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Hermitian Operators and Projections on Hardy Spaces

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HERMITIAN OPERATORS AND PROJECTIONS ON HARDY SPACES

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ABSTRACT

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We look at unbounded hermitian operators on Kolaski spaces with derivatives in a smooth space, H_N. We find the generator of one parameter groups of isometries on Kolaski spaces and provide spectral properties of the generator. We apply this result to the $S^p(D)$ spaces using results from Berkson and Porta on H^p spaces. We then consider the vector valued space $H^p_{\mathcal{H}}$ and determine the bounded hermitian operators and provide results if a particular group of disk automorphisms is used. In the second part we find that the generalized bi-circular projections on $H^p(\mathbb{T}^2)$ are given by the average of the identity and a reflection. We then find when the average of two isometries on S^p_K is a projection. This result leads to a corollary that the average of two isometries on $H^p(D)$ is a generalized bi-circular projection.

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INTRODUCTION

In this work we study two classes of operators on Banach spaces. The first class consists of unbounded hermitian operators on certain Banach spaces of analytic functions and the second class consists of generalized bi-circular projections on these spaces. The extension of the notion of hermitian operator on a Banach space goes back to the 1960's. Several mathematicians were involved in this successful and profitable extension. In particular, E. Berkson, Bonsall, G. Lumer, T. Palmer, and I. Vidav provided the basic foundations (see [5, 6, 24, 28, 29]). The key idea for the generalization of hermitian operators to Banach spaces was provided by the fundamental result of Stone which states that the generator of a strongly continuous group of unitaries on a Hilbert space is hermitian. A family of bounded linear operators, $\{T_t\}$, in a Banach space, $\mathcal{B}$, is one parameter group if it has the following properties:

1. $T_{s+t} = T_s T_t$, for all $s,t \in \mathbb{R}$
2. $T_0 = I$,

   It is further called a strongly continous one parameter group, or (C_0) group if it has the following property:

3. For all $f \in \mathcal{B}$, $T_t f$ is continuous in t.

Let

$$Af = \lim_{t\to 0} \frac{T_t f - f}{t}, \qquad f \in \mathcal{B} \qquad (0.0.1)$$

and let $\mathcal{D}(A)$ be the set of all functions f for which the limit exists. Then A is called the infinitesimal generator of the group $\{T_t\}$ and $\mathcal{D}(A)$ is the domain of A. The operator A is densely defined and closed. A is bounded if and only if the group $\{T_t\}$ is uniformly continuous. In this case you can write $T_t = e^{itA}$. An operator on a Banach space is said to be hermitian if iA is the generator of a strongly continuous one parameter group of isometries.

In the first part of this thesis , we characterize the generators of groups on a space of analytic functions which were first introduced by Kolaski in [20]. These spaces are defined as follows: Let $H(D)$ be the space of analytic functions on the open unit disc D, and let N be a norm on $H(D)$. Let H_N be the space of functions such that $N(f) < \infty$, where N is the norm on H_N. Kolaski defines S_N as the space of functions where $N(f') < \infty$, where the norm on S_N is given by $||f|| = |f(0)| + N(f')$. We call the spaces, S_N, Kolaski spaces. We also discuss the spectra and determine the resolvent operator of the generator of groups in Kolaski spaces. We then provide an application of our theorems to $S^p(D)$. The Hardy space $H^p(D)$ is the set of analytic functions, f, on the open disk equipped with the norm

$$\|f\|_{H^p} = \sup_{0<r<1} \left(\frac{1}{2\pi}\right) \left(\int_0^{2\pi} |f(re^{i\theta})| d\theta\right)^{1/p} \tag{0.0.2}$$

The Banach space $S^p(D)$ is the set of analytic functions, f, on the open disk with $f' \in H^p(D)$. The norm on this space is given by

$$\|f\|_{S^p} = |f(0)| + \|f'\|_{H^p} \tag{0.0.3}$$

Since Berkson and Porta found the generator of a (C_0) groups of isometries on $H^p(D)$ as well as providing several properties about the spectrum of the generator, we can apply our result on Kolaski spaces in order to find the generator of a (C_0) group of isometries on $S^p(D)$. We also apply our results on Kolaski spaces to give spectral properties of the generator. In addition to studying hermitian operators on Kolaski spaces, we further investigate hermitian (bounded and unbounded) operators on $H^1_{\mathcal{H}}$, a space of analytic functions taking values in a complex Hilbert space $\mathcal{H}$. We show that if a (C_0) group on $H^1_{\mathcal{H}}$ has a bounded generator, then the induced group of disk automorphisms must be the constant group. In this case we also discuss spectral properties and the resolvent operator for a (C_0) group of isomorphisms on $H^1_{\mathcal{H}}$ in the special case where the induced group of disk automorphisms has the property that there are two fixed points, $\tau \in D$ and $\overline{\tau}^{-1}$.

In the second part of the thesis we study a class of projections, called generalized bi-circular projections. A projection, P, is Hermitian if and only if $P + e^{it}(I - P)$ is an isometry for all t [5]. If a projection, P has the property that $P + e^{it}(I - P)$ is an isometry for all t, then P is a bi-circular projection. This idea was generalized to consider when $P + \lambda(I - P)$ is an isometry for $|\lambda| = 1$ with $\lambda \neq 1$. In this case, P is called a generalized bi-circular projection. Generalized bi-circular projections have been of interest because such projections are bicontractive [23]. A contractive projection is a projection such that $||P|| \leq 1$. A bicontractive projection has the further property that $||I - P|| \leq 1$. The problem of finding the contractive projections has been studied in several spaces(see [1, 11, 15, 16], and is unknown in general for Banach spaces. Finding generalized bi-circular projections provides a hint as to what contractive projections of a Banach space may look like. We find the form of generalized bi-circular projections in the H^p space of the torus. In many cases it has been shown that such a generalized bi-circular projection is the average of the identity and an isometric reflection. In the last part of the thesis, we investigate when P must be a bi-circular projection if it is the average of two isometries on the vector valued S^p space.

CHAPTER 1

HERMITIAN OPERATORS

An operator A on a Banach space X is called hermitian if iA is the generator of a one parameter (C_0) group of isometries. Uniformly continuous groups have bounded generators, while strongly continuous groups have unbounded generators, cf. [12]. The generators of uniformly continuous (C_0) groups of isometries are the bounded hermitian operators studied by Vidav [29]. In [3] Berkson, et. al. showed that the bounded hermitian operators on H^p for $p \neq 2$ are just real multiples of the identity. In this chapter, we study one parameter groups of surjective isometries on spaces, first introduced by Novinger and Oberlin [27] and studied later in more generality by Kolaski. These spaces, called Kolaski spaces, consist of analytic functions on the disk with derivative in one of the classical spaces of analytic function spaces such as H^p. In this section we determine the generators of certain (C_0) groups of isometries and describe their spectrum. We then provide an application of our results to the S^p spaces which are special Kolaski spaces. Since the spaces studied are classical spaces of analytic functions on the disk, the well known theorems on isometries of these spaces involve automorphisms of the disk. The groups of surjective isometries of these spaces are classified by the fixed point structure of the associated group of disk automorphisms.

In the first two sections we focus on the strongly continuous (C_0) groups of isometries on a Kolaski space. We determine the corresponding generator and its spectrum. We also examine the spectral properties of the generator. In the third section we provide a specific application of our theorem for Kolaski spaces. We specify the Kolaski space by requiring that the derivative of the functions belong to a Hardy space of the disk. Using the results of Berkson et. al. [3] we give a more complete description of the spectrum of the generator. In the following sections we consider a vector valued Hardy space and find the form of the generator. We are able to characterize the form of the groups in the case that the generator is bounded. We also consider a specific group of isometries based on the form of disc automorphisms and find the point spectrum and form of the resolvent operator. In the last section of the chapter, we provide an interesting example which shows that the fixed point structure of the groups of disc automorphisms can have a strong effect on the spectral properties of the generator.

1.1 (C_0) Groups of Linear Operators on S_N

C. Kolaski [20] introduced a special class of Banach spaces of analytic functions as follows. Let D be the open unit disk, and let $H(D)$ be the space of analytic functions on the unit disc, and let $N : H(D) \to [0,\infty)$ be a norm on $H(D)$. Let H_N and S_N denote the spaces of functions, f, such that$N(f) < \infty$ and $N(f') < \infty$, respectively. Assume further that H_N always contains the analytic polynomials. We will denote the norm of a function f on H_N by $\|f\|_{H_N}$ and the norm of a function g on S_N by $\|g\|_{S_N}$, where the norm on S_N is given by $\|g\|_{S_N} = |g(0)| + \|g'\|_{H_N}$. We recall that a Banach space is called smooth if its norm is weakly differentiable at every point except the origin (see [22]). Examples of smooth spaces include H^p spaces and L^p spaces. Throughout this paper H_N will always be a smooth space. Kolaski characterized the surjective linear isometries of S_N as follows:

Theorem 1.1.1. *[20, Theorem 1] Let T be an isometry of S_N onto (respectively into) S_N. If H_N is smooth, then there is a linear isometry $\mathcal{T}$ of H_N onto (respectively into) H_N and a λ with $|\lambda| = 1$ such that*

$$Tf(z) = \lambda[f(0) + \int_0^z \mathcal{T} f'(\xi) d\xi]. \tag{1.1.1}$$

We use this theorem to study the one parameter (C_0) groups on S_N and determine the respective generators. We follow standard terminology as used in [12, p.614]. A family $\{T_t\}$ $t \in \mathbb{R}$, of bounded linear operators in a Banach space X is called a strongly continuous group if

(i) $T_{s+t} = T_s T_t$

(ii) $T_0 = I$

(iii) For each $x \in X$, the map $t \to T_t x$ is continous as a function of $t \in \mathbb{R}$

These groups are called (C_0) groups if $\lim_{t\to 0} T_t f = f$ for every $f \in X$. If $\{T_t\}$ is any (C_0) group of linear operators, then the generator A of $\{T_t\}$ is defined by

$$(*) \;\; Af = \lim_{t\to 0} \frac{T_t f - f}{t} \tag{1.1.2}$$

where the domain of A, $D(A)$, is the set of all $f \in X$ for which the limit $(*)$ exists. The group $\{T_t\}$ is uniformly continuous if and only if its generator is a bounded linear operator A. If $\{T_t\}$ is strongly continuous, but not uniformly continuous, then its generator is an unbounded operator cf. [12]. We first consider (C_0) groups of surjective isometries on the space S_N and determine the generator of such a group.

Proposition 1.1.2. *Let $\{T_t\}$ be a (C_0) group of surjective linear isometries on S_N. Then there is a one-parameter group of unimodular complex numbers $\{\lambda_t\}$ and a (C_0) group of surjective linear isometries $\{\mathcal{T}_t\}$ on H_N, such that*

$$T_t f(z) = \lambda_t \left[f(0) + \int_0^z \mathcal{T}_t f'(\xi) d\xi \right]. \tag{1.1.3}$$

Proof.

From Kolaski's theorem 1.1.1, a one parameter group of surjective linear isometries on S_N, T_t is given by

$$(T_t f)(z) = \lambda_t [f(0) + \int_0^z \mathcal{T}_t f'(\xi) d\xi]. \tag{1.1.4}$$

where for each t, $\mathcal{T}_t$ is a surjective linear isometry of H_N and λ_t is a unimodular constant. By using constant functions, we obtain that T_t induces a (C_0) group of unimodular scalars, λ_t, of the form $\lambda_t = e^{i\gamma t}$ and hence,

$$T_t f(z) = e^{i\gamma t} [f(0) + \int_0^z \mathcal{T}_t f'(\xi) d\xi]. \tag{1.1.5}$$

The property that $T_0 = I$ gives

$$e^{i\gamma(0)}[f(0) + \int_0^z \mathcal{T}_0 f'(\xi)d\xi] = f(z) \tag{1.1.6}$$

. Differentiating this last equation we get that $\mathcal{T}_0 = I$. Likewise, the property that $T_s(T_t) = T_{s+t}$ gives

$$e^{i\gamma s}e^{i\gamma t}[f(0) + \int_0^z \mathcal{T}_s\mathcal{T}_t f'(\xi)d\xi] \tag{1.1.7}$$

$$= e^{i\gamma(s+t)}[f(0) + \int_0^z \mathcal{T}_{s+t} f'(\xi)d\xi]. \tag{1.1.8}$$

Differentiating this gives $e^{i\gamma s}e^{i\gamma t}\mathcal{T}_s\mathcal{T}_t f'(z) = e^{i\gamma(s+t)}\mathcal{T}_{s+t} f'(z)$. We have shown that $e^{i\gamma s}e^{i\gamma t} = e^{i\gamma(s+t)}$, thus $\mathcal{T}_s(\mathcal{T}_t) = \mathcal{T}_{s+t}$. Since T_t is strongly continuous, and

$$||(T_t - I)f||_{S_N} = ||(\mathcal{T}_t - I)f'||_{H_N} + |(e^{i\gamma t} - 1)f(0)| \geq \ ||(\mathcal{T}_t - I)f'||_{H_N}, \tag{1.1.9}$$

it follows that $\mathcal{T}_t$ is strongly continuous.

Proposition 1.1.3. *Let $\mathcal{S}$ be the generator of the (C_0) group of surjective isometries $\{T_t\}$ on S_N, let $\mathcal{R}$ be the generator of the (C_0) group of isometries $\{\mathcal{T}_t\}$ on H_N induced by $\{T_t\}$, and let $e^{i\gamma t}$ be a (C_0) group of unimodular complex numbers induced by $\{T_t\}$. Then $D(\mathcal{S}) = \{f \in S_N : f' \in D(\mathcal{R})\}$. Further, $\mathcal{S} f(z) = i\gamma f(0) + \int_0^z \mathcal{R} f'(\xi)d\xi$.*

Proof.

We will show that

$$\lim_{t\to 0}\left\|\frac{T_t f - f}{t} - \left(i\gamma f(0) + \int_0^z \mathcal{R}(f')(\xi)d\xi\right)\right\|_{S_N} = 0 \tag{1.1.10}$$

$$
\left\| \frac{T_t f - f}{t} - i\gamma f(0) - \int_0^z \mathcal{R} f'(\xi) d\xi \right\|_{S_N}
$$

$$
= \left\| \left[\frac{e^{i\gamma t} f(0) + e^{i\gamma t} \int_0^z \mathcal{T}_t f'(\xi) d(\xi) - f(z)}{t} \right] - i\gamma f(0) - \int_0^z \mathcal{R} f'(\xi) d\xi \right\|_{S_N}
$$

$$
= \left\| \left[\frac{e^{i\gamma t} f(0) + e^{i\gamma t} \int_0^z (\mathcal{T}_t f'(\xi) - f'(\xi)) d\xi + e^{i\gamma t} f(z) - e^{i\gamma t} f(0) - f(z)}{t} \right] \right.
$$

$$
\left. - i\gamma f(0) - \int_0^z \mathcal{R} f'(\xi) d\xi \right\|_{S_N}
$$

$$
\leq \left\| e^{i\gamma t} \left[\frac{(\mathcal{T}_t f'(z) - f'(z))}{t} - \mathcal{R} f'(z) \right] \right\|_{H_N} + \left\| (e^{i\gamma t} - 1) f'(z) \right\|_{H_N}
$$

$$
+ \left\| \mathcal{R} f'(z)(e^{i\gamma t} - 1) \right\|_{H_N} + \left| \left(\frac{(e^{i\gamma t} - 1)}{t} - i\gamma \right) f(0) \right|. \tag{1.1.11}
$$

Clearly the right side $\to 0$ as $t \to 0$ since $\mathcal{R}$ is the generator of the group $\{\mathcal{T}_t\}$ and $e^{i\gamma t} - 1 \to 0$ as $t \to 0$.

Corollary 1.1.4. *If A is a hermitian operator on S_N, then* $Af(z) = \gamma f(0) + \int_0^z \mathcal{R} f'(\xi) d\xi$ *where $\mathcal{R}$ is a hermitian operator on H_N and γ is a hermitian operator on D.*

1.2 Spectrum of the generator of $\{T_t\}$

We recall that the spectrum of an operator, $\mathcal{S}$ on a Banach space X, denoted by $\sigma(\mathcal{S})$ is the set of all complex numbers μ such that $\mathcal{S} - \mu I$ is non-invertible on X. The complement of this set is the resolvent and is denoted by $\rho(\mathcal{S})$. For all $\lambda \in \rho(S)$, the resolvent operator $R(\lambda, S)f = (S-\lambda I)^{-1}f$. The point spectrum of $\mathcal{S}$, $\sigma_p(S)$, is the set of all eigenvalues of $\mathcal{S}$. Hille and Phillips showed that if a generator of a one parameter group has nonempty, compact resolvent, then it has pure point spectrum cf [17, p.210]. We show how the spectrum of the generator of a (C_0) group of isometries on H_N is related to the generator of the (C_0) group of isometries on S_N. In this section, our main result is the following:

Theorem 1.2.1. *Let $\{T_t\}$ be a strongly continuous (C_0) group of isometries on S_N. Then*

$$T_t f(z) = e^{i\gamma t}[f(0) + \int_0^z \mathcal{T}_t f'(\xi)d\xi] \tag{1.2.1}$$

where $\{\mathcal{T}_t\}$ is a strongly continuous (C_0) group of isometries on H_N. Let $\mathcal{R}$ be the generator of $\{\mathcal{T}_t\}$. Let $\mathcal{S}$ be the generator of $\{T_t\}$. If $\mathcal{R}$ has compact resolvent, then $\mathcal{S}$ has compact resolvent.

We begin by showing the relationship between the point spectra for the generators of groups on H_N and S_N.

Proposition 1.2.2. *Let $\{T_t = e^{i\gamma t}[f(0) + \int_0^z \mathcal{T}_t f'(\xi)d\xi]\}$ be a strongly continuous (C_0) group of isometries on S_N with generator S, and let $\{\mathcal{T}_t\}$ is a strongly continuous (C_0) group of isometries on H_N with generator $\mathcal{R}$. Then $\sigma_p(\mathcal{R}) \subseteqq \sigma_p(\mathcal{S})$ and $\sigma_p(\mathcal{S}) \backslash \{i\gamma\} \subseteqq \sigma_p(\mathcal{R})$.*

Proof.

Let μ be an eigenvalue of $\mathcal{R}$. Then there is a function $f_\mu \in H_N$ such that $\mathcal{R} f_\mu(z) = \mu f_\mu(z)$. Define the function $f(z) = \int_0^z f_\mu(\xi)d\xi$. Then $\mathcal{S} f(z) = i\gamma f(0) + \int_0^z \mathcal{R} f'(\xi)d\xi = \int_0^z \mathcal{R} f_\mu(\xi)d\xi = \mu \int_0^z f_\mu(\xi)d\xi = \mu f(z)$, so μ is an eigenvalue of $\mathcal{S}$.

Let μ be an eigenvalue of $\mathcal{S}$. So $\exists f_\mu \in S_N$ such that $\mathcal{S} f_\mu(z) = \mu f_\mu(z)$. We first suppose that f_μ is not constant. Then, $i\gamma f_\mu(0) + \int_0^z \mathcal{R} f'(\xi)d\xi = \mu f_\mu$. Differentiation yields that $\mathcal{R} f'_\mu(z) = \mu f'_\mu$, so μ is an eigenvalue of $\mathcal{R}$. If f_μ is constant then $\mathcal{S} f_\mu = i\gamma f_\mu$ and so $\mu = i\gamma$.

To discover properties of the generator of $\{T_t\}$, it is advantageous to consider S_N as the ℓ_1 direct sum of the spaces $\mathbb{C}$ and H_N. Define the map $V : S_N \to \mathbb{C} \oplus_1 H_N$ by $Vf(z) = (f(0), f')$. It is easy to check that S_N is isometric to $\mathbb{C} \oplus_1 H_N$ since the norm of $\mathbb{C} \oplus_1 H_N$ is given by $\|(\alpha, g)\|_* = |\alpha| + \|g\|_{H_N}$ Moreover, $V^{-1}(\alpha, g) = \alpha + \int_0^z g'(\xi)d\xi$.

Proposition 1.2.3. *Let $\{T_t\}$ be a one parameter (C_0) group of isometries on S_N. Define $V_t = VT_tV^{-1}$, with $Vf = (f(0), f')$. Then $\{V_t\}$ is a (C_0) group of isometries on $\mathbb{C} \oplus_1 H_N$.*

Proof.

We will show that $\{V_t\}$ has the properties of a (C_0) group of isometries. Clearly $\{V_t\}$ is an isometry for each t.

Also, $V_0 = VT_0V^{-1} = VIV^{-1} = I$ and $V_sV_t = (VT_sV^{-1})(VT_tV^{-1}) = VT_sT_tV^{-1} = V_{s+t}$. Furthermore,

$$\begin{aligned} \|V_t(\alpha, g) - (\alpha, g)\|_* &= \|VT_tV^{-1}(\alpha, g) - (\alpha, g)\|_* \\ &= \|VT_t(\alpha + \int_0^z g(\xi)d\xi) - (\alpha, g)\|_* \\ &= \|V(e^{i\gamma t}(\alpha + \int_0^z \mathcal{T}_t g(\xi)d\xi) - (\alpha, g)\|_* \\ &= \|(e^{i\gamma t}\alpha, \mathcal{T}_t g) - (\alpha, g)\|_* \\ &= |\alpha(e^{i\gamma t} - 1)| + \|\mathcal{T}_t g - g\|_{H_N} \end{aligned} \tag{1.2.2}$$

Clearly each term on the right side of this last equation goes to 0 as $t \to 0$. Hence $\{V_t\}$ is strongly continuous and this completes the proof.

Since V_t is a (C_0) group on $\mathbb{C} \oplus_1 H_N$, we wish to determine the generator using results on the ℓ_1 sum. The fact that $S_N \cong \mathbb{C} \oplus_1 H_N$, will allow us to determine a relation between the generators of $\{T_t\}$ and $\{V_t\}$. To do this, we use application of a theorem of Fleming and Jamison [13, Thm 2.5, p 174] gives that if the factor spaces of an ℓ_1 sum have only trivial hermitian operators then the surjective isometries are diagonal. To apply this resulte, for the remainder of this section, we will assume that H_N admits only trivial bounded hermitian operators. This is the case for the H^p spaces for $p \neq 2$. So $V_t = \begin{pmatrix} \lambda_t & 0 \\ 0 & \mathcal{T}_t \end{pmatrix}$.

Proposition 1.2.4. *Let $\{T_t\}$ be a one parameter (C_0) group of isometries on S_N and suppose that H_N admits only trivial hermitian operators. Then by Proposition (2.2), $\{T_t\}$ induces (C_0) groups $\{\lambda_t = e^{i\gamma t}\}$ and $\{\mathcal{T}_t\}$ on $\mathbb{C}$ and H_N respectively. Let γ be the generator of $\{\lambda_t\}$ and let $\mathcal{R}$ be the generator of $\{\mathcal{T}_t\}$. Then $\mathcal{G} = \begin{pmatrix} \gamma & 0 \\ 0 & \mathcal{R} \end{pmatrix}$ is the generator of the induced group $\{V_t\}$ on $\mathbb{C} \oplus_1 H_N$.*

Proof.

We show that $\|V_t - \mathcal{G}\|_* \to 0$ as $t \to 0$.

$$\left\| \begin{pmatrix} \gamma & 0 \\ 0 & \mathcal{R} \end{pmatrix} \begin{pmatrix} \alpha \\ g \end{pmatrix} - \frac{1}{t}\left(V_t \begin{pmatrix} \alpha \\ g \end{pmatrix} - \begin{pmatrix} \alpha \\ g \end{pmatrix} \right) \right\|_*$$

$$= \left\| \begin{pmatrix} \gamma\alpha \\ \mathcal{R}g \end{pmatrix} - \frac{1}{t}\left(V_t \begin{pmatrix} \alpha \\ g \end{pmatrix} - \begin{pmatrix} \alpha \\ g \end{pmatrix} \right) \right\|_*$$

$$= \left(|\gamma\alpha - \frac{\lambda_t\alpha - \alpha}{t}| + \|\mathcal{R}g - \frac{\mathcal{T}_t g - g}{t}\|_{H_N} \right) \tag{1.2.3}$$

which approaches 0 as $t \to 0$, since γ and $\mathcal{R}$ are generators of $\{\lambda_t\}$ and $\{\mathcal{T}_t\}$ respectively. This completes the proof.

Proposition 1.2.5. *Let $\{T_t\}$ be a one parameter (C_0) group of isometries on S_N and $\{\lambda_t\}$ and $\{\mathcal{T}_t\}$ be the induced groups on $\mathbb{C}$ and H_N respectively. Let $\mathcal{R}$ be the generator of $\{\mathcal{T}_t\}$ and $\lambda_t = e^{i\gamma t}$. Let $\mathcal{G} = \begin{pmatrix} \gamma & 0 \\ 0 & \mathcal{R} \end{pmatrix}$ be the generator of the induced group $\{V_t\}$ on $\mathbb{C} \oplus_1 H_N$. If $\mu (\neq \gamma)$ is not an eigenvalue of $\mathcal{R}$ and $\mathcal{R} - \mu I$ is surjective, then $\mathcal{G} - \mu I$ is surjective.*

Proof.

Given $(\alpha, g) \in \mathbb{C} \oplus_1 H_N$, we want to show that there exists $(z,h) \in \mathbb{C} \oplus_1 H_N$ such that $(\mathcal{G} - \mu I)(z,h) = (\alpha, g)$ Let $(\alpha, g) \in \mathbb{C} \oplus_1 H_N$. Consider $(\mathcal{G} - \mu I)\begin{pmatrix} z \\ h \end{pmatrix} = \begin{pmatrix} \gamma - \mu & 0 \\ 0 & (\mathcal{R} - \mu I) \end{pmatrix}\begin{pmatrix} z \\ h \end{pmatrix} = \begin{pmatrix} (\gamma - \mu) z \\ (\mathcal{R} - \mu I) h \end{pmatrix} = \begin{pmatrix} \alpha \\ g \end{pmatrix}$. The surjectivity of $(\mathcal{R} - \mu I)$ and $\gamma \neq \mu$ implies that $\begin{pmatrix} \alpha \\ g \end{pmatrix}$ is in the range of $(\mathcal{G} - \mu I)$. This completes the proof.

Proposition 1.2.6. *Let $\{T_t\}$ be a one parameter (C_0) group of isometries on S_N and $\{\lambda_t\}$ and $\{\mathcal{T}_t\}$ be the induced groups on $\mathbb{C}$ and H_N respectively. Let $\mathcal{R}$ be the generator of $\{\mathcal{T}_t\}$ and $\lambda_t = e^{i\gamma t}$. Let $\mathcal{G} = \begin{pmatrix} \gamma & 0 \\ 0 & \mathcal{R} \end{pmatrix}$ be the generator of the induced group $\{V_t\}$ on $\mathbb{C} \oplus_1 H_N$. If $\mathcal{R}$ has compact resolvent, then $\mathcal{G}$ has compact resolvent.*

Proof.

Let $(\mathcal{G} - \mu I)^{-1}\begin{pmatrix} z_n \\ g_n \end{pmatrix}$ be a bounded sequence in the resolvent of $\mathcal{G}$.

$$(\mathcal{G} - \mu I)^{-1}\begin{pmatrix} z_n \\ g_n \end{pmatrix} = \begin{pmatrix} \frac{1}{\gamma - \mu} & 0 \\ 0 & (\mathcal{R} - \mu I)^{-1} \end{pmatrix}\begin{pmatrix} z_n \\ g_n \end{pmatrix} = \begin{pmatrix} \frac{z_n}{\gamma - \mu} \\ (\mathcal{R} - \mu I)^{-1} g_n \end{pmatrix} \tag{1.2.4}$$

Since $\dfrac{z_n}{\gamma - \mu}$ is a bounded sequence of complex numbers, it has a convergent subsequence, say $\dfrac{z_{n_k}}{\gamma - \mu}$. Furthermore, since g_n is a bounded sequence of H_N functions and $(\mathcal{R} - \mu I)^{-1}$ is compact on H_N, $(\mathcal{R} - \mu I)^{-1} g_{n_k}$ is a bounded sequence, so it has a convergent subsequence, say $(\mathcal{R} - \mu I)^{-1} g_{n_{k_i}}$. Thus, $(\mathcal{G} - \mu I)^{-1}\begin{pmatrix} z_n \\ g_n \end{pmatrix}$ has a convergent subsequence, and the resolvent of $\mathcal{G}$ is compact.

Theorem 1.2.1 is an immediate consequence of the preceding propositions.

1.3 Application to S^p spaces

In this section we consider a special example of a Kolaski space, namely S^p. This Banach space consists of analytic functions f on the disk with $f' \in H^p$. The norm is given by $\|f\| = |f(0)| + \|f'\|_{H^p}$. The bounded hermitian operators on this space were classified by Hornor and Jamison in [18]. In this section we will determine the generators of a certain type of strongly continuous (C_0) groups of isometries and thereby characterize certain unbounded hermitian operators associated with these groups in terms of their action on the space. Berkson and Porta cf. [3] determined the generator of the one parameter (C_0) group of isometries on H^p and this result will be crucial to our work. Since H^p is a smooth space for $1 < p < \infty$, their result together with the previous results on the Kolaski spaces can be used to find the generator of the type (i) one parameter (C_0) group of isometries on S^p.

In order to apply their result, we first introduce some notation and results. Let a one parameter group of disk automorphisms be denoted by $\{\phi_t\}$ with

$$\phi_t(z) = \frac{a_t(z - b_t)}{1 - \overline{b_t} z} \tag{1.3.1}$$

where a_t and b_t are constants such that $|a_t| = 1$ and $|b_t| < 1$. A nonconstant one parameter (C_0) group of disk automorphisms $\{\phi_t\}$ on D is said to be of type (i) if the set of common fixed points in the extended complex plane of the functions ϕ_t is a doubleton subset consisting of a point $\tau \in D$ and $\overline{\tau}^{-1}$, ($\overline{\tau}^{-1} = \infty$ if $\tau = 0$). We will call a one parameter (C_0) group, $\{T_t\}$, on S^p of type (i) if the induced one parameter (C_0) group of disk automorphisms is of type (i). To state our results we will quote the following result by Berkson and Porta on the form of the one parameter (C_0) group of disk automorphisms.

Theorem 1.3.1. *[3, Thm 1.10] Let $\{\phi_t\}$, $t \in \mathbb{R}$ be a nonconstant one parameter group of disk automorphisms of D such that $\phi_t(z)$ is continuous in t for each $z \in D$. If $\{\phi_t\}$ is of type (i), then there are uniquely determined (by $\{\phi_t\}$) constants τ and c, the former in D and* *the latter real and nonzero, such that the parameters a_t and b_t of equation (1.3.1) are given for all $t \in \mathbb{R}$ by*

$$a_t = \frac{|\tau|^2 - e^{ict}}{|\tau|^2 e^{ict} - 1}; \qquad b_t = \frac{\tau(e^{ict} - 1)}{e^{ict} - |\tau|^2}. \tag{1.3.2}$$

The constant τ is in fact the unique element of D which is left fixed by all ϕ_t for $t \in \mathbb{R}$. The constant c satisfies and is determined by the equation $e^{ict}z = (\Psi \circ \phi_t \circ \Psi)(z)$ for $t \in \mathbb{R}$ and $z \in D$ and Ψ a disk automorphism given by

$$\Psi(z) = \frac{z - \tau}{\bar{\tau}z - 1} \tag{1.3.3}$$

Conversely, if τ is any element of D and c is any nonzero real number, the parameters a_t and b_t, defined for each $t \in \mathbb{R}$ as in (1.3.2) have moduli satifying the requirements in (1.3.1), and define by formula (1.3.1) a one parameter group, $\{\phi_t\}$ of type (i).

Associated with every continuous one parameter group of disk automorphisms there is the so-called invariance polynomial. It was shown in [3] that the zeroes of this polynomial are precisely the set of fixed points of ϕ_t in the finite plane.

Corollary 1.3.2. *[3, Corollary 1.13] If $\{\phi_t\}$ is a group of type (i), then its invariance polynomial can be written in the form*

$$R(z) = \frac{-ic(\bar{\tau}z^2 - (1 + |\tau|^2)z + \tau)}{1 - |\tau|^2} \tag{1.3.4}$$

for all $z \in \mathbb{C}$, where τ is the unique point of D fixed by every ϕ_t and c is the unique constant as in Theorem 1.3.1.

We also recall the form of the surjective isometries on H^p for $1 \leq p < \infty,\ p \neq 2$ which were determined by Forelli.

Proposition 1.3.3. *[14, Theorem 2] If A is a linear isometry of H^p onto H^p, $1 \leq p < \infty$, $p \neq 2$, then there is a disk automorphism ϕ and $\lambda \in \mathbb{C}$ such that*

$$(Af)(z) = \lambda[\phi'(z)]^{1/p} f(\phi(z)) \tag{1.3.5}$$

for all $f \in H^p$ and $z \in D$.

On the other hand, if $1 \leq p < \infty$ and λ is a unimodular constant and ϕ is a disk automorphism, then (1.3.5) defines a linear isometry of H^p onto H^p.

Let $\{T_t\}$ be a one parameter (C_0) group of isometries induced by a group of disk automorphisms of type (i) and define a hermitian operator, A, to be of type (i) if it is the generator of a group of isometries where the associated group of disk automorphisms is of type (i). Berkson and Porta defined $\mathcal{R}$ with the domain, $\mathcal{D}(\mathcal{R})$, to be all functions $f \in H^p$ such that the function

$$\mathcal{R}(f) = Rf' + (1/p)R'f \tag{1.3.6}$$

is in H^p, where R is the invariance polynomial given in (1.3.4) and juxtaposition indicates multiplication. They proved the following theorem.

Theorem 1.3.4. *[3, Theorem 3.1] If A is a hermitian operator of type (i) in H^p, $1 \leq p < \infty$, $p \neq 2$, then:*

(I) there is a unique real number β such that $A = \beta I - i\mathcal{R}$, where $\mathcal{R}$ is as in (1.3.6);

(II) the eigenvalues of A are precisely the real numbers $\sigma_n = c(n+1/p)+\beta, n = 0,1,2,...$;

(III) for each n, the eigenmanifold of A corresponding to σ_n is the one dimensional span of the function $f_{n,\tau}(z)$ where, for each $z \in D$

$$f_{n,\tau}(z) = \begin{cases} (z-\tau)^n/(z-\overline{\tau}^{-1})^{n+(2/p)} & ,\text{if } \tau \neq 0 \\ \\ z^n & ,\text{if } \tau = 0. \end{cases} \tag{1.3.7}$$

and

(IV) A has compact resolvent, and hence pure point spectrum.

We now combine this theorem of Berkson and Porta along with one of the main results of this paper, Corollary (1.2.1), to find the generator and spectrum of the associated one parameter (C_0) group of isometries on S^p.

Corollary 1.3.5. *Let $\mathcal{S}$ be the generator of type (i) of a one parameter (C_0) group of isometries on S^p. Then $\mathcal{D}(\mathcal{S}) = \{f \in S^p : f' \in \mathcal{D}(\mathcal{R})\}$ and*

$$\mathcal{S} f(z) = (\gamma - \beta) f(0) + \beta f(z) - \int_0^z (\mathcal{R} f')(\xi) d\xi \tag{1.3.8}$$

where γ and β are generators of one parameter (C_0) groups of complex numbers and $\mathcal{R}$ is as given in (1.3.6). Furthermore, $\mathcal{S}$ has compact resolvent, thus pure point spectrum, so

$$\sigma(\mathcal{S}) = \{\sigma_n = c(n + (1/p)) + \beta : n = 0, 1, 2, ...\} \tag{1.3.9}$$

where the eigenmanifolds corresponding to each σ_n are spanned by the functions

$$g_{n,\tau} = \begin{cases} \int_0^z (\xi - \tau)^n / (\xi - \overline{\tau}^{-1})^{n+(2/p)} d\xi & ,if\ \tau \neq 0 \\ \\ (n+1)^{-1} z^{n+1} & ,if\ \tau = 0 \end{cases} \tag{1.3.10}$$

Proof.

The proof is a direct application of our results from sections 1 and 2 for S_N and H_N by taking $H_N = H^p$.

Remark 1.3.6. *The first two terms in $\mathcal{S}$ gives the form of the bounded hermitian operators on S^p, cf. [18].*

1.4 Hermitian operators on $H^1_{\mathcal{H}}$

In this section, we characterize hermitian operators on H^1 for vector valued functions. Let E be a finite dimensional Banach space and D be the open unit disk. Then H^p_E is the Banach space of all $F : D \to E$ such that $< F, e^* >$ belongs to the Hardy class H^p for all $e^* \in E$. The norm on H^p_E is given by

$$||F||_p = \left(\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} ||F(e^{is})||^p ds \right)^{1/p} \qquad p < \infty \tag{1.4.1}$$

$$||F||_\infty = esssup||F(e^{is})|| = sup_{z\in D}||F(z)|| \tag{1.4.2}$$

We consider the case wher E is a Hilbert space and denote it by $\mathcal{H}$.

This section relies heavily on results on disk automorphisms, so it is useful to first state some results on disk automorphisms. Recall disk automorphism can be written in the form

$$\frac{a_t(z - b_t)}{1 - b_t z}. \tag{1.4.3}$$

Write the functions $t \to a_t$ and $t \to b_t$ as $a(\cdot)$ and $b(\cdot)$ respectively. Berkson and Porta [3] gave the form of the generator of a one parameter group of disk automorphisms, or the invariance polynomial, by

$$\frac{\partial \phi_t}{\partial t}|_{t=0} R = \overline{b_t'(0)} z^2 + a'(0)z - b'(0) \tag{1.4.4}$$

It is also useful to classify one parameter groups of disk automorphisms based on the set of common fixed points. Berkson, Kaufmanm, and Porta partitioned groups of disk automorphisms into three types based on the set of common fixed points.

Proposition 1.4.1. *[2, Scholium 1.1] Let $\{\phi_t\}, t \in \mathbb{R}$ be a nonconstant one parameter group of disk automorphisms, such that for each $z \in D$, $\phi_t(z)$ is a continuous function of t. Then the set of common fixed points in the extended complex plane must be one of the* *following:*

(i) a doubleton subset of the extended complex plane consisting of a point $\tau \in D$ and $\bar{\tau}^{-1}$ (the latter taken to be ∞ if $\tau = 0$)

(ii) a singleton subset of $\mathbb{C}$, or

(iii) a doubleton subset of $\mathbb{C}$

In this section we use the result of Cambern and Jarosz to investigate one-parameter groups of isometries on $H^1_{\mathcal{H}}$. We will show that $\{T_t\}$ induces one-parameter groups of disk automorphisms and a one-parameter group of unitaries acting on $\mathcal{H}$. Cambern and Jarosz characterized the isometries of $H^p_{\mathcal{H}}$ for p=1 as follows:

Theorem 1.4.2. *[10, Theorem(p206)] Let $\mathcal{H}$ be a complex finite dimensional Hilbert space and let $T : H^1_{\mathcal{H}} \to H^1_{\mathcal{H}}$ be a surjective isometry. Then there exists a conformal map ϕ from D onto D and a fixed unitary operator $U : \mathcal{H} \to \mathcal{H}$ such that for any $F \in H^1_{\mathcal{H}}$ and any $z \in D$*

$$TF(z) = UF(\phi(z))\phi'(z) \tag{1.4.5}$$

Our first result is the following:

Theorem 1.4.3. *Let $\{T_t\}$ be a one parameter group of surjective isometries on $H^1_{\mathcal{H}}$. Then*

$$T_t f(z) = \phi_t'(z) U_t f(\phi_t(z)) \tag{1.4.6}$$

where $\{U_t\}$ is a one parameter group of unitary operators on $\mathcal{H}$ and $\{\phi_t\}$ is a one parameter group of disk automorphisms.

Proof.

The form of T_t is a direct consequence of the theorem of Cambern and Jarosz. Define $e_n = z^n \cdot v$, where $v \in \mathcal{H}$ and $\|v\| = 1$. Since $T_0 f = f$ for all f, consider $T_0(e_0) = \phi_0'(z)U_0 v = v$ and $T_0(e_1) = \phi_0'(z)\phi_0(z)U_0 v = zv$. Thus

$zT_0e_0 = T_0e_1$ which implies $z\phi_0'(z)U_0v = \phi_0(z)\phi_0'(z)U_0v$. Since $\phi_0'(z)$ and U_0v are nonzero, this yields that $\phi_0(z) = z$, which immediately yields $U_0 = Id$ Likewise,

$$\begin{aligned} T_sT_te_0 &= T_s(\phi_t'(z)U_tv) \\ &= \phi_s'(z)U_s(\phi_t'(\phi_s(z))U_tv \\ &= (\phi_t \circ \phi_s)'(z)U_sU_tv \\ &= \phi_{s+t}'(z)U_{s+t}'v = T_{s+t}e_0 \end{aligned} \tag{1.4.7}$$

and

$$\begin{aligned} T_sT_te_1 &= T_s(\phi_t'(z)U_t(\phi_t(z)v) \\ &= \phi_s'(z)U_s(\phi_t'(\phi_s(z))U_t(\phi_t(\phi_s(z))v) \\ &= (\phi_t \circ \phi_s)'(z)(\phi_t \circ \phi_s)(z)U_sU_t(v) \\ &= \phi_{s+t}'(z)\phi_{s+t}(z)U_{s+t}v = T_{s+t}e_1 \end{aligned} \tag{1.4.8}$$

.

So $\phi_{s+t}T_{s+t}e_0 = T_{s+t}e_1$, which implies

$\phi_{s+t}(\phi_t \circ \phi_s)'(z)U_sU_tv = (\phi_t \circ \phi_s)'(z)(\phi_t \circ \phi_s)(z)U_sU_tv$. Thus $\phi_t \circ \phi_s = \phi_{s+t} = \phi_{t+s}$. This implies that $U_sU_t = U_{s+t}$.

To show the strong continuity of $\phi_t(z)$, consider the inner product,

$$\begin{aligned} < T_te_1, T_te_0 > &=< \phi_t\phi_t'(z)U_tv, \phi_t'(z)U_tv > \\ &= \phi_t(z) < \phi_t'(z)U_tv, \phi_t'(z)U_tv > \\ &= \phi_t(z)\|T_te_0\|^2 \end{aligned} \tag{1.4.9}$$

Thus, since $\|v\| = 1$,

$$\phi_t(z) = \frac{< T_te_1, T_te_0 >}{\|T_te_0\|^2} \tag{1.4.10}$$

Since the inner product is continuous and $T_t f$ is continuous, it follows that ϕ_t must be continuous. Thus $\{\phi_t\}$ is a one parameter group of disk automorphisms. The fact that $t \to \phi_t'$ is strongly continuous follows from [3].

Since

$$U_t v = \frac{T_t e_0}{\phi_t'(z)} v \tag{1.4.11}$$

and ϕ_t' is never zero, $t \to U_t$ is continuous. Thus, $\{U_t\}$ is a one parameter group of unitary operators on $\mathcal{H}$.

Corollary 1.4.4. *Let $\{T_t\}$ be a (C_0) group of surjective isometries on $H^1_{\mathcal{H}}$, $\{U_t\}$ be a (C_0) group of isometries on $\mathcal{H}$, $\{\phi_t\}$ be a (C_0) group of disk automorphisms, and $e_n = z^n \ddot{v}$ where $\|v\| = 1$. Then*

$$\phi_t(z) = \frac{< T_t e_1, T_t e_0 >}{\|T_t e_0\|^2}. \tag{1.4.12}$$

Theorem 1.4.5. *Let $\{T_t\}$ be a one parameter group of isometries on $H^1_{\mathcal{H}}$, $\{\phi_t\}$ be a one parameter group of disk automorphisms with generator R as given in (1.3.1), and $\{U_t\}$ be a one parameter group of unitary operators on $\mathcal{H}$ with generator A. Then the generator of $\{T_t\}$ is given by*

$$\mathcal{G} f = \frac{\partial}{\partial t}\phi_t'|_{t=0} f + Rf' + Af \tag{1.4.13}$$

Proof.

We show that $\|\frac{T_t f - f}{t} - \mathcal{G} f\| \to 0$ as $t \to 0$.

$$\begin{aligned} \|\frac{T_t f - f}{t} - \mathcal{G}\| &= \|\frac{\phi_t' U_t f(\phi_t) - f}{t} - (\frac{\partial}{\partial t}\phi_t'|_{t=0} f + Rf' + Af)\| \\ &\leq \|(\frac{\phi_t' - \phi_0'}{t}) U_t f(\phi_t) - \frac{\partial}{\partial t}\phi_t'|_{t=0} f\| \\ &+ \|\frac{U_t f(\phi_t) - U_0 f(\phi_t}{)} t - Af\| \\ &+ \|\frac{U_0 f(\phi_t) - U_0 f(\phi_0)}{\phi_t - \phi_0} \cdot \frac{\phi_t - \phi_0}{t} - Rf'\| \end{aligned} \tag{1.4.14}$$

Each of the norms on the right $\to 0$ as $t \to 0$.

Corollary 1.4.6. *If $\mathcal{G}$ is the hermitian operator on $H^1_{\mathcal{H}}$ then*

$$\mathcal{G}f = -i\frac{\partial}{\partial t}\phi_t'|_{t=0} f - iRf' + Af \tag{1.4.15}$$

where $\{\phi_t\}$ is a one parameter group of disk automorphisms, R is as given in 1.3.1, and A is the hermitian operator on $\mathcal{H}$.

1.5 Bounded Hermitian Operators on $H^1_{\mathcal{H}}$

In this section, we show that if $\mathcal{G}$ is a bounded generator of a one parameter group of isometries on $H^1_{\mathcal{H}}$, then the induced group of disk automorphisms must be the constant group. Berkson, Kaufman, and Porta provided several results on disk automorphism. Using the classification system in Proposition 1.4.1, Berkson, Kaufman, and Porta further showed that every one parameter group of disk automorphisms can be uniquely extended to a planar group.

Theorem 1.5.1. *[2, Theorem 1.10] Every one parameter group, $\{\phi_t\}$ of disk automorphisms can be uniquely extended to a planar group of fractional linear transformations, $\{\Phi_w\}, w \in \mathbb{C}$.*

(i) If $\{\phi_t\}$ is of type (i), then

$$\Phi_w(z) = \frac{(e^{icw} - |\tau|^2)z + \tau(1 - e^{icw})}{\bar{\tau}(e^{icw} - 1)z + 1 - |\tau|^2 e^{icw}} \tag{1.5.1}$$

where $c \neq 0$ and τ and $\bar{\tau}^{-1}$ are the common fixed points of the extended complex plane and $|\tau| < 1$.

(ii) If $\{\phi_t\}$ is of type (ii), then

$$\Phi_w(z) = \frac{(1 - icw)z + icw\alpha}{-ic\bar{\alpha}z + 1 + icw} \tag{1.5.2}$$

where $c \neq 0$ and $\alpha \in \mathbb{C}$ is the common fixed point of $\{\phi_t\}$, and $|\alpha| = 1$.

(iii) If $\{\phi_t\}$ is of type (iii), then

$$\Phi_w(z) = \frac{(\beta e^{cw} - \alpha)z + \alpha\beta(1 - e^{cw})}{(e^{cw} - 1)z + \beta - \alpha e^{cw}} \tag{1.5.3}$$

where $c > 0$ and $\alpha, \beta \in \mathbb{C}$ are the distinct common fixed points of $\{\phi_t\}$ and $|\alpha| = |\beta| = 1$.

Remark 1.5.2. *By finding $\frac{\partial}{\partial t}\Phi_t|_{t=0}$ for each type, we get the invariance polynomial.*

(i) For a group of type (i):

$$R = \frac{-ic}{1 - |\tau|^2}(\bar{\tau}z^2 - (1 + |\tau|^2)z - \tau)) \tag{1.5.4}$$

where $c \neq 0$ and τ and $\bar{\tau}^{-1}$ are the common fixed points of the extended complex plane and $|\tau| < 1$.

(ii) For a group of type (ii):

$$R = i\bar{\alpha}c(z^2 - 2\alpha - \alpha^2) \tag{1.5.5}$$

where $c \neq 0$ and $\alpha \in \mathbb{C}$ is the common fixed point of $\{\phi_t\}$, and $|\alpha| = 1$.

(iii) For a group of type (iii):

$$R = \frac{c}{\beta - \alpha}(z^2 - (\alpha + \beta)z + \alpha\beta), \tag{1.5.6}$$

where $c > 0$ and $\alpha, \beta \in \mathbb{C}$ are the distinct common fixed points of $\{\phi_t\}$ and $|\alpha| = |\beta| = 1$.

We use the generator found in Theorem 1.4.5 and the forms of the disk automorphisms given to determine the form of the group on $H^1_{\mathcal{H}}$ if the generator is bounded.

Theorem 1.5.3. *Let* $\{T_t f(z) = \phi_t'(z)U_t f(\phi_t(z))\}$ *be a one parameter group of isometries on* $H^1_{\mathcal{H}}$ *where* $\{\phi_t\}$ *is a one parameter group of disk automorphisms generated by R, and* $\{U_t\}$ *is a one parameter group of isometries on* $\mathcal{H}$ *generated by A. If* $\mathcal{G}$ *is a bounded generator of* $\{T_t\}$*, then* $\{\phi_t\}$ *must be the constant group.*

Proof.

Fix $v \in \mathcal{H}$ with $\|v\|_{\mathcal{H}} = 1$. Define $e_n = z^n \cdot v$ and assume $\mathcal{G}$ is bounded. Then there is some B such that:

$$\frac{\|\frac{\partial}{\partial t}(\phi_t')|_{t=0} f + Rf' + Af\|}{\|f\|} < B \tag{1.5.7}$$

for all $f \in H^1_{\mathcal{H}}$. Notice that $\|e_n\| = \|v\|_{\mathcal{H}} = 1$ for $n = 0, 1, 2, \ldots$.

Let $f = e_0$. Then

$$\|\frac{\partial}{\partial t}(\phi_t')|_{t=0} v + Av\| < B \tag{1.5.8}$$

Let $f = e_n$ for $n \in \mathbb{N}$. Then

$$\|\frac{\partial}{\partial t}(\phi_t')|_{t=0} v + nz^{n-1}Rv + Av\| < B \tag{1.5.9}$$

Thus

$$\|nz^{n-1}Rv\| < B + \|\frac{\partial}{\partial t}(\phi_t')|_{t=0} v + Av\| < 2B \tag{1.5.10}$$

for all $n \in \mathbb{N}$. Therefore $R = 0$.

Using the form of the invariance polynomial given in equation (1.4.4), this implies that $a'(0) = b'(0) = 0$. Evaluation of the partial derivatives of a_t in types (i),(ii), and (iii) as given in 1.5.4, 1.5.5, and 1.5.6 gives:

1. For type (i):

$$R = \frac{-ic}{1-|\tau|^2}(\overline{\tau}z^2 - (1+|\tau|^2)z - \tau) = 0 \tag{1.5.11}$$

   which implies that $c = 0$ or $\tau = \overline{\tau} = 0$, both of which are contradictions.

2. For type (ii):

$$R = i\bar{\alpha}c(z^2 - 2\alpha - \alpha^2) = 0 \tag{1.5.12}$$

implies $c = 0$ or $\alpha = 0$, both of which are contradictions

3. For type (iii):

$$R = \frac{c}{\beta - \alpha}(z^2 - (\alpha + \beta)z + \alpha\beta) \tag{1.5.13}$$

which implies $\alpha = \beta$, $\alpha = \beta$ or $\alpha = \beta = 0$, all of which are contradictions.

Hence $\{\phi_t\}$ is not a group of type (i), (ii), or (iii), so must be the constant group.

The following corollary follows immediately:

Corollary 1.5.4. *If $\mathcal{G}$ is a bounded hermitian operator on $H^1_{\mathcal{H}}$, then $\mathcal{G} f = Af$, where A is the hermitian operator on $\mathcal{H}$.*

1.6 Unbounded Hermitian Operators on $H^1_{\mathcal{H}}$

In this section we examine a specific case for $\{\phi_t\}$ and find the point spectrum of the generator in this case. We leave open the form of the resolvent and spectrum as well as any examination in the general case. Let $\{\phi_t\}$ be a group of type (iii) with fixed points $\alpha = 1, \beta = -1$ defined with the notation in equation (1.4.3) by

$$a_t = 1 \qquad\qquad b_t = \frac{1 - e^t}{1 + e^t} \tag{1.6.1}$$

Then $\phi_t = \dfrac{(1 + e^{-t})z + (1 - e^{-t})}{(1 - e^{-t})z + (1 + e^{-t})}$. In this case, the generator of $\{\phi_t\}$ is $R = (1 - z^2)/2$ and $\frac{\partial}{\partial t}(\phi'(z))|_{t=0} = R' = -z$.

Applying the form of the generator found in theorem 1.4.5, this implies that

$$\mathcal{G} f = -zf + Af + (1 - z^2)f'/2 \tag{1.6.2}$$

Theorem 1.6.1. *Let $\{T_t\}$ be a one parameter group of isometries on $H^1_{\mathcal{H}}$ where $\{U_t\}$ is a one parameter group of isometries on $\mathcal{H}$ generated by A and*

$$\phi_t = \frac{(1+e^{-t})z+(1-e^{-t})}{(1-e^{-t})z+(1+e^{-t})}. \tag{1.6.3}$$

If $\mathcal{G}$ (as given in 1.6.2) is the generator of $\{T_t\}$, then $\sigma_p(\mathcal{G}) = \emptyset$.

Proof.

We want to find all λ such that $\mathcal{G} f = \lambda f$. Since A is Hermitian, we let $A = Q^*DQ$ where Q is a unitary matrix and D is a diagonal matrix with real entries $a_1, a_2, \ldots, a_n$. Define $K = (1-z^2)f/2$. Then $-zf + (1-z^2)f' + Af = \lambda f$ can be written as $K' = (\frac{2}{1-z^2})(\lambda I - Q^*DQ)K$.

Since Q is independent of z, this implies that $(QK)' = (\frac{2}{1-z^2})(\lambda I - D)QK$. Then for each entry, $g_{i,j}$, in QK, $g'_{i,j} = (\frac{2}{1-z^2})(\lambda - a_i)g_{i,j}$. So for each entry we can consider the differential equation $g'g^{-1} = 2(\lambda - a)(\dfrac{1}{1-z^2})$. Since the argument principle implies that g has no zeroes in the disk and g is analytic, we can integrate both sides, resulting in the solution $g = (\frac{1+z}{1-z})^{\lambda-a}$. Since $f = 2K/(1-z^2)$, $Qf = 2QK/(1-z^2)$. Q is unitary, thus $f \in H^1_{\mathcal{H}}$ if and only if $Qf \in H^1_{\mathcal{H}}$ if and only if $(2/(1-z^2))g_{i,j} \in H^1$ for all i, j. Hence, it remains to show whether $(\dfrac{2}{1-z^2})(\dfrac{1+z}{1-z})^{\lambda-a} \in H^1$.

Let $\lambda - a = \alpha + \beta i$. First consider the case where $\alpha = 0$. For all $t \in \mathbb{R}$, $\left(\dfrac{1+z}{1-z}\right)^{it} \in H^\infty$ [25, p. 11]. Thus, if $\left(\dfrac{2}{1-z^2}\right)\left(\dfrac{1+z}{1-z}\right)^{\beta i} \in H^1$, then $\left(\dfrac{2}{1-z^2}\right)\left(\dfrac{1+z}{1-z}\right)^{\beta i}\left(\dfrac{1+z}{1-z}\right)^{-\beta i} \in H^1$. This implies $\frac{1}{1-z} \in H^1$ which is a contradiction. Thus $\alpha \neq 0$.

Now consider $\alpha \neq 0$. Using the same argument from [25] that for all $t \in \mathbb{R}$, $\left(\dfrac{1+z}{1-z}\right)^{it} \in H^\infty$, if $\left(\dfrac{2}{1-z^2}\right)\left(\dfrac{1+z}{1-z}\right)^{\lambda-a} \in H^1$ then $\left(\dfrac{2}{1-z^2}\right)\left(\dfrac{1+z}{1-z}\right)^{\alpha+i\beta}\left(\dfrac{1+z}{1-z}\right)^{-i\beta} \in H^1$, which implies $\left(\dfrac{1+z}{1-z}\right)^{\alpha} \in H^1$, which is again a contradiction. Thus $\sigma_p(\mathcal{G}) = \emptyset$.

CHAPTER 2

GENERALIZED BI-CIRCULAR PROJECTIONS

A bounded operator, A, on a Banach space is called hermitian if and only if e^{itA} is an isometry for all $t \in \mathbb{R}$. This is equivalent to iA being the generator of a uniformly continuous group of isometries. A projection, P, is hermitian if and only if $P + e^{i\theta}(I-P)$ is an isometry for all $\theta \in \mathbb{R}$ [23]. A projection P on a Banach space X is a called a generalized bicircular projection(GBP) if $P + \lambda(I-P)$ is an isometry of X for some choice of $\lambda(\neq 1)$ with $|\lambda| = 1$. Generalized bicircular projections are also interesting since they have been shown to be bicontractive (i.e bounded linear operators P on a Banach space X with property that $\|P\| \leq 1$ and $\|I-P\| \leq 1$). In the first section of this chapter we find the form of GBPs on the H^p space of the torus using a result of Lal and Merrill. We find that they are the average of the identity and a reflection. For several other spaces, the generalized bi-circular projections have a representation as the average of the identity and an isometric reflection (see [8, 9, 23]). This observation led Botelho to consider the problem as to when a projection on $C(\Omega)$, (Ω compact Hausdorff) could be represented as the convex combination of surjective isometries [7]. In the second part of the chapter, we solve the problem of when the average of two isometries of vector valued S^p spaces of the disk is a projection. The proof of the theorem further provides results for both the vector and scalar valued H^p spaces of the disk. While in the vector cases, the average need not be a GBP, in the scalar case we find that this only happens when the projection in question is a GBP.

2.1 GBPs on H^p spaces of the Torus

Let $A(\mathbb{T}^2)$ denote the algebra of continuous , complex valued functions on the torus $\mathbb{T}^2 = \{(z,w) : |z| = |w| = 1\}$ which are uniform limits of polynomials in $z^n w^m$ where $(m,n) \in \mathcal{S} = \{(m,n) : n > 0\} \cup \{(m,0) : m \geq 0\}$. Let dm denote the normalized Haar measure on $\mathbb{T}^2$ and $\mathbf{H}^p$ denote the Banach space consisting of the closure of $A(\mathbb{T}^2)$ in $L^p(dm)$ (norm closure for $1 \le p < \infty$, w^*-closure for $p = \infty$). In this paper we characterize the generalized bicircular projections on $\mathbf{H}^p(\mathbb{T}^2)$. Berkson and Porta note that the study of this space is more involved than $H^p(D)$ due to the asymmetrical nature of the independent variables (see [4]). We recall a result of Lal and Merrill.

Theorem 2.1.1. *[21, Theorem 3] A linear operator T of $H^p(\mathbb{T}^2)$ onto $H^p(\mathbb{T}^2)$ ($1 \le p < \infty$, $p \ne 2$) is an isometry if and only if*

$$(Tf)(z,w) = \alpha(\tau'(z))^{1/p} f(\tau(z), w\sigma(z)) \tag{2.1.1}$$

for all $f \in H^p$ where $|z| = |w| = 1$, α is a complex constant of modulus 1, τ is a conformal map of the unit disk onto itself, and σ is a unimodular measurable function on the circle.

It follows from the definition of generalized bicircular projection that the associated isometry must indeed be a surjective isometry. We use this to find the generalized bi-circular projections on $H^p(\mathbb{T}^2)$.

Theorem 2.1.2. *P is a generalized bi-circular projection on $H^p(\mathbb{T}^2)$ if and only if P is trivial*

$$Pf(z,w) = \frac{1}{2}(\pm(\tau'(z))^{1/p} f(\tau(z), w\sigma(z)) + f(z,w)) \tag{2.1.2}$$

where τ is a conformal map of the unit disk onto itself such that $\tau(\tau(z)) = z$ and σ is a unimodular measurable function on the circle such that $\sigma(z)\sigma(\tau(z)) = 1$.

Proof.

P is a generalized bi-circular projection on $H^p(\mathbb{T}^2)$ if and only if $P = \dfrac{T - \lambda I}{1 - \lambda}$ where T is a surjective isometry. The equation $P^2 = P$ implies that for every $f \in H^p(\mathbb{T})$ and every

$z, w \in \mathbb{T}^2$

$$\alpha^2(\tau'(z))^{1/p}(\tau'(\tau(z)))^{1/p} f(\tau(\tau(z)), w\sigma(z)\sigma(\tau(z))) \\ -(\lambda+1)\alpha(\tau'(z))^{1/p} f(\tau(z), w\sigma(z)) + \lambda f(z,w) \\ = 0. \tag{2.1.3}$$

In particular, for $f(z,w) = 1$, (2.1.3) yields $\alpha^2(\tau'(\tau z))^{1/p} = (\lambda+1)\alpha(\tau'(z))^{1/p} - \lambda$. Then (2.1.3) reduces to

$$(\lambda+1)\alpha(\tau'(z))^{1/p}[f(\tau(\tau(z)), w\sigma(z)\sigma(\tau(z))) - f(\tau(z), w\sigma(z))] \\ = \lambda[f(\tau(\tau(z)), w\sigma(z)\sigma(\tau(z))) - f(z,w)]. \tag{2.1.4}$$

Now consider $f(z,w) = z$ and $f(z,w) = z^2$. Then (2.1.4) reduces to

$$(\lambda+1)\alpha(\tau'(z))^{1/p}[\tau(\tau(z)) - \tau(z)] = \lambda[\tau(\tau(z)) - z] \tag{2.1.5}$$

and

$$(\lambda+1)\alpha(\tau'(z))^{1/p}[(\tau(\tau(z)))^2 - (\tau(z))^2] = \lambda[(\tau(\tau(z))^2 - z^2], \tag{2.1.6}$$

respectively. By substituting (2.1.5) into (2.1.6) we get

$$(\tau(\tau(z)) - z)(\tau(z) - z) = 0. \tag{2.1.7}$$

Thus, $\forall z$, $\tau(\tau(z)) = z$ or $\tau(z) = z$. In either case $\tau(\tau(z)) = z$.

Note that by substituting $z = \tau(\tau(z))$ into (2.1.5) we get $(\lambda+1)(\tau'(z))^{1/p}[z - \tau(z)] = 0$. Therefore $\lambda = -1$ or $\forall z$, $\tau(z) = z$.

Now consider $f(z,w) = w$ and $f(z,w) = w^2$. Then (2.1.4) reduces to

$$(\lambda+1)\alpha(\tau'(z))^{1/p}[w\sigma(z)\sigma(\tau(z)) - w\sigma(z)] = \lambda[w\sigma(z)\sigma(\tau(z)) - w] \tag{2.1.8}$$

and

$$(\lambda+1)\alpha(\tau'(z))^{1/p}[(w\sigma(z)\sigma(\tau(z)))^2-(w\sigma(z))^2]=\lambda[(w\sigma(z)\sigma(\tau(z)))^2-w^2], \qquad (2.1.9)$$

respectively.

Substituting (2.1.8) into (2.1.9) we get

$$(\sigma(z)\sigma(\tau(z))-1)(\sigma(z)-1)=0. \qquad (2.1.10)$$

Thus, $\forall$z, $\sigma(z)\sigma(\tau(z))=1$ or $\sigma(z)=1$. Note that if $\tau(z)=z$, then $\sigma(z)=\pm1$. However, since we showed earlier that $\tau(\tau(z))=z$ for all z, this implies $\sigma(z)\sigma(\tau(z))=1$ for all z. Then (2.1.3) can be reduced to

$$\alpha^2 f(z,w)-(\lambda+1)\alpha(\tau'(z))^{1/p}f(\tau(z),w\sigma(z))+\lambda f(z,w)=0. \qquad (2.1.11)$$

There are two cases to consider.

1. If for all z, $\tau(z)=z$ and $\sigma(z)=1$, then the projections are $Pf(z,w)=0$ and $Pf(z,w)=f(z,w)$.

2. If $\tau(\tau(z))=z$ and $\sigma(z)\sigma(\tau(z))=1$, then recall from earlier that $\lambda=-1$. Then (2.1.11) can be written as

$$\alpha^2 f(z,w)-f(z,w)=0. \qquad (2.1.12)$$

Thus, $\alpha=\pm1$.

This yields the projections $P=\frac{1}{2}(\pm(\tau'(z))^{1/p}f(\tau(z),w\sigma(z))+f(z,w))$.

We recall a second theorem of Lal and Merrill .

Theorem 2.1.3. *[21, Theorem 1] A linear operator T of $\boldsymbol{H}^\infty(\mathbb{T}^2)$ onto $\boldsymbol{H}^\infty(\mathbb{T}^2)$ is an isometry if and only if*

$$(Tf)(z,w) = \alpha f(\tau(z), w\sigma(z)) \qquad (f \in \boldsymbol{H}^\infty(\mathbb{T}^2); |z| = |w| = 1),$$

where α is a complex constant of modulus 1, τ is a conformal map of the unit disk onto itself, and σ is a unimodular measurable function.

Proposition 2.1.4. *P is a generalized bi-circular projection on $H^\infty(\mathbb{T}^2)$ if and only if P is trivial or*

$$Pf(z,w) = \frac{1}{2}(\pm f(\tau(z), w\sigma(z)) + f(z,w)) \tag{2.1.13}$$

where $\tau(\tau(z)) = z$ and $\sigma(z)\sigma(\tau(z)) = 1$ where τ is a conformal map of the unit disk onto itself and σ is a unimodular measurable function on the circle.

Proof.

By defining $1/\infty = 0$, the proof for this is identical to the proof of Theorem 2.1.2.

2.2 The Average of Two Isometries on $S_K^p(D)$

The space $S_K^p(D)$ with $p \neq 2$, is the space of analytic functions f defined on D with values in K such that f' is in H_K^p and K is a separable complex Hilbert space. This space is equipped with the norm

$$\|f\| = \|f(0)\|_K + \|f'\|_p. \tag{2.2.1}$$

The form of generalized bi-circular projections on $S_K^p(D)$ were found by Botelho and Jamison [9]. In this section we find when the average of two isometries on $S_K^p(D)$ is a projection. We note that in some of the earlier work on generalized bi-circular projections that the projections which are the average of two isometries can be represented as the average of the identity and an isometric reflection (see [8]). Clearly, every such projection is bi-contractive. Motivated by these results, we ask when the average of two isometries on $S_K^p(D)$ is a projection. The representation of surjective isometries on this $S_K^p(D)$ is given by Hornor and Jamison.

Theorem 2.2.1. *[19, Theorem 14] Let T be a linear isometry of $S_K^p(D)$ onto $S_K^p(D)$, $p \neq 2$, and K a separable complex Hilbert space. There exist unitary operators U and V on K, and a disk automorphism ϕ such that for all $f \in S_K^p(D)$ and all $z \in D$,*

$$Tf(z) = Vf(0) + U\int_0^z \phi'(t)^{1/p} f'(\phi(t))\,dt \tag{2.2.2}$$

The next theorem is the main result of this section and gives necessary and sufficient conditions for the average of two isometries to be a projection on $S_K^p(D)$. The proof relies on the form of isometries on $S_K^p(D)$ and on a somewhat tedious case analysis involving special subsets of the disk defined by the isometry.

Theorem 2.2.2. *If P is the average of two isometries $T_j f(z) = V_j f(0) + U_j \int_0^z \phi_j'(t)^{1/p} f'(\phi_j(t))\,dt$, $j = 1,2$ on $S_K^p(D)$, where U_j and V_j are unitary operators on K and ϕ_j is a disk automorphism, then P is a projection if and only if*

$$Pf(z) = P_V f[(0)] + \int_0^z P_U(f')(t)\,dt, \tag{2.2.3}$$

where $P_V = (V_1+V_2)/2$ and P_U is trivial or it is represented in one of the following forms:

$$P_U f(z) = \frac{f(z) + [\phi'(z)]^{1/p} U f(\phi(z))}{2}, \text{ with } \phi\circ\phi(z) = z, \text{ and } U^2 = Id, \tag{2.2.4}$$

$$P_U f(z) = \frac{U_1+U_2}{2} f(z), \text{ and } \left(\frac{U_1+U_2}{2}\right)^2 = \left(\frac{U_1+U_2}{2}\right). \tag{2.2.5}$$

Proof.

Let $P = \dfrac{T_1+T_2}{2}$ be a projection on $S_K^p(D)$, and let V_1 and V_2 be the unitary operators on K associated with T_1 and T_2 respectively. By considering the projection equation

$P^2 = P$ only for the constant functions, we immediately obtain the operator equation:

$$\frac{(V_1 + V_2)^2}{2} = \frac{(V_1 + V_2)}{2}. \tag{2.2.6}$$

Using this information together with the projection equation $P^2 = P$ and differentiation, we find that

$$I_1^2 + I_1 I_2 + I_2 I_1 + I_2^2 = 2I_1 + 2I_2 \tag{2.2.7}$$

where the I_j are the associated isometries on H_K^p and are given by
$I_j f = \left(\phi_j'\right)^{1/p} U_j f\left(\phi_j\right)$ for $j = 1, 2$.

This implies that the average of I_1 and I_2 is a projection. This last equation easily leads to the following: For every $f \in H_K^p$ and $z \in D$,

$$\begin{aligned} &\left[(\phi_1 \circ \phi_1)'(z)\right]^{1/p} U_1^2 f(\phi_1 \circ \phi_1(z)) + \left[(\phi_1 \circ \phi_2)'(z)\right]^{1/p} U_1 U_2 f(\phi_1 \circ \phi_2(z)) \\ &+ \left[(\phi_2 \circ \phi_1)'(z)\right]^{1/p} U_2 U_1 f(\phi_2 \circ \phi_1(z)) + \left[(\phi_2 \circ \phi_2)'(z)\right]^{1/p} U_2^2 f(\phi_2 \circ \phi_2(z)) \\ &\qquad = 2\left(\phi_1'(z)\right)^{1/p} U_1 f(\phi_1(z)) + 2\left(\phi_2'(z)\right)^{1/p} U_2 f(\phi_2(z)) \end{aligned} \tag{2.2.8}$$

To aid in our analysis, we partition the disk as follows:

$$\begin{aligned} X_0 &= \{z \in D : \phi_1(z) = \phi_2(z) = z\} \\ X_1 &= \{z \in D : \phi_1(z) = z, \phi_2(z) \neq z\} \\ X_2 &= \{z \in D : \phi_1(z) \neq z, \phi_2(z) = z\} \\ X_3 &= \{z \in D : \phi_1(z) \neq z, \phi_2(z) \neq z\}. \end{aligned} \tag{2.2.9}$$

Since disk automorphisms must have a fixed point, then $X_3 \neq D$.

First, let $z_1 \in X_1$, so we have $\phi_1(z_1) = z_1, \phi_2(z_1) \neq z_1$. This implies that

$$(\phi_1 \circ \phi_1)(z_1) = \phi_1(z_1) = z_1, \quad (\phi_1 \circ \phi_2)(z_1) \neq z_1, \quad (\phi_2 \circ \phi_1)(z_1) = \phi_2(z_1) \neq z_1. \quad (2.2.10)$$

If $(\phi_2 \circ \phi_2)(z_1) \neq \phi_1(z_1)$, then we can find a polynomial l such that $l(\phi_1(z_1)) = l(\phi_1 \circ \phi_1(z_1)) = 1$, $l(\phi_1 \circ \phi_2(z_1)) = l(\phi_2 \circ \phi_1(z_1)) = 0$, and $l(\phi_2 \circ \phi_2(z_1)) = 0$. If we let $v \in K$ and consider $f(z) = l(z)v$, then, for $z = z_1$, equation (2.2.8) becomes

$$U_1^2 v = 2U_1 v, \quad (2.2.11)$$

which contradicts the fact that U_1 is an unitary operator on K. So, we must have $(\phi_2 \circ \phi_2)(z_1) = \phi_1(z_1) = z_1$.

If $(\phi_1 \circ \phi_2)(z_1) \neq \phi_2(z_1)$, then we can find a polynomial l such that $l(\phi_1(z_1)) = l(\phi_1 \circ \phi_1(z_1)) = l(\phi_2 \circ \phi_2(z_1)) = 0$, $l(\phi_2(z_1)) = l(\phi_2 \circ \phi_1(z_1)) = 1$, and $l(\phi_1 \circ \phi_2(z_1)) = 0$. If we let $v \in K$ and consider $f(z) = l(z)v$, then, for $z = z_1$, equation (2.2.8) becomes

$$U_2 U_1 v = 2U_2 v, \quad (2.2.12)$$

which contradicts U_1, U_2 being unitary operators on K. Thus, we have $(\phi_1 \circ \phi_2)(z_1) = \phi_2(z_1)$. But this implies that ϕ_1 has two distinct fixed points z_1, and $\phi_2(z_1)$, hence

$$\phi_1 = Id, \quad (2.2.13)$$

and (2.2.8) becomes

$$\begin{aligned} & U_1^2 f(z) + \phi_2'(z)^{1/p} \left[U_1 U_2 f(\phi_2(z)) + U_2 U_1 f(\phi_2(z)) \right] \\ & + \left[(\phi_2 \circ \phi_2)'(z) \right]^{1/p} U_2^2 f(\phi_2 \circ \phi_2(z)) \\ = \; & 2U_1 f(z) + 2\left(\phi_2'(z)\right)^{1/p} U_2 f(\phi_2(z)). \quad (2.2.14) \end{aligned}$$

If there is any $w \in D$ such that $(\phi_2 \circ \phi_2)(w) \neq w$, then we can find a polynomial l such that $l(w) = 1$, and $l(\phi_2 \circ \phi_2(w)) = l(\phi_2(w)) = 0$. If we let $v \in K$ and consider $f(z) = l(z)v$, then, for $z = w$, equation (2.2.14) becomes

$$U_1^2 v = 2U_1 v, \tag{2.2.15}$$

which is again a contradiction. Therefore, we must have $(\phi_2 \circ \phi_2)(w) = w$, for all $w \in D$, hence ϕ_2 is a reflection. Thus, the equation (2.2.8), (and hence (2.2.14)) becomes

$$\begin{aligned} & U_1^2 f(z) + \phi_2'(z)^{1/p} \left[U_1 U_2 f(\phi_2(z)) + U_2 U_1 f(\phi_2(z))\right] + U_2^2 f(z) \\ = \;& 2U_1 f(z) + 2\left(\phi_2'(z)\right)^{1/p} U_2 f(\phi_2(z)). \end{aligned} \tag{2.2.16}$$

We assume that $\phi_2 \neq Id$, so there must be some $u \in D$, such that $\phi_2(u) \neq u$. Then we consider a polynomial l such that $l(u) = 0$ and $l(\phi_2(u)) = 1$. If we let $v \in K$ and consider $f(z) = l(z)v$, then, for $z = u$, equation (2.2.16) becomes

$$\frac{U_1 U_2 + U_2 U_1}{2} v = U_2 v. \tag{2.2.17}$$

By the geometry of the unit ball of K, we must have $U_1 U_2 v = U_2 U_1 v = U_2 v$, and thus

$$U_1 v = v, \text{ for every } v \in K. \tag{2.2.18}$$

With this (2.2.16) becomes

$$\begin{aligned} & f(u) + 2\phi_2'(u)^{1/p} U_2 f(\phi_2(u)) + U_2^2 f(u) \\ & = 2f(u) + 2\phi_2'(u)^{1/p} U_2 f(\phi_2(u)). \end{aligned} \tag{2.2.19}$$

Equivalently, we have

$$U_2^2 f(u) = f(u), \tag{2.2.20}$$

which implies that U_2 is an isometric reflection, $U_2^2 = Id$.

Hence, the projection is given by

$$P_U f(z) = \frac{f(z) + \phi_2'(z)^{1/p} U_2 f(\phi_2(z))}{2}, \text{ for every } z \in D \tag{2.2.21}$$

Now, we consider $z_2 \in X_2$, so we have $\phi_1(z_2) \neq z_2$, $\phi_2(z_2) = z_2$. Similarly as before, we obtain $\phi_2 = Id$, ϕ_1 is a reflection, $U_2 v = v$, and $U_1^2 = Id$. Hence the projection becomes

$$P_U f(z) = \frac{f(z) + \phi_1'(z)^{1/p} U_1 f(\phi_1(z))}{2}, \text{ for every } z \in D \tag{2.2.22}$$

We analyze the possible cases that arise for two distinct points.

If X_0 has at least two distinct points, then both $\phi_1 = Id$, $\phi_2 = Id$. In this case, if we let $f(z) = v$ for $v \in K$, (2.2.8) becomes

$$U_1^2 v + U_1 U_2 v + U_2 U_1 v + U_2^2 v = 2U_1 v + 2U_2 v, \tag{2.2.23}$$

for any $z \in X_0$. This is equivalent to

$$\left(\frac{U_1 + U_2}{2}\right)^2 v = \left(\frac{U_1 + U_2}{2}\right) v. \tag{2.2.24}$$

Hence, as long as $|X_0| > 1$ the projection becomes

$$P_U f(z) = \frac{U_1 + U_2}{2} f(z), \text{ for every } z \in D. \tag{2.2.25}$$

The last possibility is when we consider two distinct points $z_0 \in X_0$ and $z_3 \in X_3$. Then, $\phi_1(z_0) = \phi_2(z_0) = z_0$, and $\phi_1(z_3) \neq z_3$, $\phi_2(z_3) \neq z_3$. This implies that $\phi_1(z_3) \neq \phi_1 \circ \phi_1(z_3)$, and $\phi_1(z_3) \neq \phi_1 \circ \phi_2(z_3)$.

If $\phi_1(z_3) = \phi_2(z_3)$, then $\phi_1(z) = \phi_2(z)$, for all $z \in D$. In this case (2.2.8) is reduced to

$$\left[(\phi_1 \circ \phi_1)'(z)\right]^{1/p} \left(\frac{U_1 + U_2}{2}\right)^2 f(\phi_1 \circ \phi_1(z)) \tag{2.2.26}$$

$$= \left(\phi_1'(z)\right)^{1/p} \left(\frac{U_1 + U_2}{2}\right) f(\phi_1(z)). \tag{2.2.27}$$

Suppose there is point $w \in D$ such that $\phi_1(w) \neq w$; this implies that $\phi_1 \circ \phi_1(w) \neq \phi_1(w)$. Consider the polynomial l such that $l(\phi_1 \circ \phi_1)(w) = 0$, and $l(\phi_1(w)) = 1$. If we let $f(z) = l(z)v$, for a $v \in K$, then the above equation reduces to

$$0 = \left(\phi_1'(w)\right)^{1/p} \left(\frac{U_1 + U_2}{2}\right) v. \tag{2.2.28}$$

In this case $U_1 = -U_2$, and

$$P_U f(w) = 0. \tag{2.2.29}$$

If $\phi_1(z_3) \neq \phi_2(z_3)$, then $\phi_1 \circ \phi_2(z) = \phi_2 \circ \phi_1(z)$, for all $z \in D$, cf. [26]. Hence, (2.2.8) is reduced to

$$\begin{aligned}
&\left[(\phi_1 \circ \phi_1)'(z)\right]^{1/p} U_1^2 f(\phi_1 \circ \phi_1(z)) \\
&+ \left[(\phi_1 \circ \phi_2)'(z)\right]^{1/p} (U_1U_2 + U_2U_1) f(\phi_1 \circ \phi_2(z)) \\
&\qquad + \left[(\phi_2 \circ \phi_2)'(z)\right]^{1/p} U_2^2 f(\phi_2 \circ \phi_2(z)) \\
&\qquad = 2\phi_1'(z)^{1/p} U_1 f(\phi_1(z)) + 2\phi_2'(z)^{1/p} U_2 f(\phi_2(z)).
\end{aligned} \tag{2.2.30}$$

If $\phi_2 \circ \phi_2(z_3) \neq \phi_1(z_3)$, then let l be a polynomial such that $l(\phi_1(z_3)) = 1$, $l(\phi_1 \circ \phi_1)(z_3) = l(\phi_1 \circ \phi_2)(z_3) = l(\phi_2(z_3)) = 0$, and $l(\phi_2 \circ \phi_2)(z_3) = 0$. If $f(z) = l(z)v$, for a $v \in K$, then the above equation reduces to

$$0 = 2\phi_1'(z_3)^{1/p} U_1 v, \tag{2.2.31}$$

which is a contradiction. Hence $\phi_2 \circ \phi_2(z_3) = \phi_1(z_3)$. Since $z_3 \in X_3$, then $\phi_1 \circ \phi_1(z_3) \neq \phi_1 \circ \phi_2(z_3)$. Since $\phi_1 \circ \phi_2(z) = \phi_2 \circ \phi_1(z)$ for all $z \in D$, then we can deduce that $\phi_2(z_3) \neq \phi_1 \circ \phi_2(z_3)$. Let now l be a polynomial such that $l(\phi_1 \circ \phi_2)(z_3) = 1$, and $l(\phi_1(z_3)) = l(\phi_2(z_3)) = l(\phi_2 \circ \phi_2)(z_3) = l(\phi_1 \circ \phi_1)(z_3) = 0$. If $f(z) = l(z)v$, for a $v \in K$, then equation $(2.2.30)$ is reduced to

$$\left[(\phi_1 \circ \phi_2)'(z_3)\right]^{1/p}(U_1U_2 + U_2U_1)v = 0. \tag{2.2.32}$$

Since v was arbitrary, we have that $U_1U_2 + U_2U_1 = 0$. We now use this relation in equation $(2.2.30)$ to obtain

$$\begin{aligned}
&\left[(\phi_1 \circ \phi_1)'(z)\right]^{1/p} U_1^2 f(\phi_1 \circ \phi_1(z)) \\
&+ \left[(\phi_2 \circ \phi_2)'(z)\right]^{1/p} U_2^2 f(\phi_2 \circ \phi_2(z)) \\
&\qquad = 2\phi_1'(z)^{1/p} U_1 f(\phi_1(z)) + 2\phi_2'(z)^{1/p} U_2 f(\phi_2(z)).
\end{aligned} \tag{2.2.33}$$

We recall that at this point we have $\phi_2 \circ \phi_2(z_3) = \phi_1(z_3)$. We find a polynomial l such that $l(\phi_2 \circ \phi_2(z_3)) = l(\phi_(z_3)) = 0$ and $l(\phi_1 \circ \phi_1(z_3)) = l(\phi_(z_3)) = 1$. Now let v be an arbitrary vector in K and set $f(z) = l(z)v$. Using this function in (2.2.33) we get that

$$U_1^2 v = 2U_2 v \tag{2.2.34}$$

which is absurd. This last result concludes the case analysis, and we have shown that if the average of two isometries $I_1 f = (\phi_1')^{1/p} U_1 f(\phi_1)$ and $I_2 f = (\phi_2')^{1/p} U_2 f(\phi_2)$ on H_K^p is a projection P_U, then P_U is trivial or it is represented in one of the following forms:

$$P_U f(z) = \frac{f(z) + \phi'(z)^{1/p} U f(\phi(z))}{2}, \text{ with } \phi \circ \phi(z) = z, \text{ and } U^2 = Id, \tag{2.2.35}$$

$$P_U f(z) = \frac{U_1 + U_2}{2} f(z), \text{ and } \left(\frac{U_1 + U_2}{2}\right)^2 = \left(\frac{U_1 + U_2}{2}\right). \tag{2.2.36}$$

Therefore, if P is the average of two isometries $T_j f(z) = V_j f(0) + U_j \int_0^z (\phi_j'(t))^{1/p} f'(t) dt, j = 1,2$ on S_K^p, then P is a projection if and only if

$$P f(z) = P_V f(0) + \int_0^z P_U(f')(t) dt, \tag{2.2.37}$$

where $P_V = \frac{V_1 + V_2}{2}$ is a projection and V_1 and V_2 are unitary and P_U is either trivial or as above. The converse follows immediately.

The form of the average of two isometries on $H^p(K)$ were also found in the proof of the previous theorem.

Corollary 2.2.3. *If P is the average of two isometries, T_1 and T_2, with $T_j f(z) = U_j \phi_j'(z)^{1/p} f'(\phi_j(z))$ for $j = 1,2$ on H_K^p, where U_j is a unitary operator on K and ϕ_j is a disk automorphism, then P is a projection if and only if P is trivial or can be represented in one of the following forms:*

$$P f(z) = \frac{f(z) + [\phi'(z)]^{1/p} U f(\phi(z))}{2}, \textit{ with } \phi \circ \phi(z) = z, \textit{ and } U^2 = Id, \tag{2.2.38}$$

or

$$P f(z) = \frac{U_1 + U_2}{2} f(z), \textit{ and } \left(\frac{U_1 + U_2}{2}\right)^2 = \left(\frac{U_1 + U_2}{2}\right). \tag{2.2.39}$$

The form of isometries on the scalar valued H^p spaces were found by Forelli.

Theorem 2.2.4. *[14, Theorem 2] If $1 \leq p < \infty$, $p \neq 2$ and T is a linear isometry of H^p onto H^p, then there exists a disk automorphism ϕ and a unimodular constant α such that*

$$Tf = \alpha(\phi')^{1/p} f(\phi). \tag{2.2.40}$$

These isometries are unsurprisingly quite similar to those of the vector valued H^p spaces, and thus, using the same techniques as in this proof, we have the following corollary.

Corollary 2.2.5. *If P is the average of two isometries, T_1 and T_2, with $T_j f(z) = \phi_j'(z)^{1/p} f'\left(\phi_j(z)\right)$ for $j = 1, 2$ on H^p, where ϕ_j is a disk automorphism, then P is a projection if and only if P is trivial or can be represented as follows:*

$$Pf(z) = \frac{f(z) \pm [\phi'(z)]^{1/p} f(\phi(z))}{2}, \text{ with } \phi \circ \phi(z) = z \tag{2.2.41}$$

Remark 2.2.6. *In the scalar H^p case, projections given as the average of two isometries are precisely the GBPs.*

CHAPTER 3

PROBLEMS FOR FUTURE WORK

1. What are the unbounded hermitian operators on $L^p(\mu, \mathcal{H})$, $p \neq 2$, where $\mathcal{H}$ is a Hilbert space and (Ω, Σ, μ) is a finite measure space?

2. What are the unbounded hermitian operators on $H^p(B)$, $p \neq 2$, where $B = \{(z_1, z_2, ..., z_n) | \sum_{k=1}^n |z_k| < 1\}$?

   We have used the forms of the isometries on S^p and $H^1_{\mathcal{H}}$ to discover the unbounded hermitian operators on these spaces. In general, it is unkonwn if a given Banach space supports unbounded hermitian operators. The isometries of $L^p(\mu, \mathcal{H})$ are known and similar to those on $H^p(D)$, but determination of the unbounded hermitian operators will require different techniqes, because we must deal with measurable functions and set transformations rather than analytic functions and disk automorphisms. In the case of $H^p(B)$, we would have to study (C_0) group automorphisms of the ball rather than disk automorphisms, which is considerably more complicated.

3. When is the average of two isometries on $L^p(\mu, X)$, $p \neq 2$, a projection if X has trivial ℓ^p structure?

4. When is the average of two weighted composition operators on $H^p(D)$ a projection?

   Again, solving the problem on $L^p(\mu, X)$, will require us to deal with measurable functions and set transformations, which requires different techniques from analytic functions and disk automorphisms. Finding when the average of two weighted compostition operators on $H^p(D)$ is a generalization of a theorem by Botelho and Jamison. In several spaces, the isomorphisms are weighted composition operators, and they found when their average gave an isometry. In $H^p(D)$, the isometries are not weighted composition operators, so assumptions that can be made about isometries wouldn't apply.

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