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A Survey of Computed-Based Simulation of Multi-Rigidbody Dynamics

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A SURVEY OF COMPUTED-BASED SIMULATION OF MULTI-RIGIDBODY DYNAMICS

by

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To all of you who contributed to make this a success, I say “medamoase paa”.

Abstract

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In real time physics simulation and visualization, we have a dilemma. Consumers want to experience realism as objects interact during simulation. Satisfying the user involves a trade-off between performance and accuracy on the part of the designer. The designer is aware of the dynamic nature of physics and so errs on the side of cautious representation of physical objects.

This project summarizes the mechanics and the algorithms necessary to simulate, optimize the arbitrary motion of multi-rigidbodies. This work is not intended to be a statical primer; rather it provides a sufficient basis from which to stake out more advanced methods of proving to the society the importance of multi-rigidbody formalisms in various engineering fields.

This work improves and develops methods for generalizing, simulating and rendering various objects. The result is the state of the art rendering and simulation of Moby which suits both in-game and story driven environment.

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LIST OF VARIABLE NOTATIONS

$\mathbf{m}$:	Mass of a body
$\mathbf{P}$:	Position of a body
$\mathbf{r}_{cm}$:	Position of center of mass
$\mathbf{v}_{cm}$:	Velocity of r_{cm}
$\mathbf{d}(t)$:	Distance w.r.t (time)
e :	coefficient of restitution
anc:	anchor point
f^c:	contact force
ω:	Angular velocity
t_{mc}:	time of maximum compression
W_a:	work done
f_w:	initial reference frame
cor:	correction
θ_{err}:	error associated with joint
p_a:	normal impulse
α :	angular acceleration
$\mathbf{Q}$:	The orientation matrix
$\mathbf{q}$:	The orientation as a quaternion vector
$\mathbf{p}$:	The linear momentum
$\mathbf{l}$:	The angular momentum w.r.t. to center of mass
$\mathbf{v}$:	The linear velocity
ω:	Angular velocity
$\mathbf{Y}(t)$:	The state function
$\mathbf{R}(t)$:	Rotation matrix
$\mathbf{I}$:	Inertial tensor
$\mathbf{f}$:	The force
$\mathbf{f}^e$:	The external force
$\mathbf{a}$:	The linear acceleration of the center of mass
$\mathbf{f}^c$:	contact force
$\mathbf{r}_i(t)$:	The position vector of $i'th$ element
$\mathbf{J}$:	The Jacobian matrix

1 Introduction to Rigid Bodies

A rigid body is a collection of masses of a fixed size in which deformation during simulation is negligible. Multi body dynamics models the dynamic behavior of interconnected rigid bodies, each of which may be subjected to series of physics and mathematically based complex calculations that we will talk about in the proceeding chapters.

The figure 1 below is a screenshot of a typical rigid body (robot) in action during the simulation.

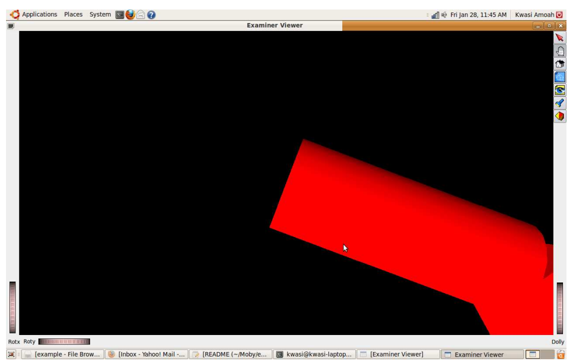

Figure 1: A figure showing a screenshot of the robot

Simulation Paradigms and Related Work

Current simulation methods for rigid and multi-body systems are classified in (Erleben, 2005) into several groups: penalty-based method, constraint-based method, impulse-based method and a hybrid method. Each method has its strength and limitation. The method we discuss is based on the impulse-based method which will be comprehensively discussed in the following chapters.

The penalty based paradigm is the most frequently used in real time simulation of multi bodies because of the ease at which it models rigid bodies and the simplicity at which it can be expanded to model soft bodies. In spite of its simplicity in implementation, it is possible to allow penetration among the rigid-bodies.

Constraint Based Simulation. There are several formulations (approaches) for which the constrained based method can apply; however, the most popular are the acceleration and the velocity approach, as used by Erleben's (2004) thesis. Interestingly his simulation is limited to his rigid-bodies. For instance the velocity based constrained method by Anitescu and Potra in their (1997) publication guarantees an acceptable output, yet it may allow penetration especially if the motion of the bodies are on concave boundaries.

Collision Synchronization. This method has adapted an effective numerical computational base a technique introduced in the thesis work of (Erleben, 2005), which computes the quadratic problems that normally occur during simulation. Employing this approach will give with precision an accurate positions of the objects in the simulation. Collision Synchronization is believed to adapt well in most simulators, making it a utility paradigm.

Hybrid. This is another utility paradigm. It selects the effective portions of all the paradigms, therefore it has many of the qualities of the previously mentioned simulators.

This thesis employs the impulse-based approach which will be discussed comprehensively in the rest of the chapters.

Overview

In addition to the believability issue on the part of the consumer, the designer also has to be able to devise an efficient algorithm, that will be able to solve the task associated with the simulation, since in the real world, an unlimited number of factors are known to interrupt the stability of the smooth motion or even the static objects. These factors include attraction forces, material constituents of the object in question, blowing wind, motion of other adjacent objects, etc.These interruptions in simulation parlance are sometimes termed external factors. Also as the rigid-bodies interact or get in contact with one another during simulation, there are abrupt changes in velocities due to collision induced forces which destabilize the rigid-bodies in question. These are the major tasks designers modeling a completely realistic physical world have to face.

This problem could be algorithmically stated: ”Given either a rigid-body or a multi rigid-body in an ideal simulation, how do designers stabilize the system to ensure a continuous simulation?”.

The main goal of this thesis is summarizing the research necessary to build a multi-rigid body dynamic simulation which includes computation, analyzation and optimization of the arbitrary motion of multi-rigidbody dynamics to improve and develop the characteristics of the computed-based simulation of multi-rigidbody, as we use the existing physical theorems to model (which includes formulation of equations to determine all the forces acting on the rigid-bodies as they get in contact with one another resulting in changes of states and apply these force through the joints that hold the rigid-bodies connected in terms of multi rigid-body).

Literature Review

The systematical modeling of linked rigid bodies has opened numerous important multi body techniques and explorations of significant theories in the areas of physics, mathematics, etc. For example, Newton's laws of motion speaks about elements of multi body dynamics, i.e., a free particle. Euler and Lagrange introduced reaction forces between bodies and formalisms based on minimal coordinates together with constraints on the bodies in question.

The motion of bodies are described by their kinematic behavior. The dynamics behavior of an object in motion has different forces acting on it at every point in time as it changes momentum. Modeling a single rigid body is not as tricky as modeling multi bodies, especially if the body in question is stationary.

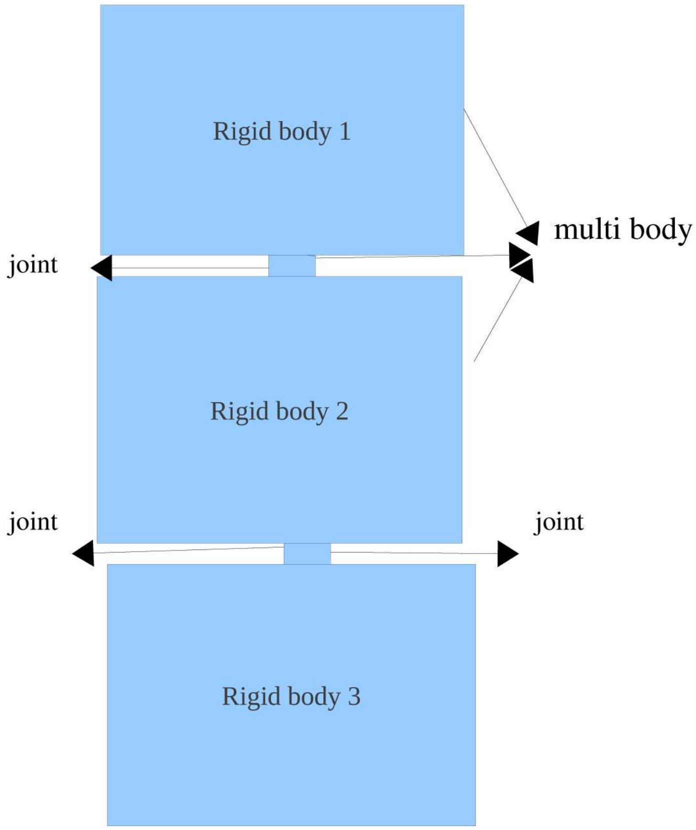

Figure 2: A typical rigid body joined to another to form a multibody

Calculating the position and other forces is easy and less painful, but the situation with multi bodies connected to each other, rotating with variable forces acting on the multi bodies varying with time, then the equation to specify the position and orientation at any given time becomes difficult.

This is difficult because although the mathematics and physics are well established, it is not easy to calculate on the computer and there is a constant demand to simulate more rigid bodies at faster rates. Since 1980 there have been papers on rigid body simulation in nearly every *ACM Siggraph Proceedings*, and subsequently multi body dynamics has ascended in the research domain leading to a category of numerous fields in the engineering sector of research, especially in military, robotics, and vehicle dynamics industries, to name but a few.

As an important feature, multi body dynamics simulations sometimes gives a pseudo, computer-aided approach to models that analyze, simulate and optimize the motion of multi bodies in a robust, versatile and stable manner. The demand for fast and real simulations is what has driven and opened avenues for computer gaming industry, likewise the video industry, not forgotten, the military and vehicle industry, to mention only a few.

This fits quite well with animation because we need to show or render the scene regularly with precision to give viewers the correct sense of motion. This thesis is a complete depiction of the state of the art in multi body dynamics.

Concepts

There are many options to choose from when it comes to multi body dynamic simulation; all the methods have their pros and cons, an implication of the fact that no one method suits all situation. The choice of a particular method is solely dependent on what the programmer wants to achieve. The choice ranges from less precision to high precision or simpler computations to highly complex computations respectively. We begin our simulation of multi body by considering two basic concepts needed to enable physical objects to be modeled by a computer.

1. *Kinematics.* Kinematics is the study of motion without taking into consideration its mass, position, orientation, angular velocity and acceleration (Magnus, 1978). Kinematics is inturn categorized into inverse and forward kinematics. For example,

we can calculate the end point of a chain being simulated (Haug, 1989) when given the angle of all of its joints. Likewise, when we are given the end point of any of its links we can calculate the angles of the specified joints.

2. *Dynamics.* Dynamics is the study of motion with consideration to the mass of the body as well as its rotation, forces, momentum, etc. Dynamics, is also subdivided into inverse and forward dynamics. For instance, with the forward dynamics the starting point of the motion of a given body is known and the end point must be predicted, unlike the inverse dynamics in which both the starting and the end points of the motion of a given body is given, in which we need to find a solution to get to the start from the end point or the vice-versa.

Organization of the chapters

Chapter 2: Generally focusses on the derivation of dynamic equations of an individual rigid-body from the classical mechanics point of view. The chapter considers all the necessary components of motion that could be experienced by any rigid-body in the Newtonian coordinates.

Chapter 3: We derive the equations of motion for the unconstrained and constrained multi rigid-body, starting with a brief review of the work done by earlier researchers in the dynamic simulation field. The introduction of constraints (holonomic) as bodies are connected together by joint leads us to the derivative of the Jacobian matrix. The chapter also talks about how joints are modeled to ensure stability of the rigid-bodies as they interact with one another during a simulation.

Chapter 4: This section describes the mathematical models of the contact kinematics of non-linear multi rigid-body in an ideal simulation, with special preference to the time-stepping scheme that combines the Coulomb friction laws as well as the Signorini contact conditions. And finally talks about how the Gauss-Seidel iteration approach is employed as an optimization method for contact resolution.

Chapter 5: Implementation of the algorithm is explained, the merits of our processes is discussed. We concluded this thesis with further discussions and future work suggestions.

Appendix A: Establish the classical mechanics equation $\mathbf{f} = m\mathbf{a}$.

2 Equations of motion of Rigidbody

Introduction to Rigid Body Dynamics

Rigid Body Dynamics:

A rigid body is a collection of masses of a fixed size in which deformation is nonexistent. I will remind readers that all motion of bodies are described relative to the world coordinates system (WCS). For convenience and identification purposes, vectors used here are represented with bold typeface letters, matrices are capitalized and bold typeface and scalars are normal typeface letters. Since we are dealing with three-dimensional $(3 - D)$ models, unless otherwise stated all matrices are in $m \times n$; $(\mathbb{R}^{m \times n})$.

The presentation given here, is similar to the theory used in (Barraff, 1990), Erleben (2005), and Mirtich (1996).

Unconstrained Rigid Body: Ideally, the state of a rigid body consist of its center of mass $x(t)$, orientation matrix $\mathbf{R}(\mathbf{t})$, linear momentum $\mathbf{p}(t)$, angular momentum $\mathbf{l}(t)$, $\mathbf{v}_{cm}$ is the velocity of the center of mass, from the position center of mass of $\mathbf{r}_{cm}$,and ω represents the angular velocity. The motion of a rigid body of mass m in a simulation, can be described by a state vector $X(t)$ where

$$X(t) = \left\{ \begin{array}{c} \mathbf{x}(t) \\ \mathbf{R}(t) \\ \mathbf{p}(t) \\ \mathbf{l}(t) \end{array} \right\}.$$

The differential of the state vector $d/dtX(t)$

$$= \left\{ \begin{array}{r} \mathbf{v}(t) \\ \omega(t)\ R(t) \\ \mathbf{f}(t) \\ \tau(t) \end{array} \right\}.$$

At any point in time, we can compute for

$$\mathbf{I} = \mathbf{R}(t)\mathbf{I}\mathbf{R}^T(t)$$

$$\omega(t) = (\mathbf{I}^{-1}(t)(\mathbf{L}(t))$$

$$\mathbf{v}(t) = \mathbf{p}(t)/M$$

$$\tau(t) = dl(t)/dt$$

$$\mathbf{l}(t) = \mathbf{I}(t)\omega(t)$$

The above equations sets and describes the motion of an unconstrained rigid-body in an ideal simualtion. Given the mass m_i of a rigid-body i, with an initial position x_0, velocity

$\mathbf{v}$, the corresponding initial tensor $\mathbf{I}_i$, and angular velocity ω_i. A displacement in the position of the body from x_0 to x_1 at time t_n resulting from an applied force $\mathbf{f}$. The displacement defined in classical mechanics, as $\mathbf{f}$ = m $\mathbf{a}$, and could be modified and written as $\mathbf{v}(t)$ = $\mathbf{p}/m$. Where $\mathbf{a}$ is the acceleration or displacement of the position of the body, with its derivative given as $\mathbf{v}(t)$ while $\mathbf{p}$ denotes the linear momentum. We know that displacement could entail rotation and translation (Neil, 2004), given as $\mathbf{v}(t)$ = $\mathbf{v}_{cm} + \omega_i \times \mathbf{r}_i$. Where $\mathbf{r}_i$ is a vector of the center of mass of the body and the ω_i being the angular velocity.

As the body moves under the influence of forces, it rotates. This we sometimes term the angular motion of the body. Here we seek to describe the rotation of the rigid-body under the influence of external forces by defining the three associated components; the angular momentum **l**, which is the easiest way, to describe the rotation of a rigd-body, the torque τ, is commonly used to calculate for the angular momentum when we know the angular velocity, just like the inertia tensor **I**, can easily be related to the rotational matrix $\mathbf{R}$. We reserve the last part of this chapter for the derivation and inter-relationship amonge the above mentioned parameters pertaining to the unconstrained rigid-body.

Alternatively, we can define the state of a rigid body as

$$S(t) = \left\{ \begin{array}{c} \mathbf{x}(t) \\ \mathbf{R}(t) \\ \mathbf{v}(t) \\ \omega(t) \end{array} \right\}$$

also differentiated as $d/dt(S(t))$ where

$$
= \left\{ \begin{array}{r} \mathbf{v}(t) \\ \omega(t)\ R(t) \\ \mathbf{f}(t)/m \\ \mathbf{I}^{-1}((t)(\mathbf{I}(t)\omega(t) \times (-\omega(t)) + \tau(t)) \end{array} \right\}.
$$

Newton's second law of motion is given as $\mathbf{f} = m\mathbf{a}$, where $\mathbf{f}$ is the force applied to a body with mass m and an acceleration $\mathbf{a}$. Similarly the 3rd law intuitively means $f_j body$ = $-f_k body$, thus the force applied by j onto k is consequently met with equal but opposite force from k onto j, assuming that j and k are two different bodies.

A rigid body has mass and can be described by its density or volume. The description of motion of a rigid body in the world coordinate system is easier using the affected body construct (frame).This frame accounts for the fixed constituents of the rigid body, with its origin at the center of mass of the body and also with corresponding axes to the angular motion of the body, which it governs. Thus, the state of the body is determined by the velocity and orientation of the body's construct in the world coordinate system (WCS). Let $\mathbf{R}$ be a rotational matrix that coordinates the rigid body from the body construct onto the world coordinate system. Any part (x_i) of the body (x) such that $x = (x_i, x_j, \ldots, x_z)$, in the body construct can be traced in the world coordinate system by moving and rotating (x_i) in relation to position center of mass ($\mathbf{r}_{cm}$). Thus

$$
x_{wcs}(t) = \mathbf{R}(t)(x_i) + \mathbf{r}_{cm}(t). \tag{1}
$$

The system can account for the change in velocity of any part (x_i) of a rigid body in motion at any given time by formulating

$$\mathbf{v}(x_i)(t) = \mathbf{v}_{cm}(t) + \omega(t) \times \mathbf{r}(x_i)(t) \tag{2}$$

given that $\mathbf{r}$ is a vector in the world coordinate system from the $\mathbf{r}_{cm}$ to x_{wcs}. This is the classical approach as stated by (Neil, 2004) to find the position and the velocities of any particle in a body, given $\mathbf{R}^{3\times3}$ rotational matrix for ω, $\mathbf{r}_{cm}$ and $\mathbf{v}_{cm}$.

Linear Motion of a Rigid Body

Recall from Newton's 2nd law of motion that $\mathbf{f} = m\mathbf{a}$. Let the force $\mathbf{f}$, be both internal and external forces. Note that the 3rd law of motion implies that $(\mathbf{f}_k = -\mathbf{f}_j)$, where $\mathbf{f}_k$ and $\mathbf{f}_j$ denotes equal but opposite forces, a situation which leads the existing internal force to be nullified. Assume we take the j^{th} particle in the body, then we can formulate Newton's 2nd law as $\mathbf{f}_j(t)$ = $m_j\mathbf{a}_j(t)$. We can also formulate the equation with the center of mass of the body as $\mathbf{f} = m\mathbf{r}_{cm}$. If we can find the linear momentum of a part x_i of a rigid body x given as

$$(x_i) = m\mathbf{v} \tag{3}$$

then as shown in (Neils, 2004), we can find the linear momentum of the entire rigid body using similar approach. Therefore we can deduce the linear momentum for a rigid body with respect to time as

$$\mathbf{p}(t) = m\mathbf{v}_{cm}(t). \tag{4}$$

The instantaneous change in linear momentum in relation to the second law of motion will be $dp/dt = \mathbf{f}(t)$.

Angular Motion of a Rigid Body

Let $\mathbf{l}(t)$ be the angular momentum and τ be the torque. We could say that the torque exerted on the particle is given as $\tau(t) = \mathbf{r}(t) \times \mathbf{f}(t)$, then

$$\mathbf{l}(t) = \mathbf{r}(t) \times \mathbf{p}(t) \tag{5}$$

Where $\mathbf{r}(t)$ is a vector of the position of a particle from a fixed point in the wcs, $\mathbf{p}(t)$ as the linear momentum of the particle in the wcs. By differentiating the angular momentum with respect to time, we establish a relationship between $\mathbf{l}(t)$ and τ as $dl/dt = dl/dt(\mathbf{r}(t)) \times \mathbf{p}(t) = \mathbf{p}(t)\mathbf{v}(t) + \mathbf{r}(t) \times \mathbf{f}(t)$ and expressed as $\mathbf{v}(t) \times mv + \mathbf{r}(t) \times \mathbf{f}(t)$ as the product of parallel vector equals zero, therefore the equation results in $\mathbf{r}(t) \times \mathbf{f}(t)$ = τ, as shown in (Neils, 2004).

Rotational Motion of a Rigid Body

We would deduce from the Newton's second law of motion by integrating $\mathbf{f} = m\mathbf{a}$ with respect to time, as a way of describing the rotational motion of a rigid body and its relationship with other components of the body in the wcs. If an applied external force on a rigid body results in the change of position at the center of the mass of the body with respect to time, depending on the force applied, various parts of the body will move.

As a body rotates on its axes the angular momentum given as $\mathbf{l}(t)$ is proportionally related to the angular velocity ω and the inertia tensor $\mathbf{I}$ leading to the formulation $\mathbf{l}(t) = \mathbf{I}\omega$. All things being equal, the angular momentum keeps the same magnitude and stays in the same direction. Should the rotation be influenced by an external torque leading to a change in momentum as a function of time, then the formulation could be given as $\tau = d\mathbf{l}/dt$. If τ and $\mathbf{l}(t)$ follow a path along a fixed axis, then $\mathbf{l}(t)$ and ω move along similar path, resulting in a change in magnitude of $\mathbf{l}(t)$, while the inertial tensor is constant. This will lead to a formulation given as $\tau = d\mathbf{l}/dt = \mathbf{I}d\omega/dt = \mathbf{I}\alpha(t)$. Note that α is the angular acceleration.

Similarly we could formulate a relationship between $\tau, \mathbf{l}(t), \mathbf{I}, \omega$, and $\alpha(t)$, when the body construct in the wcs is not rotating by letting $\tau_0(t) = d/dt[I_0(t)\omega_0(t)]$. On the other hand if τ acts contrary to the direction of $\mathbf{l}$, with time, $\mathbf{l}$ changes direction, resulting in a different formulation. Readers should reference from (Signell, 2001) for detailed proof of the above.

Inertial Tensor

Let $\mathbf{R}(t)$ be a rotation matrix coordinating vectors from the body construct to the corresponding vectors in wcs. Mass matrix describes how mass is distributed in the body with respect to the body's construct. The body construct originates at the body's center of mass and rotates with the body. From the above relation we realize that the inertia tensor, otherwise called mass matrix, is a function of the body's orientation matrix and must be recalculated whenever the body changes orientation. We could form an equation with this information such that $\mathbf{I} = R(t) I_{body} R(t)^T$. It is worth to note that $R^T R = 1$, since $R(t)$ is an orthogonal matrix and $\mathbf{I}(t)$ is a 3×3 matrix. We could rewrite the angular momentum as

$$\mathbf{l}(t)_x = \mathbf{I}_{xx}(t)\omega_x(t) + \mathbf{I}_{xy}(t)\omega_y(t) + I_{xz}(t)\omega_z(t) \tag{6}$$

$$\mathbf{l}(t)_x = \mathbf{I}(t)\omega(t). \tag{7}$$

Recall that $\mathbf{I}(t)$ is a 3×3 matrix

$$\mathbf{R}\tau(t) = d/dt(\mathbf{R}(t) IRT(t)\mathbf{R}(t)\omega(t)) \tag{8}$$

$$= (\mathbf{R}(t)\mathbf{I}\omega(t) + \mathbf{R}(t)\mathbf{I}\omega(t) \tag{9}$$

$$= (\mathbf{R}(t)I\omega(t) + \mathbf{R}(t)I\dot{\omega}(t), \tag{10}$$

where

$$\dot{\omega}(t) = \begin{pmatrix} 0 & -\omega_z(t) & -\omega_y(t) \\ \omega_z(t) & 0 & -\omega_y(t) \\ \omega_y(t) & -\omega_x(t) & 0 \end{pmatrix},$$

Multiplying both sides by $\mathbf{R}^T$ then making $\dot{\omega}$ the subject

$$\mathbf{R} = [\omega(t) \times i(t) | \omega(t) \times j(t) | \omega(t) \times k(t)] \tag{11}$$

$$= \omega(t)\mathbf{R}(t) \tag{12}$$

where the standard basis i, j, k are the units vectors in various coordinates of the frames that depicts the bodies during the simulation. We can compute $\omega(t)$. If D is a diagonal matrix which can be inverted (Mirtich, 1996), then similarly I can be inverted. Therefore,

$$\omega(t) = \mathbf{I}^{-1}(t)\mathbf{l}(t). \tag{13}$$

We have all the necessary derivatives that will enable us define the state of a rigid body as

$$S(t) = \left\{ \begin{array}{r} \mathbf{x}(t) \\ \mathbf{R}(t) \\ \mathbf{v}(t) \\ \omega(t) \end{array} \right\}$$

and the differential as introduced in (Neils, 2004) of $d/dt(S(t))$ is

$$= \left\{ \begin{array}{r} \mathbf{v}(t) \\ \omega(t)\ R(t) \\ \mathbf{f}(t)/m \\ \mathbf{I}^{-1}(t)(\mathbf{I}(t)(-\omega(t)) \times \omega(t) + \tau(t)) \end{array} \right\}$$

3 Equations of Multi Rigid Body

Dynamic Equations of Multi Rigid Bodies

Erleben's method for constraint equations of motion forms the basis for the formulation of the equations described in this section. An outline of this formulation is presented, however, considering the complexity of this method, we will give a summary in this thesis; thus some of the derivation are partially proven but thoroughly enough for interested readers to understand.

We formulate the dynamic equations of multi rigid-bodies by generalizing the coordinates and all motion variables of the rigid-bodies in the world coordinate system (wcs) into a system vector which is denoted $\mathbf{q}(t)$, and the associated components such as the positions and rotations of the affected rigid-bodies placed in a single generalized matrix $\mathbf{s}\epsilon\mathbb{R}^{4\times4}$. Similarly the velocity components such as the linear and angular velocities are placed in a single vector $\mathbf{u}\epsilon\mathbb{R}^{6n}$, and the force (f) components such as the torques and inertia are also placed in a vector $\mathbf{f}\epsilon\mathbb{R}^{6n}$.

The construction of a dynamical model for the system $\mathbf{Y}$ requires variables which will give the location of a reference point and orientation of a reference frame that is fixed for each body in $\mathbf{Y}$. Lagrange, as shown in Kane and Levinson (1985) calls these variables generalized coordinates (which indicates the location of reference frames defined for $\mathbf{Y}$ relative to one another) and denotes them as $q_i(i = 1, \ldots, n)$. Where i is the subscript for the i^{th} body and n is the degree of freedom (DOF) that a body can experience.

Since Lagrange's equations are second-order differential equations in the generalized coordinates, conversion of an equation to first-order differential equation calls for additional motion variables derivatives known as generalized velocities.

In general, both Lagrange and Kane's methods Kane et al.(1985) are related in ideas in the sense that the generalized coordinates used by Lagrange is termed as generalized speed by Kane. The system could be described analytically or numerically (differentially). For instance, the differential approach leads us into solving a set of differential equations, as follows:

$$d/dt(\mathbf{q}(t)) = \mathbf{f}(t, \mathbf{q}(t)) \tag{14}$$

where in the right hand side of the equation $\mathbf{f}(t, \mathbf{q}(t))$,is a derivation from the Newton's law of motion, $\mathbf{f}$ represents the net force applied on the rigid-bodies in the system while $\mathbf{q}$ is a component of motion. The above described dimensions are configured in a control vector $\mathbf{w}$ which leads to a formulation of the differential equation as

$$d/dt(\mathbf{q}(t)) = \mathbf{f}(t, \mathbf{q}(t), \mathbf{w}), \tag{15}$$

without ignoring their masses, we can define the linear and angular velocities of the affected bodies with respect to time as

$$d/dt(\mathbf{u}(t)) = \mathbf{M}(t)\mathbf{f}(t) \tag{16}$$

where $\mathbf{M}$ is the mass matrix of the rigid-bodies. Also from basic differentiation, we can formulate this

$$d/dt(\mathbf{p}(t)) = \mathbf{u}(t),$$

where $\mathbf{p}$ denotes positions of the rigid-bodies in the system, whereas, $\mathbf{u}(t)$ is the velocity time derivative for the computation of $\mathbf{p}$. Therefore

$$\mathbf{u} = (\mathbf{v}_1(t), \omega_1(t)\mathbf{R}_1(t), \mathbf{v}_2(t), \omega_2(t)\mathbf{R}_2(t) \ldots)^T \tag{17}$$

where $\mathbf{v}$ and ω defines the linear and angular velocities respectively. Integrating equation (15) yields

$$\mathbf{q}(t) = \mathbf{q}_0(\mathbf{w}) + \{\int_{t_0}^{t} \mathbf{f}(t, \mathbf{q}(t), \mathbf{w})dt\}. \tag{18}$$

As mentioned earlier in chapter one, the simulation system is described by the Newtion-Euler equations of motion through the positions and the velocities of the rigid-bodies. Consider the above situation in which we are given the state vector $\mathbf{q}(t)$ and the initialized positions of the rigid-bodies at $\mathbf{q}_0(\mathbf{w})$, a change in the intialized state of the positions resulting from changes in the control vector $\mathbf{w}$ (as forces, vis-a-vis velocities which are sub functions of the control vector) and the integration of the deduced function $\mathbf{f}(t, \mathbf{q}(t), \mathbf{w})$, which must be modified in order to enable an accurate determination of the next desirable positions of the rigid-bodies in the system. Manipulation of the above defined equation yields the general equation for an unconstrained multi rigid-body as

$$M(\mathbf{q}, t)\mathbf{u} = \varpi(\mathbf{q}, \mathbf{u}, t) \tag{19}$$

where $\mathbf{q}$ = $n \times 1$ is the matrix of the generalized coordinates of the system, $\mathbf{u} = n \times 1$ matrix represents the generalized velocities ($\dot{q}_1 \ldots \dot{q}_n$), with M and ϖ denotes $n \times n$ mass matrix, and $n \times 1$ vector consisting of inertia forces that are non-linear functions of the velocities. Examples of such forces include, the corilois, centrifugal, etc.

Remember in Chapter 1, we established a relationship between the mass and the inertia force. Based on this idea, we could formulate a generalized inverse mass matrix M as follows,

$$M = \begin{pmatrix} m_1^{-1} & 0 & 0 & 0 & 0 & 0 & 0\ldots \\ 0 & I_1^{-1} & 0 & 0 & 0 & 0 & 0\ldots \\ 0 & 0 & m_2^{-1} & 0 & 0 & 0 & 0\ldots \\ 0 & 0 & 0 & I_2^{-1} & 0 & 0 & 0\ldots \\ 0 & 0 & 0 & 0 & m_3^{-1} & 0 & 0\ldots \\ 0 & 0 & 0 & 0 & 0 & I_3^{-1} & 0\ldots \\ \vdots & \vdots & & & & & \end{pmatrix}.$$

In an ideal simulation we can derive the equation of motion of a multi rigid-body given, for instance, the i^{th} body of B rigid-body's mass as m_i, the inertia tensor $\mathbf{I}_i$, the position of the center of mass as $\mathbf{r}_i$, its velocity $\mathbf{v}_i$, with $\mathbf{q}_i$ and ω_i as the orientation in quaternion, and angular velocity repectively, τ^e as the total external torque, and $\mathbf{f}^e$

denoting the external forces acting on the center of mass of the affected bodies. The equation of motion for the body i starting from the initial position of the $i'th$ rigid-body (at the start of a time step in the simulation), as shown

$$\mathbf{r}_{i(0)}(t) = \mathbf{v}_{i(0)} = \omega_{i(0)}. \tag{20}$$

This means that the derivative of the position of the center of mass $\mathbf{r}_i$ could be equal to the velocity $\mathbf{v}_i$ of the $i'th$ rigid-body since no force has been applied to the bodies yet. In other words, the simulation has not been set into motion.

A change in the position of the $i'th$ rigid-body due to an applied force induces a motion of the rigid-body from point r_0 to point r_1, given as

$$\mathbf{v}_i(t+1) = (1/m_i\mathbf{f}_i + (\omega_i \times \mathbf{r}_i))_{r_1} - (1/m_i\mathbf{f}_i + (\omega_i \times \mathbf{r}_i))_{r_0}. \tag{21}$$

or

$$\mathbf{v}_i(t+1) = (\mathbf{v}_i(t) + \triangle t \quad \mathbf{f}_i(\mathbf{r}_{(t)} \times \mathbf{v}_{(t)}) \times m_i^{-1}$$

As the body moves, from its initial position r_0 to the next successive state r_1, it rotates thus termed angular velocity, which is given in consideration to its substantive rigid-body j as

$$\omega_i(t) = \mathbf{I}_i^{-1}(\mathbf{r}_i \times \mathbf{f}_i)_{r_1} - \mathbf{I}_i^{-1}(\mathbf{r}_i \times \mathbf{f}_i)_{r_0} \tag{22}$$

This event occurs within every step-time (h) usually termed an iteration of the simulation, remember the rigid-body i is connected by a joint to its substantive rigid-body

j to form a multi rigid-body (i, j). This connection, induces a general reaction force to sustain the continous motion of the affected rigid-bodies in the system. And it is this general induced force that leads to the introduction of Lagrange multiplier and the Jacobian which will be discuss in the subsequent pages.

The above provisional notations can be use to formulate the equations of motion of a multi rigid-body in the Newton-Euler form as

$$\dot{\mathbf{s}} = \mathbf{S}\mathbf{u} \tag{23}$$

$$\dot{\mathbf{u}} = \mathbf{M}^{-1}\mathbf{f} + \mathbf{f}^{e} \tag{24}$$

Where $\mathbf{s}$ is a generalized position and orientation vector, $\mathbf{u}$ is a generalized velocity vector and $\mathbf{S}$ denoted as angular (rotation) matrix, given as

$$\mathbf{S} = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0\ldots \\ 0 & Q_i & 0 & 0 & 0 & 0 & 0\ldots \\ 0 & 0 & I & 0 & 0 & 0 & 0\ldots \\ 0 & 0 & 0 & Q_j & 0 & 0 & 0\ldots \\ 0 & 0 & 0 & 0 & I & 0 & 0\ldots \\ 0 & 0 & 0 & 0 & 0 & Q_k & 0\ldots \\ \vdots & \vdots & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & Q_n \end{bmatrix}$$

$$\mathbf{Q}_i = \begin{bmatrix} -q_{i1} & -q_{i2} & -q_{i3} \\ -q_{i0} & q_{i3} & q_{i1} \\ -q_{i3} & q_{i0} & q_{i2} \\ q_{i2} & -q_{i1} & q_{i0} \end{bmatrix}$$

$\mathbf{Q}_i \epsilon \mathbb{R}^{4\times 3}$ rotational matrix of q_i, an orientation matrix which is described as quaternion and given as $q_i = [s_i, x_i, y_i, z_i] \epsilon \mathbb{R}^4$. Also $I \epsilon \mathbb{R}^{3\times 3}$ identity matrix, as introduced in (Erleben, 2005), these parameters together form the position update of the simulation. Consideration of a change in time from $t_{(0)}$ to $t_{(n)}$ could lead the above equations to be rewritten as

$$\mathbf{s}^{t+\triangle t} = \mathbf{s}^t + \triangle t \mathbf{S} \mathbf{u}^{t+\triangle t} \tag{25}$$

$$\mathbf{u}^{t+\triangle t} = \mathbf{u}^t + \triangle t \mathbf{M}^{-1} \mathbf{f}^{t+\triangle t} + \mathbf{f}^e. \tag{26}$$

The parameters of all the components form the general ideas behind an ideal simulation. The task ahead is to determine the forces associated with the motion of the bodies as they are being restricted by the joints that keep them together .

Joints

The rest of this section focuses on joints, with special references to holonomic and non-holonomic constraints, with examples on the hinge and the ball-in-socket joints. A joint is simply an area at which two ends of surfaces are attached. It is worth knowing that the joints are what keep the bodies in stable condition to enable proper functionality of the bodies which are connected.

Holonomic constraints: A holonomic constraint removes degree(s) of freedom from a system to which it is applied. Given a system of n bodies, if the system's constraints are bilateral at time (t), then the coordinates $(x_1, y_1, z_1, \ldots, x_n, y_n, z_n)$ points depicting the system in the world coordinate system must meet certain equations at time (t), such that

$$F_1(x_1, y_1, z_1 \ldots, x_n, y_n, z_n, t) = 0,$$

$$\vdots$$

$$F_m(x_1, y_1, z_1, \ldots x_n, y_n, z_n, t) = 0$$

written as

$$F_k(x_i, y_i, z_i, \ldots, x_n, y_n, z_n, t) = 0, \qquad (k = 1, 2, \ldots, m) \tag{27}$$

and in short as $\phi(s, t)$, if the constraints, are unilateral at the time (t). Then in addition to equation (27) the relations

$$\psi_s(x_1, y_1, z_1, \ldots x_w, y_w, z_w, t) \geq 0, \qquad (s = 1, 2, \ldots w) \tag{28}$$

or in brief $\psi(s, t)$ hold. If a system's constraint is represented by the above two equations (27) and (28), it is called holonomic. If the given function is independent of time, then the system is known as scleronomic, written as $\phi(s)$. Even, if just one of the constraints is time dependent, the system is termed as rheonomic. An example of a rheonomic constraint is

$$F_i(x_1, y_1, z_1 \ldots, z_n, t) = 0, \qquad (i = 1, 2, \ldots, m) \tag{29}$$

$$\psi_s(x_1, y_1, z_1 \ldots, z_n, t) \geq 0, \qquad (s = 1, 2, \ldots, r). \tag{30}$$

Generally constraint set limits on the motion of the bodies to which they may be related. A holonomic constraint applies force and torque to bodies, and is quite distinct from the external force. The constraint forces and torques have to be calculated and applied to the system which it is subjected.

A rigid body without limitations has six degrees of freedom. In the case of spatial motion, three of the DOF are meant for the position of the rigid-body it represents and the other three for the body's rotation in the world coordinates (x, y, z) axis. Two rigid-bodies will have 12 degrees of freedom; three rigid-bodies will have eighteen DOFs. This trend, continues in that pattern, but remember that the connected rigid bodies with joints will have some of the degrees of freedom removed.

The constraint-based method uses holonomic and non holonomic constraints forces and Jacobians, to stabilize its rigid-bodies during the simulation. Consider a system with n degrees of freedom, that is subjected to m holonomic constraint of the form $\delta(s, t) = 0$. Each holonomic constraint will remove one DOF, an implication of a reduction of the size of the n DOF written as n - m. We know from classical mechanics and differentiation that the derivative of position leads to velocity and the derivative of velocity gives acceleration, with these notions when given a vector $\mathbf{r}$ as the position of a rigid-body as depicted in equation (20), derived with respect to time, this is written as

$$dr/dt = (d(r_x)/dt, d(r_y)/dt, d(r_z)/dt) \tag{31}$$

which can briefly be written as

$$d(r)/dt = u(t) \tag{32}$$

recall from chapter one that

$$\mathbf{u} = [\mathbf{v}(x, y, z), \omega(x, y, z)]^T.$$

we could manipulate the above equations to obtain residual quantities of ω since $\mathbf{u} = (\mathbf{v}, \omega)$ as shown in equation (17). We use the integral residues of ω given as ϕ = (x, y, z) in order to determine the changes in the positions from the given vector $\mathbf{r}$ as $\mathbf{r}_i(\phi_i)$ of rigid-body i and $\mathbf{r}_j(\phi_j)$ of body j. In the Newtonian reference frame (wcs), with the existence of a common anchor point for various coordinates, ϕ is defined as

$$\phi_x(s) = x_i - x_j = 0$$

$$\phi_y(s) = y_i - y_j = 0$$

$$\phi_z(s) = z_i - z_j = 0$$

differentiating each joint into kinematic constraint as stated in (Erleben, 2005) as

$$\frac{d\phi(s)}{dt} = \frac{d\phi}{ds}\frac{ds}{dt} = \frac{d\phi}{ds}Su,$$

a second derivative being

$$\frac{d^2}{dt^2}s = \frac{d}{dt}(\mathbf{J}_\phi)u + \mathbf{J}_\phi\frac{d}{dt}(u)$$

therefore

$$\mathbf{J}_\phi\dot{u} + \dot{\mathbf{J}}_\phi u = 0 \qquad (33)$$

where $\frac{d\phi}{dt}(s(t)) = \mathbf{J}_\phi$. In this way, we have defined the Jacobian matrix $\mathbf{J}\phi\epsilon\mathbb{R}^{m\times 6}$, which describes how a rigid body with m number of holonomic constraints reduces the associated n number of degrees of freedom. The system's motion according to the Newton's law of motion is given as

$$\dot{\mathbf{u}} = M^{-1} \times \mathbf{f}^e$$

could be rewritten in relation to the above defined Jacobian as

$$\dot{\mathbf{u}} = \mathbf{M}^{-1}(\mathbf{J}^T\lambda + \mathbf{f}^e). \qquad (34)$$

Such that

$$\mathbf{J}\mathbf{u} + \hat{\mathbf{f}} = 0. \tag{35}$$

For readability we omit the symbol ϕ. Here the vector $\hat{\mathbf{f}}$ is the constraint force that is being imposed on the bodies in the system in a way that the acceleration condition will be maintained. An expansion of the formulated velocity equation above, equation (35) in consideration to equation (34), results in the form

$$\mathbf{J}(\mathbf{M}^{-1}\mathbf{J}^{T}\lambda + \mathbf{M}^{-1}\mathbf{f}^{e}) + \hat{\mathbf{f}} = 0 \tag{36}$$

which could be rewritten as

$$\mathbf{J}^{T}\lambda\mathbf{J}\mathbf{M}^{-1} = -(\mathbf{J}M^{-1}\mathbf{f}^{e} + \hat{\mathbf{f}}) \tag{37}$$

By principle of virtual work (Baraff, 1996), the permitted velocities are those which do not alter the constraint yet satisfy the formulation given below

$$\mathbf{J}\dot{\mathbf{u}} = 0 \tag{38}$$

where $\mathbf{u}$ denotes a velocity. An implication that the applied constraint force $\hat{\mathbf{f}}$ contribution is negligible, for all $\mathbf{J}\dot{\mathbf{u}} = 0$, in which all vectors $\hat{\mathbf{f}}$ that satisfy this condition could be formulated as

$$\mathbf{f} = \mathbf{J}^T \lambda \tag{39}$$

where $\lambda \epsilon \mathbb{R}^m$. λ is a vector of the constraints called the Lagrange multipliers that defines the forces of the constraints. The Lagrange multipliers, allow the computation of forces induced by the constraints in the system.

Based on the Newton-Euler equations, we can conclude that the generalized constraint force exerted by a holonomic constraint is $\mathbf{f}_\phi = \mathbf{J}_\phi^T \lambda_\phi$. with a few modifications to the above derived equations (19 and 26), we will be able to rewrite the equation of motion to include the known forces as

$$\dot{\mathbf{u}} = M^{-1}(\mathbf{J}_\phi^T \lambda_\phi + \mathbf{f}^e) \tag{40}$$

where $\mathbf{J}_\phi^T \lambda_\phi$ are the joint forces between the i and j rigid-bodies respectively and the $\mathbf{f}^e$ represents the external effects of forces such as the gravitational force exerted on the center of mass of the rigid bodies. We can formulate the acceleration of each body through Newton's first law of motion, since force is equivalent to mass of the rigid-body multiplied by the acceleration, written as $\mathbf{f} = m\mathbf{a}$ where $\mathbf{a}$ is the acceleration of the rigid-body, while m is the individual mass of each rigid-body. By modification of the second Newton's law of motion as $\mathbf{f} \times m^{-1} = \mathbf{a}$, we can write the acceleration in terms of the Jacobian and the Lagrange as introduced in (Barraf, 1994), as

$$\mathbf{a} = M^{-1}(\mathbf{J}_{\phi}^{T}\lambda_{\phi} + \mathbf{f}^{e}). \tag{41}$$

Non-holonomic: Given an inequality kinematic constraint, we say it is nonholonomic if its differential constraint cannot be integrated. There are many of these kinds that will be discussed in the next chapter under the contact formulation.

Joint Modeling

Recall that a joint is the connection (link) between two or more bodies or a body defined by a kinematical constraint, which restricts the relative motion of the bodies to which it connects. Therefore for a given joint type, the constraint could be written

$$\mathbf{J}\dot{\mathbf{u}} = 0.$$

The joint constraint is said to move at the same velocity as the bodies with which it is joined. This ensures precision and accuracy in the modeling of the joint. Different Jacobian matrices are then derived for each joint type within the system. For two rigid bodies (a and b), we will normally have

$$\mathbf{J}_a \qquad \begin{bmatrix} \mathbf{v}_a \\ \omega_a \end{bmatrix}$$

for the part of body a and similarly

$$\mathbf{J}_b \qquad \begin{bmatrix} \mathbf{v}_b \\ \\ \omega_b \end{bmatrix}$$

represents the part of the body b, where $\mathbf{v}_a$ and ω_b denote the linear velocity and angular velocity respectively, details of the derivation is shown in (Erleben, 2005).

Error

Usually errors may occur in the positions of the bodies, since the constraints are exerted on the velocities of the bodies in the simulation. The most common errors result from numerical approximations that leads into a twist or misalignment of the joint bearings which eventually destabilizes the bodies. We correct this kind of error by the method called stabilization approach. This approach basically increases or reduces the velocity of the joint bearing upon which the error occurred, in the simulation. The increment or reduction of the velocity is dependent on the magnitude of the constraint. Whichever situation that might confront the simulation, care must be taken not to introduce kinetic energy into the system during the process of correcting the error. Usually it is very real to increase the velocity of the joint bearing. When it comes to the classical approach, the velocity error rectification could be given as;

$$\mathbf{J}\dot{\mathbf{u}} = b$$

where b is the error correction term. Usually the correction term itself might experience some issues: sometimes kinetic energy is added into the system in an attempt to correct an error. When that occurs an error reduction parameter which governs the rate of rectification is used to bring stability into the system.

Connectivity

Fundamentally, the connected bodies could be described by a joint axis and anchor (anc) points. An anchor point is simply a point in the world coordinate system where two bodies meet, and are align. The related body (a) at that point in space could have a vector $\mathbf{r}^a$ and the position of the point in relation to the body is $\mathbf{r}_{anc} = \mathbf{r}^a + \mathbf{R}(q_a)\mathbf{r}^a{}_{anc}$, where $\mathbf{R}(\mathbf{q})$ is the rotation matrix of the quaternion $\mathbf{q}$ while the joint axis mainly describes the permitted direction of motion as shown in (Erleben, 2005), such as the axis of translation in a 3-dimensional vector.

Joint Types

Joint types are modeled by using a kinematic constraint formulation of the Jacobian matrices derived by differentiation of holonomic constraints. Different Jacobian matrices are then derived for each joint type. Let K_{cor} , θ_{err} represent vector of correcting term, and angle of error, $\mathbf{r}_{off}$ as off set vector so that $c - \mathbf{r}_{off}$ which represent error condition could be formulated, where $c = \mathbf{r}^b - \mathbf{r}^a$, as stated in (Erleben, 2004).

Ball-In-Socket

A ball-in-socket joint allows one part to rotate at almost any angle with respect to one another. It actually removes three (3) constraints, hence it is a three by twelve (3 x 12) matrix. The sub-matrix in relation with the Jacobian is

$\mathbf{J}_{ball} = [\mathbf{J}^{a}_{linear}, \mathbf{J}^{b}_{linear}, \mathbf{J}^{a}_{angular}, \mathbf{J}^{b}_{angular}]$,

$$\begin{bmatrix}\mathbf{J}^{a}_{linear}\end{bmatrix} = \begin{bmatrix}1, 0, 0 \\ 0, 1, 0 \\ 0, 0, 1\end{bmatrix}$$

$$\begin{bmatrix}\mathbf{J}^{b}_{linear}\end{bmatrix} = \begin{bmatrix}-1, 0, 0 \\ 0, -1, 0 \\ 0, 0, -1\end{bmatrix}$$

$$\mathbf{J}^{a}_{angular} = (-(\mathbf{R}(q_a) r^{a}_{anc}))^{x}$$

and

$$\mathbf{J}^{b}_{angular} = (\mathbf{R}(q_b) r^{b}_{anc})^{x}$$

and the velocity matrix given as

$$[\mathbf{J}^{a}_{linear}, \mathbf{J}^{a}_{angular}, \mathbf{J}^{b}_{linear}, \mathbf{J}^{b}_{angular}]$$

and the velocity error correction term

$$\mathbf{b}_{ball}$$

given as

$$\mathbf{b}_{ball} = k_{cor}(\mathbf{r}^{b} + \mathbf{R}(q_b)\mathbf{r}^{b}_{anc} - \mathbf{r}^{a} - \mathbf{R}(q)\mathbf{r}^{a}_{anc})$$

as depicted by, (Erleben, 2005).

Revolute Joint

There are five constraints in the Revolute joint thus allowing only one (1) DOF, therefore the Jacobian matrix is a (5×12) type of matrix. The procedure is similar to the ball-in-socket with two more rows added to enable the revolute joint to be modeled (Erleben, 2004). We do not want the relative angular velocity to be zero, considering the joint axis, meaning $\mathbf{s}(\omega_a - \omega_b) \neq 0$ (can not be zero), where $\mathbf{s}$ is a unit vector on a joint axis, s_a and s_b are vectors representation of body a and body b respectively. Assume the two bodies in the hinge joint have vectors t_1 , $t_2 \epsilon R^3$, and the bodies are perpendicular to the joint axis. Then we can use these two equations as the constraints for the hinge Jacobian: $t_1(\omega_a - \omega_b) = 0$, $t_2(\omega_a - \omega_b) = 0$

$$s_a = \mathbf{R}(q_a)s,\ s_b = \mathbf{R}(q_b)s.$$

Slide Joint

The slide joint is uni-directional in terms of rotation and therefore has 1 DOF given a (5×12) Jacobian matrix. The initial three rows prevent the bodies from running into one another while they maintain similar angular velocity, and the last rows are meant for the motion of the connected bodies along the joint axis. This mimics the same approach as the hinge joint to a formulation as

$$t_1(-v_{slider}) = 0, \quad (42)$$

$$t_2(-v_{slider}) = 0 \quad (43)$$

We can use the above equations to formulate the error condition given in (Erleben, 2005) as

$$\begin{bmatrix} \mathbf{b}_{slider} \end{bmatrix} = K_{cor} \begin{bmatrix} 2v \\ t_a(c - \mathbf{r}_{off}) \\ t_b(c - \mathbf{r}_{off}) \end{bmatrix}$$

4 Equations of Contacts

Contact Points

In rigid body simulation interaction among bodies are generated by modeling contact constraints and impact dynamics. The timely recognition of possible contacts which it impacts could lead to a successful response as far as collisions or unilateral contact are concerned. And with the experimental laws like the Signorini and the Coulomb friction laws employed (Lotstedt, 1982) and (Pang, and Trinkle, 1996), we are able to describe the physical behavior of the rigid bodies in question during a simulation.

Contact Model

This model shows the description of the unilateral contacts between rigid-bodies including the impact hypotheses, such as Newton's law of impact, which states that the relation of the normal velocity at the contact point after impact on a rigid-body to the same velocity before impact is equal to the coefficient of restitution. This can be stated in a simple form as

$$\mathbf{e} = -\mathbf{v}_a(t_f)/\mathbf{v}_a(_0),$$

where e is the coefficient of restitution and $\mathbf{v}_a$ is the normal relative velocity between $\mathbf{v}_a(t) \leq 0$, thus, when bodies are moving towards each other at time t, and to the contrary $\mathbf{v}_a(t) \geq 0$, where bodies are said to be separating from each other this was a technique used in Mirtich, (1996). Collision starts at time zero t_0 and ends at time (t_f). Other hypotheses described by the model are the Energy and the Poisson impulse.

The Poisson Impulse

The Poission impulse states that the normal component of impulse delivered during the restitution phase is e times the normal component of impulse given during the compression time. This statement is simplified as

$$\mathbf{p}_a(t_f) - (\mathbf{p}_a(t_{m_c})/\mathbf{p}_a(t_{m_c})) = \mathbf{e}$$

Here $\mathbf{p}_a$ is the normal impulse and t_{m_c} is the time of maximum compression during collision at which the normal velocity changes sign and also exhibits a phase change from a compression to restitution phase.

The Stronge hypothesis according to Mirtich, (1996), ties the coefficient of restitution to the energy released at the point of contact during the restitution period to the energy absorbed by the deformation at the compression time. This statement could briefly be defined as

$$(\mathbf{W}_a(t_f) - \mathbf{W}_a(t_{m_c}))/\mathbf{W}_a(t_{m_c}) = -\mathbf{e}^2$$

where $\mathbf{w}_a$ denotes work done during the restitutiton period. In effect, the contact model is not limited to the relationship of the geometry of the rigid-bodies alone, rather it gives descriptions of all the events that might occur during the contact phase (Mirtich, 1996).

Contact Condition

For rigid-bodies to be in contact, the closest points of the rigid-bodies in question should touch one another. Through the study of constraints, we know that contact restricts a rigid-body's degree of freedom. Generally, almost all the conditions establish the idea that, bodies in contact should not penetrate into one another. If we denote g as a gap and consider perfectly regular time steps, then the unilateral contact condition between rigid bodies could be formulated as; $g \geq 0$.

For the sake of completeness, we shall give a numerical model of the multi rigid body dynamics, characterised with unilateral contacts, as presented by (Moreau, 1999). This presentation bears resemblance to that of Wang et al. (2005) but varies in approach in some instances. To describe the kinematics of the contacting surfaces of the affected bodies, suppose body i and body j are in contact. Let frame i (f_i) be attached to body i and equally frame j, (f_j) be attached to body j, and also denote an inertia reference frame w (f_w) onto which the corresponding positions of the two bodies can be referenced after they have been rotated and translated. In the simulation, unless otherwise stated, we let the frames be located at the center of mass with their axes aligned with the principal axes of rotation, all together forming a coordinate system of the surfaces, where various configurations are represented.

Contact Coordinates

At the point of contact, let the function N: (u_i, v_i) and D :(u_j, v_j) describe the surface coordinates of the gemoetrical shapes in the reference frames of rigid-bodies i and j in relation to inertia frame w. N, is the vector othonormal to the contact point p with coordinates (x,y,z) in frame i at origin $N(w)$. Subsequent points could be traced on $N(p)$=$[N(x_i, y_i, z_i)]$. For ease and convenience we ignore the identifying notation i. Coordinate axes for body i could be expressed as

$$x_i = N, u/\|N, u\|$$

$$y_i = ((N, u{\times}N, u)N, v){-}((N, u{\times}N, v)N, u)/\|((N, u{\times}N, u)N, v){-}((N, u{\times}N, v)N, u)\|$$

$$z_i = N, u \times N, v/\|N, u \times N, v\|$$

in which the vectors x,y,z form the orthonomal axes provided $(N, u \times N, v \neq 0)$, Wang, and Liu, (2005). We define body j using similar approach $D\ (u_j, v_j)$.

Let $\mathbf{v}_i$ and $\mathbf{v}_{i+1}$ be initial and final velocities respectively, then we can formulate $p_i(\mathbf{v}_{i+1}, \mathbf{v}_i)$ and $p_j(v_{j+1}, v_j)$ as the parametric functional representation of both body i and body j respectively . Also let the gap between the nearest points of the two bodies relative to their coordinates in their dimensional frames be given as g, and the angle made on the y,x and z-axis in relationship to both bodies at the point of contact be denoted as Φ as shown in figure

We generalize the parameters of the two bodies by concatenating them into $\Theta = [\mathbf{v}_{i+1}, \mathbf{v}_i, \mathbf{v}_{j+1}, \mathbf{v}_j, g, \Phi]^T \epsilon \mathbb{R}^{12\times1}$.

Consider the two rigid bodies coming into contact with each body i and j has a known mass m, inertia tensor $\mathbf{J}$, angular velocity ω with r as the off-set of the contact point relative to body $i's$ center of mass, then the change in velocity v of body i at the

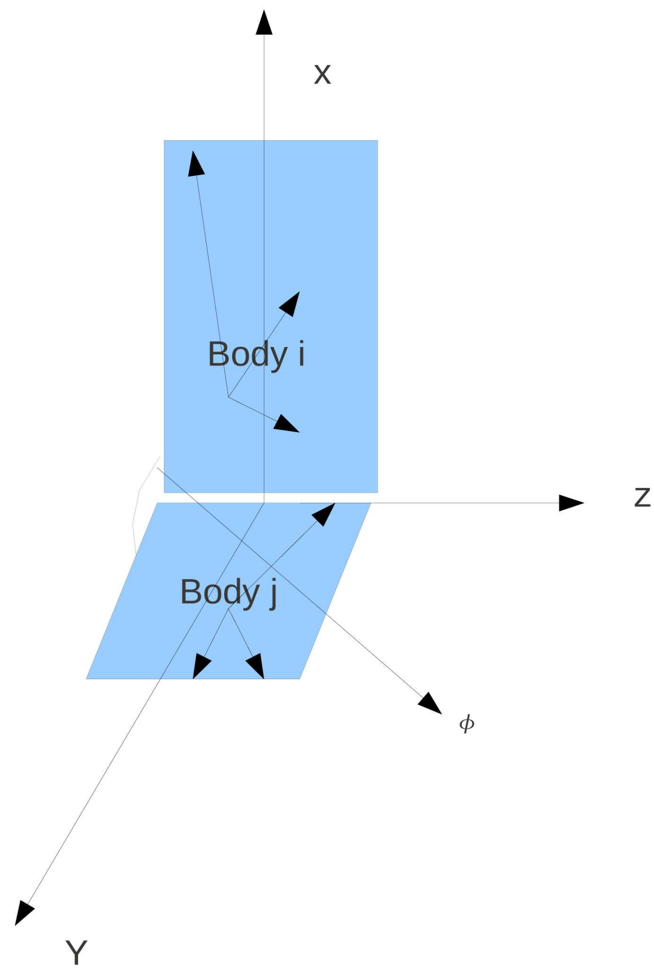

Figure 3: Typical rigid bodies i and j in contact at an angle ϕ on (x, y, z) axis

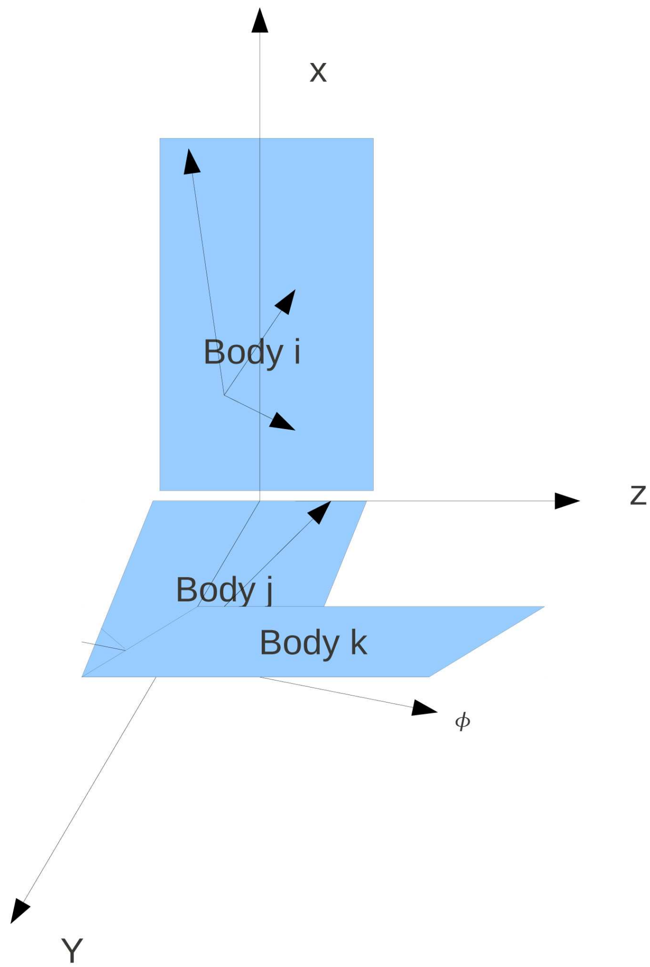

Figure 4: An angle ϕ made between rigid bodies j and k on (y and z) axis

contact point p_c is given as:

$$\triangle\ \mathbf{u}_i = \triangle\ \mathbf{v}_i + \triangle\ \omega_i \times \mathbf{r}_i. \tag{44}$$

and also for body j as

$$\triangle\ \mathbf{u}_j = \triangle\ \mathbf{v}_j + \triangle\ \omega_j \times \mathbf{r}_j.$$

We can formulate the relative contact velocity of both bodies as

$$\mathbf{u}_{i,j} = \triangle\ \mathbf{v}_j - \triangle\ \mathbf{v}_i \rightarrow v_{i,j} = A\dot{\Theta}, \tag{45}$$

where A is the geometrical properties of the rigid bodies. Given that the spatial velocity ($\Theta(j,i) = \omega, \mathbf{v}$) of a body is described by the motion of the body in f_i in respect to the body in f_j, subsequently we consider the change in velocity resulting from the correspondent change in force associated with body j given as

$$\triangle\ \mathbf{v}_j = \mathbf{P}/m_j$$

and the change in the angular velocity of body j also given as

$$\triangle\ \omega_j = \mathbf{J}^{-1} \times (\mathbf{r}_j \times \mathbf{f})$$

Newton's law stated by Mirtch, (1994), that the force applied to a body is equally met with similar magnitudinal reaction force. Based on this principle, the rigid-body i will exert negative reactional force onto body j as well as the angular velocity.

Adjoint Transformation

The spatial motion and wrench transform according to the adjoint transformation (Ad_x^y) as introduced in Moreau (1996). The superscript and subscript indicate the coordinates of three- dimensional vectors of frames i. From the above, we can write the $Ad_j^i \epsilon \mathbb{R}^{6\times6}$ as:

$$Ad_{E_j^i} = \begin{bmatrix} \theta & [p]\theta \\ 0 & \theta \end{bmatrix}$$

and

$$E_j^i = \begin{bmatrix} \theta & p \\ 0 & 1 \end{bmatrix}$$

where as p defines a 3×1 displacement, θ defines a 3×3 rotational matrix. These parameters suffice to formulate the equation of a contact velocity in relation to the velocity of body i and body j through adjoint transformation as:

$$\mathbf{v}_{ij}^j = -(Ad_{E_i^j}\mathbf{v}_{ji}^j)$$

in relation to the above definitions, with the notion that both bodies i and j are fixed in their frames

$$\mathbf{v}_{ij} = (Ad_E^{-1j}{}_i(\mathbf{v}_i j))$$

modified as

$$Ad_{E^i{}_j}\mathbf{v}^i{}_j = \mathbf{v}_{\omega i} - Ad^i_{Ej}\mathbf{v}_{\omega j}$$

And finally, rewrite the relative contact velocity through the adjoint transformation formulated by Tong and Michael (2005) as

$$\mathbf{v}_{j,i} = Ad_{E^j_i}(\mathbf{v}_{w_i} - Ad_{E^j_i}\mathbf{v}_{w_j}). \tag{46}$$

Contact Dynamics

Our goal in this section is to derive the equations of motion that can be used to describe the motion of rigid bodies in terms of their kinematic variables such as: velocities, displacements, time, and the forces that act on the rigid-bodies as they come in contact with one another.

Given the familiar classical equation of motion $\mathbf{f} = m\mathbf{a}$, where $\mathbf{a}$ denotes acceleration, $\mathbf{f}$ denotes force and m denotes mass of the rigid-body in question, also $\mathbf{f}^c$ denotes forces resulting from contacts, $\mathbf{f}^e$ as the external force exerted on the rigid-body, τ^e the external torque, and τ^c as contact torque, the equation of motion for a multi rigid-body with n DOF subjected to unilateral constraints can be written as

$$m(d\mathbf{v})/dt = \mathbf{f}^c + \mathbf{f}^e. \tag{47}$$

Recall from Chapter 1, where we defined angular momentum as $\mathbf{l}$ and angular velocity as ω. Let's consider these two parameters to be the coordinates of the rigid body's angular velocity and angular momentum in relation to the frame attached to the body i. Note that the coordinate vectors are related to each other through an inertia matrix $\mathbf{J}$, which gives us the opportunity to formulate a familiar equation $\mathbf{l} = \mathbf{J}\omega$ with $\mathbf{J}\epsilon\mathbb{R}^{3\times3}$. We know that matrices have to be transformed in order to operate on vectors located in various bases. Given the body's angular momentum and the angular velocity coordinate vectors in their distinct fixed frames ω_0 and l_0 , we can derive a relationship through the matrix $\mathbf{J}$ from the relation $\mathbf{J} = \mathbf{R}\mathbf{J}^{-1}\mathbf{R}^T$, where $\mathbf{R}$ denotes the rotation matrix that transfers vectors in the body frame to the coordinate in the fixed frame. With the above given parameters we can formulate the Newton-Euler equation as

$$d\mathbf{l}/dt = \tau^e + \mathbf{f}^c \times pc \tag{48}$$

where pc is the point (position) of contact with respect to centre of mass. Considering the variant nature of forces in regards to friction from rigid body contacts, equation (48) could be rewritten as

$$d\mathbf{l}/dt = \tau^e + \tau_c + \mathbf{f}^c \times pc. \tag{49}$$

similarly the angular momentum and angular velocity of the i body could also be formulated as:

$$l_i = (\tau_e + \tau_c + \mathbf{f}^c \times pc)_i \bigtriangleup t.$$

Therefore a change in linear momentum could be formulated as

$$(l_{i+1} - l_i) = (\tau_e + \tau_c + \mathbf{f}^c \times pc)_i \triangle t.$$

Remember in an ideal simulation with frictional forces, due to body contact, bodies are bound to experience the effects of τ_c andτ_e with .

$$\mathbf{l} = \mathbf{J}(\omega) \tag{50}$$

In regards to a time-step scheme $\triangle t = [t^{i+1} - t^i]$, we could formulate the change in velocity as

$$m(\mathbf{v}_{i+1} - \mathbf{v}_i) = (\mathbf{f}^c + \mathbf{f}^e)_i \triangle t \tag{51}$$

where $\mathbf{v}_{i+1}$ and $\mathbf{v}_i$ are the final and initial velocities of the rigid-body$i$. Therefore we make the final velocity the subject of the above equation, which yeilds

$$\mathbf{v}_{i+1} = (\mathbf{f}^c + \mathbf{f}^e)_i \triangle t \quad m^{-1} + \mathbf{v}_i \tag{52}$$

for a single contact. We know that velocity consist of rotation and translation from chapter one, remember we could derive angular momentum when given linear momentum $\mathbf{l}_i$ with the subscript i signifying the $i'th$ body. Given that

$\mathbf{J}$ could be written as $\mathbf{R}J^{-1}\mathbf{R}^T$, and

$$l_i = (\tau_e + \tau_c + \mathbf{f}^c \times pc)_i \triangle t. \tag{53}$$

Remember, in the velocity based-impulse simulation of rigid bodies, forces are calculated by time averaging impulses over a short interval, thus when the step-size is very small, we could work with the impulse forces. Subsequently a change in angular velocity could be derived as $\triangle \omega_i = \mathbf{J}^{-1} l$ can be written as

$\omega_{i+1} - \omega_i = \mathbf{J}^{-1} l$ and written in relation to equation (59) as:

$$\omega_{i+1} - \omega_i = (\tau_e + \tau_c + \mathbf{f}^c \times pc)_i \triangle t \tag{54}$$

$$\omega_{i+1} = \triangle t(\tau_e + \tau_c + \mathbf{f}^c \times pc)_i(\mathbf{R}J{-}1R^T) + \omega_i) \tag{55}$$

The task now is to determine the values for τ_c and $\mathbf{f}^c$. And an efficient approach is through adjoint transformation.

Transformation of Forces and Velocities through Adjoint Transformation (Ad)

Let us consider that the two rigid bodies (i, j) that are in contact are also known to have regular surface (smooth) functions. We can generalize the contact forces configuration as $f = [\mathbf{f}_c, \mathbf{f}_f, \mathbf{f}_n, \tau_m]^T$, which denotes contact force, frictional force, normal force as well as frictional moments respectively. Similarly, we concatenate the velocities into $\mathbf{v} = [\mathbf{v}_c, \mathbf{v}_f, \mathbf{v}_n, \omega_m]$. In rigid body simulation involving unilateral contacts, the transformation of parametric configurations and the computation of relative changes in contact velocities with respect to the reference frame f_w is as shown below:

$$v_{i,j} = Ad_{E_i} \begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix} \begin{bmatrix} \mathbf{v}_{i+1} \\ \omega_{i+1} \end{bmatrix} - Ad_{Ei} \quad Ad_{Ej} \begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix} \begin{bmatrix} v_{j+1} \\ \omega_{j+1} \end{bmatrix}$$

In view of the above equation, we can rewrite the relative contact velocity of rigid-body i and body j in consideration of all forces that may occur during the contact, as formulated in equation(47) as:

$$v_{i,j} = Ad_{E_i} \begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix} \begin{bmatrix} \triangle\, t \quad m^{-1}((\mathbf{f}^e) + \mathbf{f}^c)_i + \mathbf{v}_i \\ (\triangle\, t(\tau^e + \tau^c \\ +\mathbf{f}^c \times pc)_i \mathbf{J}) + (\omega_i) \end{bmatrix} - Ad_{Ei} Ad_{Ej} \begin{bmatrix} \theta & 0 \\ 0 & \theta \end{bmatrix}$$

$$\begin{bmatrix} \triangle\ t \quad m_j^{-1}((\mathbf{f}^e) + \mathbf{f}^c)_j + \mathbf{v}_j \\ (\triangle\ t(\tau^e) + \tau^c + \mathbf{f}^c \times pc)_j(\mathbf{J}) + (\omega_j) \end{bmatrix}.$$

Interested readers should refer to Murray, Li , and Sastry (1994), for the detailed proof of the relative contact velocity. The equation above is categorized into non-contact force (γ) variables and contact force(ρ) variables. Knowing the torques, contact forces and their magnitude, we can determine the velocity of a body and as such given an initial position of a rigid-body we can determine with high precision various successive positions at any given time. There are many possible cases of impact which depends on the rigid body in contact and with which body the said contact occured. It is fairly simple, for instance, when the bodies involved are just i and j with a single contact. It is different in an ideal simulation, where there are multiple rigid-bodies, it can be hard, since many deductions have to be made, thus exploring every possible contact in other to arrive at a stable solution for the resolution of the dynamic simulation.

Contact Constraints

Signorini laws: In this section, we present a new method of tackling a unilateral contact and its resultant frictional forces by employing the Signorini conditions and the Coulomb friction laws.

For simplicity we consider a system consisting of two colliding rigid bodies. Existence of a relative normal velocity at the contact area implies that there will be no penetration among the two rigid bodies which may be either separate from each other or stuck together after the contact.

A simple approach to deal with contact between the two rigid bodies is to build a complementarity problem between the separation distance $\mathbf{v}_n$ and the reaction force of magnitude $\mathbf{f}^c$. This is the condition which the Signorini law stipulated the normal contact velocity and the contact force must meet. For simplicity, the motion of body i and body j must be different in terms of direction $v_n \leq 0$. The force $\mathbf{f}^c$ exerted on body 1 by body 2 must be less than zero $\mathbf{f}^c \leq 0$ and the $\mathbf{v}_n\mathbf{f}^c = 0$. If the Signorini conditions are satisfied as stated in Wang et al. (2005) at at any given time (t) at which the force $\mathbf{f}^c = 0$ and the normal velocity $\mathbf{v}_n < 0$.

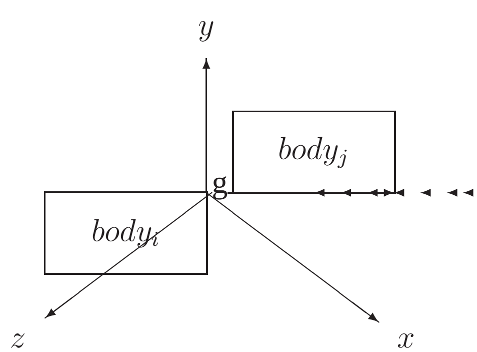

Figure 5: Two rigid-bodies rotating towards one another in a three dimensional-axis.

Coulomb's Friction law : Friction forces counter relative velocity in the tangential plane when there is only a single point of contact existing between two bodies. The frictional model is needed to determine if the bodies in collision will slip or stick.

The Coulomb friction law states that the tangential force is limited by the coefficient

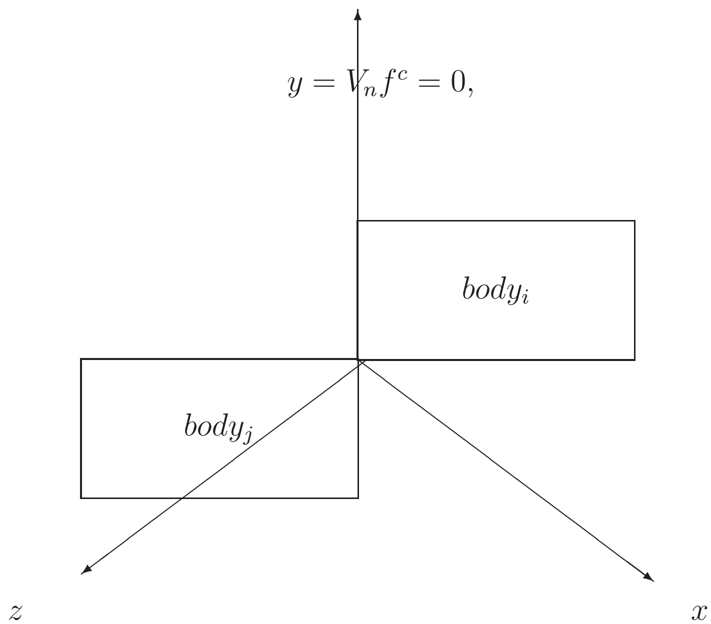

Figure 6: Two rigid-bodies in contact.

of the friction , multiplied by the normal force which is simply

$$\|\mathbf{f}_t^c\| \leq \mu \| f_n^c \| \tag{56}$$

where μ , $\mathbf{f}_t^c$ and $\mathbf{f}_n^c$ are said to be the coefficient of friction, tangential and the normal frictional components of force exerted on body j by body i respectively. μ is the constant of proportionality between the friction force and the normal force. Its value is determined by the constituents of the bodies in contact such as the adhesion existing between the contacting surfaces, microscopic topology as well as relative deformation of the bodies in contact as depicted in (Mirtich, 1996).

When the contact point velocity $\mathbf{V}_c = 0$ and remains as such, then the point(s) on the plane that satisfy the condition in the equation just mentioned stick to the surface, while points on the plane that do not satisfy the equation are said to separate or detach from the surface either by sliding $\mathbf{v}_{i+1} \neq 0$ or rolling along the surface as a result of indentations formed, resulting in one surface to roll on the other. We term point(s) that do not satisfy the equation as the dynamic friction force(s), thus $\mathbf{v} \neq 0$ (Mirtich, 1996), generally defined as

$$\| f_t^c \| = -\mu \mathbf{f}_n^c. \tag{57}$$

Multiple Contacts

Until now we have been discussing an individual contact between rigid-body j and rigid-body i as a single contact. We expand the formulation to n rigid-bodies with n contacts, where n denotes two or more contacts.

Multiple contacts can be incorporated into this frame work by including additional variables together with the generalized velocities and forces. The necessary data includes all corresponding scalar variables for locating the boundary of various contacts p_c. In addition, at each contact, the average normal force for the contact over the time step, the coefficient for the friction impulse for various contacts, and the matrices formed from the vector spanning the frictional force space at various contact be transformed to generalized coordinates are all needed. The general idea here is to couple the forces and the related data for the multiple contact of the rigid bodies simulation.

Similar to the individual contact, we start with the relative contact velocity at the k’th contact as

$$\triangle\ \mathbf{u}_k = \triangle\ \mathbf{u}_{jk} - \triangle\ \mathbf{u}_{ik}, \tag{58}$$

and subsequently the changes in the velocities resulting from multi contacts as

$$\triangle\ \mathbf{u}_{ik} = \triangle\ \mathbf{v}_{ik} + \triangle\ \omega_{ik} \times \mathbf{r}_{ik} \tag{59}$$

and also for $k'th$ body contact as

$$\triangle\ \mathbf{u}_{jk} = \mathbf{v}_{jk} + \triangle\ \omega_{jk} \times \mathbf{r}_{jk}. \tag{60}$$

Here is the twist. Previously we worked with just a contact and had to get the forces and torques that operated at the contact around the rigid-bodies i and j but now, we must incoporate the effect of contact contribution at the k’th rigid-body into the equation. In effect, we can rewrite the modified form of equation (48) as

$$\triangle\ v_{jk} = \frac{\left[(\sum_{c,jc=jk}^{n} f^c - \sum_{c,ic=ik}^{n} f^c) + f^e_{jk} \right]}{m_{jk}} \tag{61}$$

and subject the result to a step-time h. This gives us almost similar equation as equation (57) of the individual contact:

$$\triangle\ \mathbf{v}_{jk} = \frac{\left[(\sum_{c,jc=jk}^{n} \mathbf{f}^c - \sum_{c,ic=ik}^{n} \mathbf{f}^c) + \mathbf{f}^e_{jk} \right]_{jk} h}{m_{jk}} \tag{62}$$

Similarly the changes in the angular velocities also formulated as

$$\triangle\ \omega_{jk} = \frac{(\tau^{e}_{jk} + (\sum_{c,jc=jk}^{n} \tau^{c}_{jk} - \sum_{c,ic=ik}^{n} \tau^{c}_{ik}) + (\sum_{c,jc=jk}^{n} \mathbf{f}^{c} \times pc_{c,jc} - \sum_{c,ic=ik}^{n} \mathbf{f}^{c} \times pc_{c,ic}))}{\mathbf{J}_{jk}} \quad (63)$$

which could also represent the individual contact equation (61) in the n contact rigid-bodies system.

The aforementioned equations are the n multiple contact's counter-part to the single contact, which undergoes similar approaches as the single contact in other to calculate for the torques and the forces of the rigid-bodies. Details of contact points are thoroughly discussed in (Mitirch, 1996) and (Erleben, 2004).

Gauss–Seidel Iteration Scheme

As a criteria for optimality, we employed the Gauss-Seidel iteration scheme for the propagation of the contact among the rigid bodies contacts in the system. Unlike other optimality methods, the Gauss-Seidel solver solves each constraint equation separately at every iteration step, an implication of the fact that only one contact is updated, while the rest of the other contacts in the system are assumed to have already been resolved.

5 Numerical Implementations

Implementation

The transfer equation coupled with the Gauss-Seidel scheme determines the contact forces and the contact velocities (contact resolution), whereas the contact laws (the Sigorini and the Coulomb friction) govern the rigid-bodies to ensure stability of the simulation respectively. The information gathered so far permits us to proceed with the simulation as follows:

1. Given the position, the velocity data and the time step, initialize the positions and the velocities of the rigid-bodies. $p_i = 0; v_i = 0; \omega_i = 0; t = 0$

2. The simulation loop starts at time t_0 by contact detection, thus measure the distances between the bodies, tracking the potential (closest) distance.

3. We perform forward motion of the bodies, through Euler integration step. Through the Gauss-Seidel scheme contacts are resolved iteratively.

4. Update the positions of the bodies by integration of each bodies velocity within the step time.

Conclusion

A survey for research necessary to build a multi-rigid body dynamics which includes computing, analyzing and optimizing a dynamic simulation in a 3-dimentional coordinates, for an individual rigid-body and a multi-rigid-body systems with a chain structure and a robot figure as means of showing the benefit of the numerical approach of dynamic simulation. We have obtained a deeper understanding of the physics underlying the computation and simulation of dynamical multi rigid bodies, the analysis is a huge benefit to mostly the auto industry as a whole.

The incorporation of the Gauss-Seidel, Coulomb friction and the Signorini conditions into the work coupled with the fixed and smaller step-time, ensured a timely determination of the contact forces, for explicit determination of contacts out of which the normal, and the much needed relative velocity was determined, to ensure not just the continuity of the simulation but also stability of the affected rigid-bodies during the simulation. Unlike other optimality methods, the Gauss-Seidel solver solves each constraint equation separately at every iteration step without resorting to any linearization of the friction cone. The computational method introduced can simulate several thousands of bodies with ease, in a situation where precision and accuracy is needed, designers can incorporate other integrational methods with ease to satisfy consumers demand. This method is applicable to open and/or closed chain which is usually associated with kinematical joints.

Finally we realize that, the interactive simulation can withstand vigorous and robust real-time simulation, because the resulting ordinary differential equation can be easily integrated using explicit integrators without the need for constraint stabilization, and special collision detection scheme associated with several simulation models.

Future Work

Even though the evaluation on the simulation gave precise data as demanded, I will want to experiment the numerical approach using other optimization methods such as the multi-grid with a variable time-steps coupled with other integrational procedures in a non-linear situation and compare the numerical stability and robustness with the above discussed approach.

Appendix A

From classical mechanics,$\mathbf{f} = m\mathbf{a}$. Thus a single vector of all external force exerted on a body $\mathbf{f}$ of mass m, given the moments of torque τ on the body. The first order differential equation

$$\mathbf{f}(t) = d/dt(m\dot{\mathbf{r}}(t) = m\ddot{\mathbf{r}}(t)$$

. $\dot{\mathbf{r}}(t) = \mathbf{v}(t)$ and $\mathbf{v}(t) = m^{-1}\mathbf{f}(t)$. At the initial frame of coordinates (x, y, z), we could formulate an equation in relation to the position center of mass as $\mathbf{r}_x(0) + \mathbf{V}_x(0)$ at the x axis,$\mathbf{r}_y(0) + \mathbf{V}_y(0)$ with the y axis and $\mathbf{r}_z(0) + \mathbf{V}_z(0) - g$ at the z axis,with the assumption that gravitational force pulls this body. In general we can write $\sum \mathbf{f}(t) = m\mathbf{a}$ and also the torque $\sum \tau(t) = \mathbf{I}(t) + \omega(t).\mathbf{I}\omega(t)$; from the $\mathbf{I} = \mathbf{RIR}^T$, (Mirtich, 1996)

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