilon}\mu(\infty; x)$. Clearly,

$$a_{-} = |x|\, e^{|x|}(\beta,\infty) + (1-\epsilon)\, |x|\, e^{|x|}[0,\beta],$$

and

$$a_{+} = |x|\, e^{|x|}(\beta,\infty) + (1+\epsilon)\, |x|\, e^{|x|}[0,\beta].$$

Hence $0 \le a_{-} \le |x|$, and so $\mu(a_{-}) \le \mu(x)$. Furthermore, as it was shown in the proof of Lemma 1.18, $\mu(a_{+}) = \mu(x)$. Now observe that

$$\begin{aligned} |b_{-}|^2 = |b_{+}|^2 = b_{-}^{*}b_{-} = b_{+}^{*}b_{+} = |x|^2 + \epsilon^2\, |x|^2\, e^{|x|}[0,\beta] \\ \le |x|^2 + \epsilon^2\, |x|^2\, e^{|x|}[0,\beta] + 2\epsilon\, |x|^2\, e^{|x|}[0,\beta] = (|x| + \epsilon\, |x|\, e^{|x|}[0,\beta])^2 = a_{+}^2. \end{aligned}$$

Hence

$$\mu^2(b_{-}) = \mu^2(b_{+}) = \mu(|b_{+}|^2) = \mu(|b_{-}|^2) \le \mu^2(a_{+}) = \mu^2(x)$$

and $\mu(b_{+}) = \mu(b_{-}) \le \mu(x)$.

Thus $|x| + \lambda\epsilon\, |x|\, e^{|x|}[0,\beta] \in B_{E(\mathcal{M},\tau)}$, and therefore $x + \lambda\epsilon u\, |x|\, e^{|x|}[0,\beta] \in B_{E(\mathcal{M},\tau)}$ for all $\lambda = \pm 1, \pm i$. By the assumption that x is a $\mathbb{C}$-extreme point, $u\, |x|\, e^{|x|}[0,\beta] = 0$. But $u^{*}u\, |x| = |x|$ and therefore $|x|\, e^{|x|}[0,\beta] = |x|\, e^{|x|}\left[0, \dfrac{1}{1+\epsilon}\mu(\infty; x)\right] = 0$ for every $0 < \epsilon < 1$. Since ϵ can be arbitrarily small, $|x|\, e^{|x|}\, [0, \mu(\infty; x)) = 0$. Hence $0 = \displaystyle\int_0^{\mu(\infty;x)} \lambda de^{|x|}(\lambda)$, which implies that $e^{|x|}(0, \mu(\infty; x)) = 0$. Therefore $s(x) = e^{|x|}(0,\infty) = e^{|x|}[\mu(\infty; x), \infty)$ and finally,

$$|x| = \int_{[\mu(\infty;x),\infty)} \lambda de^{|x|}(\lambda) \ge \mu(\infty; x) e^{|x|}[\mu(\infty; x), \infty) = \mu(\infty; x) s(x).$$

□

Lemma 3.9. *If x is a $\mathbb{C}$-extreme point of the unit ball $\mathrm{B}_{E(\mathcal{M},\tau)}$ then $\mu(\infty; x) = 0$ or*

$n(x)\mathcal{M}n(x^*) = 0$.

Proof. Assume for a contrary that $n(x)\mathcal{M}n(x^*) \neq 0$ and $\mu(\infty; x) > 0$, while x is a complex extreme point. By [73, Chapter 5, Lemma 1.7] there exist nonzero projections $p, q \in \mathcal{P}(\mathcal{M})$ such that $p \leq n(x)$, $q \leq n(x^*)$ and $p \sim q$, that is there exists a partial isometry v such that $p = v^*v$ and $q = vv^*$.

Let $x = u\,|x|$ be the polar decomposition of x. Set $w = u + v$. We claim that w is a partial isometry and $x = w\,|x|$. Indeed, since $\operatorname{Ker}(u^*) = \operatorname{Ker}(x^*)$ we have that $u^*n(x^*) = 0$ and $u^*q = 0$. Now $0 = u^*qu = u^*vv^*u = (v^*u)^*(v^*u) = |v^*u|^2$. Therefore $v^*u = 0$ and $u^*v = 0$.

Hence

$$\begin{aligned} |w|^2 = |u+v|^2 = (u^* + v^*)(u+v) = u^*u + u^*v \\ + v^*u + v^*v = u^*u + v^*v = s(x) + p, \end{aligned}$$

and thus w is a partial isometry, since w^*w is a projection. Now, since $0 = (|x|\,n(x)p)^* = (|x|\,p)^* = p\,|x| = v^*v\,|x|$ and v^*v is a projection on the $\operatorname{Ker}^{\perp} v$, it follows that $\operatorname{Range}|x| \subset \operatorname{Ker} v$ and therefore $v\,|x| = 0$. Hence $x = u\,|x| = u\,|x| + v\,|x| = w\,|x|$.

Note that $||x| + \lambda\mu(\infty; x)n(x)| = |x| + \mu(\infty; x)n(x)$ for all $\lambda = \pm 1, \pm i$, since $|x|\,n(x) = n(x)\,|x| = 0$. Furthermore, by Proposition 1.5(1), $\mu(|x| + \mu(\infty; x)n(x)) = \mu(x)$. Hence $|x| + \lambda\mu(\infty; x)n(x) \in B_{E(\mathcal{M},\tau)}$ for all $\lambda = \pm 1, \pm i$. Moreover,

$$\begin{aligned} \mu(x + \lambda\mu(\infty; x)wn(x)) &= \mu(w\,|x| + \lambda\mu(\infty; x)wn(x)) \\ &\leq \mu(|x| + \lambda\mu(\infty; x)n(x)) = \mu(x), \end{aligned}$$

which implies that $x + \lambda\mu(\infty; x)wn(x) \in B_{E(\mathcal{M},\tau)}$ for all $\lambda = \pm 1, \pm i$. Applying the assumption that x is a $\mathbb{C}$-extreme point, $\mu(\infty; x)wn(x) = 0$. But $\mu(\infty; x) > 0$ and therefore $wn(x) = 0$. Since $x = u\,|x|$, we know that $\operatorname{Ker} u = \operatorname{Ker} x$, and so $un(x) = 0$.

Hence $0 = wn(x) = (u + v)n(x) = vn(x)$. But then

$$p = pn(x) = v^* vn(x) = 0,$$

which contradicts the fact that $p \neq 0$. $\square$

Theorem 3.10. *Suppose that the trace τ on $\mathcal{M}$ is σ-finite. If x is a complex extreme point of $\mathrm{B}_{E(\mathcal{M},\tau)}$ then $\mu(x)$ is a complex extreme point of B_E and either*

(i) $\mu(\infty; x) = 0$ *or*

(ii) $n(x)\mathcal{M}n(x^*) = 0$ *and* $|x| \geq \mu(\infty; x)s(x)$.

Proof. Note first that by Proposition 1.8 without loss of generality, we can assume that the von Neumann algebra $\mathcal{M}$ is non-atomic.

Consider first the case $\tau(s(x)) = \infty$. By Lemma 3.8, $|x| \geq \mu(\infty; x)s(x)$. Also $\mu(|x| - \mu(\infty; x)s(x)) = \mu(x) - \mu(\infty; x)$ by Proposition 1.5(2), and consequently $\mu(\infty; |x| - \mu(\infty; x)s(x)) = 0$. Clearly,

$$s(x)(|x| - \mu(\infty; x)s(x)) = (|x| - \mu(\infty; x)s(x))s(x) = |x| - \mu(\infty; x)s(x).$$

Applying now Proposition 1.14 to the operator $|x| - \mu(\infty; x)s(x)$ and $p = s(x)$, there exists a $*$-isomorphism W from $S([0, \infty), m)$ into $s(x)S(\mathcal{M}, \tau)s(x)$ such that

$$W(\mu(|x| - \mu(\infty; x)s(x))) = |x| - \mu(\infty; x)s(x) \text{ and } \mu(W(f)) = \mu(f),$$

for all $f \in S([0, \infty), m)$. Since $W(1) = s(x)$,

$$\begin{aligned} |x| - \mu(\infty; x)s(x) &= W(\mu(|x| - \mu(\infty; x)s(x))) = W(\mu(x) - \mu(\infty; x)) \\ &= W(\mu(x)) - \mu(\infty; x)W(1) = W(\mu(x)) - \mu(\infty; x)s(x) \end{aligned}$$

and consequently $W(\mu(x)) = |x|$.

Let now $\mu(x)+\lambda f \in B_E$ for all $\lambda = \pm 1, \pm i$, where $f \in B_E$. Since W is an isometry, $\| |x| + \lambda W(f)\|_{E(\mathcal{M},\tau)} = \|W(\mu(x)) + \lambda W(f)\|_{E(\mathcal{M},\tau)} \leq 1$, for $\lambda = \pm 1, \pm i$. Let $x = u|x|$ be the polar decomposition of x. Clearly,

$$x + \lambda uW(f) = u|x| + \lambda uW(f) \in B_{E(\mathcal{M},\tau)}, \quad \lambda = \pm 1, \pm i.$$

Since x is a $\mathbb{C}$-extreme point, $uW(f) = 0$. Recall that $u^*u = s(x)$. Hence $s(x)W(f) = 0$. Note also that $W(f) \in s(x)S(\mathcal{M},\tau)s(x)$. Therefore $W(f) = s(x)W(f) = 0$, and consequently $f = 0$.

Consider now the case when $\tau(s(x)) < \infty$, and hence $\mu(\infty; x) = 0$. By Proposition 1.11 applied to $|x|$, there exists a $*$-isomorphism V from $S([0,\tau(\mathbf{1})), m)$ into $S(\mathcal{M},\tau)$, such that $V(\mu(x)) = |x|$ and $\mu(V(f)) = \mu(f)$ for all $f \in S([0,\tau(\mathbf{1})), m)$. Furthermore, since $s(x) \sim s(x^*)$, $\tau(s(x^*)) = \tau(s(x)) < \infty$ and it follows from [73, Chapter 5, Proposition 1.38] that $n(x) \sim n(x^*)$. Hence by Lemma 2.5, there exists an isometry w such that $x = w|x|$.

Let $\mu(x) + \lambda f \in B_E$, where $\lambda = \pm 1, \pm i$ and $f \in B_E$. Then for $\lambda = \pm 1, \pm i$,

$$\begin{aligned}\|x + \lambda wV(f)\|_{E(\mathcal{M},\tau)} &= \|w|x| + \lambda wV(f)\|_{E(\mathcal{M},\tau)} = \| |x| + \lambda V(f)\|_{E(\mathcal{M},\tau)} \\ &= \|V(\mu(x)) + \lambda V(f)\|_{E(\mathcal{M},\tau)} = \|\mu(x) + \lambda f\|_E \leq 1.\end{aligned}$$

Now by the assumption that x is a $\mathbb{C}$-extreme point, $wV(f) = 0$. Since both w and V are injective, $f = 0$. $\square$

We summarize this chapter with complete characterization of $\mathbb{C}$-extreme points in $B_{E(\mathcal{M},\tau)}$. The first result is an immediate consequence of Theorems 3.7 and 3.10.

Theorem 3.11. *Let E be a symmetric space on $[0,\tau(\mathbf{1}))$ and $\mathcal{M}$ be a semifinite von Neumann algebra with a faithful, normal, σ-finite trace τ. An operator x is a complex extreme point of $\mathrm{B}_{E(\mathcal{M},\tau)}$ if and only if $\mu(x)$ is a complex extreme point of B_E and* *one of the following, not mutually exclusive, conditions holds:*

(i) $\mu(\infty; x) = 0$

(ii) $n(x)\mathcal{M}n(x^*) = 0$ *and* $|x| \geq \mu(\infty; x)s(x)$.

By Theorem 3.11 applied to the commutative von Neumann algebra $\mathcal{M} = L_\infty[0, \tau(\mathbf{1}))$, we can characterize complex extreme functions in terms of their decreasing rearrangements. Since in the commutative settings for any operator x, $n(x) = n(x^*)$, the assertion $n(x)\mathcal{M}n(x^*) = 0$ reduces to the condition $n(x) = 0$.

Corollary 3.12. *Let E be as symmetric function space. The following conditions are equivalent:*

(i) *f is a $\mathbb{C}$-extreme point of B_E;*

(ii) *$\mu(f)$ is a $\mathbb{C}$-extreme point of B_E and $|f| \geq \mu(\infty; f)$.*

For Banach lattices complex rotundity is equivalent to strict monotonicity of the norm. To establish analogous result for noncommutative spaces, we need first the following two facts.

Lemma 3.13. *If the norm on a symmetric space E is strictly monotone then $E = E_0$.*

Proof. Suppose that $E \neq E_0$. Hence, there exists a function $f \in E$ such that $\mu(\infty; f) > 0$ and $m((\operatorname{supp} f)^c) = m\{t : f(t) = 0\} > 0$. Then

$$|f| + \mu(\infty; f)\chi_{(\operatorname{supp} f)^c} \geq |f| \text{ and } |f| + \mu(\infty; f)\chi_{(\operatorname{supp} f)^c} \neq |f|.$$

Since $\mu(|f| + \mu(\infty; f)\chi_{(\operatorname{supp} f)^c}) = \mu(f)$, we have that

$$\| \, |f| + \mu(\infty; f)\chi_{(\operatorname{supp} f)^c}\|_E = \|f\|_E,$$

and so E is not strictly monotone. □

Corollary 3.14. *Symmetric space E is complex rotund if and only if $E(\mathcal{M}, \tau)$ is complex rotund.*

Proof. If E is complex rotund, then by Theorem 1.19 (3), E is strictly monotone. Therefore by Lemma 3.13, $E = E_0$ and consequently Theorem 3.7 implies that $E(\mathcal{M}, \tau)$ is complex rotund.

Suppose now that $E(\mathcal{M}, \tau)$ is complex rotund. By the argument in Proposition 1.8, we can assume that $\mathcal{M}$ is non-atomic. It is easy to check that if $E(\mathcal{M}, \tau)$ is complex rotund, then $E(\mathcal{M}_p, \tau_p)$ is complex rotund for any projection $p \in P(\mathcal{M})$. Let $p \in P(\mathcal{M})$ be a σ-finite projection with $\tau(p) = \tau(\mathbf{1})$ as in Lemma 1.13. By Proposition 1.11, E is isometrically embedded in $E(\mathcal{M}_p, \tau_p)$, and therefore E inherits from it the complex rotundity. □

Theorem 3.15. *Let E be a symmetric function space. Then the norm $\|\cdot\|_E$ on E is strictly monotone if and only if the norm $\|\cdot\|_{E(\mathcal{M},\tau)}$ on $E(\mathcal{M}, \tau)$ is strictly monotone.*

Proof. Suppose that $\|\cdot\|_E$ on E is strictly monotone. Then by Lemma 3.13, $E = E_0$. Let $x, y \in E(\mathcal{M}, \tau)$, $0 \le x \le y$ and $x \ne y$. Clearly $\mu(x) \le \mu(y)$. We have that $x, y - x \in S^+(\mathcal{M}, \tau)$, $x \ge \mu(\infty; x)\mathbf{1} = 0$ and $y - x \ne 0$. Hence by Proposition 1.5(3) applied to x and $y - x$, there exists $s > 0$ such

$$\mu(s; y) = \mu(s; (y - x) + x) > \mu(s; x).$$

Since singular value function is right continuous, there exists a set A of non-zero measure, such that $\mu(t; y) > \mu(t; x)$, for all $t \in A$. Consequently, using the assumption that $\|\cdot\|_E$ is strictly monotone,

$$\|x\|_{E(\mathcal{M},\tau)} = \|\mu(x)\|_E < \|\mu(y)\|_E = \|y\|_{E(\mathcal{M},\tau)},$$

proving that $\|\cdot\|_{E(\mathcal{M},\tau)}$ on $E(\mathcal{M}, \tau)$ is strictly monotone.

Assume now that $\|\cdot\|_{E(\mathcal{M},\tau)}$ on $E(\mathcal{M}, \tau)$ is strictly monotone, and let $f, g \in E$ be such that $0 \le f \le g$, $f \ne g$. In view of Proposition 1.8, we can assume that the von

Neumann algebra $\mathcal{M}$ is non-atomic. By Lemma 1.13 there exists a σ-finite projection in $\mathcal{M}$ with $\tau(p) = \tau(\mathbf{1})$, and so τ_p is σ-finite on $\mathcal{M}_p$. Thus we can assume that τ on $\mathcal{M}$ is σ-finite. By Proposition 1.11 there exists a $*$-isomorphism V from $S([0,\tau(\mathbf{1})), m)$ into $S(\mathcal{M},\tau)$ such that $\mu(V(g)) = \mu(g)$, for all $g \in S([0,\tau(\mathbf{1})), m)$. We have now that $0 \leq V(f) \leq V(g)$, and since V is one-to-one $V(f) \neq V(g)$. Applying the assumption that $\|\cdot\|_{E(\mathcal{M},\tau)}$ is strictly monotone it follows now that

$$\|f\|_E = \|V(f)\|_{E(\mathcal{M},\tau)} < \|V(g)\|_{E(\mathcal{M},\tau)} = \|g\|_E.$$

□

Since complex rotundity of Banach lattice is equivalent with strict monotonicity of its norm, the following result is an immediate consequence of Theorem 3.15 and Corollary 3.14.

Corollary 3.16. *$E(\mathcal{M},\tau)$ is complex rotund if and only if the norm $\|\cdot\|_{E(\mathcal{M},\tau)}$ on $E(\mathcal{M},\tau)$ is strictly monotone.*

Theorem 3.17. *Let $E \subset c_0$ be a symmetric sequence space. Then C_E is complex rotund if and only if E is complex rotund.*

Proof. Since E is isometrically embedded in C_E [3], the claim that E is a complex rotund space if C_E has this property is instant. By the relations $C_E = G(B(H), \mathrm{tr})$ and $\|x\|_{C_E} = \|x\|_{G(B(H),\mathrm{tr})}$ for any compact operator x (see Proposition 1.6), the proof of the reverse implication is conducted analogously to the proof of Theorem 3.7. □

4 Complex Local Uniform Rotundity

4.1 Complex Local Uniform Points and Complex Local Uniform Rotundity

In this section, we study the relations between complex local uniform rotundity of the symmetric function space E and complex local uniform rotundity of the corresponding symmetric space $E(\mathcal{M},\tau)$ of measurable operators. The content of this chapter is included in [11].

Theorem 4.1. *Let E be strongly symmetric and x be an order continuous element of $E(\mathcal{M},\tau)$. If $\mu(x)$ is a $\mathbb{C}-LUR$ point of B_{E_0} then x is a $\mathbb{C}-LUR$ point of $\mathrm{B}_{E_0(\mathcal{M},\tau)}$.*

Proof. Let $x \in S_{E_0(\mathcal{M},\tau)}$ and suppose that $\mu(x)$ is a $\mathbb{C}-LUR$ point of B_{E_0}.

Case 1^0. Let $x \geq 0$ and the sequence $\{y_n\} \subset B_{E_h(\mathcal{M},\tau)}$, $\{y_n\} \subset E_0(\mathcal{M},\tau)$, be such that $\|x + iy_n\|_{E(\mathcal{M},\tau)} \to 1$. By Proposition 1.5(4), $\mu(t; x + iy_n) \geq \mu(t; x)$ for all $t > 0$. Since $\mu(x)$ is an $ULUM$ point in E_0 by Theorem 1.19(2), we have

$$\|\mu(x) - \mu(x + iy_n)\|_E \to 0. \tag{4.1}$$

Also $\mu(t; x+iy_n/2) \geq \mu(t; x)$ for all $t > 0$ and $\|x+iy_n/2\|_{E(\mathcal{M},\tau)} \to 1$. The latter follows from the inequality $1 \leq \|x + iy_n/2\|_{E(\mathcal{M},\tau)} \leq \|x + iy_n\|_{E(\mathcal{M},\tau)}/2 + \|x\|_{E(\mathcal{M},\tau)}/2 \to 1$ as $n \to \infty$. Again, using the fact that $\mu(x)$ is an $ULUM$ point we can conclude that

$$\|\mu(x) - \mu\left(x + iy_n/2\right)\|_E \to 0. \tag{4.2}$$

Denote $a_n = x + iy_n$. By (4.2) it follows that $\|\mu(x) - \mu\left((x + a_n)/2\right)\|_E \to 0$, and by (4.1) that $\|\mu(x) - \mu(a_n)\|_E \to 0$. Applying now Lemma 1.9 we get $a_n \xrightarrow{\tau} x$. Finally, by Proposition 2.3 we have $\|y_n\|_{E(\mathcal{M},\tau)} = \|x - a_n\|_{E(\mathcal{M},\tau)} \to 0$.

Case 2^0. Let $\|x + \lambda y_n\|_{E(\mathcal{M},\tau)} \to 1$ for $\lambda = \pm 1, \pm i$, where $x \geq 0$ and $\{y_n\} \subset$

$B_{E_0(\mathcal{M},\tau)}$. Recall that for any $n \in \mathbb{N}$, $\mathrm{Re}\,(y_n) = (y_n + y_n^*)/2$, $\mathrm{Im}\,(y_n) = (y_n - y_n^*)/(2i)$ are self-adjoint operators and $y_n = \mathrm{Re}\,(y_n) + i\,\mathrm{Im}\,(y_n)$. Note that by Proposition 1.5(4), $\mu\,(x + i\,(y_n + y_n^*)/2) \geq \mu(x)$, and thus

$$\begin{aligned} 1 \leq \|x + i\,\mathrm{Re}\,(y_n)\|_{E(\mathcal{M},\tau)} &\leq \frac{1}{2}\|x + iy_n\|_{E(\mathcal{M},\tau)} + \frac{1}{2}\|x + iy_n^*\|_{E(\mathcal{M},\tau)} \\ &= \frac{1}{2}\|x + iy_n\|_{E(\mathcal{M},\tau)} + \frac{1}{2}\|x - iy_n\|_{E(\mathcal{M},\tau)}. \end{aligned}$$

Then $\|x + i\,\mathrm{Re}\,(y_n)\|_{E(\mathcal{M},\tau)} \to 1$, and by Case 1^0, $\|\mathrm{Re}\,(y_n)\|_{E(\mathcal{M},\tau)} \to 0$. Similarly, by the inequality

$$\begin{aligned} 1 \leq \|x + i\,\mathrm{Im}\,(y_n)\|_{E(\mathcal{M},\tau)} &\leq \frac{1}{2}\|x + y_n\|_{E(\mathcal{M},\tau)} + \frac{1}{2}\|x - y_n^*\|_{E(\mathcal{M},\tau)} \\ &= \frac{1}{2}\|x + y_n\|_{E(\mathcal{M},\tau)} + \frac{1}{2}\|x - y_n\|_{E(\mathcal{M},\tau)} \end{aligned}$$

it follows that $\|x + i\mathrm{Im}\,(y_n)\|_{E(\mathcal{M},\tau)} \to 1$, and consequently $\|\mathrm{Im}\,(y_n)\|_{E(\mathcal{M},\tau)} \to 0$. Therefore $\|y_n\|_{E(\mathcal{M},\tau)} \to 0$.

Case 3^0. Suppose now that x is an arbitrary element of $S_{E_0(\mathcal{M},\tau)}$ and let $\|x + \lambda y_n\|_{E(\mathcal{M},\tau)} \to 1$, for $\lambda = \pm 1, \pm i$ and $\{y_n\} \subset B_{E_0(\mathcal{M},\tau)}$. Let $x = u|x|$ be a polar decomposition of x. Then for all $\lambda = \pm 1, \pm i$, $\||x| + \lambda u^* y_n\|_{E(\mathcal{M},\tau)} = \|u^*x + \lambda u^* y_n\|_{E(\mathcal{M},\tau)}$ $\leq \|x + \lambda y_n\|_{E(\mathcal{M},\tau)}$, and so $\overline{\lim_n}\|\,|x| + \lambda u^* y_n\|_{E(\mathcal{M},\tau)} \leq 1$, $\lambda = \pm 1, \pm i$. By Lemma 1.20, it follows that for $\lambda = \pm 1, \pm i$,

$$\lim_n \|\,|x| + \lambda u^* y_n\|_{E(\mathcal{M},\tau)} = 1.$$

Similarly, using the polar decomposition $x^* = u^*|x^*|$ of x^*, one can show that for $\lambda = \pm 1, \pm i$,

$$\lim_n \||x^*| + \lambda u y_n^*\|_{E(\mathcal{M},\tau)} = 1.$$

Since $\mu(|x|) = \mu(|x^*|)$ is a $\mathbb{C} - LUR$ point, by Case 2^0 we can conclude that

$$\|u^* y_n\|_{E(\mathcal{M},\tau)} \to 0 \text{ and } \|u y_n^*\|_{E(\mathcal{M},\tau)} = \|y_n u^*\|_{E(\mathcal{M},\tau)} \to 0.$$

Hence, in view of $s(x) = u^*u$ and $s(x^*) = uu^*$ we have

$$\|s(x^*)y_n\|_{E(\mathcal{M},\tau)} \to 0 \quad \text{and} \quad \|y_n s(x)\|_{E(\mathcal{M},\tau)} \to 0.$$

It is also easy to check that for $\lambda = \pm 1, \pm i$,

$$|x + \lambda n(x^*) y_n n(x)| = \left| |x| + \lambda \left| n(x^*) y n(x) \right| \right|.$$

Combining the above and the equality $\mathbf{1} = n(x^*) + s(x^*) = n(x) + s(x)$, we get

$$\begin{aligned}
\| |x| + \lambda |n(x^*) y_n n(x)| \|_{E(\mathcal{M},\tau)} &= \|x + \lambda n(x^*) y_n n(x)\|_{E(\mathcal{M},\tau)} \\
&\leq \|x + \lambda y_n\|_{E(\mathcal{M},\tau)} + \|s(x^*) y_n s(x)\|_{E(\mathcal{M},\tau)} \\
&+ \|s(x^*) y_n n(x)\|_{E(\mathcal{M},\tau)} \\
&+ \|n(x^*) y_n s(x)\|_{E(\mathcal{M},\tau)} \to 1.
\end{aligned}$$

Again, applying Lemma 1.20, it follows that $\lim_n \| |x| + \lambda |n(x^*) y_n n(x)| \|_{E(\mathcal{M},\tau)} = 1$ for $\lambda = \pm 1, \pm i$, and by the first case, $\|n(x^*) y_n n(x)\|_{E(\mathcal{M},\tau)} \to 0$. Hence

$$\begin{aligned}
\|y_n\|_{E(\mathcal{M},\tau)} \leq \|n(x^*) y_n n(x)\|_{E(\mathcal{M},\tau)} + 2\|s(x^*) y_n\|_{E(\mathcal{M},\tau)} \\
+ \|y_n s(x)\|_{E(\mathcal{M},\tau)} \to 0,
\end{aligned}$$

and the proof is complete. □

Theorem 4.2. *Suppose that the trace τ on $\mathcal{M}$ is σ-finite. If x is a $\mathbb{C} - LUR$ point in $B_{E(\mathcal{M},\tau)}$ then $\mu(x)$ is a $\mathbb{C} - LUR$ point in B_E and either*

(i) $\mu(\infty; x) = 0$ *or*

(ii) $n(x)\mathcal{M}n(x^*) = 0$ *and* $|x| \geq \mu(\infty; x)s(x)$.

Proof. Since every $\mathbb{C}$-LUR point is a $\mathbb{C}$-extreme point, by Theorem 3.10, (i) or (ii) is satisfied. The fact that $\mu(x)$ is a $\mathbb{C} - LUR$ point whenever x is $\mathbb{C} - LUR$ can be proved analogously as the corresponding statement about strongly extreme points in Theorem 2.6, replacing $\lambda = \pm 1$ with $\lambda = \pm 1, \pm i$. □

Corollary 4.3. *Let E be a symmetric function space on $[0, \tau(\mathbf{1}))$.*

(i) *If x is an order continuous element of a strongly symmetric space E then x is a $\mathbb{C} - LUR$ point of $B_{E_0(\mathcal{M},\tau)}$ if and only if $\mu(x)$ is a $\mathbb{C} - LUR$ point of B_{E_0}.*

(ii) *If f is an order continuous function in a strongly symmetric space E, then f is a $\mathbb{C} - LUR$ point of B_{E_0} if and only if $\mu(f)$ is a $\mathbb{C} - LUR$ point of B_{E_0}.*

(iii) *Suppose that E is order continuous. Then E is a $\mathbb{C} - LUR$ space if and only if $E(\mathcal{M}, \tau)$ is a $\mathbb{C} - LUR$ space.*

Proof. As explained in the proof of Corollary 2.7, we can assume that the trace τ on $\mathcal{M}$ is σ-finite. Therefore condition (i) holds by Theorems 4.1 and 4.2. Assertion (ii) follows from (i) applied to commutative von Neumann algebra $\mathcal{M} = L_\infty[0, \tau(\mathbf{1}))$.

Recall that if E is order continuous then $E = E_0$, and the norm on E is strongly symmetric. Therefore statement (iii) is also a consequence of (i). □

Note that in fact an arbitrary symmetric function space E, not necessarily order continuous, inherits complex local uniform rotundity from $E(\mathcal{M}, \tau)$.

Remark 4.4. If $E(\mathcal{M}, \tau)$ is a $\mathbb{C} - LUR$ space then E is a $\mathbb{C} - LUR$ space.

Proof. Suppose that $E(\mathcal{M}, \tau)$ is $\mathbb{C} - LUR$. In view of Proposition 1.8, we can assume that $\mathcal{M}$ is non-atomic. It is standard to check then that $E(\mathcal{M}_p, \tau_p)$ is a $\mathbb{C} - LUR$ space, for any projection $p \in P(\mathcal{M})$. Consider a σ-finite projection p with $\tau(p) = \tau(\mathbf{1})$ (see Lemma 1.13). Then $E(\mathcal{M}_p, \tau_p)$ is $\mathbb{C} - LUR$ and the trace τ_p on $\mathcal{M}_p$ is σ-finite.

Therefore we can assume that the trace τ on $\mathcal{M}$ is σ-finite. Hence by Proposition 1.11, E is isometrically embedded in $E(\mathcal{M},\tau)$. Consequently, E inherits from $E(\mathcal{M},\tau)$ complex local uniform rotundity. $\square$

Recall that if the Banach lattice E is $\mathbb{C}-LUR$, then the norm $\|\cdot\|_E$ on E is upper locally uniformly monotone, Theorem 1.19 (4). It was shown in [29, Theorem 2.8], that the norm on E is upper locally uniformly monotone if and only if the norm on $E(\mathcal{M},\tau)$ is upper locally uniformly monotone. Hence in view of Remark 4.4, we have the following result, which relates complex local uniform rotundity of $E(\mathcal{M},\tau)$ with the upper local uniform monotonicity of its norm.

Corollary 4.5. *If $E(\mathcal{M},\tau)$ is $\mathbb{C}-LUR$ then the norm $\|\cdot\|_{E(\mathcal{M},\tau)}$ on $E(\mathcal{M},\tau)$ is upper locally uniformly monotone.*

Theorem 4.6. *Let $E \subset c_0$ be an order continuous symmetric sequence space. Then C_E is a $\mathbb{C}-LUR$ space if and only if E is a $\mathbb{C}-LUR$ space.*

Proof. Since E is isometrically embedded in C_E if C_E is a $\mathbb{C}-LUR$ space then so is E. Now by Proposition 1.6, $C_E = G\,(B(H),\mathrm{tr})$ and $\|x\|_{C_E} = \|x\|_{G(B(H),\mathrm{tr})}$ for any compact operator x. Hence the proof of the fact that if E is a $\mathbb{C}-LUR$ space then C_E is a $\mathbb{C}-LUR$ space can be conducted analogously to the proof of Theorem 4.1. $\square$

4.2 $\mathbb{C}-LUR$ and $\mathbb{C}-MLUR$ properties

Let us discuss here the notions of complex strongly extreme points and complex midpoint locally uniformly rotund spaces. One can define a modulus of complex strong extremality [6] analogously as the modulus of strong extremality in the real case, introduced by C. Finet in [34]. Let $(X,\|\cdot\|)$ be a Banach space over the field of complex numbers. For $x \in S_X$ and $\epsilon > 0$, the *modulus of complex strong extremality* at x is the number

$$\Delta(x,\epsilon) = \inf \{1 - |\lambda| : \quad \exists y, \|y\| > \epsilon \quad \|\lambda x \pm y\| \leq 1, \text{ and } \|\lambda i x \pm y\| \leq 1\}.$$

An element $x \in S_X$ is said to be a $\mathbb{C} - MLUR$ point in B_X, or *complex strongly extreme* point of the unit ball B_X, if for any $\epsilon > 0$, the modulus of complex extremality $\Delta(x,\epsilon) > 0$. A Banach space X is said to be *complex midpoint locally uniformly rotund* or $\mathbb{C} - MLUR$ space, if every element from the unit sphere S_X is a $\mathbb{C} - MLUR$ point [6].

We will demonstrate that the notions of $\mathbb{C} - LUR$ and $\mathbb{C} - MLUR$ points, and hence the notions of $\mathbb{C} - LUR$ and $\mathbb{C} - MLUR$ spaces, are equivalent in any complex Banach space. Consequently, in complex Banach spaces these complex properties are not distinguishable contrary to their corresponding properties LUR and $MLUR$ [53]. We need first the following lemma.

Lemma 4.7. *An element $x \in S_X$ is a $\mathbb{C} - MLUR$ point if and only for any $\{x_n\} \subset X$, $\lambda = \pm 1, \pm i$, $\|x + \lambda x_n\| \to 1$ implies that $\|x_n\| \to 0$.*

Proof. Suppose that x is a $\mathbb{C} - MLUR$ point, that is for all $\epsilon > 0$, the modulus $\Delta(x,\epsilon) > 0$. Let $\|x \pm x_n\| \to 1$ and $\|x \pm i x_n\| \to 1$, where $\{x_n\} \subset X$. Set

$$c_n = \max_{\lambda \in \{\pm 1, \pm i\}} \|x + \lambda x_n\|.$$

Clearly, $c_n \to 1$. If for some n, $c_n \leq 1$ then $\|x + \lambda x_n\| \leq 1$ for all $\lambda = \pm 1, \pm i$, and consequently $x_n = 0$. Indeed, suppose that $x_n \neq 0$. Hence, there exists an $\epsilon > 0$ such that $\|x_n\| > \epsilon$, $\|x \pm x_n\| \leq 1$ and $\|ix \pm x_n\| \leq 1$. But then $\Delta(x,\epsilon) = 0$, which leads to a contradiction. Therefore without lost of generality, we can assume that $c_n > 1$ for all $n \in \mathbb{N}$. Clearly, for all $n \in \mathbb{N}$,

$$\|c_n^{-1} x \pm c_n^{-1} x_n\| \leq 1 \quad \text{and} \quad \|i c_n^{-1} x \pm c_n^{-1} x_n\| \leq 1.$$

Denote $\lambda_n = c_n^{-1}$, $n \in \mathbb{N}$. Then for each λ_n, there exists an element $a_n = c_n^{-1}x_n$ such that $\|\lambda_n x \pm a_n\| \leq 1$ and $\|i\lambda_n x \pm a_n\| \leq 1$. Hence $\|a_n\| \to 0$ and consequently $\|x_n\| \to 0$. If not, then there exists an $\epsilon > 0$ and a subsequence a_{n_k} such that $\|a_{n_k}\| > \epsilon$, and since $\lambda_n \to 1$,

$$\Delta(x, \epsilon) = \inf \{1 - |\lambda| : \exists y, \|y\| > \epsilon, \ \|\lambda x \pm y\| \leq 1, \text{ and } \|\lambda i x \pm y\| \leq 1\} = 0.$$

To prove the reverse implication, assume that $\Delta(x, \epsilon) = 0$ for some $\epsilon > 0$. Therefore there exists a sequence $\{\lambda_n\} \subset \mathbb{C}$ satisfying $|\lambda_n| \uparrow 1$ and for all $n \in \mathbb{N}$, there is an element $x_n \in B_X$, $\|x_n\| \geq \epsilon$ such that $\|\lambda_n x \pm x_n\| \leq 1$ and $\|i\lambda_n x \pm x_n\| \leq 1$. Therefore, for all $n \in \mathbb{N}$ we have

$$\|x \pm \lambda_n^{-1} x_n\| \leq |\lambda_n|^{-1} \quad \text{and} \quad \|x \pm i\lambda_n^{-1} x_n\| \leq |\lambda_n|^{-1},$$

and since $|\lambda_n| \to 1$, $\overline{\lim}_n \|x \pm \lambda_n^{-1} x_n\| \leq 1$ and $\overline{\lim}_n \|x \pm i\lambda_n^{-1} x_n\| \leq 1$. In view of $\|x\| = 1$, by Lemma 1.20 it follows that $\lim_n \|x \pm \lambda_n^{-1} x_n\| = 1$ and $\lim_n \|x \pm i\lambda_n^{-1} x_n\| = 1$. Hence there exists a subsequence $\{\lambda_{n_k}^{-1} x_{n_k}\}$ with $\lim\limits_k \|\lambda_{n_k}^{-1} x_{n_k}\| \neq 0$ such that $\lim\limits_k \|x \pm \lambda_{n_k}^{-1} x_{n_k}\| = 1$ and $\lim\limits_k \|x \pm i\lambda_{n_k}^{-1} x_{n_k}\| = 1$. □

Now we can state the equivalence result of $\mathbb{C} - LUR$ and $\mathbb{C} - MLUR$ properties.

Proposition 4.8. *Let $x \in S_X$. The following conditions are equivalent:*

(i) *An element $x \in S_X$ is a $\mathbb{C}$-LUR point of B_X,*

(ii) *For all $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for all $y \in X$,*

$\sup_{\lambda = \pm 1, \pm i} \|x + \lambda y\| < 1 + \delta$ *implies that* $\|y\| < \epsilon$,

(iii) *For all $\{y_n\} \subset X$, $\sup_{\lambda = \pm 1, \pm i} \|x + \lambda y_n\| \to 1$ implies $\|y_n\| \to 0$,*

(iv) *For all $\{y_n\} \subset X$, $\|x \pm y_n\| \to 1$ and $\|x \pm i y_n\| \to 1$ implies $\|y_n\| \to 0$,*

(v) *An element $x \in S_X$ is a $\mathbb{C} - MLUR$ point of B_X.*

Proof. Let $x \in S_X$. It is clear that (ii) implies (i), (iii) implies (iv) and conditions (ii) and (iii) are equivalent. By Lemma 4.7, conditions (iv) and (v) are also equivalent. It remains to show implication from (i) to (ii) and from (iv) to (iii).

Let $\epsilon > 0$ and suppose that condition (i) holds true. Hence for $\epsilon/2 > 0$, there exists $\delta(\epsilon) > 0$ such that for any $y \in X$, $\sup_{|\lambda|\le 1} \|x + \lambda y\| < 1 + \delta(\epsilon)$ implies that $\|y\| < \epsilon/2$. Assume now that $\sup_{\lambda=\pm 1,\pm i} \|x + \lambda y\| < 1 + \delta$. Then for $-1 \le c \le 1$ and $\lambda = \pm 1, \pm i$ we have

$$\|x + c\lambda y\| \le (1 + c)/2\|x + \lambda y\| + (1 - c)/2\|x - \lambda y\| < 1 + \delta.$$

Hence for all $c \in \mathbb{R}$ with $|c| \le 1$, $\|x + cy\| < 1 + \delta$ and $\|x + ciy\| < 1 + \delta$. Let $\lambda = a + bi \in \mathbb{C}$, with $|\lambda| \le 1$. Then, since $|a|, |b| \le 1$ it follows that

$$\|x + \lambda y/2\| = \|x + ay/2 + biy/2\| \le \|x + ay\|/2 + \|x + biy\|/2 < 1 + \delta.$$

Consequently, $\sup_{|\lambda|\le 1} \|x + y/2\| < 1 + \delta$ and by (i), $\|y/2\| < \epsilon/2$ and so $\|y\| < \epsilon$. Therefore (ii) is satisfied.

To show that (iv) implies (iii), suppose that $\sup_{\lambda=\pm 1,\pm i} \|x + \lambda y_n\| \to 1$, $\{y_n\} \subset X$. Then $\overline{\lim}_n \|x \pm y_n\| \le 1$ and $\overline{\lim}_n \|x \pm iy_n\| \le 1$. Thus by Lemma 1.20, for all $\lambda = \pm 1, \pm i$ we have $\lim_n \|x + \lambda y_n\| = 1$, and so by (iv), $\|y_n\| \to 0$.

□

We finish with examples of Banach spaces showing that complex uniform rotundity, complex local uniform rotundity and complex rotundity do not coincide.

Example 4.9. (1) The space $E = (\ell_\infty, \|\cdot\|)$ equipped with the norm $\|x\| = \|x\|_\infty + \sum_{n=1}^{\infty} |x(n)|/2^{n-1}$ is complex rotund but not complex locally rotund. One can show easily [50] that the unit vector $2^{-1}e_1 = (2^{-1}, 0, \dots)$ is an UM-point in B_{E_r}, so it is $\mathbb{C}$-extreme point in B_E, but not $ULUM$ point so not $\mathbb{C} - LUR$ point (Theorem 1.19).

(2) Orlicz-Lorentz spaces $\Lambda_{\varphi,w}$ are locally uniformly rotund spaces and hence $\mathbb{C} - LUR$ whenever φ is strictly convex and φ satisfies condition Δ_2 [5]. However if they

are complex uniformly rotund then in addition to those conditions on φ, the weight w must be regular [10]. So there exist Orlicz-Lorentz spaces that are $\mathbb{C} - LUR$ but not complex uniformly rotund.

5 Smoothness and Fréchet Smoothness

The results on smoothness and Fréchet smoothness presented in this chapter will appear in [12].

5.1 Smooth Points

In this section we shall discuss smooth points of $B_{E(\mathcal{M},\tau)}$. It was shown in [3] that $x \in S_{C_E}$ is a smooth point of B_{C_E} if and only if the sequence $s(x) = \{s_n(x)\}$ of singular numbers of x is a smooth point of B_E, where $E \neq \ell_1$ is a symmetric sequence space. Our goal here is to show that similar result holds true in the space $E(\mathcal{M},\tau)$.

Notice that although the next result was proved in [8] only for a non-atomic von Neuman algebra, by the discussion in preliminaries (Proposition 1.8), we in fact have a characterization of extreme points of $B_{E(\mathcal{M},\tau)}$ in full generality (see also [31]).

Theorem 5.1. *An operator $x \in S_{E(\mathcal{M},\tau)}$ is an extreme point of $B_{E(\mathcal{M},\tau)}$ if and only if $\mu(x)$ is an extreme point of B_E and one of the following conditions holds:*

(i) $\mu(\infty; x) = 0$,

(ii) $n(x)\mathcal{M}n(x^*) = 0$, $|x| \geq \mu(\infty; x)s(x)$.

Before we state our first result we will need the following elementary lemma.

Lemma 5.2. *Let $x \in B_{E(\mathcal{M},\tau)}$ and $y \in B_{E^\times(\mathcal{M},\tau)}$. If $\tau(xy) = 1$ and $\mu(\infty, y) > 0$ then $s(y) \geq s(x^*)$.*

Proof. Note first that in view of Proposition 1.17, if $\tau(xy) = 1$ then $\tau(|x^*|\,|y|) = 1$. Indeed, this relation follows from inequalities $0 \leqslant \tau(|x^*|\,|y|), \tau(|x|\,|y^*|) \leq 1$, and

$$1 = \tau(xy) \leq \tau(|x^*|\,|y|)^{\frac{1}{2}} \tau(|x|\,|y^*|)^{\frac{1}{2}} \leq 1.$$

Since $\mu(y + \mu(\infty, y)n(y)) = \mu(y)$, by Proposition 1.5(1), and $|y + \mu(\infty, y)n(y)| =$

$|y| + \mu(\infty, y)n(y)$, we have that

$$1+\mu(\infty,y)\tau(|x^*|\,n(y)) = \tau(|x^*|\,|y|)+\mu(\infty,y)\tau(|x^*|\,n(y)) = \tau(|x^*|\,(|y|+\mu(\infty,y)n(y)) \leq 1.$$

In view of $\tau(|x^*|\,n(y)) = \tau(n(y)\,|x^*|\,n(y)) \geq 0$, if $\mu(\infty, y) > 0$ then $n(y)\,|x^*|\,n(y) = 0$. Taking now any $\xi \in \operatorname{Ker} y \cap D(x^*)$,

$$\langle |x^*|\,\xi, \xi\rangle = \langle |x^*|\,n(y)\xi, n(y)\xi\rangle = \langle n(y)\,|x^*|\,n(y)\xi, \xi\rangle = 0,$$

which implies that $\operatorname{Ker} x^* \geq \operatorname{Ker} y$ and $s(y) \geq s(x^*)$. □

Now we are ready to prove the first main result of this section.

Theorem 5.3. *Suppose that E is order continuous. Let $\mu(x)$ be a smooth point of B_E and $F(h) = \int h(t)f(t)dt$, $h \in E$, for some $f \in S_{E^\times}$, be the functional supporting $\mu(x)$. If*

(i) $\mu(\infty; f) = 0$, *or*

(ii) $s(x^*) = \mathbf{1}$

then x is a smooth point of $B_{E(\mathcal{M},\tau)}$.

Proof. Let $x \in S_{E(\mathcal{M},\tau)}$. If $\mu(x)$ is a smooth point of B_E, then there exists unique functional F that supports B_E at $\mu(x)$ such that $F(h) = \int h(t)f(t)dt$, $h \in E$, for some (unique) $f \in E^\times$ with $F(\mu(x)) = 1$. Observe that in view of the inequality $1 = \int \mu(t;x)f(t)dt \leqslant \int \mu(t;x)\mu(t;f)dt \leqslant \|\mu(x)\|_E\|\mu(f)\|_{E^\times} \leq 1$, $\int \mu(t;x)\mu(t;f)dt = 1$ and therefore by uniqueness of the functional F supporting $\mu(x)$, it follows that $f = \mu(f)$. Furthermore, as it was observed at the beginning of this section, f is an extreme point of $B_{E^\times}$.

Let $y \in S_{E^\times(\mathcal{M},\tau)}$ be any operator such that $\tau(xy) = 1$. We have that

$$1 = |\tau(xy)| \leq \tau(|xy|) = \int \mu(t;xy)dt \leqslant \int \mu(t;x)\mu(t;y)dt \leq \|\mu(x)\|_E\|\mu(y)\|_{E^\times} \leq 1.$$

Hence $\int \mu(t;x)\mu(t;y)dt = 1$, and by the uniqueness of f we get that $\mu(y) = f$.

Let $\tau(xy_1) = 1$ for $y_1 \in B_{E^\times(\mathcal{M},\tau)}$. Then $\tau(xy_1) = \tau(xy) = \tau(x(y_1 + y)/2) = 1$. Repeating the same argument as above, it follows that $\mu((y_1+y)/2) = \mu(y) = \mu(y_1) = f$.

If $\mu(\infty; f) = 0$ and hence $\mu(\infty; y) = \mu(\infty; y_1) = 0$, then by Proposition 1.5(6), $y = y_1$.

Suppose now that $\mu(\infty; f) > 0$ and $s(x^*) = \mathbf{1}$. Then in view of Lemma 5.2, $s(y) = s(y_1) = \mathbf{1}$. Since $\mu(\infty, y) = \mu(\infty; y_1) = \mu(\infty; f)$, $|y| \geq \mu(\infty; f)\mathbf{1}$ and $|y_1| \geq \mu(\infty; f)\mathbf{1}$, by Lemma 1.18.

In view of $0 \leqslant \tau(|x^*|\,|y|), \tau(|x|\,|y^*|) \leq 1$, the relation

$$1 = \tau(xy) \leq \tau(|x^*|\,|y|)^{\frac{1}{2}} \tau(|x|\,|y^*|)^{\frac{1}{2}} \leq 1,$$

implies that $\tau(|x^*|\,|y|) = 1$. Similarly, we have that $\tau(|x^*|\,|y_1|) = 1$, and hence $\tau(|x^*|\,|y|) = \tau(|x^*|\,|y_1|) = \tau(|x^*|\,(|y| + |y_1|)/2) = 1$.

Applying the similar argument as previously, one can show that

$$\mu((|y| + |y_1|)/2) = \mu(|y|) = \mu(|y_1|) = f.$$

Therefore, by the fact that $\mu(\infty; (|y|+|y_1|)/2) = \mu(\infty; f)$ and in view of Proposition 1.5(2), we have

$$\mu((|y| - \mu(\infty; f)\mathbf{1} + |y_1| - \mu(\infty; f)\mathbf{1})/2) = \mu(|y| - \mu(\infty; f)\mathbf{1}) = \mu(|y_1| - \mu(\infty; f)\mathbf{1}).$$

Note that if the space $E^\times(\mathcal{M},\tau)$ contains an operator which does not belong to $S_0(\mathcal{M},\tau)$, then $\mathbf{1} \in E^\times(\mathcal{M},\tau)$. Applying now Proposition 1.5(6) to the operators

$|y| - \mu(\infty; f)\mathbf{1}, |y_1| - \mu(\infty; f)\mathbf{1} \in (L_1(\mathcal{M}, \tau) + \mathcal{M}) \cap S_0(\mathcal{M}, \tau),$

$$|y| - \mu(\infty; f)\mathbf{1} = |y_1| - \mu(\infty; f)\mathbf{1},$$

and so $|y| = |y_1|$.

We have shown that if $\tau(xy_1) = 1$ for $y_1 \in B_{E^\times(\mathcal{M},\tau)}$, then $|y_1| = |y|$. Since $\tau(x(y+y_1)/2) = 1$ and $(y+y_1)/2 \in B_{E^\times(\mathcal{M},\tau)}$, it follows that $|y+y_1|/2 = |y| = |y_1|$ and by Corollary 1.10, $y = y_1$. □

In order to show the converse of the above theorem, we need some preparatory lemmas. The first one describes a connection between a smooth point and its supporting functional.

Lemma 5.4. *Let E be order continuous. If $x \in S_{E(\mathcal{M},\tau)}$ is a smooth point of $B_{E(\mathcal{M},\tau)}$ and the functional $\Phi_y(z) = \tau(zy)$, $z \in E(\mathcal{M}, \tau)$, $y \in E^\times(\mathcal{M}, \tau)$, supports x, then either*

(i) $\mu(\infty; y) = 0$, *or*

(ii) $s(x) = s(x^*) = s(y) = s(y^*) = \mathbf{1}$, $|y| \geq \mu(\infty; y)\mathbf{1}$ *and* $|y^*| \geq \mu(\infty; y)\mathbf{1}$.

Proof. Let x be a smooth point of $B_{E(\mathcal{M},\tau)}$ and suppose that Φ_y, for some $y \in S_{E^\times(\mathcal{M},\tau)}$, is a unique functional supporting x. Then, in view of $1 = \tau(xy) = \tau(s(x^*)xy) = \tau(xys(x^*))$, it follows that $ys(x^*) = y$ and so $s(y) \leq s(x^*)$. Moreover, if $\mu(\infty; y) > 0$, then by Lemma 5.2, $s(y) \geq s(x^*)$. This, combined with previously obtained reversed inequality, implies that $s(y) = s(x^*)$, whenever $\mu(\infty; y) > 0$. Thus if $\mu(\infty; y) > 0$, then $n(y) = n(x^*)$ and in view of $x \in E(\mathcal{M}, \tau)$ and $y + \mu(\infty; y)n(y) \in E^\times(\mathcal{M}, \tau)$, by Proposition 1.15 we have that

$$\begin{aligned}\Phi_{y+\mu(\infty;y)n(y)}(x) &= \tau(x(y + \mu(\infty; y)n(y))) = \tau((y + \mu(\infty; y)n(y))x) \\ &= \tau(yx) + \mu(\infty; y)\tau(n(x^*)x) = \tau(xy) = 1.\end{aligned}$$

Since x is smooth, $y+\mu(\infty;y)n(y)=y$ and if $\mu(\infty;y)>0$, $n(x^*)=n(y)=0$. Thus if $\mu(\infty;y)>0$ then $s(y)=s(x^*)=\mathbf{1}$ and by Lemma 1.18 it follows that $|y|\geq\mu(\infty;y)\mathbf{1}$.

Clearly, x is a smooth point if and only if x^* is a smooth point. Therefore, by what was proved above, if $\mu(\infty;y^*)=\mu(\infty;y)>0$ then $s(y^*)=s(x)=\mathbf{1}$ and $|y^*|\geq\mu(\infty;y)\mathbf{1}$. □

The next lemma will allow us to reduce the proof of Theorem 5.7 to positive operators.

Lemma 5.5. *Let E be order continuous. If $x\in S_{E(\mathcal{M},\tau)}$ is a smooth point of $B_{E(\mathcal{M},\tau)}$ then $|x|$ is a smooth point of $B_{E(\mathcal{M},\tau)}$.*

Proof. Suppose that $x\in S_{E(\mathcal{M},\tau)}$ is a smooth point of $B_{E(\mathcal{M},\tau)}$. Since E is order continuous, there exists a unique functional Φ_y, $y\in S_{E^\times(\mathcal{M},\tau)}$, such that $\Phi_y(x)=1$. Moreover, since x is smooth if and only if x^* is smooth, by the proof of Lemma 5.4 we have that $s(y^*)\leq s(x)$, equivalently $n(x)\leq n(y^*)$, and $\tau(|x|\,|y^*|)=\tau(|x^*|\,|y|)=1$.

We will show that the functional $\Phi_{|y^*|}$ is a unique functional supporting $|x|$. Suppose there exists an operator $\tilde{y}\in S_{E^\times(\mathcal{M},\tau)}$ and the functional $\Phi_{\tilde{y}}$ such that $\Phi_{\tilde{y}}(|x|)=1$. We shall prove that $\tilde{y}=|y^*|$. Let $x=u\,|x|$ be the polar decomposition of x. Then

$$1=\Phi_{\tilde{y}}(|x|)=\tau(|x|\,\tilde{y})=\tau(u^*x\tilde{y})=\tau(x\tilde{y}u^*)=\Phi_{\tilde{y}u^*}(x),$$

and by the fact that Φ_y is a unique functional supporting x we get the equality $\tilde{y}u^*=y$.

Since Φ_y uniquely supports x, so y is an extreme point of $B_{E^\times(\mathcal{M},\tau)}$ and then by Theorem 5.1, $\mu(y)$ is an extreme point of $B_{E^\times}$. Consider the function $f=\mu(\tilde{y})-\mu(y)$. Since $\mu(y)=\mu(\tilde{y}u^*)\leq\mu(\tilde{y})$, $f\geqslant 0$. Thus $|\mu(y)\pm f|\leq\mu(y)+f$ and $\mu(y)+f=\mu(\tilde{y})\in B_{E^\times}$. Therefore $\mu(y)\pm f\in B_{E^\times}$, which implies that $f=0$ and $\mu(y)=\mu(\tilde{y})$.

Applying now the same reasoning as above to any $b \in B_{E^\times(\mathcal{M},\tau)}$ satisfying $\Phi_b(|x|) = 1$, we can conclude that $bu^* = y$ and $\mu(b) = \mu(y)$. Hence $\Phi_{|y^*|}(|x|) = 1$ implies that $|y^*|\, u^* = y$. Moreover it is easy to observe that $\Phi_{|\tilde{y}|}(|x|) = 1$. Indeed, again using Proposition 1.17, we get that

$$1 = \Phi_{\tilde{y}}(|x|) = \tau(|x|\, \tilde{y}) \leq \tau(|x|\, |\tilde{y}|)^{\frac{1}{2}} \tau(|x|\, |\tilde{y}^*|)^{\frac{1}{2}} \leq 1,$$

and consequently $\Phi_{|\tilde{y}|}(|x|) = \tau(|x|\, |\tilde{y}|) = 1$. Thus, $\Phi_{(|y^*|+|\tilde{y}|)/2)}(|x|) = 1$ and by the argument above $\mu\left((|y^*| + |\tilde{y}|)/2\right) = \mu(y)$. Since we showed earlier that $\mu(y) = \mu(\tilde{y}) = \mu(|\tilde{y}|)$ and it is clear that $\mu(y) = \mu(y^*) = \mu(|y^*|)$, it follows that

$$\mu\left((|y^*| + |\tilde{y}|)/2\right) = \mu(|y^*|) = \mu(|\tilde{y}|).$$

If $\mu(\infty; y) = 0$ and hence $\mu(\infty; |y^*|) = \mu(\infty; |\tilde{y}|) = 0$, then by Proposition 1.5(6), $|y^*| = |\tilde{y}|$.

If $\mu(\infty; y) > 0$, then by Lemma 5.4, $s(x) = s(y^*) = s(|y^*|) = \mathbf{1}$. Consequently, Lemma 5.2 implies that $s(|\tilde{y}|) = \mathbf{1}$, and since $\mu(\infty, |y^*|) = \mu(\infty; |\tilde{y}|)$, in view of Lemma 1.18, $|y^*| \geq \mu(\infty; |y^*|)\mathbf{1}$ and $|\tilde{y}| \geq \mu(\infty; |y^*|)\mathbf{1}$. Repeating the same argument as in the proof of Theorem 5.3, it follows that also in this case $|y^*| = |\tilde{y}|$.

Therefore $n(\tilde{y}) = n(y^*) \geq n(x)$ and $|y^*|\, n(x) = \tilde{y}n(x) = 0$. Now, since $y = \tilde{y}u^* = |y^*|\, u^*$, we have $\tilde{y}s(x) = \tilde{y}u^*u = |y^*|\, u^*u = |y^*|\, s(x)$, proving that $\tilde{y} = |y^*|$ and that $\Phi_{|y^*|}$ is a unique functional supporting $|x|$. $\square$

Lemma 5.6. *Let E be order continuous. If $x \in S_{E(\mathcal{M},\tau)}$ is a smooth point of $B_{E(\mathcal{M},\tau)}$ then x_p is a smooth point of $B_{E(\mathcal{M}_p,\tau_p)}$ for $p \in P(\mathcal{M})$ satisfying $p \geqslant s(x^*)$.*

Proof. Let x be a smooth point of $B_{E(\mathcal{M},\tau)}$ and Φ_y, $y \in S_{E^\times(\mathcal{M},\tau)}$, be a unique functional supporting x.

Let $p \in P(\mathcal{M})$ and $p \geqslant s(x^*)$. Then $px = x$, and for all $z \in E^{\times}(\mathcal{M}, \tau)$ we have

$$\tau(xz) = \tau(pxz) = \tau(pxzp) = \tau_p((xz)_p) = \tau_p(x_p z_p).$$

Since $E(\mathcal{M}_p, \tau_p)$ is order continuous by [21, Theorem 2.9], it is now easy to observe that the functional $\Psi_{y_p}(z') = \tau_p(z' y_p)$, $z' \in E(\mathcal{M}_p, \tau_p)$, is the unique functional in $E^*(\mathcal{M}_p, \tau_p)$ supporting x_p. □

Theorem 5.7. *Let E be order continuous. If x is a smooth point of $B_{E(\mathcal{M},\tau)}$ then $\mu(x)$ is a smooth point of B_E.*

Proof. Note first that by Proposition 1.8 and in view of the equality $\mu(x) = \tilde{\mu}(\tilde{\pi}(x))$, we can assume that the von Neumann algebra $\mathcal{M}$ is non-atomic.

Since $\mu(\infty, x) = 0$, for $p = s(x) \vee s(x^*)$, by Lemma 1.12 τ_p is a σ-finite trace on $\mathcal{M}_p$. Moreover, $pxp = x$. In view of the equality $\mu^{\tau_p}(x_p) = \mu(pxp) = \mu(x)$ and Lemma 5.6 stating that x_p is smooth in $E(\mathcal{M}_p, \tau_p)$, we can assume that the trace τ is σ-finite.

Let x be a smooth point of $B_{E(\mathcal{M},\tau)}$. Since by Lemma 5.5, $|x|$ is also a smooth point of $B_{E(\mathcal{M},\tau)}$ and $\mu(x) = \mu(|x|)$ we can assume that $x \geqslant 0$.

Applying now Proposition 1.11 to the operator x, there exists a $*$-isomorphism V acting from $S([0, \tau(\mathbf{1})), m)$ into $S(\mathcal{M}, \tau)$, such that $V(\mu(x)) = x$ and $\mu(V(f)) = \mu(f)$ for all $f \in S([0, \tau(\mathbf{1})), m)$.

Since E is order continuous and x is smooth, there exists a unique $y \in S_{E^{\times}(\mathcal{M},\tau)}$ such that $\Phi_y(x) = \tau(xy) = 1$. Note that $s(y) \leq s(x^*) = s(x)$ (see the proof of Lemma 5.4).

Define functional $F \in E^*$ by setting $F(g) = \int g(t)\mu(t; y)dt$, $g \in E$. Clearly,

$$1 \geqslant F(\mu(x)) = \int \mu(t; x)\mu(t; y)dt \geqslant \int \mu(t; xy)dt = \tau(|xy|) \geq |\tau(xy)| = 1,$$

and so $F(\mu(x)) = 1$.

Moreover, since every $*$-isomorphism is positive, $xV(\mu(y)) = V(\mu(x)\mu(y)) \geqslant 0$ and

$$\tau(xV(\mu(y))) = \|V(\mu(x)\mu(y))\|_{L_1(\mathcal{M},\tau)} = \|\mu(x)\mu(y)\|_{L_1(0,\tau(\mathbf{1}))} = F(\mu(x)) = 1.$$

By uniqueness of the functional Φ_y supporting x, it follows that $V(\mu(y)) = y$.

Suppose now that $G(\mu(x)) = 1$, where $G(h) = \int h(t)g(t)dt$, $h \in E$, for some $g \in B_{E^\times}$. Then it is not difficult to see that also $\int \mu(t;x)\,|g|\,(t)dt = 1$.

By the fact that V as a $*$-homomorphism is positive, we have that

$$xV(|g|) = V(\mu(x))V(|g|) = V(\mu(x)\,|g|) \geq 0,$$

and consequently

$$\begin{aligned} 1 &= \int \mu(t;x)\,|g|\,(t)dt = \|\mu(x)\,|g|\,\|_{L_1(0,\tau(\mathbf{1}))} = \|V(\mu(x)\,|g|)\|_{L_1(\mathcal{M},\tau)} \\ &= \|V(\mu(x))V(|g|)\|_{L_1(\mathcal{M},\tau)} = \|xV(|g|)\|_{L_1(\mathcal{M},\tau)} = \tau(xV(|g|)) = \Phi_{V(|g|)}(x). \end{aligned}$$

Since Φ_y is a unique functional supporting x, $V(|g|) = y = V(\mu(y))$. By the fact that V is one-to-one, $|g| = \mu(y)$ and it is left to show that $|g| = g$.

Applying the above argument to the positive function $(|g| + g)/2$ for which $\int \mu(t;x)(|g| + g)/2(t)dt = 1$, it follows that $\Phi_{V((|g|+g)/2)}(x) = 1$ and therefore $V\left((|g| + g)/2\right) = y = V(\mu(y))$. Hence $(|g| + g)/2 = \mu(y)$ and since we have shown earlier that $|g| = \mu(y)$, it follows now that $g = |g| = \mu(y)$.

□

The next theorem combines the results of Theorem 5.3, Theorem 5.7 and Lemma 5.4.

Theorem 5.8. *Let E be order continuous. Then x is a smooth point of $B_{E(\mathcal{M},\tau)}$ if and only if $\mu(x)$ is a smooth point of B_E, and either*

(i) $\mu(\infty; f) = 0$, *where* $f \in S_{E^\times}$ *supports* $\mu(x)$, *or*

(ii) $s(x^*) = \mathbf{1}$.

Proof. By Theorem 5.3 if $\mu(x)$ is a smooth point of B_E and either (i) or (ii) holds, then x is a smooth point of $B_{E(\mathcal{M},\tau)}$.

Conversely, if x is a smooth point of $B_{E(\mathcal{M},\tau)}$ and the operator $y \in S_{E^\times(\mathcal{M},\tau)}$ supports x, then by Theorem 5.7, $\mu(x)$ is a smooth point of B_E and $f = \mu(y)$ supports $\mu(x)$. Hence the conditions (i) or (ii) follow by Lemma 5.4. $\square$

Considering the commutative von Neumann algebra $\mathcal{M} = L_\infty[0,\alpha)$, $0 < \alpha \leqslant \infty$, we obtain the corresponding result for the symmetric function spaces.

Corollary 5.9. *Let E be an order continuous symmetric function space on $[0,\alpha)$, $0 < \alpha \leqslant \infty$. Then the function x is a smooth point of B_E if and only if its decreasing rearrangement $\mu(x)$ is a smooth point of B_E, and either*

(i) $\mu(\infty; f) = 0$, *where* $f \in S_{E^\times}$ *supports* $\mu(x)$, *or*

(ii) $\operatorname{supp}(x) = [0,\alpha)$ *a.e.*

The theory developed in this section implies that the above results hold true for the symmetric sequence space $E \neq \ell_1$ and the unitary matrix space C_E. In fact, the following can be easily observed.

Lemma 5.10. *Let E be an order continuous symmetric sequence space and $x \in C_E$. If $\mu(x)$ is a smooth point in G then $s(x) = \{s_n(x)\}$ is a smooth point in E.*

Proof. Let $F(g) = \int g(t)f(t)dt$, $g \in G$, for some $f \in S_{G^\times}$ be a unique functional supporting $\mu(x)$. By the uniqueness of the functional F, $f = \mu(f)$, and hence $\sum_{n=1}^\infty s_n(x)\pi_n(f) = \int \mu(t;x)f(t)dt = 1$, where by Lemma 1.7 we have that $\pi(f) \in S_{E^\times}$. Suppose that $\sum_{n=1}^\infty s_n(x)b_n = 1$, for some $b = \{b_n\} \in S_{E^\times}$. Setting $g =$

$\sum_{n=1}^{\infty} b_n \chi_{[n-1,n)}$, $\int \mu(t;x)g(t)dt = \sum_{n=1}^{\infty} s_n(x)b_n = 1$ and $g \in S_{G^\times}$ by Lemma 1.7. Since $\mu(x)$ is smooth in G, $g = f$. Therefore g is decreasing and $b = \mu(b) = \pi(g) = \pi(f)$, proving that $s(x)$ is a smooth point in E. $\square$

Using Proposition 1.6 in the preliminary section about an identification of the space C_E with the symmetric operator space $G(B(H), \mathrm{tr})$ and previous lemma we can show the following result, established earlier by J. Arazy in [3].

Theorem 5.11. *Let $E \neq \ell_1$ be an order continuous symmetric sequence space. Then x is a smooth point of B_{C_E} if and only if $s(x) = \{s_n(x)\}$ is a smooth point of B_E.*

Proof. Suppose that x is a smooth point of the unit ball in $C_E = G(B(H), \mathrm{tr})$. Then by Theorem 5.7, $\mu(x)$ is a smooth point in G and by Lemma 5.10, $s(x)$ is smooth in E.

To show that x is a smooth point in C_E whenever $s(x) = \{s_n(x)\}$ is a smooth point in the order continuous symmetric sequence space $E \neq \ell_1$, one can repeat the same argument as in the proof of Theorem 5.3, replacing the singular value functions with the sequences of singular numbers. Note that the assumption $E \neq \ell_1$ implies that $E^\times \subset c_0$. Hence condition (i) of Theorem 5.8 is always satisfied. $\square$

The following corollaries are direct consequences of the results above.

Corollary 5.12. *Let E be an order continuous symmetric function space such that $E^\times = (E^\times)_0$. Then E is smooth if and only if $E(\mathcal{M}, \tau)$ is smooth.*

Corollary 5.13. *Let E be an order continuous symmetric sequence space. Then space E is smooth if and only if C_E is smooth.*

Note that $E = \ell_1$ is not smooth, since an element $a = \{a_n\} \in \ell_1$ is a smooth point of B_{ℓ_1} if and only if $a_n \neq 0$ for all $n \in \mathbb{N}$. Moreover, $x \in C_{\ell_1}$ is a smooth point in $B_{C_{\ell_1}}$ if and only if either x or x^* is one-to-one [41]. Hence also C_{ℓ_1} is not smooth and the above corollary holds true for any order continuous symmetric sequence space E.

5.2 Strongly Smooth Points

In this section, we will characterize strongly smooth points in $E\left(\mathcal{M},\tau\right)$ in connection with their singular value functions. We also prove the analogous result for order continuous symmetric sequence space E and unitary matrix space C_E.

We start this section with a simple observation about the operator $y \in E^{\times}\left(\mathcal{M},\tau\right)$ which strongly supports an element $x \in S_{E(\mathcal{M},\tau)}$.

Proposition 5.14. *Let E be order continuous. If $x \in S_{E(\mathcal{M},\tau)}$ is strongly smooth and the operator $y \in S_{E^{\times}(\mathcal{M},\tau)}$ is such that the functional $\Phi_y(z) = \tau(zy), z \in E(\mathcal{M},\tau)$, strongly supports x, then y is an order continuous element of $E^{\times}\left(\mathcal{M},\tau\right)$.*

Proof. Let x be a strongly smooth element of $E(\mathcal{M},\tau)$ and Φ_y, for some $y \in S_{E^{\times}(\mathcal{M},\tau)}$, be the functional that strongly supports x. Let $\{p_n\} \subset P(\mathcal{M})$ be such that $p_n \downarrow 0$. Since x is order continuous $\|p_n x\|_{E(\mathcal{M},\tau)} \to 0$. By the fact that $xy \in L_1\left(\mathcal{M},\tau\right)$, we have

$$\tau(xyp_n) = \tau(p_n xy) \le \|p_n x\|_{E(\mathcal{M},\tau)} \|y\|_{E^{\times}(\mathcal{M},\tau)} \to 0,$$

and therefore $\Phi_{yp_n^{\perp}}(x) = \tau(xyp_n^{\perp}) = \tau(xy) - \tau(xyp_n) = 1 - \tau(xyp_n) \to 1$. Using the assumption that x is strongly smooth it follows that

$$\|yp_n\|_{E^{\times}(\mathcal{M},\tau)} = \|y - yp_n^{\perp}\|_{E^{\times}(\mathcal{M},\tau)} = \|\Phi_y - \Phi_{yp_n^{\perp}}\|_{E^{*}(\mathcal{M},\tau)} \to 0,$$

and therefore y is order continuous in $E^{\times}\left(\mathcal{M},\tau\right)$. □

We are ready now for the first part of the main claim.

Theorem 5.15. *Let E be order continuous and the trace τ on $\mathcal{M}$ be σ-finite. If $\mu(x) \in S_E$ is a strongly smooth point of B_E then x is a strongly smooth point of $B_{E(\mathcal{M},\tau)}$.*

Proof. Suppose that $\mu(x)$ is a strongly smooth point of B_E. Hence, by Proposition 5.14 applied to the commutative von Neumann algebra $\mathcal{M} = L_\infty[0, \tau(\mathbf{1}))$, there exists an order continuous function $f \in S_{E^\times}$ and a functional $F(g) = \int g(t)f(t)dt$, $g \in E$, such that F strongly supports $\mu(x)$. Thus F supports $\mu(x)$ and as it was shown in the proof of Theorem 5.3, we have $\mu(f) = f$. Following further the proof of Theorem 5.3, there exists an operator $y \in B_{E^\times(\mathcal{M},\tau)}$, such that $\mu(y) = f$ and the functional Φ_y supports x. We will show that it supports x strongly.

Let $\Phi_{y_n}(x) = \tau(xy_n) \to 1$, where $\{y_n\} \subset B_{E^\times(\mathcal{M},\tau)} \cap S_0(\mathcal{M},\tau)$. Define the sequence of functionals $\{F_n\} \subset B_{E^*}$ by $F_n(g) = \int g(t)\mu(t;y_n)dt$, $g \in E$. Then

$$1 \leftarrow \tau(xy_n) \leq \int \mu(t;x)\mu(t;y_n)dt = F_n(\mu(x)) \leq 1,$$

and so $F_n(\mu(x)) \to 1$. Since $\mu(x)$ is strongly smooth, $\|\mu(y) - \mu(y_n)\|_{E^\times} = \|F - F_n\|_{E^*} \to 0$.

By linearity of the trace τ, $\tau\left(x\left(y_n + y\right)/2\right) \to 1$ and repeating the same argument as above, it follows that $\|\mu\left(\left(y_n + y\right)/2\right) - \mu(y)\|_{E^\times} \to 0$. As a consequence of Lemma 1.9 we have that $y - y_n \xrightarrow{\tau} 0$. Note that since E as an order continuous space is fully symmetric, also $E^\times$ is fully symmetric. Since $\mu(y)$ is order continuous, y is an order continuous element of $E^\times(\mathcal{M},\tau)$ by Proposition 2.2. Consequently, Proposition 2.3 implies that $\|y - y_n\|_{E^\times(\mathcal{M},\tau)} \to 0$.

Suppose now that $\Phi_{y_n}(x) = \tau(xy_n) \to 1$, where $\{y_n\} \in B_{E^\times(\mathcal{M},\tau)}$ is arbitrary, not necessarily in $S_0(\mathcal{M},\tau)$. Since τ is σ-finite, there exists a sequence of projections $\{p_n\} \subset P(\mathcal{M})$, satisfying $p_n \uparrow \mathbf{1}$ and $\tau(p_n) < \infty$, for $n \in \mathbb{N}$. Since E is order continuous and $\{y_n\} \in B_{E^\times(\mathcal{M},\tau)}$, we have that $\tau(xy_np_n^\perp) = \tau(p_n^\perp xy_n) \leq \|p_n^\perp x\|_{E(\mathcal{M},\tau)} \to 0$, and consequently $\tau(xy_np_n) = \tau(xy_n) - \tau(xy_np_n^\perp) \to 1$. In view of $\tau(p_n) < \infty$, $\{y_np_n\} \subset S_0(\mathcal{M},\tau)$, and by the previous argument $\|y - y_np_n\|_{E^\times(\mathcal{M},\tau)} \to 0$. Since y is order continuous, $\|yp_n^\perp\|_{E^\times(\mathcal{M},\tau)} \to 0$, and $\|(y - y_n)p_n\|_{E^\times(\mathcal{M},\tau)} \to 0$.

Let $\{q_k\} \subset P(\mathcal{M})$, $q_k \uparrow \mathbf{1}$ as $k \to \infty$ and $\tau(q_k) < \infty$, $k \in \mathbb{N}$. Then for all $n \in \mathbb{N}$, $q_k |y - y_n| q_k \uparrow |y - y_n|$ as $k \to \infty$. Since the Köthe dual $E^\times(\mathcal{M}, \tau)$ has the Fatou property, for all $n \in \mathbb{N}$, $\lim_k \|q_k |y - y_n| q_k\|_{E^\times(\mathcal{M},\tau)} = \|y - y_n\|_{E^\times(\mathcal{M},\tau)}$. Taking $\epsilon > 0$, for all $n \in \mathbb{N}$ there exists k_n such that $q_{k_n} \uparrow \mathbf{1}$ as $n \to \infty$, and

$$\|y - y_n\|_{E^\times(\mathcal{M},\tau)} \leq \|q_{k_n} |y - y_n| q_{k_n}\|_{E^\times(\mathcal{M},\tau)} + \epsilon/2 \leqslant \|(y - y_n)q_{k_n}\|_{E^\times(\mathcal{M},\tau)} + \epsilon/2.$$

Since we have shown earlier that for any sequence of projections $\{p_n\} \subset P(\mathcal{M})$, such that $p_n \uparrow \mathbf{1}$ and $\tau(p_n) < \infty$, $\|(y - y_n)p_n\|_{E^\times(\mathcal{M},\tau)} \to 0$, it follows that there exists N such that $\|(y-y_n)q_{k_n}\|_{E^\times(\mathcal{M},\tau)} \leq \epsilon/2$, for all $n \geqslant N$. Consequently, $\|y-y_n\|_{E^\times(\mathcal{M},\tau)} \leq \epsilon$, for all $n \geqslant N$, and $\|y - y_n\|_{E^\times(\mathcal{M},\tau)} \to 0$. □

We will need the next two lemmas to prove Theorem 5.19, a converse statement of the previous result.

Lemma 5.16. *Suppose that the trace τ on $\mathcal{M}$ is σ-finite and x is an order continuous element of $E(\mathcal{M}, \tau)$. Then x is a strongly extreme point of $B_{E(\mathcal{M},\tau)}$ if and only if $|x|$ is a strongly extreme point of $B_{E(\mathcal{M},\tau)}$.*

Proof. We will prove only implication in one direction. The converse statement can be shown using the same technique.

Let x be a strongly extreme point of $B_{E(\mathcal{M},\tau)}$ and suppose that $\| |x| \pm y_n\|_{E(\mathcal{M},\tau)} \to 1$. By the assumption τ is σ-finite, and hence there exists a sequence $\{p_n\} \subset P(\mathcal{M})$, such that $p_n \uparrow \mathbf{1}$ and $\tau(p_n) < \infty$, $n \in \mathbb{N}$. Note that by the order continuity of the operator x, and in view of $\left|xp_n^\perp\right| = \left||x|\, p_n^\perp\right|$, $\lim_n \|xp_n^\perp\|_{E(\mathcal{M},\tau)} = \lim_n \| |x|\, p_n^\perp\|_{E(\mathcal{M},\tau)} = 0$. We have now that

$$\begin{aligned}\| |x| \pm y_n\|_{E(\mathcal{M},\tau)} - \| |x|\, p_n^\perp\|_{E(\mathcal{M},\tau)} &\leq \| |x|\, p_n \pm y_n\|_{E(\mathcal{M},\tau)} \\ &\leq \| |x| \pm y_n\|_{E(\mathcal{M},\tau)} + \| |x|\, p_n^\perp\|_{E(\mathcal{M},\tau)},\end{aligned}$$

and hence $\lim_n \| \, |x| \, p_n \pm y_n \|_{E(\mathcal{M},\tau)} = 1$.

Note that $s(|x| \, p_n) \le p_n$, and so $\tau(s(|x| \, p_n)) \le \tau(p_n) < \infty$. Since $s(|x| \, p_n) \sim s\left((|x| \, p_n)^*\right)$, we have that $s(|x| \, p_n)$ and $s\left((|x| \, p_n)^*\right)$ are finite, equivalent projections in $\mathcal{M}$. Therefore by [73, Chapter 5, Proposition 1.38], $n(|x| \, p_n) \sim n\left((|x| \, p_n)^*\right)$ and consequently by Lemma 2.5, for all $n \in \mathbb{N}$ there exists an isometry w_n, such that $|x| \, p_n = w_n \, |xp_n|$, where $|xp_n| = ||x| \, p_n|$. Thus

$$\| \, |xp_n| \pm w_n^* y_n \|_{E(\mathcal{M},\tau)} = \|w_n^* \, |x| \, p_n \pm w_n^* y_n \|_{E(\mathcal{M},\tau)} = \| \, |x| \, p_n \pm y_n \|_{E(\mathcal{M},\tau)} \to 1.$$

Repeating the argument above, in view of $\tau(s(xp_n)) \le \tau(p_n) < \infty$, $n \in \mathbb{N}$, for each $n \in \mathbb{N}$ there exists an isometry v_n such that $xp_n = v_n \, |xp_n|$. Hence,

$$\|xp_n \pm v_n w_n^* y_n \|_{E(\mathcal{M},\tau)} = \|v_n \, |xp_n| \pm v_n w_n^* y_n \|_{E(\mathcal{M},\tau)} = \| \, |xp_n| \pm w_n^* y_n \|_{E(\mathcal{M},\tau)} \to 1,$$

and

$$\begin{aligned} \|xp_n \pm v_n w_n^* y_n \|_{E(\mathcal{M},\tau)} - \|xp_n^\perp\|_{E(\mathcal{M},\tau)} &\le \|x \pm v_n w_n^* y_n \|_{E(\mathcal{M},\tau)} \\ &\le \|xp_n \pm v_n w_n^* y_n \|_{E(\mathcal{M},\tau)} + \|xp_n^\perp\|_{E(\mathcal{M},\tau)}. \end{aligned}$$

Therefore $\|x \pm v_n w_n^* y_n \|_{E(\mathcal{M},\tau)} \to 1$, and since x is strongly extreme,

$$\|y_n\|_{E(\mathcal{M},\tau)} = \|v_n w_n^* y_n \|_{E(\mathcal{M},\tau)} \to 0.$$

□

Lemma 5.17. *Let E be order continuous and the trace τ on $\mathcal{M}$ be σ-finite. If $x \in S_{E(\mathcal{M},\tau)}$ is a strongly smooth point of $B_{E(\mathcal{M},\tau)}$ then $|x|$ is a strongly smooth point of $B_{E(\mathcal{M},\tau)}$.*

Proof. Let $x \in S_{E(\mathcal{M},\tau)}$ be a strongly smooth point of $B_{E(\mathcal{M},\tau)}$ and $\Phi_y(z) = \tau(zy)$,

$z \in E(\mathcal{M},\tau)$, for some $y \in B_{E^\times(\mathcal{M},\tau)}$, be a unique functional strongly supporting x. Let $x = u\,|x|$ be the polar decomposition of x. Then as shown in Lemma 5.5, $\Phi_{|y^*|}(z) = \tau(z\,|y^*|)$, $z \in E(\mathcal{M},\tau)$, is a unique functional supporting $|x|$. Moreover, since $\Phi_{yu}(|x|) = \tau(|x|\,yu) = \tau(u\,|x|\,y) = \tau(xy) = \Phi_y(x) = 1$, $|y^*| = yu$. Observe that $\operatorname{Ker} u = \operatorname{Ker} x$, which implies that $us(x) = u$. Hence, $|y^*|\,s(x) = yus(x) = yu = |y^*|$ and $s(y^*) = s(|y^*|) \le s(x)$.

Suppose now that $\Phi_{y_n}(|x|) \to 1$ as $n \to \infty$, where $\{y_n\} \subset B_{E^\times(\mathcal{M},\tau)}$. Hence,

$$\Phi_{y_n u^*}(x) = \tau(x y_n u^*) = \tau(u^* x y_n) = \tau(|x|\,y_n) = \Phi_{y_n}(|x|) \to 1,$$

and since x is strongly smooth, $\|y_n u^* - y\|_{E^\times(\mathcal{M},\tau)} = \|\Phi_{y_n u^*} - \Phi_y\|_{E^*} \to 0$. Thus

$$\|\,|y^*| - y_n s(x)\|_{E^\times(\mathcal{M},\tau)} = \|yu - y_n u^* u\|_{E^\times(\mathcal{M},\tau)} \le \|y - y_n u^*\|_{E^\times(\mathcal{M},\tau)} \to 0.$$

Now

$$\|\,|y^*| + y_n n(x)\|_{E^\times(\mathcal{M},\tau)} \le \|\,|y^*| - y_n s(x)\|_{E^\times(\mathcal{M},\tau)} + \|y_n\|_{E^\times(\mathcal{M},\tau)} \le \|\,|y^*| - y_n s(x)\|_{E^\times(\mathcal{M},\tau)} + 1.$$

Also in view of $s(y^*) \le s(x)$, $|y^*|\,n(x) = 0$, and hence

$$|(|y^*| \pm y_n n(x))^*|^2 = ||y^*| \pm n(x) y_n^*|^2 = (|y^*| \pm y_n n(x))(|y^*| \pm n(x) y_n^*) = |y^*|^2 + |n(x) y_n|^2 .$$

Thus $||y^*| + y_n n(x)| = ||y^*| - y_n n(x)|$, and so

$$\|\,|y^*| - y_n n(x)\|_{E^\times(\mathcal{M},\tau)} = \|\,|y^*| + y_n n(x)\|_{E^\times(\mathcal{M},\tau)} \le \|\,|y^*| - y_n s(x)\|_{E^\times(\mathcal{M},\tau)} + 1.$$

Consequently, $\overline{\lim}_n \|\,|y^*| \pm y_n n(x)\|_{E^\times(\mathcal{M},\tau)} \le 1$ and so $\|\,|y^*| \pm y_n n(x)\|_{E^\times(\mathcal{M},\tau)} \to 1$ by Lemma 1.20.

From the fact that Φ_y is a unique functional strongly supporting operator x, y

and y^* are strongly extreme points of $B_{E^\times(\mathcal{M},\tau)}$. Since y^* is order continuous by Proposition 5.14, and τ is σ-finite, Lemma 5.16 implies that $|y^*|$ is a strongly extreme point of $E^\times(\mathcal{M},\tau)$. Therefore $\|y_n n(x)\|_{E^\times(\mathcal{M},\tau)} \to 0$ and

$$\|\Phi_{|y^*|}-\Phi_{y_n}\|_{E^*} = \|\,|y^*|-y_n\|_{E^\times(\mathcal{M},\tau)} = \|\,|y^*|-y_n s(x)\|_{E^\times(\mathcal{M},\tau)}+\|y_n n(x)\|_{E^\times(\mathcal{M},\tau)} \to 0,$$

proving that $|x|$ is a strongly smooth point of $B_{E(\mathcal{M},\tau)}$. □

Lemma 5.18. *Let E be order continuous. If $x \in S_{E(\mathcal{M},\tau)}$ is a strongly smooth point of $B_{E(\mathcal{M},\tau)}$ then x_p is a strongly smooth point of $B_{E(\mathcal{M}_p,\tau_p)}$, for $p \in P(\mathcal{M})$ satisfying $p \geqslant s(x^*)$.*

Proof. Let $x \in S_{E(\mathcal{M},\tau)}$ be a strongly smooth point of $B_{E(\mathcal{M},\tau)}$ and Φ_y, for some $y \in S_{E^\times(\mathcal{M},\tau)}$, be a functional that strongly supports x. As shown in Lemma 5.6, for all $z \in E^\times(\mathcal{M},\tau)$ we have that $\tau(xz) = \tau_p(x_p z_p)$, and the functional $\Psi_{y_p}(z') = \tau_p(z' y_p)$, $z' \in E(\mathcal{M}_p,\tau_p)$, supports x_p. To show it strongly supports x_p, suppose that $\{\Psi_n\} \subset B_{E^*(\mathcal{M}_p,\tau_p)}$ and $\Psi_n(x_p) \to 1$. Since E is order continuous, for all $n \in \mathbb{N}$ we have that $\Psi_n(x_p) = \tau_p(x_p y_n')$, for some $y_n' \in B_{E^\times(\mathcal{M}_p,\tau_p)}$. Hence $y_n' = (y_n)_p$, for some $y_n \in B_{E^\times(\mathcal{M},\tau)}$, and we have that

$$\Phi_{y_n}(x) = \tau(xy_n) = \tau_p(x_p(y_n)_p) = \Psi_n(x_p) \to 1.$$

Applying the fact that x is strongly smooth, $\|y-y_n\|_{E^\times(\mathcal{M},\tau)} = \|\Phi_y-\Phi_{y_n}\|_{E^*(\mathcal{M},\tau)} \to 0$. Consequently,

$$\begin{aligned}\|\Psi_{y_p}-\Psi_n\|_{E^*(\mathcal{M}_p,\tau_p)} &= \|y_p-(y_n)_p\|_{E^\times(\mathcal{M}_p,\tau_p)} = \|\mu^{\tau_p}((y-y_n)_p)\|_{E^\times} \\ &= \|\mu(p(y-y_n)p)\|_{E^\times} \leq \|y-y_n\|_{E^\times(\mathcal{M},\tau)} \to 0,\end{aligned}$$

proving that x_p is strongly smooth. □

Theorem 5.19. *Let E be order continuous. If x is a strongly smooth point of $B_{E(\mathcal{M},\tau)}$ then $\mu(x)$ is a strongly smooth point of B_E.*

Proof. Note first that by Remark 1.8 and in view of the equality $\mu(x) = \tilde{\mu}(\tilde{\pi}(x))$, we can assume that the von Neumann algebra $\mathcal{M}$ is non-atomic.

Since $\mu(\infty, x) = 0$, for $p = s(x) \vee s(x^*)$, τ_p is a σ-finite trace on $\mathcal{M}_p$ by Lemma 1.12. In view of the equality $\mu^{\tau_p}(x_p) = \mu(pxp) = \mu(x)$ and Lemma 5.18 stating that x_p is strongly smooth in $E(\mathcal{M}_p, \tau_p)$, we can assume that the trace τ is σ-finite. In view of Lemma 5.17 and the fact that $\mu(|x|) = \mu(x)$ we can assume now that $x \geqslant 0$.

Applying Proposition 1.11 to the operator x, there exists a $*$-isomorphism V from $S\left([0, \tau(\mathbf{1})), m\right)$ into $S\left(\mathcal{M}, \tau\right)$, such that $V(\mu(x)) = x$ and $\mu(V(f)) = \mu(f)$ for all $f \in S\left([0, \tau(\mathbf{1})), m\right)$.

Suppose that the functional $\Phi_y(z) = \tau(zy)$, $z \in E(\mathcal{M}, \tau)$, strongly supports x, where $y \in S_{E^\times(\mathcal{M},\tau)}$. We will show that the functional $F(g) = \int g(t)\mu(t; y)dt$, $g \in E$, strongly supports $\mu(x)$. By Theorem 5.7 and its proof, it supports $\mu(x)$ and $V(\mu(y)) = y$.

Suppose that $F_n(x) = \int \mu(t; x) f_n(t)dt \to 1$, for $\{f_n\} \subset B_{E^\times}$. The goal is to show that then $\|F_n - F\|_{E^*} = \|f_n - \mu(y)\|_{E^\times} \to 0$. If is clear that $\int \mu(t; x) \left|f_n\right|(t)dt \to 1$. Since V as a $*$-homomorphism is positive, it follows that

$$xV(|f_n|) = V(\mu(x))V(|f_n|) = V(\mu(x)\left|f_n\right|) \geq 0$$

and

$$\begin{aligned}\Phi_{V(|f_n|)}(x) &= \tau(xV(|f_n|)) = \|xV(|f_n|)\|_{L_1(\mathcal{M},\tau)} = \|\mu(x)\left|f_n\right|\|_{L_1(0,\tau(\mathbf{1}))} \\ &= \int \mu(t; x)\left|f_n\right|(t)dt \to 1.\end{aligned}$$

Thus since x is strongly smooth,

$$\begin{aligned}\|\mu(y)-|f_n|\,\|_{E^\times} &= \|V(\mu(y))-V(|f_n|)\|_{E^\times(\mathcal{M},\tau)} = \|y-V(|f_n|)\|_{E^\times(\mathcal{M},\tau)} \\ &= \|\Phi_y-\Phi_{V(|f_n|)}\|_{E^*}\to 0.\end{aligned}$$

Now by $(|f_n|+f_n)/2\geqslant 0$ and $\int\mu(t;x)(|f_n|(t)+f_n(t))/2dt\to 1$, again it follows that $\|\mu(y)-(|f_n|+f_n)/2\|_{E^\times}\to 0$. Hence

$$\|\mu(y)-f_n\|_{E^\times}\leq \|2\mu(y)-f_n-|f_n|\,\|_{E^\times}+\|\mu(y)-|f_n|\,\|_{E^\times}\to 0,$$

which shows that $\mu(x)$ is a strongly smooth point of B_E. □

Let us summarize the above results.

Theorem 5.20. *Let E be an order continuous symmetric function space and the trace τ on $\mathcal{M}$ be σ-finite. Then x is a strongly smooth point of $B_{E(\mathcal{M},\tau)}$ if and only if $\mu(x)$ is a strongly smooth point of B_E.*

Considering the commutative von Neumann algebra $\mathcal{M}=L_\infty[0,\alpha)$, $0<\alpha\leqslant\infty$, we obtain the following consequence of the previous theorem.

Corollary 5.21. *Let E be an order continuous symmetric function space. Then the function x is a strongly smooth point of B_E if and only if its decreasing rearrangement $\mu(x)$ is a strongly smooth point of B_E.*

Before we will give a version of Theorem 5.15 and Theorem 5.19 for the unitary matrix space let us state the following simple result. For the definition of the symmetric function space G and the identification of C_E with $G(B(H),\mathrm{tr})$, we refer to Proposition 1.6.

Lemma 5.22. *Let E be an order continuous symmetric sequence space and $x\in C_E$. If $\mu(x)$ is a strongly smooth point in G then $s(x)=\{s_n(x)\}$ is a strongly smooth point in E.*

Proof. Let $\mu(x)$ be a strongly smooth point in G and $F(g) = \int g(t)f(t)dt$, $g \in G$, for some $f \in S_{G^\times}$, be a functional that strongly supports $\mu(x)$. As shown in Lemma 5.10, $f = \mu(f)$ and the functional $G(a) = \sum_{n=1}^{\infty} a_n \pi_n(f)$, $a = \{a_n\} \in E$, supports $s(x)$, where $\pi(f) \in S_{E^\times}$. Suppose now that $\sum_{i=1}^{\infty} s_i(x) b_{ni} \to 1$ as $n \to \infty$, where $b_n = \{b_{ni}\}_{i=1}^{\infty}$ and $\{b_n\} \subset B_{E^\times}$. Setting $g_n = \sum_{i=1}^{\infty} b_{ni}\chi_{[i-1,i)}$, $n \in \mathbb{N}$, we have that $\int \mu(t;x) g_n(t)dt = \sum_{i=1}^{\infty} s_i(x) b_{ni} \to 1$ as $n \to \infty$, and $\{g_n\} \subset B_{G^\times}$ by Lemma 1.7. Using the assumption that $\mu(x)$ is strongly smooth in G, we get that $\|f - g_n\|_{G^\times} \to 0$ as $n \to \infty$. Applying Lemma 1.7, it follows that for all $n \in \mathbb{N}$,

$$\|\pi(f) - b_n\|_{E^\times} = \left\| \left\{ \int_{n-1}^{n} (f - g_n)(t)dt \right\} \right\|_{E^\times} \leq \|f - g_n\|_{G^\times}.$$

Hence $\|\pi(f) - b_n\|_{E^\times} \to 0$ and $s(x)$ is a strongly smooth point in E. □

Theorem 5.23. *Let E be an order continuous symmetric sequence space. Then the sequence of singular numbers $s(x) = \{s_n(x)\}$ is a strongly smooth point of B_E if and only if x is a strongly smooth point of B_{C_E}.*

Proof. Suppose first that x is a strongly smooth point of the unit ball $B_{C_E} = B_{G(B(H),\mathrm{tr})}$. Then by Theorem 5.19, $\mu(x)$ is strongly smooth in G, and in view of Lemma 5.22, $s(x)$ is strongly smooth in E.

It is known and standard to check that there are no strongly smooth points in ℓ_1. Therefore by the preceding argument, C_{ℓ_1} has no strongly smooth points.

Suppose now that $E \neq \ell_1$ and $s(x)$ is a strongly smooth point of B_E. Let $F(a) = \sum_{n=1}^{\infty} a_n b_n$, $a = \{a_n\} \in E$, for some $b = \{b_n\} \in S_{E^\times}$, be a functional that strongly supports $s(x)$. By Theorem 5.11 and its proof, if $s(x)$ is a smooth point of B_E then x is a smooth point of B_{C_E} and moreover, the functional $\Phi_y(x) = \mathrm{tr}(xy) = \sum_{n=1}^{\infty} s_n(xy)$ supports x, for $y \in C_{E^\times}$ whose sequence $s(y)$ of singular numbers satisfies the condition $s_n(y) = b_n$, $n \in \mathbb{N}$.

Suppose that $\Phi_{y_n}(x) \to 1$, for the sequence $\{y_n\} \subset B_{C_{E^\times}}$. In view of the inequality

$\sum_{i=1}^{\infty} s_i(xy_n) \leq \sum_{i=1}^{\infty} s_i(x)s_i(y_n) \leq 1$, we have that $\sum_{i=1}^{\infty} s_i(x)s_i(y_n) \to 1$ as $n \to \infty$. Applying the assumption that $s(x)$ is strongly smooth, it follows that $\|s(y) - s(y_n)\|_{E^\times} \to 0$. As outlined in the proof of Theorem 5.15, one can show that also $\|s(y) - s\left((y+y_n)/2\right)\|_{E^\times} \to 0$. Since $E^\times \neq \ell_\infty$, by Lemma 1.9, $y_n \xrightarrow{\text{tr}} y$.

Similarly as in the proof of Proposition 5.14, it can be shown that if $F(a) = \sum_{n=1}^{\infty} a_n s_n(y)$, $a = \{a_n\} \in E$, is a functional that strongly supports $s(x)$, then $s(y) = \{s_n(y)\}$ is order continuous in $E^\times$. Since $E^\times \neq \ell_\infty$ one can identify the space $C_{E^\times}$ with the symmetric space of operators, and apply the analogous argument as in the proof of Proposition 2.2, to show that y is an order continuous element of $C_{E^\times}$. Finally, Proposition 2.3 implies that $\|y - y_n\|_{C_{E^\times}} \to 0$. Consequently, x is a strongly smooth point of B_{C_E} and the proof is complete. $\square$

We finish with the following important consequences.

Corollary 5.24. *Let E be an order continuous symmetric function space and the trace τ on $\mathcal{M}$ be σ-finite. Then E is Fréchet smooth if and only if $E(\mathcal{M}, \tau)$ is Fréchet smooth.*

Corollary 5.25. *Let E be order an continuous symmetric sequence space. Then E is Fréchet smooth if and only if C_E is Fréchet smooth.*

6 Exposed and Strongly Exposed Points

The content of this chapter will be published in [13].

6.1 Exposed Points

In this section, we explore exposed points of the unit ball in the order continuous symmetric function space E and corresponding to it noncommutative space $E(\mathcal{M},\tau)$.

We start with a result on existence of a functional $F \in E^*$ that exposes decreasing function $g \in E$ and satisfies several useful and convenient conditions.

Lemma 6.1. *Let E be an order continuous symmetric space on $[0,\alpha)$, $0 < \alpha \leqslant \infty$. If $g = \mu(g) \in S_E$ is an exposed point of B_E, then there exists a linear functional $F \in E^*$ that exposes B_E at g which is of the form $F(h) = \int h(t)f(t)dt$, $h \in E$, where $f \in E^\times$, $f = \mu(f)$, $\operatorname{supp} f = \operatorname{supp} g$ and f is constant on intervals of constancy of g.*

Proof. Let $g = \mu(g) \in S_E$ be an exposed point of B_E. Since E is order continuous, the functional F_0 that exposes B_E at g must be of the form $F_0(h) = \int h(t)f_0(t)\,dt$, $h \in E$, for some $f_0 \in S_{E^\times}$. It is easy to check that $f_0 \geqslant 0$ on $\operatorname{supp} g$.

We will show first that for $f_1 = f_0\chi_{\operatorname{supp} g}$ the functional $F_1(h) = \int h(t)f_1(t)dt$, $h \in E$, also exposes the ball B_E at g. Clearly $\operatorname{supp} f_1 \subset \operatorname{supp} g$, and $F_1(g) = \int g(t)f_0(t)dt = F_0(g) = 1$. Now, observe that for any $h \in B_E$ with $h\chi_{\operatorname{supp} g} = g$ we get that $h = g$. Indeed, denoting $h_1 = g + h\chi_{\operatorname{supp} g^c}$ and $h_2 = g - h\chi_{\operatorname{supp} g^c}$, we get that $|h_1| = |h_2| = |h|$ and $g \pm h\chi_{\operatorname{supp} g^c} \in B_E$. Since g is an extreme point of B_E, $h\chi_{\operatorname{supp} g^c} = 0$ and $h = g$. Next, if $h \in B_E$ is such that $h\chi_{\operatorname{supp} g} \neq g$ then $F_1(h) = \int h(t)f_1(t)\,dt = \int h(t)\chi_{\operatorname{supp} g}(t)f_0(t)\,dt \neq 1$. Thus F_1 exposes B_E at g. Observe that $\operatorname{supp} g = \operatorname{supp} f_1$. Indeed,

$$1 = F_1(g) = \int g(t)f_1(t)dt = \int g(t)\chi_{\operatorname{supp} f_1}(t)f_1(t)dt = F_1(g\chi_{\operatorname{supp} f_1}),$$

and since F_1 exposes B_E at g, $g\chi_{\operatorname{supp} f_1} = g$. Hence $\operatorname{supp} g \subset \operatorname{supp} f_1$, and since $\operatorname{supp} f_1 \subset \operatorname{supp} g$, it follows that $\operatorname{supp} g = \operatorname{supp} f_1$. Clearly $\|f_1\|_{E^\times} = 1$ and $f_1 \geqslant 0$.

We claim next that also the functional $F_2(h) = \int h(t)\mu(t; f_1)\, dt$, $h \in E$, exposes B_E at g. Clearly $\operatorname{supp} \mu(f_1) = \operatorname{supp} f_1 = \operatorname{supp} g$. We have that

$$1 = F_1(g) = \int g(t) f_1(t)\, dt \leqslant \int g(t)\mu(t; f_1)\, dt = F_2(g) \leq \|g\|_E \|\mu(f_1)\|_{E^\times} \leqslant 1,$$

and hence $F_2(g) = 1$.

Observe that $f_1 \geqslant \mu(\infty; f_1)\chi_{\operatorname{supp} f_1}$. Indeed, if there was a set $A \subset \operatorname{supp} f_1$ with $mA > 0$ and $f_1 < \mu(\infty; f_1)\chi_{\operatorname{supp} f_1}$ on A then there would exist $B \subset A$ with $mB > 0$ and $0 < \epsilon < \mu(\infty; f_1)$ such that $f_1 \leqslant (\mu(\infty; f_1) - \epsilon)\chi_{\operatorname{supp} f_1}$ on B. Clearly $\mu(f_1 + \epsilon\chi_B) = \mu(f_1)$, and so

$$\begin{aligned} 1 + \epsilon \int g(t)\chi_B(t)\, dt &= \int f_1(t) g(t)\, dt + \int \epsilon\chi_B(t) g(t)\, dt \\ &= \int (f_1(t) + \epsilon\chi_B(t)) g(t) \leq \int \mu(t; f_1) g(t)\, dt \leqslant 1. \end{aligned}$$

Hence, we would get $\int g(t)\chi_B(t)\, dt = 0$, that is $g = 0$ on B which is impossible since $B \subset \operatorname{supp} f_1 = \operatorname{supp} g$.

We claim that $f_1 = \mu(f_1) \circ \sigma$ where $\sigma : \operatorname{supp} f_1 \to [0, m(\operatorname{supp} f_1))$ is a measure preserving transformation. Indeed, if $\mu(\infty; f_1) = 0$ then the existence of such transformation follows directly from Corollary 7.6 [4, Chapter 2].

Consider now the case when $\mu(\infty; f_1) > 0$, and therefore $\operatorname{supp} f_1 = \operatorname{supp} g = [0, \infty)$. Let $B = \{t \geqslant 0 : f_1(t) = \mu(\infty; f_1)\}$.

Assume first that $mB^c = \infty$. We claim then that $f_1(t) > \mu(\infty; f_1)$ on $[0, \infty)$. If $mB = 0$ then the claim follows immediately. Hence, we can assume that $mB > 0$. Denote $\tilde{f}_1 = f_1 - \mu(\infty; f_1)\chi_{[0,\infty)}$. Clearly $\operatorname{supp} \tilde{f}_1 = B^c$, $\operatorname{supp} \mu(\tilde{f}_1) = [0, \infty)$ and $\mu(\infty; \tilde{f}_1) = 0$. Thus by Corollary 7.6 [4, Chapter 2] there exists a measure preserving transformation $\sigma : B^c \to [0,\infty)$ such that $\tilde{f}_1 = \mu(\tilde{f}_1) \circ \sigma$. Since $\mu(\tilde{f}_1) = \mu(f_1) - \mu(\infty; f_1)\chi_{[0,\infty)}$ we get that $f_1 = \mu(f_1) \circ \sigma$ on B^c. Therefore

$$\begin{aligned} 1 &= \int_0^\infty \mu(t; f_1) g(t)\, dt = \int_{B^c} \mu(\sigma(u); f_1) g(\sigma(u))\, du \\ &= \int_{B^c} f_1(u) g(\sigma(u))\, du = \int_0^\infty f_1(t)\tilde{g}(t)\, dt = F_1(\tilde{g}), \end{aligned}$$

where $\tilde{g}(t) = g(\sigma(t))$ if $t \in B^c$ and $\tilde{g}(t) = 0$ if $t \in B$. Since F_1 exposes g, it follow that $\tilde{g} = g$ which is impossible since $mB > 0$ and $\operatorname{supp} g = [0,\infty)$. Thus in this case we have that $f_1 > \mu(\infty; f_1)$ on $[0,\infty)$, $\operatorname{supp} \tilde{f}_1 = [0,\infty)$, $\operatorname{supp} \mu(\tilde{f}_1) = [0,\infty)$ and $\mu(\infty; \tilde{f}_1) = 0$. By Corollary 7.6 [4, Chapter 2] applied to the function $\tilde{f}_1$ we conclude that $f_1 = \mu(f_1) \circ \sigma$, where $\sigma : [0;\infty) \to [0,\infty)$ is a measure preserving transformation.

Now consider the remaining case when $mB^c < \infty$. Clearly $mB = \infty$ and $\mu(t; f_1) = \mu(\infty; f_1)$ for all $t \geqslant mB^c$. By Theorem 7.5 [4, Chapter 2] there exists a measure preserving transformation $\sigma_1 : B^c \to [0, mB^c)$ such that $f_1 = \mu(f_1) \circ \sigma_1$ on B^c. By non-atomicity of the Lebesgue measure it is easy to show that there exists a measure preserving transformation $\sigma_2 : B \to (mB^c, \infty)$ such that $f_1 = \mu(f_1) \circ \sigma_2$ on B. Letting $\sigma = \sigma_1 \chi_{B^c} + \sigma_2 \chi_B$, we get that $f_1 = \mu(f_1) \circ \sigma$.

Consequently, we can write $f_1 = \mu(f_1) \circ \sigma$ for some measure preserving transformation $\sigma : \operatorname{supp} f_1 \to [0, m(\operatorname{supp} f_1))$, regardless of the value of $\mu(\infty; f_1)$.

Now, since

$$\begin{aligned} 1 = F_2(g) &= \int g(t)\mu(t; f_1)\, dt = \int_{\operatorname{supp} f_1} g(\sigma(t))\mu(\sigma(t); f_1)\, dt \\ &= \int_{\operatorname{supp} f_1} g(\sigma(t)) f_1(t)\, dt = F_1(\tilde{g}), \end{aligned}$$

and F_1 exposes B_E at g, it follows that $\tilde{g} = g$ where $\tilde{g}(t) = g \circ \sigma(t)$ for $t \in \operatorname{supp} g$ and $\tilde{g}(t) = 0$ for $t \in (\operatorname{supp} g)^c$. Denote by I_i, $i \in K \subset \mathbb{N}$, intervals of constancy of $g = \mu(g)$. That is $g(t) = c_i$ for $t \in I_i$, $i \in K \subset \mathbb{N}$, where $c_i \neq 0$ for all $i \in K$ and

$c_i \neq c_j$ whenever $i \neq j$ for all $i, j \in K$. Since $g(\sigma(t)) = g(t)$ for $t \in \operatorname{supp} g$ and g is one-to-one on $J := \operatorname{supp} g \setminus \bigcup_{i\in K} I_i$, $\sigma(t) = t$ for $t \in J$. Also σ acts on each interval of constancy I_i separately as a measure preserving transformation since $c_i \neq c_j$ for all $i, j \in K$. Hence, we can write

$$\sigma(t) = t\chi_J(t) + \sum_{i\in K} \sigma_i(t)\chi_{I_i}(t), \quad t \in \operatorname{supp} g,$$

where $\sigma_i : I_i \to I_i$ is a measure preserving transformation for each $i \in K$.

Suppose now that $F_2(h) = 1$, where $h \in B_E$. We will show that $h = g$. Observe first that

$$\begin{aligned} 1 = F_2(h) &= \int_0^{m(\operatorname{supp} f_1)} h(t)\mu(t; f_1)\, dt = \int_{\operatorname{supp} f_1} h(\sigma(t))\mu(\sigma(t); f_1)\, dt \\ &= \int_J h(t) f_1(t)\, dt + \int_{\bigcup_{i\in K} I_i} h(\sigma(t)) f_1(t)\, dt = F_1(\tilde{h}), \end{aligned}$$

where $\tilde{h}(t) = h(t)\chi_J(t) + h(\sigma(t))\chi_{\bigcup_{i\in K} I_i}(t)$. Since F_1 exposes B_E at g, $\tilde{h} = g$. Therefore

$$h(t) = g(t) \text{ for } t \in J, \text{ and } h(\sigma_i(t)) = g(t) = c_i \text{ for } t \in I_i, i \in K.$$

But $\sigma_i : I_i \to I_i$, $i \in K$, is a measure preserving transformation, and so

$$\int_G h(u)\, du = \int_{\sigma_i^{-1}(G)} h(\sigma_i(t))\, dt = m\sigma_i^{-1}(G)c_i = c_i mG.$$

Hence $(mG)^{-1} \int_G h(u)\, du = c_i$ for all measurable subsets $G \subset I_i$, $i \in K$, and therefore $h(t) = g(t) = c_i$ for $t \in I_i$, $i \in K$. Consequently $h = g$, proving that F_2 exposes B_E at g.

Finally, let

$$f(t)=\begin{cases}\mu(t;f_1) & \text{if } t\in J\\ (mI_i)^{-1}\int_{I_i}\mu(u;f_1)\,du & \text{if } t\in I_i, i\in K\end{cases},$$

and define the functional $F(h)=\int h(t)f(t)dt$, $h\in E$. Clearly $\operatorname{supp} f=\operatorname{supp} g$. Note that $f\in B_{E^\times}$ since $f\prec f_1$ and $E^\times$ is fully symmetric [51, Chapter II, Theorem 4.10]. We claim that F exposes B_E at g. We have

$$\begin{aligned}F(g)&=\int_J g(t)\mu(t;f_1)\,dt+\int_{\bigcup_{i\in K}I_i} g(t)\left((mI_i)^{-1}\int_{I_i}\mu(u;f_1)\,du\right)\chi_{I_i}(t)dt\\&=\int_J g(t)\mu(t;f_1)\,dt+\sum_{i\in K}(mI_i)^{-1}\int_{I_i}c_i dt\int_{I_i}\mu(u;f_1)\,du\\&=\int_J g(t)\mu(t;f_1)\,dt+\sum_{i\in K}c_i\int_{I_i}\mu(u;f_1)\,du\\&=\int_J g(t)\mu(t;f_1)\,dt+\int_{\bigcup_{i\in K}I_i} g(u)\mu(u;f_1)\,du=\int g(t)\mu(t;f_1)\,dt=F_2(g)=1.\end{aligned}$$

Now, let $h\in B_E$ be such that $F(h)=1$. By [51, Property 18°, page 72] and in view of $\mu(f)=f\prec\mu(f_1)$,

$$1=F(h)\le\int\mu(t;h)f(t)\,dt\leqslant\int\mu(t;h)\mu(t;f_1)\,dt=F_2(\mu(h))\le 1,$$

and so $\mu(h)=g$. Since F_2 induced by $\mu(f_1)$ exposes B_E at g, and

$$\begin{aligned}F(h)&=\int_J h(t)\mu(t;f_1)\,dt+\sum_{i\in K}(mI_i)^{-1}\int_{I_i}h(t)\,dt\int_{I_i}\mu(u;f_1)\,du\\&=\int_J h(t)\mu(t;f_1)\,dt+\sum_{i\in K}\int_{I_i}\left((mI_i)^{-1}\int_{I_i}h(t)\,dt\right)\mu(u;f_1)\,du,\end{aligned}$$

it follows that $h(t)=g(t)$ for $t\in J$ and $(mI_i)^{-1}\displaystyle\int_{I_i}h(t)\,dt=g(t)$, for $t\in I_i$, $i\in K$. Hence for $t\in I_i$, $i\in K$, $\int_{I_i}h(t)dt=g(t)mI_i=c_i mI_i=\int_{I_i}g(t)\,dt$. Since previously we have shown that $\mu(h) = g$, it follows now that $h = g$ and F exposes B_E at g. $\square$

Recall that $\tau(s(x)) = \inf\{t \geqslant 0 : \mu(t;x) = 0\}$. The following lemma, as well as its proof, is a modification of Proposition 2.8 [31, Chapter 3]. We state it here together with its proof for the reader's convenience.

Lemma 6.2. *Let $x \in S_0^+(\mathcal{M}, \tau)$, and let $\phi : [0,\infty) \to [0,\infty]$ be an increasing function which is finite on $[0, \mu(0;x))$, where $\mu(0;x) = \lim_{t\to 0^+} \mu(t;x)$, left-continuous at all points of continuity of $d(x)$ and $\phi(0) = 0$. Suppose also that ϕ satisfies the following conditions*

(i) *If $\mu(0;x) < \infty$ and $m\{t \geqslant 0 : \mu(t;x) = \mu(0;x)\} > 0$ then $\phi(\mu(0;x)) < \infty$,*

(ii) *$\phi(t) > 0$ for all $t \in (\beta, \infty)$ and $\phi(\beta) > 0$ if $e^x\{\beta\} \neq 0$,*

where $\beta = \lim_{s\to\tau(s(x))^-} \mu(s;x)$.

Then $\mu(\phi(x)) = \phi \circ \mu(x)$ holds on $[0,\infty)$, $\phi(x) \geq 0$ and $s(\phi(x)) = s(x)$.

Proof. Let $C = \{c_1, c_2, \ldots\}$ be the set of all points at which ϕ is not left-continuous, and $\beta = \lim_{s\to\tau(s(x))^-} \mu(s;x)$. By the assumption that ϕ is left-continuous at all points of continuity of $d(x)$, $c_i > 0$ and $c_i \in [\beta, \mu(0;x)]$.

Let $s \in \bigcup_i [\phi(c_i), \phi(c_i^+)] \cup \phi[\beta, \mu(0;x)) \setminus \bigcup_i [\phi(c_i^-), \phi(c_i))$. Then the inverse image $\phi^{-1}(s,\infty) \supseteq (\alpha, \mu(0;x)]$ or $\phi^{-1}(s,\infty) \supseteq (\alpha, \mu(0;x))$ if $\phi(\mu(0;x)) = \infty$, where $\alpha = \inf\{t > 0 : \phi(t) \geq s\}$. In any case, by the assumption (i) and since $\mu(t;x) > \alpha$ if and only if $\phi(\mu(t;x)) > s$ we get that

$$d(s;\phi(x)) = \tau(e^{\phi(x)}(s,\infty)) = \tau(e^x(\phi^{-1}(s,\infty)) = \tau(e^x(\alpha,\infty))$$
$$= d(\alpha;x) = d(\alpha;\mu(x)) = d(s;\phi\circ\mu(x)).$$

Now suppose that $s \in [\phi(c_i^-), \phi(c_i))$ for some i. Clearly $\phi^{-1}(s,\infty) \supseteq [c_i, \mu(0;x))$ or $\phi^{-1}(s,\infty) \supseteq [c_i, \mu(0;x)]$, and it is easy to check that $\mu(t;x) \geq c_i$ if and only if

$\phi(\mu(t;x)) > s$. On one hand, we get that

$$d(s;\phi(x)) = \tau(e^x(\phi^{-1}(s,\infty)) = \tau(e^x[c_i,\infty)),$$

on the other hand,

$$d(s;\phi\circ\mu(x)) = m\{t\in[0,\tau(\mathbf{1})) : \mu(t;x)\geq c_i\} = \tau(e^x[c_i,\infty)),$$

since $\tau(e^x(B)) = m\{t\in(0,\tau(\mathbf{1})) : \mu(t;x)\in B\}$ for any Borel set $B\subset(0,\infty)$ by Lemma 3.6 [31, Chapter 3].

If $\mu(0;x)=\infty$ and $\lim_{t\to\infty}\phi(t)<\infty$ then for $s\geqslant\lim_{t\to\infty}\phi(t)$, $\phi^{-1}(s,\infty)=\emptyset$ and hence $d(s;\phi(x)) = \tau(e^x(\phi^{-1}(s,\infty)))=0$. Also $d(s;\phi\circ\mu(x)) = m\{t\in[0,\tau(\mathbf{1})) : \phi(\mu(t;x))>s\} = m(\emptyset)=0$.

If $\mu(0;x)<\infty$ and $\phi(\mu(0;x))<\infty$ then for any $s\geqslant\phi(\mu(0;x))$ we see that $\phi^{-1}(s,\infty)\subseteq(\mu(0;x),\infty)$. Hence $d(s;\phi(x)) = \tau(e^x(\phi^{-1}(s,\infty)))=0$ and $d(s;\phi\circ\mu(x)) = m\{t\in[0,\tau(\mathbf{1})) : \mu(t;x)>\mu(0;x)\}=0$.

Finally, consider the remaining case, when $0\leqslant s<\phi(\beta)$. Again by the assumption that ϕ is finite on $[0,\mu(0;x))$, $\phi^{-1}(s,\infty)\supseteq(\alpha,\mu(0;x)]$ or $\phi^{-1}(s,\infty)\supseteq(\alpha,\mu(0;x))$, where $\alpha=\inf\{t>0:\phi(t)\geq s\}$ and $\alpha<\beta$. Moreover, since ϕ is left-continuous on $[0,\beta)$, $\mu(t;x)>\alpha$ if and only if $\phi(\mu(t;x))>s$. Therefore, in view of the condition (i),

$$d(s;\phi(x)) = \tau(e^x(\phi^{-1}(s,\infty))) = \tau(e^x(\alpha,\infty)) = m\{\in[0,\tau(\mathbf{1})) : \mu(t;x)>\alpha\} = \tau(s(x)),$$

and

$$\begin{aligned} d(s;\phi\circ\mu(x)) &= m\{t\in[0,\tau(\mathbf{1})) : \phi(\mu(t;x))>s\} = m\{\in[0,\tau(\mathbf{1})) : \mu(t;x)>\alpha\} \\ &= \tau(s(x)). \end{aligned}$$

We have shown that $d(s;\phi(x)) = d(s;\phi\circ\mu(x))$ for all $s \geqslant 0$. Now since $\phi\circ\mu(x)$ is decreasing and right-continuous on $[0,\infty)$, we get that $\mu(\phi(x)) = \phi\circ\mu(x)$ on $[0,\infty)$.

Note that $\phi^{-1}\{0\} = \{0\}$ or $\phi^{-1}\{0\} = [0,\gamma)$ or $\phi^{-1}\{0\} = [0,\gamma]$, for some $\gamma \leqslant \beta$. Moreover, we always have that $e^x(0,\gamma] = 0$, whenever $\gamma < \beta$. If in addition $\gamma = \beta$, that is $\phi(\beta) = 0$, then by assumption (ii) it follows that also $e^x(0,\gamma] = 0$. Thus in any case

$$n(\phi(x)) = e^{\phi(x)}\{0\} = e^x(\phi^{-1}\{0\}) = e^x\{0\} = n(x),$$

and so $s(\phi(x)) = s(x)$. Clearly $\phi(x) \geq 0$. □

Now we show that the exposed points of $B_{E(\mathcal{M},\tau)}$ can be characterized in terms of $\mu(x)$.

Theorem 6.3. *Let E be order continuous and $x \in S_{E(\mathcal{M},\tau)}$. If $\mu(x)$ is an exposed point of B_E then x is an exposed point of $B_{E(\mathcal{M},\tau)}$.*

Proof. Suppose that $\mu(x)$ is an exposed point of B_E. Then by Lemma 6.1 there exist a functional F that exposes B_E at $\mu(x)$ and $f \in S_{E^\times}$, such that $F(h) = \int h(t)f(t)dt$, $h \in E$, where $f = \mu(f)$, $\operatorname{supp} f = \operatorname{supp}\mu(x) = [0,\tau(s(x)))$ and f is constant on the intervals of constancy of $\mu(x)$.

Assume first that $x \geqslant 0$. Let $x = \int_{[0,\infty)} \mu(t;x)d\tilde{e}(t)$, where $\tilde{e}(B) = e^x(d(x)^{-1}(B))$ for any Borel set $B \subset [0,\infty)$, be a Schmidt representation of x [25]. Let the function $\phi : [0;\infty) \to [0,\infty]$ be given as $\phi(t) = (f\circ d(x))(t)$ for $t > 0$ and $\phi(0) = 0$. Define the operator y as

$$y = \phi(x) = \int_{(0,\infty)} \phi(t)de^x(t) = \int_{(0,\infty)} (f\circ d(x))(t)de^x(t) = \int_{d(x)(0,\infty)} f(t)d\tilde{e}(t).$$

The last equality follows from Theorem 1.22 [31, Chapter I].

It is standard to check that x and ϕ satisfy assumptions of Lemma 6.2. Hence, we conclude that $y \geqslant 0$, $s(y) = s(x)$ and

$$\mu(y) = \mu\left(\phi(x)\right) = \phi(\mu(x)) = f \circ d(x) \circ \mu(x) \text{ holds on } [0,\infty).$$

Let I_i, $i \in K \subset \mathbb{N}$, be the intervals of constancy of $\mu(x)$. That is $I_i = [a_i, b_i)$ or $I_i = [a_i, b_i]$, $i \in K$, and $\mu(t;x) = c_i$ for $t \in I_i$, where $c_i > 0$ for all $i \in K$ and $c_i \neq c_j$ whenever $i \neq j$, $i, j \in K$. Note that $(d(x) \circ \mu(x))(t) = d(\mu(t;x);x) = t$ for $t \in J := \operatorname{supp} \mu(x) \setminus \bigcup_{i \in K} I_i$. Consequently for $t \in J$, $\mu(t;y) = f(t)$. Now, if $t \in [a_i, b_i)$, for some $i \in K$, then $(d(x) \circ \mu(x))(t) = a_i$ and $\mu(t;y) = f(a_i) = f(t)$, since f is constant on each interval I_i. Therefore $\mu(t;y) = f(t)$ for all $t \in \operatorname{supp} \mu(x)$ except possibly at some b_i's. Since $\tau(s(x)) = \tau(s(y))$ we have that $\mu(t;y) = 0$ for all $t \geqslant \tau(s(x))$ when $\tau(s(x)) < \infty$. Also, in this case $f \circ d(x) \circ \mu(x)(t) = f(\tau(s(x))) = 0$ since $\operatorname{supp} f = \operatorname{supp} \mu(x)$. By right continuity of $\mu(y)$ and f we obtain that $\mu(y) = f$ on $[0,\infty)$. It follows that $y \in E^{\times}(\mathcal{M},\tau)$ and $\|y\|_{E^{\times}(\mathcal{M},\tau)} = 1$.

Recall that for any Borel set $B \subset (0,\infty)$, $\tau(e^x(B)) = m\{t > 0 : \mu(t;x) \in B\}$ [31]. Hence $\tau(e^x(B)) = 0$ whenever $B \cap \{\mu(s;x) : s \geqslant 0\} = \emptyset$, which gives $e^x(B) = 0$. In particular $e^x(\{\mu(s;x) : s \geqslant 0\}^c) = 0$. Since $t = (\mu(x) \circ d(x))(t)$ for all real $t \in \{\mu(s;x) : s \geqslant 0\}$,

$$\begin{aligned} xy &= \int_{(0,\infty)} t\phi(t)de^x(t) = \int_{(0,\infty)} (\mu(x) \circ d(x))(t)(f \circ d(x))(t)de^x(t) \\ &= \int_{d(x)(0,\infty)} \mu(t;x)f(t)d\tilde{e}(t). \end{aligned}$$

We claim that the functional Φ_y exposes $B_{E(\mathcal{M},\tau)}$ at x. Let $(\tau\tilde{e})(B) = \tau(\tilde{e}(B))$ for any Borel set $B \subset [0,\infty)$. Note first that by Lemma 3.6 [31, Chapter III], for any Borel set $B \subset (0,\infty)$ we have

$$(\tau\tilde{e})(B) = m\{t \in [0,\tau(\mathbf{1})) : \quad \mu(t;x) \in d(x)^{-1}(B)\}.$$

If $B \subset J$, since $d(\mu(t;x);x) = t$ for $t \in B$, we have that

$$(\tau\tilde{e})(B) = m\{t \in [0, \tau(\mathbf{1})) : \quad t \in B\} = mB,$$

and so $\tau\tilde{e}$ is a Lebesgue measure on J.

For $a_i > 0$, $i \in K$, we get

$$(\tau\tilde{e})\{a_i\} = m\{t \in [0, \tau(\mathbf{1})) : \ \mu(t;x) = c_i\} = mI_i.$$

Also, if $a_1 = 0$ then

$$(\tau\tilde{e})\{0\} = \tau(e^x(d(x)^{-1}\{0\})) = \tau(e^x(\mu(0;x), \infty)) + \tau(e^x\{\mu(0;x)\}) = mI_1.$$

If $B \subset (a_i, b_i)$, $i \in K$, or $B = \{b_i\}$ for $i \in K$ such that $I_i = [a_i, b_i]$, or $B \subset (\operatorname{supp}\mu(x))^c$, then $d(x)^{-1}(B) = \emptyset$ and hence

$$(\tau\tilde{e})(B) = m\{t \in [0, \tau(\mathbf{1})) : \quad \mu(t;x) \in \emptyset\} = 0.$$

We also have $d(x)(0,\infty)\setminus\{0\} = J \cup \bigcup_{i\in K}\{a_i\}\setminus\{0\}$. Note that $\tilde{e}\{0\} = 0$ whenever $\mu(0;x) = \lim_{t\to 0^+}\mu(t;x) = \infty$ or $f(0) = \lim_{t\to 0^+} f(t) = \infty$. The second case follows from the fact that $\mu(x)$ is not constant in any positive neighborhood of zero, since f

is constant on intervals of constancy of $\mu(x)$. Thus,

$$\begin{aligned}
\Phi_y(x) &= \tau(xy) = \tau\left(\int_{d(x)(0,\infty)} \mu(t;x)f(t)d\tilde{e}(t)\right) \\
&= \tau\left(\int_{\{0\}} \mu(t;x)f(t)d\tilde{e}(t) + \int_{d(x)(0,\infty)\setminus\{0\}} \mu(t;x)f(t)d\tilde{e}(t)\right) \\
&= \tau\left(\mu(0;x)f(0)\tilde{e}\{0\}\right) + \tau\left(\int_J \mu(t;x)f(t)d\tilde{e}(t)\right) \\
&+ \tau\left(\sum_{i\in K, a_i\neq 0} \int_{\{a_i\}} \mu(t;x)f(t)d\tilde{e}(t)\right) = \int_J \mu(t;x)f(t)d(\tau\tilde{e}) \\
&+ \mu(0;x)f(0)(\tau\tilde{e})\{0\} + \sum_{i\in K, a_i\neq 0} c_i f(a_i)(\tau\tilde{e})\{a_i\} \\
&= \int_J \mu(t;x)f(t)dt + \sum_{i\in K} c_i f(a_i) mI_i = \int_J \mu(t;x)f(t)dt \\
&+ \sum_{i\in K}\int_{I_i} \mu(t;x)f(t)dt = \int \mu(t;x)f(t)dt = F(\mu(x)) = 1.
\end{aligned}$$

Suppose next that $\Phi_y(z) = 1$, where $z \in B_{E(\mathcal{M},\tau)}$. We have

$$1 = \tau(zy) \leq \tau(|zy|) = \int \mu(t;zy)dt \leqslant \int \mu(t;z)\mu(t;y)dt \leqslant \|z\|_{E(\mathcal{M},\tau)}\|y\|_{E^\times(\mathcal{M},\tau)} \leqslant 1.$$

Hence $F(\mu(z)) = \int \mu(t;z)\mu(t;y)dt = 1$ and so $\mu(z) = \mu(x)$. Thus for any operator $z \in B_{E(\mathcal{M},\tau)}$, if $\Phi_y(z) = 1$ then $\mu(z) = \mu(x)$. By linearity of Φ_y, $\Phi_y((z+x)/2) = 1$ and hence $\mu((z+x)/2) = \mu(z) = \mu(x)$. Thus by Proposition 1.5(6) we obtain that $z = x$.

We have proven so far, that if $\mu(x)$ is an exposed point of B_E and $x \geqslant 0$ then x is an exposed point of $B_{E(\mathcal{M},\tau)}$. Thus if $\mu(|x|) = \mu(x)$ is an exposed point of B_E then $|x|$ is an exposed point of $B_{E(\mathcal{M},\tau)}$.

Suppose now that x is arbitrary and the functional $F(g) = \int g(t)f(t)dt$ exposes B_E at $\mu(x)$. Then by the above, there exists $y \in S_{E^\times(\mathcal{M},\tau)}$ such that $y \geqslant 0$, $\mu(y) = f$, $s(y) = s(x)$, and the functional Φ_y exposes $B_{E(\mathcal{M},\tau)}$ at $|x|$. Let $x = u\,|x|$ be the polar decomposition of x. Set $y_1 = yu^*$ and consider the functional Φ_{y_1}. Then, since

$\tau(xy) \leq \|x\|_{E(\mathcal{M},\tau)}\|y\|_{E^\times(\mathcal{M},\tau)} \leq 1$, $xyu^*, u^*xy \in L^1(\mathcal{M},\tau)$, and by Proposition 1.15 we have

$$\Phi_{y_1}(x) = \tau(xy_1) = \tau(xyu^*) = \tau(u^*xy) = \tau(|x|\,y) = \Phi_y(|x|) = 1.$$

Furthermore, observe that $\mu(y_1) = \mu(yu^*) \leq \mu(y) = \mu(ys(y)) = \mu(ys(x)) = \mu(yu^*u) \leq \mu(yu^*) = \mu(y_1)$, and so $\mu(y_1) = \mu(y) = f$.

We claim that Φ_{y_1} exposes $B_{E(\mathcal{M},\tau)}$ at x. To show it, suppose that $\Phi_{y_1}(z) = 1$ for some $z \in B_{E(\mathcal{M},\tau)}$. Applying again Proposition 1.15 to $u^*zy, zyu^* \in L^1(\mathcal{M},\tau)$, it follows that

$$1 = \Phi_{y_1}(z) = \tau(zy_1) = \tau(zyu^*) = \tau(u^*zy) = \Phi_y(u^*z).$$

Since Φ_y exposes $B_{E(\mathcal{M},\tau)}$ at $|x|$, $u^*z = |x|$. Moreover $s(x^*)z = uu^*z = u\,|x| = x$. Note also that

$$1 = \Phi_{y_1}(z) = \tau(zyu^*) \leq \tau(|zyu^*|) = \int \mu(t; zyu^*)dt \leqslant \int \mu(t;z)\mu(t;y)dt = F(\mu(z)) \leqslant 1.$$

Hence $F(\mu(z)) = 1$ and since F exposes B_E at $\mu(x)$, we must have that $\mu(z) = \mu(x)$. Thus $\mu(x) = \mu(z) = \mu(s(x^*)z + n(x^*)z) = \mu(x + n(x^*)z)$. It is not hard to observe that $|x + n(x^*)z| = |x - n(x^*)z|$ and therefore $\mu(x) = \mu(x \pm n(x^*)z)$ and $x \pm n(x^*)z \in S_{E(\mathcal{M},\tau)}$. Since every exposed point is an extreme point, $\mu(x)$ is an extreme point of B_E. By order continuity of E we also have that $\mu(\infty; x) = 0$. Therefore by Theorem 5.1, x is an extreme point of $B_{E(\mathcal{M},\tau)}$ and thus $n(x^*)z = 0$. Finally $z = n(x^*)z + s(x^*)z = s(x^*)z = x$, proving that x is an exposed point of $B_{E(\mathcal{M},\tau)}$. □

Lemma 6.4. *Let E be order continuous. If $x \in S_{E(\mathcal{M},\tau)}$ is an exposed point of $B_{E(\mathcal{M},\tau)}$ then $|x|$ is an exposed point of $B_{E(\mathcal{M},\tau)}$.*

Proof. Suppose that x is an exposed point of $B_{E(\mathcal{M},\tau)}$. Hence there exists a functional Φ_y, where $y \in S_{E^\times(\mathcal{M},\tau)}$, exposing $B_{E(\mathcal{M},\tau)}$ at x. Let $x = u\,|x|$ be the polar decomposition of x and set $y_1 = yu$. We claim that the functional Φ_{y_1} exposes $B_{E(\mathcal{M},\tau)}$ at $|x|$. Indeed, since $u\,|x|\,y, |x|\,yu \in L^1(\mathcal{M},\tau)$, applying Proposition 1.15 we get

$$\Phi_{y_1}(|x|) = \tau(|x|\,y_1) = \tau(|x|\,yu) = \tau(u\,|x|\,y) = \tau(xy) = \Phi_y(x) = 1.$$

Let now $z \in B_{E(\mathcal{M},\tau)}$ be such that $\Phi_{y_1}(z) = 1$. Hence

$$1 = \Phi_{y_1}(z) = \tau(zy_1) = \tau(zyu) = \tau(uzy) = \Phi_y(uz),$$

and since Φ_y exposes $B_{E(\mathcal{M},\tau)}$ at x, $uz = x$. Thus $s(x)z = u^*uz = u^*x = |x|$. We have now that $|x| + n(x)z = s(x)z + n(x)z = z \in B_{E(\mathcal{M},\tau)}$. Note that

$$||x| \pm n(x)z|^2 = (|x| \pm z^*n(x))(|x| \pm n(x)z) = |x|^2 + |n(x)z|^2,$$

and so $||x| + n(x)z| = ||x| - n(x)z|$. Consequently, $|x| \pm n(x)z \in B_{E(\mathcal{M},\tau)}$. Since x is an exposed point, it is an extreme point of $B_{E(\mathcal{M},\tau)}$. In view of E being order continuous, $\mu(\infty; x) = 0$ and by Theorem 5.1, $|x|$ is an extreme point. Thus $n(x)z = 0$ and $z = s(x)z + n(x)z = s(x)z = |x|$, proving that $|x|$ is an exposed point of $B_{E(\mathcal{M},\tau)}$ □

Lemma 6.5. *Let E be order continuous. If $x \in S_{E(\mathcal{M},\tau)}$ is an exposed point of $B_{E(\mathcal{M},\tau)}$ then there exists an operator $y \in S_{E^\times(\mathcal{M},\tau)}$ such that $s(y) = s(x^*)$ and the functional $\Phi_y(z) = \tau(zy)$, $z \in E(\mathcal{M},\tau)$, exposes $B_{E(\mathcal{M},\tau)}$ at x.*

Proof. Assume that $x \in S_{E(\mathcal{M},\tau)}$ is an exposed point of $B_{E(\mathcal{M},\tau)}$. Then there exists a functional Φ_{y_1}, where $y_1 \in S_{E^\times(\mathcal{M},\tau)}$, that exposes $B_{E(\mathcal{M},\tau)}$ at x.

Set $y = y_1 s(x^*)$. We will show that Φ_y also exposes $B_{E(\mathcal{M},\tau)}$ at x. Clearly, $s(x^*)x = x$ and by Proposition 1.15 applied to $s(x^*)xy_1, xy_1s(x^*) \in L^1(\mathcal{M},\tau)$, it follows that

$$\Phi_y(x) = \tau(xy) = \tau(xy_1 s(x^*)) = \tau(s(x^*)xy_1) = \tau(xy_1) = \Phi_{y_1}(x) = 1.$$

Let now $\Phi_y(z) = 1$, where $z \in B_{E(\mathcal{M},\tau)}$. Thus,

$$1 = \Phi_y(z) = \tau(zy) = \tau(zy_1 s(x^*)) = \tau(s(x^*)zy_1) = \Phi_{y_1}(s(x^*)z),$$

and since Φ_{y_1} exposes $B_{E(\mathcal{M},\tau)}$ at x, $s(x^*)z = x$. Hence $x + n(x^*)z = s(x^*)z + n(x^*)z = z \in B_{E(\mathcal{M},\tau)}$. It is easy to verify that $|x + n(x^*)z| = |x - n(x^*)z|$, which implies that $x \pm n(x^*)z \in B_{E(\mathcal{M},\tau)}$. Since every exposed point is an extreme point, x is an extreme point of $B_{E(\mathcal{M},\tau)}$, and so $n(x^*)z = 0$. Thus $z = s(x^*)z = x$, proving that Φ_y exposes $B_{E(\mathcal{M},\tau)}$ at x, where $y = y_1 s(x^*)$. Hence $s(y) \leq s(x^*)$. We claim that in fact $s(y) = s(x^*)$. Indeed,

$$1 = \Phi_y(x) = \tau(xy) = \tau(xys(y)) = \tau(s(y)xy) = \Phi_y(s(y)x),$$

which implies that $x = s(y)x$ and so $s(x^*) \leq s(y)$.

□

We give now a similar result proved in Lemma 6.1 for a function space E. For a positive exposed operator $x \in E(\mathcal{M},\tau)$ we find a functional in $E^*(\mathcal{M},\tau)$ which exposes x and possesses some desired properties.

Lemma 6.6. *Let E be order continuous. If $x \in S_{E^+(\mathcal{M},\tau)}$ is an exposed point of $B_{E(\mathcal{M},\tau)}$ then there exists an operator $y \in E^\times(\mathcal{M},\tau)$ such that $s(y) = s(x)$, $y \geqslant \mu(\infty; y)s(y)$ and the functional $\Phi_y(z) = \tau(zy)$, $z \in E(\mathcal{M},\tau)$, exposes $B_{E(\mathcal{M},\tau)}$ at x.*

Proof. Assume that $x \in S_{E^+(\mathcal{M},\tau)}$ is an exposed point of $B_{E(\mathcal{M},\tau)}$. By Lemma 6.5 there exists functional Φ_{y_1} that exposes $B_{E(\mathcal{M},\tau)}$ at x and $s(y_1) = s(x)$. We will show that $y = |y_1|$ satisfies our hypothesis. It is clear that $s(y) = s(y_1) = s(x)$. It remains to show that Φ_y exposes $B_{E(\mathcal{M},\tau)}$ at x, and that $|y| \geq \mu(\infty; y)s(y)$.

Note first that by Proposition 1.17, we have

$$1 = \Phi_{y_1}(x) = \tau(xy_1) \leq \tau(|x|\,|y_1^*|)^{\frac{1}{2}}\tau(|x^*|\,|y_1|)^{\frac{1}{2}} = \tau(x\,|y_1^*|)^{\frac{1}{2}}\tau(x\,|y_1|)^{\frac{1}{2}} \leq 1,$$

and consequently $\Phi_y(x) = \tau(x\,|y_1|) = 1$.

It remains to prove that $\Phi_y(z) \neq 1$ for all $z \in B_{E(\mathcal{M},\tau)} \setminus \{x\}$. To show it, suppose that $\Phi_y(z) = 1$, for some $z \in B_{E(\mathcal{M},\tau)}$. Let $y_1 = v\,|y_1|$ be the polar decomposition of y_1. Then, applying Proposition 1.15, it follows that

$$1 = \Phi_y(z) = \tau(z\,|y_1|) = \tau(|y_1|\,z) = \tau(v^* y_1 z) = \tau(y_1 z v^*) = \tau(z v^* y_1) = \Phi_{y_1}(zv^*),$$

and hence $zv^* = x$. Recall that x as an exposed point is an extreme point of $B_{E(\mathcal{M},\tau)}$. Consequently by Theorem 5.1, $\mu(x)$ is an extreme point of B_E. Consider the function $f = \mu(z) - \mu(x)$. Since $\mu(x) = \mu(zv^*) \leq \mu(z)$, $f \geqslant 0$ and $|\mu(x) \pm f| \leq \mu(x) + f = \mu(z) \in B_E$. Thus, $\mu(x) \pm f \in B_E$ and so $f = 0$ and $\mu(x) = \mu(z)$.

We have shown so far that if for $z \in B_{E(\mathcal{M},\tau)}$, $\Phi_y(z) = 1$ then $zv^* = x$ and $\mu(z) = \mu(x)$. Observe next, by applying Proposition 1.17, that

$$1 = \Phi_y(z) = \tau(z\,|y_1|) \leq \tau(|z|\,|y_1|)^{\frac{1}{2}}\tau(|z^*|\,|y_1|)^{\frac{1}{2}} \leq 1,$$

and since $0 \leqslant \tau(|z|\,|y_1|), \tau(|z^*|\,|y_1|) \leq 1$ we have that $\Phi_y(|z|) = \tau(|z|\,|y_1|) = 1$. It follows that $|z|\,v^* = x$ and in view of $s(x) = s(y)$, $|z|\,s(x) - zs(x) = |z|\,s(y) - zs(y) = |z|\,v^*v - zv^*v = (|z|\,v^* - zv^*)v = 0$ and $|z|\,s(x) = zs(x)$. Moreover, by linearity of Φ_y, $\Phi_y\left((|z| + x)/2\right) = 1$ and so $\mu\left((|z| + x)/2\right) = \mu(x) = \mu(|z|)$. Thus by Proposition 1.5(6), $|z| = x$ and $s(x) = s(|z|) = s(z)$. Hence $zn(x) = zn(z) = 0$ and $|z|\,n(x) = |z|\,n(z) = 0$, and the condition $|z|\,s(x) = zs(x)$ ensures that $|z| = z$

and consequently $x = z$. Thus the functional Φ_y exposes $B_{E(\mathcal{M},\tau)}$ at x. Finally, since $s(x) = s(y)$, where $x, y \geqslant 0$, Lemma 1.18 implies that $y \geqslant \mu(\infty; y)s(y)$. $\square$

Lemma 6.7. *Let E be order continuous. If $x \in S_{E(\mathcal{M},\tau)}$ is an exposed point of $B_{E(\mathcal{M},\tau)}$ then x_p is an exposed point of $B_{E(\mathcal{M}_p,\tau_p)}$, where $p = s(x) \vee s(x^*)$.*

Proof. If x is an exposed point of $B_{E(\mathcal{M},\tau)}$ then by Lemma 6.5 there exists an operator $y \in S_{E^\times(\mathcal{M},\tau)}$ with $s(y) = s(x^*)$ such that the functional Φ_y exposes the unit ball $B_{E(\mathcal{M},\tau)}$ at x.

Since $s(y) = s(x^*)$, $yp = y$ and then by Proposition 1.15, the following relation holds for all $z \in E(\mathcal{M}, \tau)$,

$$\tau(zy) = \tau(zyp) = \tau(pzyp) = \tau_p((zy)_p) = \tau_p(z_p y_p).$$

Therefore, it is easy to verify that the functional $\Psi_{y_p}(z') = \tau_p(z' y_p)$, $z' \in E(\mathcal{M}_p, \tau_p)$, exposes the unit ball $B_{E(\mathcal{M}_p,\tau_p)}$ at x_p. $\square$

Now we are ready to prove a counterpart of Theorem 6.3.

Theorem 6.8. *Let E be order continuous. If $x \in S_{E(\mathcal{M},\tau)}$ is an exposed point of $B_{E(\mathcal{M},\tau)}$ then $\mu(x)$ is an exposed point of B_E.*

Proof. As it was discussed before we can assume that the von Neumann algebra $\mathcal{M}$ is non-atomic (see Proposition 1.8).

Since $\mu(\infty; x) = 0$, setting $p = s(x) \vee s(x^*)$, by Lemma 1.12 the von Neumann algebra $\mathcal{M}_p$ has a σ-finite trace τ_p. Notice that $pxp = x$. Moreover by Lemma 6.7, x_p is an exposed point of $B_{E(\mathcal{M}_p,\tau_p)}$, where the singular value function μ^{τ_p} for x_p computed with respect to $(\mathcal{M}_p, \tau_p)$ satisfies the relation, $\mu^{\tau_p}(x_p) = \mu(pxp) = \mu(x)$. Therefore we may also assume that the trace τ is σ-finite.

By Lemma 6.4, if x is an exposed point then so is $|x|$. In view of $\mu(|x|) = \mu(x)$ we can assume without loss of generality that $x \geqslant 0$. It follows now by Lemma

6.6 that there exists a functional Φ_y exposing $B_{E(\mathcal{M},\tau)}$ at x, where $y \in S_{E^\times(\mathcal{M},\tau)}$, $y \geqslant \mu(\infty; y)s(y)$ and $s(x) = s(y)$.

Note that by Proposition 1.5(2), $\mu(y - \mu(\infty; y)s(y)) = \mu(y) - \mu(\infty; y)$. Thus $\mu(\infty; y - \mu(\infty; y)s(y)) = 0$. Moreover, by $y - \mu(\infty; y)s(y) \geq 0$ and $s(x)y = s(y)y = ys(y) = ys(x)$, it follows that

$$s(x)\left(y - \mu(\infty; y)s(y)\right) = \left(y - \mu(\infty; y)s(y)\right)s(x) = y - \mu(\infty; y)s(y).$$

Suppose that $\tau(s(x)) = \tau(\mathbf{1})$. Then by Proposition 1.14, applied to the operator $y - \mu(\infty; y)s(y)$, there exists a $*$-isomorphism V from $S\left([0, \tau(\mathbf{1})), m\right)$ into $s(x)S\left(\mathcal{M}, \tau\right)s(x)$ such that $V(\mu(y - \mu(\infty; y)s(y))) = y - \mu(\infty; y)s(y)$ and $\mu(V(f)) = \mu(f)$ for all $f \in S\left([0, \tau(\mathbf{1})), m\right)$. Hence,

$$\begin{aligned} V(\mu(y)) - \mu(\infty; y)s(y) &= V(\mu(y)) - \mu(\infty; y)V\left(\chi_{[0,\tau(\mathbf{1}))}\right) = V(\mu(y) - \mu(\infty; y)) \\ &= V\left(\mu\left(y - \mu(\infty; y)s(y)\right)\right) = y - \mu(\infty; y)s(y), \end{aligned}$$

and so $V(\mu(y)) = y$.

Suppose now that $\tau(s(x)) = \tau(s(y)) < \tau(\mathbf{1})$ and so $\mu(\infty; y) = 0$. Then by Proposition 1.11 there exists a $*$-isomorphism V from $S\left([0, \tau(\mathbf{1})), m\right)$ into $S\left(\mathcal{M}, \tau\right)$ such that $V(\mu(y)) = y$ and $\mu(V(f)) = \mu(f)$ for all $f \in S\left([0, \tau(\mathbf{1})), m\right)$.

Thus a $*$-isomorphism satisfying conditions above exists in either case, independent of the value of $\tau(s(x))$.

Define the functional $F \in E^*$ by setting $F(g) = \int g(t)\mu(t; y)dt$, for $g \in E$. We will show that F exposes B_E at $\mu(x)$. Clearly,

$$1 = \Phi_y(x) = \tau(xy) \leq \tau(|xy|) = \int \mu(t; xy)dt \leqslant \int \mu(t; x)\mu(t; y) = F(\mu(x)) \leq 1,$$

and hence $F(\mu(x)) = 1$.

Consequently, using the fact that V as a $*$-homomorphism is positive,

$$\Phi_y(V(\mu(x))) = \tau(V(\mu(x))y) = \tau(V(\mu(x)\mu(y))) = \|V(\mu(x)\mu(y))\|_{L^1(\mathcal{M},\tau)}$$
$$= \|\mu(x)\mu(y)\|_{L^1(0,\tau(\mathbf{1}))} = F(\mu(x)) = 1,$$

and so $V(\mu(x)) = x$.

Suppose now that $F(g) = 1$, for some $g \in B_E$. From the inequality

$$1 = F(g) = \int g(t)\mu(t;y)dt \leqslant \int |g|(t)\mu(t;y)dt \leqslant 1,$$

it follows that $F(|g|) = 1$.

Observe that since $|g|\mu(y) \geq 0$ and V is positive as a $*$-homomorphism, $V(|g|)y = V(|g|)V(\mu(y)) = V(|g|\mu(y)) \geq 0$. Thus

$$1 = F(|g|) = \int |g|(t)\mu(t;y)dt = \| |g|\mu(y)\|_{L^1(0,\tau(\mathbf{1}))} = \|V(|g|\mu(y))\|_{L^1(\mathcal{M},\tau)}$$
$$= \|V(|g|)y\|_{L^1(\mathcal{M},\tau)} = \tau(V(|g|)y) = \Phi_y(V(|g|)).$$

Functional Φ_y exposes $B_{E(\mathcal{M},\tau)}$ at x, and so $V(|g|) = x = V(\mu(x))$. Since V is one-to-one, $|g| = \mu(x)$.

Consider now a positive function $(|g| + g)/2$. Then $F((|g| + g)/2) = 1$ and by the argument above $V((|g| + g)/2) = x = V(\mu(x))$ and so $(|g| + g)/2 = \mu(x)$. Consequently, $g = |g|$ and hence $g = \mu(x)$.

□

Finally we state the main result of this section, which combines Theorems 6.3 and 6.8.

Theorem 6.9. *Let E be order continuous and $x \in S_{E(\mathcal{M},\tau)}$. Then x is an exposed point of $B_{E(\mathcal{M},\tau)}$ if and only if $\mu(x)$ is an exposed point of B_E.*

If we take the commutative von Neumann algebra $\mathcal{M} = L_\infty[0,\alpha)$, $0 < \alpha \leqslant \infty$, we obtain the corresponding result for the symmetric function spaces.

Theorem 6.10. *Let E be an order continuous function space on $[0,\alpha)$. Then function x is an exposed point of B_E if and only if its decreasing rearrangement $\mu(x)$ is an exposed point of B_E.*

6.2 Strongly exposed points

In this section we characterize strongly exposed points of the unit ball in $E(\mathcal{M},\tau)$ in relation to their singular value functions.

Our first result states that if $\mu(x)$ is a strongly exposed point, then x is also a strongly exposed point.

Theorem 6.11. *Let E be order continuous and $x \in S_{E(\mathcal{M},\tau)}$. If $\mu(x)$ is a strongly exposed point of B_E then x is a strongly exposed point of $B_{E(\mathcal{M},\tau)}$.*

Proof. Suppose that $\mu(x)$ is a strongly exposed point of B_E and let $F(g) = \int g(t)f(t)\,dt$, $g \in E$, be the functional that strongly exposes B_E at $\mu(x)$. By Theorem 6.3 we can find the functional $\Phi_y \in E^*(\mathcal{M},\tau)$ such that $y \in S_{E^\times(\mathcal{M},\tau)}$, $\mu(y) = f$, $\Phi_y(x) = 1$ and $\Phi_y(z) \neq 1$ for all $z \in B_{E(\mathcal{M},\tau)} \setminus \{x\}$.

Let $\Phi_y(x_n) \to 1$, $\{x_n\} \subset B_{E(\mathcal{M},\tau)}$. Then

$$1 \leftarrow \tau(x_n y) \leq \tau(|x_n y|) = \int \mu(t;x_n y)dt \leqslant \int \mu(t;x_n)\mu(t;y)dt = F(\mu(x_n)) \leq 1,$$

and so $F(\mu(x_n)) \to 1$. Since F strongly exposes B_E at $\mu(x)$, $\|\mu(x) - \mu(x_n)\|_E \to 0$. Note that $\Phi_y\left((x_n + x)/2)\right) = \Phi_y(x_n)/2 + \Phi_y(x)/2 = \Phi_y(x_n)/2 + 1/2 \to 1$. Thus repeating the same argument as above, $\|\mu\left((x_n + x)/2\right) - \mu(x)\|_E \to 0$. Now by Lemma 1.9 and Proposition 2.3 it follows immediately that $\|x - x_n\|_{E(\mathcal{M},\tau)} \to 0$ and thus x is strongly exposed point of $B_{E(\mathcal{M},\tau)}$. □

In order to prove the converse result, we need some preparatory lemmas.

Lemma 6.12. *Let E be order continuous. If $x \in S_{E(\mathcal{M},\tau)}$ is a strongly exposed point of $B_{E(\mathcal{M},\tau)}$ then $|x|$ is also a strongly exposed point of $B_{E(\mathcal{M},\tau)}$.*

Proof. Suppose that x is a strongly exposed point of $B_{E(\mathcal{M},\tau)}$ and let $x = u\,|x|$ be the polar decomposition of x. Let Φ_y, for some $y \in S_{E^\times(\mathcal{M},\tau)}$, be a functional which strongly exposes $B_{E(\mathcal{M},\tau)}$ at x. Setting $y_1 = yu$, by Lemma 6.4 and its proof we get that Φ_{y_1} exposes $B_{E(\mathcal{M},\tau)}$ at $|x|$. Assume now that $\Phi_{y_1}(x_n) \to 1$, where $\{x_n\} \subset B_{E(\mathcal{M},\tau)}$. Then, since $ux_ny, x_nyu \in L^1(\mathcal{M},\tau)$, by Proposition 1.15,

$$\Phi_y(ux_n) = \tau(ux_ny) = \tau(x_nyu) = \Phi_{y_1}(x_n) \to 1.$$

By applying the assumption that x is strongly exposed, it follows that $\|x - ux_n\|_{E(\mathcal{M},\tau)} \to 0$. Thus $\|\,|x| - s(x)x_n\|_{E(\mathcal{M},\tau)} = \|u^*x - u^*ux_n\|_{E(\mathcal{M},\tau)} \le \|x - ux_n\|_{E(\mathcal{M},\tau)} \to 0$. Also

$$\begin{aligned}\|\,|x| + n(x)x_n\|_{E(\mathcal{M},\tau)} &= \|\,|x| - s(x)x_n + x_n\|_{E(\mathcal{M},\tau)} \le \|\,|x| - s(x)x_n\|_{E(\mathcal{M},\tau)} \\ &\quad + \|x_n\|_{E(\mathcal{M},\tau)} \le \|\,|x| - s(x)x_n\|_{E(\mathcal{M},\tau)} + 1,\end{aligned}$$

which implies that $\overline{\lim}_n\|\,|x| + n(x)x_n\|_{E(\mathcal{M},\tau)} \le 1$. It can be easily observed that $|\,|x| + n(x)x_n| = ||x| - n(x)x_n|$, and hence $\overline{\lim}_n\|\,|x| \pm n(x)x_n\|_{E(\mathcal{M},\tau)} \le 1$. Now by Lemma 1.20 it follows that $\lim_n \|\,|x| \pm n(x)x_n\|_{E(\mathcal{M},\tau)} = 1$. Since x is strongly exposed it is a strongly extreme point of $B_{E(\mathcal{M},\tau)}$. By order continuity of E, $\mu(\infty; x) = 0$ and therefore Corollary 2.7 implies that $|x|$ is also a strongly extreme point of $B_{E(\mathcal{M},\tau)}$, and so $\|n(x)x_n\|_{E(\mathcal{M},\tau)} \to 0$. Consequently,

$$\|\,|x| - x_n\|_{E(\mathcal{M},\tau)} \le \|\,|x| - s(x)x_n\|_{E(\mathcal{M},\tau)} + \|n(x)x_n\|_{E(\mathcal{M},\tau)} \to 0,$$

proving that $|x|$ is a strongly exposed point of $B_{E(\mathcal{M},\tau)}$. □

The next three lemmas are analogies to Lemmas 6.5, 6.6 and 6.7 for exposed points. They characterize functionals strongly exposing elements of the unit sphere of $E(\mathcal{M},\tau)$.

Lemma 6.13. *Let E be order continuous. If $x \in S_{E(\mathcal{M},\tau)}$ is a strongly exposed point of $B_{E(\mathcal{M},\tau)}$ then there exists an operator $y \in E^\times(\mathcal{M},\tau)$ such that $s(y) = s(x^*)$ and the functional $\Phi_y(z) = \tau(zy)$, $z \in E(\mathcal{M},\tau)$, strongly exposes $B_{E(\mathcal{M},\tau)}$ at x.*

Proof. Let $x \in S_{E(\mathcal{M},\tau)}$ be a strongly exposed point of $B_{E(\mathcal{M},\tau)}$ and Φ_{y_1} be a functional that strongly exposes the unit ball at x. Setting $y = y_1 s(x^*)$, by Lemma 6.5 and its proof, the functional Φ_y exposes the unit ball $B_{E(\mathcal{M},\tau)}$ at x and $s(y) = s(x^*)$.

To show that it strongly exposes $B_{E(\mathcal{M},\tau)}$ at x, let $\{x_n\} \subset B_{E(\mathcal{M},\tau)}$ be such that $\Phi_y(x_n) \to 1$. Observe that

$$\Phi_{y_1}(s(x^*)x_n) = \tau(s(x^*)x_n y_1) = \tau(x_n y_1 s(x^*)) = \tau(x_n y) = \Phi_y(x_n) \to 1,$$

which, in view of the fact that Φ_{y_1} strongly exposes $B_{E(\mathcal{M},\tau)}$ at x, implies that $\|x - s(x^*)x_n\|_{E(\mathcal{M},\tau)} \to 0$. Since $|x + n(x^*)x_n| = |x - n(x^*)x_n|$, we have that

$$\|x - n(x^*)x_n\|_{E(\mathcal{M},\tau)} = \|x + n(x^*)x_n\|_{E(\mathcal{M},\tau)} \leq \|x - s(x^*)x_n\|_{E(\mathcal{M},\tau)} + 1.$$

Thus $\overline{\lim}_n \|x \pm n(x^*)x_n\|_{E(\mathcal{M},\tau)} \leq 1$ and by Lemma 1.20, $\lim_n \|x \pm n(x^*)x_n\|_{E(\mathcal{M},\tau)} = 1$. Since x is a strongly extreme point of $B_{E(\mathcal{M},\tau)}$, $\|n(x^*)x_n\|_{E(\mathcal{M},\tau)} \to 0$. Consequently,

$$\|x - x_n\|_{E(\mathcal{M},\tau)} \leq \|x - s(x^*)x_n\|_{E(\mathcal{M},\tau)} + \|n(x^*)x_n\|_{E(\mathcal{M},\tau)} \to 0,$$

proving that Φ_y strongly exposes $B_{E(\mathcal{M},\tau)}$ at x. □

Lemma 6.14. *Let E be order continuous. If $x \in S_{E^+(\mathcal{M},\tau)}$ is a strongly exposed point of $B_{E(\mathcal{M},\tau)}$ then there exists an operator $y \in E^\times(\mathcal{M},\tau)$ such that $y \geqslant \mu(\infty; y)s(y)$,*

$s(y) = s(x)$ and the functional $\Phi_y(z) = \tau(zy)$, $z \in E(\mathcal{M}, \tau)$, strongly exposes $B_{E(\mathcal{M},\tau)}$ at x.

Proof. Let $x \in S_{E^+(\mathcal{M},\tau)}$ be a strongly exposed point of $B_{E(\mathcal{M},\tau)}$. Then by Lemma 6.13 there exists a functional Φ_{y_1} strongly exposing $B_{E(\mathcal{M},\tau)}$ at x and such that $s(y_1) = s(x)$. Set $y = |y_1|$. We will show next that the functional Φ_y strongly exposes $B_{E(\mathcal{M},\tau)}$ at x. By Lemma 6.6 and its proof, Φ_y exposes $B_{E(\mathcal{M},\tau)}$ at x and $y \geqslant \mu(\infty; y)s(y)$. Suppose that $\Phi_y(x_n) \to 1$, for $\{x_n\} \subset B_{E(\mathcal{M},\tau)}$. Let $y_1 = v\,|y_1|$ be the polar decomposition of y_1. Then

$$\Phi_{y_1}(x_n v^*) = \tau(x_n v^* y_1) = \tau(x_n\,|y_1|) = \Phi_y(x_n) \to 1,$$

and since Φ_{y_1} strongly exposes $B_{E(\mathcal{M},\tau)}$ at x, $\|x_n v^* - x\|_{E(\mathcal{M},\tau)} \to 0$.

Consider now the function sequence $f_n = \mu(x_n) - \mu(x_n v^*)$, $n \in \mathbb{N}$. Then, the relation $\mu(x_n v^*) \le \mu(x_n)$ implies that $f_n \geqslant 0$ and $\mu(x) \le \mu(x) + f_n$ for all $n \in \mathbb{N}$. Therefore, in view of $\mu(x) - \mu(x_n v^*) \prec \mu(x - x_n v^*)$, we have that

$$\begin{aligned} 1 = \|\mu(x)\|_E &\le \|\mu(x) + f_n\|_E = \|\mu(x) + \mu(x_n) - \mu(x_n v^*)\|_E \\ &\le \|\mu(x) - \mu(x_n v^*)\|_E + \|\mu(x_n)\|_E \le \|x - x_n v^*\|_{E(\mathcal{M},\tau)} + 1 \to 1. \end{aligned}$$

Consequently, $\|\mu(x) + f_n\|_E \to 1$. Note also that $|\mu(x) - f_n| \le \mu(x) + f_n$. Hence $\overline{\lim}_n \|\mu(x) - f_n\|_E \le 1$ and by Lemma 1.20, $\lim_n \|\mu(x) \pm f_n\|_E = 1$. Since x is a strongly extreme point of $B_{E(\mathcal{M},\tau)}$, by Theorem 2.7 in [11], $\mu(x)$ is a strongly extreme point of B_E. The latter implies that

$$\|\mu(x_n) - \mu(x_n v^*)\|_E = \|f_n\|_E \to 0.$$

Since also $\|\mu(x) - \mu(x_n v^*)\|_E \le \|x - x_n v^*\|_{E(\mathcal{M},\tau)} \to 0$, it follows that $\|\mu(x_n) - \mu(x)\|_E \to 0$.

We have shown that for any sequence $\{x_n\} \subset B_{E(\mathcal{M},\tau)}$, if $\Phi_y(x_n) \to 1$ then $\|\mu(x) - \mu(x_n)\|_E \to 0$. Thus in view of $\Phi_y((x + x_n)/2) \to 1$, we have that $\|\mu((x + x_n)/2) - \mu(x)\|_E \to 0$. Corollary 3.6 implies then that $\|x - x_n\|_{E(\mathcal{M},\tau)} \to 0$, proving that Φ_y strongly exposes $B_{E(\mathcal{M},\tau)}$ at x. □

Lemma 6.15. *Let E be order continuous. If $x \in S_{E(\mathcal{M},\tau)}$ is a strongly exposed point of $B_{E(\mathcal{M},\tau)}$ then x_p is a strongly exposed point of $B_{E(\mathcal{M}_p,\tau_p)}$, where $p = s(x) \vee s(x^*)$.*

Proof. By Lemma 6.13, if x is a strongly exposed point of $B_{E(\mathcal{M},\tau)}$, there exists a functional Φ_y that strongly exposes the unit ball at x, where $y \in S_{E^\times(\mathcal{M},\tau)}$ and $s(y) = s(x^*)$. As explained in the proof of Lemma 6.7, $\tau(zy) = \tau_p(z_p y_p)$ for all $z \in E(\mathcal{M},\tau)$, and the functional $\Psi_{y_p}(z') = \tau_p(z' y_p)$, $z' \in E(\mathcal{M}_p,\tau_p)$, exposes the unit ball $B_{E(\mathcal{M}_p,\tau_p)}$ at x_p.

Suppose that $\Psi_{y_p}(z_n') \to 1$, for the sequence $\{z_n'\} \subset B_{E(\mathcal{M}_p,\tau_p)}$. Hence for all $n \in \mathbb{N}$, $z_n' = (z_n)_p$, for some $z_n \in B_{E(\mathcal{M},\tau)}$ and

$$\Phi_y(z_n) = \tau(z_n y) = \tau_p((z_n)_p y_p) = \Psi_{y_p}(z_n') \to 1.$$

Since Φ_y strongly exposes the unit ball at x, $\|x - z_n\|_{E(\mathcal{M},\tau)} \to 0$, and therefore

$$\|x_p - (z_n)_p\|_{E(\mathcal{M}_p,\tau_p)} = \|\mu^{\tau_p}((x - z_n)_p)\|_E = \|\mu(p(x - z_n)p)\|_E \le \|x - z_n\|_{E(\mathcal{M},\tau)} \to 0.$$

□

We are ready now to state the following theorem.

Theorem 6.16. *Let E be order continuous. If $x \in S_{E(\mathcal{M},\tau)}$ is a strongly exposed point of $B_{E(\mathcal{M},\tau)}$ then $\mu(x)$ is a strongly exposed point of B_E.*

Proof. By Proposition 1.8 we assume without loss of generality that the von Neumann algebra $\mathcal{M}$ is non-atomic.

By Lemma 6.15, for $p = s(x) \vee s(x^*)$, x_p is a strongly exposed point of $B_{E(\mathcal{M}_p,\tau_p)}$, where the trace τ_p on $\mathcal{M}_p$ is σ-finite. Moreover, $\mu^{\tau_p}(x_p) = \mu(x)$. Therefore, we can also assume that the trace τ is σ-finite.

Suppose that x is a strongly exposed point of $B_{E(\mathcal{M},\tau)}$. Since by Lemma 6.12, $|x|$ is a strongly exposed point of $B_{E(\mathcal{M},\tau)}$ and $\mu(|x|) = \mu(x)$, we can further assume that $x \geqslant 0$. Then in view of Lemma 6.14, there exists the functional Φ_y which strongly exposes $B_{E(\mathcal{M},\tau)}$ at x, where $y \in S_{E^\times(\mathcal{M},\tau)}$, $y \geqslant \mu(\infty; y)s(y)$ and $s(y) = s(x)$.

Similarly as in the proof of Theorem 6.8 one can show that there exists a $*$-isomorphism V from $S([0,\tau(\mathbf{1})), m)$ into $S(\mathcal{M},\tau)$ such that $V(\mu(y)) = y$ and $\mu(V(f)) = \mu(f)$ for all $f \in S([0,\tau(\mathbf{1})), m)$. Moreover, if x is exposed, then $V(\mu(x)) = x$.

We claim that $F \in E^*$ defined as $F(g) = \int g(t)\mu(t;y)dt$, for $g \in E$, strongly exposes B_E at $\mu(x)$. In the proof of Theorem 6.8 we showed that F exposes B_E at x. Suppose now that $F(f_n) \to 1$, where $\{f_n\} \subset B_E$.

It is easy to observe that also $\lim_n F(|f_n|) = 1$. Indeed,

$$\begin{aligned}1 = \lim_n F(f_n) &= \lim_n \int f_n(t)\mu(t;y)dt \\ &\leq \underline{\lim}_n \int |f_n|(t)\mu(t;y)dt \leqslant \overline{\lim}_n \int |f_n|(t)\mu(t;y)dt \leqslant 1,\end{aligned}$$

and so $\lim_n F(|f_n|) = \lim_n \int |f_n|(t)\mu(t;y)dt = 1$. Since V is positive, we have that $V(|f_n|)y = V(|f_n|)V(\mu(y)) = V(|f_n|\mu(y)) \geq 0$ and

$$\begin{aligned}1 = \lim_n F(|f_n|) &= \lim_n \| |f_n|\mu(y)\|_{L^1(0,\tau(\mathbf{1}))} = \lim_n \|V(|f_n|\mu(y))\|_{L^1(\mathcal{M},\tau)} \\ &= \lim_n \|V(|f_n|)y\|_{L^1(\mathcal{M},\tau)} = \lim_n \tau(V(|f_n|)y) = \lim_n \Phi_y\left(V(|f_n|)\right).\end{aligned}$$

By the fact that Φ_y strongly exposes $B_{E(\mathcal{M},\tau)}$ at x and $V(\mu(x)) = x$,

$$\lim_n \| \, |f_n| - \mu(x) \|_E = \lim_n \|V(|f_n|) - x\|_{E(\mathcal{M},\tau)} = 0.$$

In view of $(|f_n| + f_n)/2 \geqslant 0$ and $F\left((|f_n| + f_n)/2\right) \to 1$, applying the similar argument as above, $\Phi_y\left(V\left((|f_n| + f_n)/2\right)\right) \to 1$, and $\|V\left((|f_n| + f_n)/2\right) - x\|_{E(\mathcal{M},\tau)} \to 0$. Thus

$$\begin{aligned} \|f_n - |f_n|\,\|_E/2 = \|(|f_n| + f_n)/2 - |f_n|\,\|_E &= \|V\left((|f_n| + f_n)/2\right) - V(|f_n|)\|_{E(\mathcal{M},\tau)} \\ &\leq \|V\left((|f_n| + f_n)/2\right) - x\|_{E(\mathcal{M},\tau)} + \|x - V(|f_n|)\|_{E(\mathcal{M},\tau)} \to 0, \end{aligned}$$

and so $\|f_n - \mu(x)\|_E \to 0$, which completes the proof.

□

The final result follows from Theorems 6.11 and 6.16.

Theorem 6.17. *Let E be order continuous and $x \in S_{E(\mathcal{M},\tau)}$. An operator x is a strongly exposed point of $B_{E(\mathcal{M},\tau)}$ if and only if $\mu(x)$ is a strongly exposed point of B_E.*

The next result is an immediate consequence of the previous theorem, taking for $\mathcal{M}$ the commutative von Neumann algebra $L_\infty[0, \alpha)$.

Theorem 6.18. *Let E be an order continuous function space on $[0, \alpha)$. Then function x is a strongly exposed point of B_E if and only if its decreasing rearrangement $\mu(x)$ is a strongly exposed point of B_E.*

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