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Raluca Ioana Pop Clendenen

WENTZELL BOUNDARY CONDITIONS WITH GENERAL WEIGHTS AND ASYMPTOTIC PARABOLICITY FOR STRONGLY DAMPED WAVES

by

Raluca Ioana Pop Clendenen

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This dissertation is dedicated to Matthew and to all my family.

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Abstract

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In using dynamic or Wentzell boundary conditions in parabolic and hyperbolic problems in partial differential equations, a positive function β on the boundary of the underlying domain arises naturally. The relevant space of functions on the boundary is $L^2(\partial\Omega, dS/\beta)$. In all previous studies, β was positive, continuous, and both β and $1/\beta$ were bounded. In this Thesis, β is merely positive and Lebesgue measurable. In particular, both β and $1/\beta$ can be essentially unbounded. The construction of the appropriate selfadjoint operator on the L^2 space involving both Ω and its boundary is based on quadratic form methods. This allows for more general coefficients in the basic elliptic operators. This leads to new wellposedness results which are applied to give significant extensions of recent asymptotic parabolicity results for strongly damped wave equations.

Contents

1 EXISTENCE OF WEAK SOLUTIONS

1.1 INTRODUCTION

We begin with a bit of recent history. A systematic study of the problem

$$\frac{\partial u}{\partial t} = \Delta u, \quad x \in \Omega \subset\subset R^N, \quad t \in [0, \infty), \tag{1.1}$$

$$u(x, 0) = f(x), \quad x \in \Omega, \tag{1.2}$$

$$\Delta u + \beta \frac{\partial u}{\partial n} + \gamma u - q \Delta_{LB} u = 0, \quad x \in \partial\Omega, \quad t \in [0, \infty), \tag{1.3}$$

was begun by Favini, Goldstein, Goldstein and Romanelli in [8] in 2002. If (1.1) holds in $\Omega \times [0, \infty)$, then the Wentzell boundary condition (1.3) becomes a dynamic boundary condition when in it we replace Δu by $\frac{\partial u}{\partial t}$. As will be explained in detail shortly, the problem (1.1) – (1.3) can be rewritten, for $U = (u|_\Omega, tr(u))$ in

$$H = L^2(\Omega, dx) \oplus L^2(\partial\Omega, dS/\beta),$$

as

$$\frac{dU}{dt} + AU = 0, \; U(0) = F,$$

where

$$H = L^2(\Omega, dx) \oplus L^2(\partial\Omega, dS/\beta),$$

$$A = \begin{pmatrix} -\Delta & 0 \\ \beta \frac{d}{dn} & \gamma - q\Delta_{LB} \end{pmatrix}$$

with A having a suitable domain. Then the Laplacian Δ and the Laplace-Beltrami operator Δ_{LB} can be replaced by more general second order symmetric operators. In [7] (see also [6]), Ω was taken to be a smooth enough bounded domain with $0 < \beta \in C(\partial\Omega)$, $q \in [0, \infty)$. Now we comment on the term "suitable domain" for A.

Let K be the operator A having domain the C^2 functions v on $\overline{\Omega}$ in H for which Av is in H, and v satisfies the boundary condition (1.3). It was shown in [7], [6] that the closure of K is a selfadjoint operator on H, bounded below, and that $(1.1) - (1.3)$ is governed by a (C_0) quasicontractive positive semigroup of operators on H. In other words, the provisional domain of A is a core for the selfadjoint version of A, which is the closure of the original operator. Using the closure (or core) idea specifies the "suitable" domain of A precisely, but this is an abstract characterization and it does not explain it in "everyday objects" such as Sobolev spaces. However, in the bounded domain case in which the boundary $\partial\Omega$ and all the coefficients (including β and γ) are of class C^∞, then Coclite, Favini, Goldstein, Goldstein and Romanelli in [5] proved that the domain which makes A seldadjoint is

$$H^2(\Omega) \text{ if } q = 0, \quad \text{and}$$

$$H^2_*(\Omega) = \{u \in H^2(\Omega) : \ tr(u) \in \ H^2(d\Omega)\} \quad \text{if } q > 0.$$

In this Thesis we allow β to be a positive Lebesgue measurable function on $d\Omega$, and both β and $\frac{1}{\beta}$ can be unbounded. This is a serious complication which seems to prevent the use of the "core" approach used in [7]. Instead we must use the quadratic form approach to construct the desired selfadjoint operator. Use of the

Lax-Milgram Lemma gives an abstract specification of the domain of the selfadjoint version of A on H, but there does not seem to be a way to characterize it concretely, in terms of Sobolev spaces. The extension to allow such a general function β is the main abstract result of the thesis. This permits us to extend significantly the recent results on asymptotic parabolicity of dissipative wave equations to a much more general context.

Let Ω be a proper domain in $\mathbb{R}^N$, that is Ω is an open connected set $\Omega \subset \mathbb{R}^N$, where $\Omega \neq \emptyset$, $\Omega \neq \mathbb{R}^N$ and with a sufficiently regular boundary $\partial\Omega$. We assume that the boundary $\partial\Omega$ is a disjoint union of sufficiently smooth $N-1$ dimensional surfaces. We let β be a positive function on $\partial\Omega$. We assume β is Borel measurable, but it need not be continuous. Thus $0 < \beta(x) < \infty$ for all $x \in \partial\Omega$, but both β and $\frac{1}{\beta}$ can be unbounded.

We start with the Hilbert space triple,

$$V \hookrightarrow H \hookrightarrow V^*, \tag{1.4}$$

where

$$H = L^2(\Omega, dx) \oplus L^2(\partial\Omega, dS/\beta), \tag{1.5}$$

and where

$$V = H^1(\Omega, dx) \oplus L^2(\partial\Omega, dS/\beta). \tag{1.6}$$

Here we use standard Sobolev space notation.

Recall that $u \in H^1(\Omega)$ implies u has a trace, $tr(u) \in L^2_{loc}(\partial\Omega, dS)$. If $\partial\Omega$ bounded then $tr(u) \in L^2(\partial\Omega, dS)$.

The arrow $\hookrightarrow$ indicates that the injection is dense and continuous. If $u \in H^s(\Omega)$, then u has a trace $tr(u)$ in $H^{s-\frac{1}{2}}_{loc}(\partial\Omega)$, and we are only interested in the case $\frac{1}{2} < s \leq 3$. Moreover, $u \longrightarrow tr(u)$ is continous from $H^s_{loc}(\Omega)$ to $H^{s-\frac{1}{2}}_{loc}(\partial\Omega)$, and from $H^s(\Omega)$ to $H^{s-\frac{1}{2}}(\partial\Omega)$ if $\partial\Omega$ is bounded.

Note that $u|_{\partial\Omega} \in C^2(\partial\Omega)$ implies $w = \Delta_{LB} u \in C(\partial\Omega)$, where $\Delta_{LB} = \nabla_\tau \cdot \nabla_\tau$ is the Laplace-Beltrami operator and ∇_τ is the tangential gradient on $\partial\Omega$. It may or may not be true that $w \in L^2(\partial\Omega, dS/\beta)$.

We denote

$$H = L^2(\Omega, dx) \oplus L^2(\partial\Omega, dS/\beta) \text{ by } L^2\left(\overline{\Omega}, d\mu\right), \tag{1.7}$$

where dx denotes the Lebesgue measure on Ω and dS denotes the natural induced surface measure on the boundary $\partial\Omega$. Thus dS/β is the natural surface measure with weight $\frac{1}{\beta(x)}$, $x \in \partial\Omega$. Also, by definition, $d\mu|_{\overline{\Omega}} = dx|_{\Omega} \otimes \frac{dS}{\beta}|_{\partial\Omega}$ (product measure).

Define

$$V_q := \left\{ U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in H^1(\Omega, dx) \oplus H^1_q(\partial\Omega, dS/\beta) : \; u_2 = tr(u_1) \right\} \tag{1.8}$$

where

$$H^1_q(\partial\Omega, dS/\beta) = L^2(\partial\Omega, dS/\beta) \text{ if } q = 0, \tag{1.9}$$

and for $q > 0$,

$$H^1_q(\partial\Omega, dS/\beta) = H^1(\partial\Omega, dS/\beta), \tag{1.10}$$

with norm

$$\|w\|_{H_q^1} = \left(\int\limits_{\partial\Omega} |w(x)|^2 \frac{dS(x)}{\beta(x)} + \int\limits_{\partial\Omega} q \left|\nabla_\tau w(x)\right|^2 dS \right)^{\frac{1}{2}} \tag{1.11}$$

for $q \geq 0$. Here ∇_τ is the tangential gradient, and this gradient is not present when $q = 0$.

We assume $0 < \beta(x) < \infty$ for almost every $x \in \partial\Omega$ relative to surface measure dS.

If we had assumed

$$\int_{\partial\Omega} \mathcal{X}_{\{x\in\partial\Omega:\ \beta(x)=0\}}(y)\, dS_y > 0,$$

then letting

$$\Gamma_0 := \{x \in \partial\Omega : \ \beta(x) = 0\}, \quad dS/\beta \equiv \infty \ \text{ on } \Gamma_0.$$

Thus any $U \in H$ must satisfy $u_2 = 0$ on Γ_0, which implies that U satisfies the Dirichlet boundary condition on Γ_0, and Γ_0 is a nonnull subset of $\partial\Omega$.

Similarly, if

$$\Gamma_\infty = \{x \in \partial\Omega : \ \beta(x) = \infty\}$$

had positive surface measure, then $dS/\beta \equiv 0$ on Γ_∞, so any two choices of $u_2 : \Gamma_\infty \to \mathbb{C}$ are equal a.e., and thus Γ_∞ is irrelevant to our boundary value problem. Thus our desire to impose a Wentzell boundary condition on all of $\partial\Omega$ implies that $\Gamma_0 \cup \Gamma_\infty$ must be a null set relative to the surface measure dS.

Thus $u \in H_q^1(\partial\Omega, dS/\beta)$ if and only if $u \in L^2(\partial\Omega, dS/\beta)$ and $q\nabla_\tau u \in L^2(\partial\Omega, dS)$, and

$$\langle v,w\rangle_{V_q} := \langle v,w\rangle_{H^1(\Omega)} + \langle v,w\rangle_{L^2(\partial\Omega,dS/\beta)} + \int\limits_{\partial\Omega} q\left(\nabla_\tau v\right)\cdot\left(\nabla_\tau \overline{w}\right)dS. \tag{1.12}$$

These spaces are all complex Hilbert spaces.

We want to be certain that $L^2\left(\partial\Omega, dS/\beta\right)$ contain enough functions, so that for instance,

$$L^2\left(\partial\Omega, dS/\beta\right)\cap L^2\left(\partial\Omega, dS\right)$$

is dense in both $L^2\left(\partial\Omega, dS/\beta\right)$ and $L^2\left(\partial\Omega, dS\right)$. For this is sufficient to assume

$$\left\{\begin{array}{c} \widetilde{\delta}=\beta \text{ or } \frac{1}{\beta} \text{ is equal a.e. } [dS] \text{ to} \\ \text{a function } \delta:\partial\Omega\longrightarrow\mathbb{C} \text{ such that} \\ \delta\in L^2_{loc}\left(\partial\Omega\backslash K\right) \text{ where} \\ K=\overline{K} \text{ and } \int_{\partial\Omega}\chi_K\left(x\right)dS\left(x\right)=0. \end{array}\right\} \tag{1.13}$$

Next let

$$Lu = \nabla\cdot\left(\mathcal{A}\left(x\right)\nabla u\right) \tag{1.14}$$

define an operator L on functions $u\in H^2\left(\Omega\right)$, where we assume, for each $x\in\Omega$,

$$\alpha_0 I \le \mathcal{A}\left(x\right) = \mathcal{A}\left(x\right)^* = \left(a_{ij}\left(x\right)\right) \le \alpha_1 I,$$

$$\mathcal{A}\in C^1\left(\overline{\Omega}, Mat_{N\times N}\right),$$

$$\mathcal{A}\left(x\right) \text{ is a real Hermitian } N\times N \text{ matrix}, \tag{1.15}$$

$$0<\alpha_0\le\alpha_1<\infty.$$

We define the Wentzell Boundary Condition to be

$$Lu + \beta \partial_\nu^{\mathcal{A}} u + \gamma u - q\beta \Delta_{LB} u = 0 \ a.e. \ on \ \partial\Omega. \tag{1.16}$$

Here ν is the unit outer normal on $\partial\Omega$,

$$\partial_\nu^{\mathcal{A}} u = (\mathcal{A} \nabla u) \cdot \nu$$

is the conormal derivative with respect to $\mathcal{A}$, and γ is a real function in $L^\infty(\partial\Omega)$.

More generally, we may replace Δ_{LB} by the operator defined by

$$L_\partial u = \nabla_\tau \cdot (\mathcal{B}(x) \nabla_\tau u), \ x \in \partial\Omega,$$

where ∇_τ is the tangential gradient on $\partial\Omega$, and where we assume, for each $x \in \partial\Omega$,

$$\alpha_0 I \leq \mathcal{B}(x) = (b_{ij}(x)) \leq \alpha_1 I,$$

$$\mathcal{B}(x) \ \textit{is a real Hermitian} \ (N-1) \times (N-1) \ \textit{matrix}. \tag{1.17}$$

Here α_0, α_1 are as before.

We temporarily assume $\mathcal{A}$, $\mathcal{B}$ are sufficiently smooth as functions of x, and also that $\partial\Omega$ is sufficiently smooth.

With L we associate the general Wentzell Boundary Condition

$$Lu + \beta \partial_\nu^{\mathcal{A}} u + \gamma u - q\beta L_\partial u = 0 \ a.e. \ on \ \partial\Omega. \tag{1.18}$$

Because $tr\,(u) \in L^2(\partial\Omega, dS/\beta)$, it follows that $\gamma\; tr\,(u) \in L^2(\partial\Omega, dS/\beta)$ when $\gamma \in L^\infty(\partial\Omega)$.

Hypothesis 1.1. *Let each component a_{ij} of the real symmetric $N \times N$ matrix $\mathcal{A}$ be in $L^\infty(\Omega)$. Similarly assume each component b_{ij} of the real symmetric $(N-1) \times (N-1)$ matrix $\mathcal{B}$ is in $L^\infty(\partial\Omega)$. Let $0 < \alpha_0 \leq \alpha_1 < \infty$ satisfy*

$$\alpha_0 I \leq \mathcal{B}(x) \leq \alpha_1 I,$$

$$\alpha_0 I \leq \mathcal{A}(y) \leq \alpha_1 I,$$

for all $x \in \partial\Omega$, $y \in \Omega$.

Let Z consist of all $u \in C^2\left(\overline{\Omega}\right) \cap H$ satisfying the boundary condition (1.18), and by identifying u with $U \in H$ above we view Z as a subspace of H, considering H as $L^2\left(\overline{\Omega}, \mu\right)$ as in (1.7). We view A_0 as the operator $-L$ on H satisfying the Wentzell boundary condition (1.18). That is $D(A_0) = Z$. Then for $u, v \in Z$, identified with $U, V \in H$,

$$\begin{aligned}
\langle A_0 U, V\rangle_H &= \left\langle \begin{pmatrix} -L & 0 \\ 0 & tr\,(-L) \end{pmatrix} U, V \right\rangle \\
&= \langle -Lu, v\rangle_{L^2(\Omega)} + \langle tr\,(-Lu), v\rangle_{L^2\left(\partial\Omega, \frac{dS}{\beta}\right)} \\
&= \int\limits_{\Omega} (\mathcal{A}\nabla u) \cdot \nabla \overline{v} dx + \int\limits_{\partial\Omega} \left(-Lu - \beta \partial_\nu^{\mathcal{A}} u\right) \overline{v} \frac{dS}{\beta}
\end{aligned}$$

by the divergence theorem

$$= \int\limits_{\Omega} (\mathcal{A}\nabla u) \cdot \nabla \overline{v} dx + \int\limits_{\partial\Omega} \gamma u \overline{v} \frac{dS}{\beta} - \int\limits_{\partial\Omega} q\,(L_\partial u)\, \overline{v} dS$$

by the boundary condition (1.18)

$$= \int\limits_{\Omega} (\mathcal{A}\nabla u) \cdot \nabla \overline{v} dx + \int\limits_{\partial\Omega} \frac{\gamma}{\beta} u\overline{v} dS + \int\limits_{\partial\Omega} q \left(\mathcal{B}\nabla_\tau u\right) \cdot \nabla_\tau \overline{v} dS \tag{1.19}$$

by Stokes' theorem on the boundary.

Several conclusions follow now. By Hermitian symmetry of the equality (1.19) in u and v, A_0 is symmetric on H. Moreover, A_0 can be represented by the operator matrix

$$A_0 = \begin{pmatrix} -L & 0 \\ \beta\partial_\nu^{\mathcal{A}} & \gamma - q\beta L_\partial \end{pmatrix}.$$

For $\lambda \in \mathbb{R}$,

$$A_\lambda := A_0 + \lambda I$$

defines the sesquilinear form

$$B_\lambda(U,V) := \int_\Omega (\mathcal{A}\nabla u) \cdot \nabla \overline{v} dx + \int_\Omega \lambda u \overline{v} dx$$

$$+ \int\limits_{\partial\Omega} \left[\frac{(\gamma+\lambda)}{\beta} u\overline{v} + q\left(\mathcal{B}\nabla_\tau u\right) \cdot \nabla_\tau \overline{v} \right] dS \tag{1.20}$$

on $Z \times Z$, which agrees with (1.19) when $\lambda = 0$ and which extends uniquely by continuity to a map $B_\lambda : V_q \times V_q \longrightarrow \mathbb{C}$ (see (1.8)).

Then the sesquilinear form B_λ satisfies, for all U, $V \in V_q$,

$$B_\lambda(U,V) = \overline{B_\lambda(V,U)}. \tag{1.21}$$

1.2 THE MAIN EXISTENCE THEOREM IN HILBERT SPACE

Note that A_0 in Section 1.1 is bounded below. The symmetric operator G constructed in Theorem 1.1 is bounded above.

We begin this section by stating our main theorem.

Theorem 1.1. *Let Ω be a proper domain in $\mathbb{R}^N$ with a sufficiently smooth nonempty boundary $\partial\Omega$. Assume Hypothesis 1.1. Let B_λ be as in* (1.20). *Then $B = B_0$ defines a Hermitian sesquilinear form on $V_q \oplus V_q$ which uniquely determines a selfadjoint operator G on the Hilbert space H. [For $U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, formally $GU = V$ where $v_1 = \nabla \cdot \mathcal{A}\nabla u_1$; and*

$$G = \begin{pmatrix} L & 0 \\ \beta\partial_\nu^{\mathcal{A}} & \gamma - q\beta L_\partial \end{pmatrix}$$

with suitable domain]. The selfadjoint operator G on the Hilbert space H is bounded above, and G generates an analytic (C_0) positive quasicontractive semigroup.

In Theorem 1.1, the domain of G is $D(G) = D(A_\lambda)$, which is specified by the last sentence of the statement of the Lax-Milgram Lemma on the next page.

We remark that if in Theorem 1.1, Ω is bounded, $0 < \beta \in C(\delta\Omega)$ and all the coefficients are smooth enough, then the basic calculation which includes (1.18) can be done in reverse order. Thus if u is the (generalized) solution of the parabolic problem constructed in Theorem 1.1, and if (using obvious notation) $u \in C^2(\overline{\Omega}) \cap BC$, then $u(t)$ is in the core discussed at the beginning of this section for all $t \geq 0$, and u is a strong solution.

Proof. We consider a Hilbert space triple

$$W_0 \hookrightarrow H_0 \hookrightarrow W_0^* \tag{1.22}$$

where the injections are continuous and dense, and W_0^* is the (variational) dual space of W_0 with respect to H_0.

Now we recall the following well known result.

Lax-Milgram Lemma. *Let $B_0 : W_0 \times W_0 \to \mathbb{K}$ $(= \mathbb{R}$ or $\mathbb{C})$ be sesquilinear. Suppose $\lambda \in I_0 := (\lambda_0, \infty)$ or $[\lambda_0, \infty)$, for some $\lambda_0 \in \mathbb{R}$, and let*

$$B_\lambda(u, v) = B_0(u, v) + \lambda \langle u, v \rangle_{H_0}.$$

Assume that for all $\lambda \in I_0$, there exists positive constants C_1, C_2 (which depend on λ) such that, for all $u, v \in W_0$,

1) $B_\lambda(u, v) = \overline{B_\lambda(v, u)}$,

2) $|B_\lambda(u, v)| \leq C_1 \|u\|_{W_0} \|v\|_{W_0}$ *for all* $u, v \in W_0$

3) $\operatorname{Re}(B_\lambda(u, u)) \geq C_2 \|u\|_{W_0}^2$ *for all* $u \in W_0$.

Then for each $f \in H_0$ *there is a unique* $u \in W_0$ *such that*

$$B_\lambda (u, v) = \langle f, v \rangle_{H_0} \quad \textit{for all } v \in W_0 \textit{ and for any fixed } \lambda \in I_0.$$

Furthermore there exists a selfadjoint operator $A_\lambda = A_\lambda^*$ *mapping* $D(A_\lambda) \subset W_0$ *to* H_0*, with* $w \in D(A_\lambda)$ *if and only if* $w \in W_0$ *and* $A_\lambda w \in H_0$*, in which case* $\langle A_\lambda w, v \rangle_{H_0} = B_\lambda (w, v)$ *for each* $v \in W_0$.

The operators $A_{\lambda_1} - \lambda_1 I$ and $A_{\lambda_2} - \lambda_2 I$ coincide for all $\lambda_1, \lambda_2 \in I_0$. Thus we can define $A = A_\lambda - \lambda I$ for $\lambda \in I_0$. Then $D(A) = D(A_\lambda)$ and $A = A^*$ on H_0. Furthermore, $I_0 \subset \rho(A)$. There is a more general version of the Lax-Milgram Lemma in which B_0 need not be Hermitian.

We want to solve the elliptic problem with the general Wentzell boundary conditions using the Lax-Milgram Lemma.

That is, we want to solve

$$\begin{gathered} \lambda u_1 - L u_1 = f_1 \textit{ in } \Omega \\ \beta \partial_\nu^{\mathcal{A}} u_1 + (\lambda + \gamma) u_2 - q\beta L_\partial u_2 = f_2 \textit{ on } \partial\Omega \end{gathered} \tag{1.23}$$

in a suitable generalized sense. If u and f are smooth enough, then the basic calculation on page 7 says that $Lu = \lambda u + f$ in $\overline{\Omega}$ and the boundary condition (1.16) holds, then the divergence theorem applies to show $B_\lambda (U, V) = \langle U, F \rangle_H$. Conversely, if this holds and if u, f are smooth enough, then the steps in this argument can be done in reverse order. The last equation (for all V) is what we mean by saying that u is a generalized solution of the original problem. But at the level of generality of Theorem 1.1, the generalized solution need not be a strong solution, and we do not expect u to belong to $H^2(\Omega)$.

We emphasize that we are assuming no regularity on $\mathcal{A}$, $\mathcal{B}$, γ, β except Lebesgue measurability, plus (1.13), (1.16), (1.17), $\gamma \in L^{\infty}(\partial\Omega, \mathbb{R})$ and $0 < \beta(x) < \infty$ for a.e. $x \in \partial\Omega$. In this case, u is a solution of (1.23) means $U = \begin{pmatrix} u \\ tr(u) \end{pmatrix}$ maps $[0, \infty)$ into V_q and satisfies

$$B_\lambda (U, V) = \langle F, V \rangle_H \quad \text{for all } v \in V_q$$

when $F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in H$. (In fact, this makes sense in the more general case when $F \in V_q^*$.) This is our definition of generalized weak solution. The Lax-Milgram lemma supplies us with existence and uniqueness of weak (or generalized) solutions.

Our derivation of (1.19) shows that the Hypothesis 1) of the Lax-Milgram Lemma is satisfied by B_λ; see (1.20).

Now we check Hypotheses 2) and 3). Then it will follow that there exists a unique weak solution $u_1 \in H^1(\Omega, dx) \oplus H_q^1(\partial\Omega, dS/\beta)$ of problem (1.23) for λ real and large enough. We have

$$B_\lambda (U, V) = \int_\Omega (\mathcal{A}\nabla u) \cdot \overline{\nabla v} dx + \int_{\partial\Omega} (\lambda + \gamma) u\overline{v} dS/\beta$$

$$+ \int_\Omega \lambda u\overline{v} dx + \int_{\partial\Omega} q (\mathcal{B}\nabla_\tau u) \cdot \overline{\nabla_\tau v} dS .$$

By the Cauchy-Schwarz inequality we have

$$|B_\lambda (U, V)| \leq \max\{\alpha_1, |\lambda|\} \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$$

$$+\left(|\lambda|+(ess\sup\gamma)\right)\cdot\|u\|_{L^2(\partial\Omega,dS/\beta)}\|v\|_{L^2(\partial\Omega,dS/\beta)}$$

$$+q\|\nabla_\tau u\|_{L^2(\partial\Omega,dS)}\|\nabla_\tau v\|_{L^2(\partial\Omega,dS)}$$

$$\leq C_1\|u\|_W\|v\|_W \quad (\text{for all } u,\ v\in W)\,.$$

Thus Hypothesis 2) holds.

Next, for λ real, $B_\lambda(U,U)$ is real and

$$B_\lambda(U,U)=\int\limits_\Omega(\mathcal{A}\nabla u)\cdot\overline{\nabla u}dx+\int\limits_{\partial\Omega}(\lambda+\gamma)\,u\overline{u}dS/\beta$$

$$+\int\limits_\Omega\lambda u\overline{u}dx+\int\limits_{\partial\Omega}q\,(\mathcal{B}\nabla_\tau u)\cdot\overline{\nabla_\tau u}dS\ .$$

Furthermore,

$$B_\lambda(U,U)\geq\alpha_0\|\nabla u\|^2_{L^2(\Omega)}+\lambda\|u\|^2_{L^2(\partial\Omega,dS/\beta)}+\int\limits_{\partial\Omega}\gamma|u|^2\,dS/\beta$$

$$+\lambda\|u\|^2_{L^2(\Omega)}+\alpha_0\|\sqrt{q}\nabla_\tau u\|^2_{L^2(\partial\Omega,dS)}$$

$$\geq\min(\alpha,\lambda)\|u\|^2_{H^1(\Omega)}$$

$$+\lambda\|u\|^2_{L^2(\Omega)}+\left(\lambda+\left(ess\inf_{\partial\Omega}\gamma\right)\right)\|u\|^2_{L^2(\partial\Omega,dS/\beta)}$$

$$+\left\|\sqrt{q}\nabla_\tau u\right\|^2_{L^2(\partial\Omega,dS)} \geq C_2\left\|u\right\|^2_{W_0} \ ,$$

and $C_2 > 0$ if $\lambda > 0$ and $\lambda > ess\inf_{\partial\Omega}\gamma$. Thus we may take $I_0 = \left(0 \vee \left(ess\inf_{\partial\Omega}\gamma\right), \infty\right)$.

Then Hypothesis 3) holds.

The G in Theorem 1.1 is the negative of A constructed above, using the Lax-Milgram Lemma.

This completes the proof of Theorem 1.1. □

2 PARABOLIC APPROXIMATION OF THE TELEGRAPH EQUATION

Let Ω, H, $\mathcal{A}$, $\mathcal{B}$, L, L_∂, β, γ, q be as before. We now specify Ω to be an unbounded domain in $\mathbb{R}^N$, with a sufficiently regular boundary $\partial\Omega \neq \emptyset$. We assume that the boundary $\partial\Omega$ of Ω consists of a number of sufficiently smooth $N-1$ dimensional surfaces. We let β be a positive a.e. Borel function on $\partial\Omega$. We assume β is measurable and (1.13), but it need not be continuous.

Formally, $L_\lambda u = \nabla \cdot \mathcal{A}\nabla - \lambda I$, by (1.14),where $\lambda \geq 0$. Here $L = L_0$.

Define, on $V_q \times V_q$,

$$B_\lambda (U, V) := \int_\Omega (\mathcal{A}\nabla u) \cdot \nabla \overline{v} dx + \int_\Omega \lambda u \overline{v} dx$$

$$+ \int_{\partial\Omega} (\gamma + \lambda) u\overline{v} dS/\beta + \int_{\partial\Omega} q (\mathcal{B}\nabla_\tau u) \cdot \nabla_\tau \overline{v} dS \ ,$$

$$V_{q_\lambda} := \left\{ U = \begin{pmatrix} u \\ tr(u) \end{pmatrix} : \ u \in H^1(\Omega), \right.$$

$$\left. tr(u) \in L^1(\partial\Omega), \quad q \ tr(u) \in H^1(\partial\Omega) \right\},$$

with inner product which depends on λ

$$\langle w, z \rangle_{V_{q_\lambda}} := \int_\Omega \left(\mathcal{A}(x) \nabla w \cdot \nabla \overline{z} + \lambda w \overline{z} \right) dx + \int_{\partial\Omega} (\gamma + \lambda) w \overline{z} dS / \beta + \int_{\partial\Omega} q \nabla_\tau w \cdot \nabla_\tau \overline{z} dS \; .$$

Then, by Theorem 1.1, for $\lambda > \left\| \gamma_- \right\|_\infty$, there exists a positive invertible selfadjoint operator A_λ on H such that

$$B_\lambda (u, v) = \langle A_\lambda u, v \rangle \quad \text{for all } v \in V_{q_\lambda} \;,\; u \in \mathcal{D}(A_\lambda) \, .$$

We can extend A_λ to $\widehat{A}_\lambda$, where $\widehat{A}_\lambda : V_{q_\lambda} \to V_{q_\lambda}^*$ is bounded. Here $A = A_\lambda - \lambda I = A_0$ (and $G = G_\lambda + \lambda I = - A_\lambda + \lambda I = -A$). In fact, $B = B_0$ is Hermitian since $B(u, v) = \overline{B(v, u)}$ if and only if $A = A^*$. A is bounded below by $-\left\| \gamma_- \right\|_\infty$. Thus $A = A^* \geq 0$ if $\gamma \geq 0$ on $\partial\Omega$. Here let $\widetilde{A}_\lambda$ be the restriction of $\widehat{A}_\lambda$ defined by $D\left(\widetilde{A}_\lambda\right) = \{u \in V_{q_\lambda} : \widehat{A}_\lambda u \in H\}$. Then $\widetilde{A}_\lambda = A_\lambda$. Similarly for $A = A_0$, and A_μ is a selfadjoint operator on H for all $\mu \in \mathbb{R}$.

The telegraph and heat equations we consider are, with a a positive constant,

$$\begin{gathered} \frac{\partial^2 u}{\partial t^2} + 2a \frac{\partial u}{\partial t} - Lu = 0 \quad in \;\; \mathbb{R}^+ \times \Omega \\ Lu + \beta \partial_\nu^{\mathcal{A}} u + \gamma u - q \beta L_\partial u = 0 \; on \;\; \mathbb{R}^+ \times \partial\Omega \\ u(0, x) = f_1(x) \, , \quad \frac{\partial u}{\partial t}(0, x) = f_2(x) \, , \quad x \in \overline{\Omega} \end{gathered} \tag{2.1}$$

where $\mathbb{R}^+ = [0, \infty)$ and

$$2a\frac{\partial v}{\partial t} - Lv = 0 \quad in \;\; \mathbb{R}^+ \times \Omega$$

$$Lv + \beta\partial_\nu^{\mathcal{A}} v + \gamma v - q\beta L_\partial v = 0 \; on \;\; \mathbb{R}^+ \times \partial\Omega \tag{2.2}$$

$$v(0, x) = h(x), \quad x \in \overline{\Omega}.$$

Here L is as in (1.14).

Consider the abstract telegraph equation (or dissipative wave equation) with initial conditions

$$u''(t) + 2au'(t) + Au(t) = 0, \quad t \in \mathbb{R}^+, \tag{2.1'}$$

$$u(0) = f_1, \quad u'(0) = f_2.$$

It is wellposed for $a > 0$. This can be shown using the spectral theorem and its associated functional calculus in the space $L^2(\Omega, dx) \oplus L^2(\partial\Omega, dS/\beta) =: H$.

The corresponding heat equation problem

$$2av'(t) + Av(t) = 0, \quad t \in \mathbb{R}^+, \tag{2.2'}$$

$$v(0) = h$$

is also wellposed for $a > 0$, again by the spectral theorem.

We want to show that, under suitable hypotheses, given f_1, f_2 there is an $h =$

$h(a, f_1, f_2)$ such that the solution u of $(2.1')$ and the solution v of $(2.2')$ satisfy

$$u(t) = v(t)(1 + o(1))$$

as $t \to \infty$, i.e.,

$$\|u(t) - v(t)\|_H = \|v(t)\|_H (o(1))$$

as $t \to \infty$. This condition requires that $h \neq 0$.

Hypothesis 2.1. *Let Ω be a proper unbounded domain in $\mathbb{R}^N$ containing arbitrarily large balls, i.e. given $R > 0$ there is an $x_R \in \Omega$ such that the ball*

$$B(x_R, R) := \left\{ y \in \mathbb{R}^N \;:\; |y - x_R| < R \right\}$$

is in Ω.

Hypothesis 2.2. *Consider, as before, formally*

$$Lu = \nabla \cdot \mathcal{A} \nabla u \quad in\ \Omega,$$

$$Lu + \beta \partial_\nu^{\mathcal{A}} u + \gamma u - q\beta L_\partial u = 0 \ on\ \partial\Omega.$$

Suppose $\mathcal{A}$, $\mathcal{B}$ are as before but with L^∞ entries in the matrices. Then Theorem 1.1 defines a selfadjoint operator $A = A^* \geq 0$ in this context on $H = L^2\left(\overline{\Omega}, \mu\right)$, provided $\gamma \geq 0$.

Hypotheses 2.1 and 2.2 imply, by Theorem 1.1, that this determines a selfadjoint operator $A = A^* \geq 0$ on $H = L^2\left(\overline{\Omega}, \mu\right)$.

Hypothesis 2.3. *Suppose* $0 \in \sigma(A)$ *and* A *is injective,* $a^2 I - A$ is *injective, i.e.* a^2 *is not an eigenvalue of* A*, where* $2a$ *is the friction coefficient. Let*

$$K_\delta = \mathcal{X}_{[\delta, a^2-\delta]}(A) + \mathcal{X}_{[a^2,\infty]}(A)$$

for $\delta > 0$ *and let*

$$\mathcal{K} = \bigcup_{\delta>0} Range\,(K_\delta)\,.$$

Assume

$$f_2 + af_1 \in Range\left(\left(a^2 I - A\right)^{\frac{1}{2}}\right) \cap \mathcal{K}$$

and define

$$h := \mathcal{X}_{(0,a^2)}(A)\left[\frac{f_2}{2} + \left(a^2 I - A\right)^{-\frac{1}{2}}\left(\frac{f_2 + af_1}{2}\right)\right] \neq 0. \tag{2.3}$$

Note that $\mathcal{K}$ *is dense in* H*, as is the set of* h_1 *defined by the version of* (2.3) *obtained by deleting* $\mathcal{X}_{(0,a^2)}(A)$.

Theorem 2.1. *Let Hypotheses 2.1, 2.2 and 2.3 hold. Let* u *solve* $(2.1')$ *and* v *solve* $(2.2')$ *and we view* $u, v : \mathbb{R}^+ \longrightarrow H$.

Then

$$u(t) = v(t)(1 + o(1))$$

with h *given by* (2.3) *and* $h \neq 0$.

That is, both $u(t) \longrightarrow 0$, $v(t) \longrightarrow 0$ *and*

$$\frac{\|u(t) - v(t)\|}{\|v(t)\|} \longrightarrow 0$$

as $t \to \infty$.

Proof. We show that A is injective and $\inf \sigma(A) = 0$.

Assume $AU = 0$. Then

$$\nabla \cdot (\mathcal{A} \nabla u) = 0 \quad in \quad \overline{\Omega},$$

$$\nabla \cdot (\mathcal{A} \nabla u) + \beta \partial_\nu^{\mathcal{A}} u + \gamma u - q\beta L_\partial u = 0 \quad on \quad \partial\Omega.$$

Taking the inner product $\langle AU, U \rangle_H = 0$ yields

$$-\int\limits_{\Omega} (\mathcal{A} \nabla u) \cdot \nabla \overline{u} dx - \int\limits_{\Omega} |u|^2 \, dx - \int\limits_{\partial\Omega} \gamma \, |u|^2 \, \frac{dS}{\beta} - q \int\limits_{\partial\Omega} \left| \mathcal{B}^{\frac{1}{2}} \nabla_\tau u \right|^2 dS = 0.$$

Since $\gamma \geq 0$ we conclude that u coincides with a constant on Ω. Since $u|_{\partial\Omega} = trace\,(u|_\Omega)$, u is a constant on $\overline{\Omega}$. In addition, since $u \in L^2(\Omega)$ and $\int_\Omega dx = \infty$ by Hypothesis 2.1, it follows that $u \equiv 0$. Thus A is injective.

Let $R > 0$ be given. Choose $x_R \in \Omega$ so that the ball $B(x_R, R) \subset \Omega$. Assume further, without loss of generality, that $B(x_R, R)$ is compactly contained in Ω.

Any function supported in $B(x_R, R)$ will satisfy the boundary condition

$$\nabla \cdot (\mathcal{A} \nabla u) + \beta \partial_\nu^{\mathcal{A}} u + \gamma u - q\beta L_\partial u = 0 \quad on \quad \partial\Omega,$$

since the function vanishes on and near $\partial\Omega$. Let

$$\begin{aligned} \psi_1(x) &= e^{-\frac{1}{x}} \quad \text{for} \quad x > 0, \\ \psi_1(x) &= 0 \qquad \text{for} \quad x \leq 0. \end{aligned}$$

Then $\psi_1 \in C^\infty(\mathbb{R})$. In $\mathbb{R}^N$, let $r = |x|$ and let $\widetilde{\psi_2}(x) = \psi_2(r) = \psi_1(r)\,\psi_1(1-r)$. Then $\psi_2 \in C_c^\infty(\mathbb{R})$, $\psi_2 > 0$ inside $B(0,1)$ and $\psi_2(r) = 0$ for $r \geq 1$. Given

$R > 1$, let $r = |x|$ and

$$\widetilde{\psi_R}(x) = \begin{cases} \psi_2(r) & for \;\; 0 < r < \frac{1}{2} \\ \psi_2\left(\frac{1}{2}\right) & for \;\; \frac{1}{2} \leq r < R - \frac{1}{2} \\ \psi_2(R-r) & for \;\; R - \frac{1}{2} \leq r < R \\ 0 & for \qquad r \geq R. \end{cases}$$

Finally, let

$$\phi(x) = \widetilde{\psi_R}(x - x_R),$$

which is defined on $\mathbb{R}^N$, be viewed as a function on Ω. Let $\omega_N = \int_{\partial B(0,1)} dS$ be the surface area of the unit sphere in $\mathbb{R}^N$. Then

$$\begin{aligned} \langle \phi, \phi \rangle_H &= \omega_N \int_0^{\frac{1}{2}} [\psi_2(r)]^2 r^{N-1} dr + \omega_N \int_{\frac{1}{2}}^{R-\frac{1}{2}} \left[\psi_2\left(\frac{1}{2}\right)\right]^2 r^{N-1} dr \\ & \quad + \omega_N \int_{R-\frac{1}{2}}^{R} [\psi_2(R-r)]^2 r^{N-1} dr. \end{aligned}$$

It is easily seen that there are positive constants k_1, k_2, such that

$$k_1 \left(R - \frac{1}{2}\right)^N \leq \langle \phi, \phi \rangle_H \leq k_2 R^N. \tag{2.4}$$

Next,

$$
\begin{aligned}
0 &< \langle A\phi, \phi\rangle_H = \int_\Omega (\mathcal{A}\nabla\phi)\cdot\nabla\phi dx \\
&\le \alpha_1 \int_\Omega |\nabla\phi|^2 dx \\
&= \alpha_1\omega_N \int_0^R \left|\frac{\partial}{\partial r}\widetilde{\psi_R}(x)\right|^2 r^{N-1}dr \\
&= \alpha_1\omega_N \left[\int_0^{\frac{1}{2}} \left|\psi_2'(r)\right|^2 r^{N-1}dr + \int_{R-\frac{1}{2}}^R \left|\psi_2'(R-r)\right|^2 r^{N-1}dr\right] \\
&\le \alpha_1\omega_N \left\|\psi_2'\right\|_\infty \left[\left(\frac{R^N-\left(R-\frac{1}{2}\right)^N}{N}\right)+\frac{2^{-N}}{N}\right].
\end{aligned}
$$

By Taylor's formula,

$$
R^N - \left(R-\frac{1}{2}\right)^N = \frac{N}{2}\xi^{N-1} \le \frac{N}{2}R^{N-1}
$$

for some $\xi \in \left(R-\frac{1}{2}, R\right)$.

Thus

$$
0 < \langle A\phi,\phi\rangle_H \le \alpha_1\omega_N \left\|\psi_2'\right\|_\infty \left(\frac{R^{N-1}+2^{-N}}{2}\right). \tag{2.5}
$$

Combining (2.4) and (2.5) yields

$$
0 < \frac{\langle A\phi,\phi\rangle_H}{\langle\phi,\phi\rangle_H} \le \frac{\alpha_1\omega_N \left\|\psi_2'\right\|_\infty \left(\frac{R^{N-1}+2^{-N}}{2}\right)}{k_1\left(R-\frac{1}{2}\right)^N} \to 0
$$

as $R\to\infty$.

In the multiplicative representation of A, $A = U_0^{-1}M_mU_0$ for a $\sum$−measurable function $m:\Lambda\to\mathbb{R}^+$, where U_0 is unitary from H to $L^2(\Lambda,\Sigma,\lambda)$. Rewriting ϕ as

ϕ_R, we have, for $\widehat{\phi}_R = U_0\phi_R$,

$$0 < \frac{\langle A\phi_R, \phi_R\rangle_H}{\langle \phi_R, \phi_R\rangle_H} = \frac{\int_\Lambda m \left|\widehat{\phi}_R\right|^2 d\lambda}{\int_\Lambda \left|\widehat{\phi}_R\right|^2 d\lambda} \to 0$$

as $R \to \infty$. Thus m must take arbitrarily small positive values on a set of positive $\lambda-$measure, since $\lambda\left(\{\omega \in \Lambda : m(\omega) = 0\}\right) = 0$ since A is injective. But taking into account that $essRange\,(m) = \sigma(A)$, it follows that $\inf \sigma(A) = 0$.

Recall the theorem of Clarke, Eckstein and Goldstein [1].

Theorem 2.2. (CEG) *Let* $A = A^* \geq 0$ on H,

$$0 = \inf \sigma(A), \quad 0, a^2 \notin \sigma_\rho(A). \tag{2.6}$$

Let $a > 0$, and let u, v satisfy

$$u'' + 2au' + Au = 0, \quad t \geq 0,$$

$$u(0) = f_1, \quad u'(0) = f_2,$$

$$2av' + Av = 0, \quad t \geq 0, \tag{2.2'}$$

$$v(0) = h$$

when $f_1 \in \mathcal{D}(A)$, $f_2 \in \mathcal{D}\left(A^{\frac{1}{2}}\right)$ and h is as in (2.3) and $h \neq 0$. Then

$$u(t) = v(t)\left(1 + o(1)\right)$$

as $t \to \infty$.

Theorem 2.1 now follows from Theorem 2.2 and the above argument which proved (2.6). □

3 THE STRONGLY DAMPED WAVE EQUATION

3.1 INTRODUCTION

After I proved Theorem 2.1, I generalized it to Theorem 4.1 below which replaces the damping term $-2au'(t)$ by a more general damping term $-2Bu'(t)$ where B is a positive selfadjoint operator, not necessarily a multiple of the identity.

The corresponding theorem was recently published by G. Fragnelli, G.R. Goldstein, J.A. Goldstein and S. Romanelli [9], and our extension allows for the case of general β, γ and only L^∞ regularity of $\mathcal{A}$ and $\mathcal{B}$.

Let S be an injective nonnegative selfadjoint operator on a complex Hilbert space H. That is $S = S^* \geq 0,\ 0 \notin \sigma_\rho(S)$. Consider the damped wave equation

$$u''(t) + 2Bu'(t) + S^2 u(t) = 0, \quad t \geq 0, \tag{3.1}$$

with initial conditions

$$u(0) = f, \quad u'(0) = g; \tag{3.2}$$

here $' = \frac{d}{dt}$. When $B = 0$, (3.1) reduces to the wave equation. When $B \neq 0$ the corresponding heat equation is

$$2Bv'(t) + S^2 v(t) = 0.$$

We take $B = F(S)$ to be a positive selfadjoint operator which commutes with S

and is smaller than S in some sense. More precisely, we assume that

$$0 = \inf \sigma(S) \notin \sigma_\rho(S), \tag{3.3}$$

i.e., 0 is in the spectrum of $S = S^* \geq 0$, but is not an eigenvalue, F is a continuous function from $(0, +\infty)$ to $(0, +\infty)$, and F satisfies: there exists $\gamma_0 > 0$ such that

$$\begin{cases} F(x) > x & for\ 0 < x < \gamma_0, \\ F(\gamma_0) = \gamma_0, & \\ F(x) < x & for\ x > \gamma_0, \\ \limsup_{x \longrightarrow 0^+} F(x) < +\infty, & \\ \liminf_{x \longrightarrow +\infty} ((1-\delta)x - F(x)) \geq 0, & for\ some\ \delta > 0. \end{cases} \tag{3.4}$$

The operator B represents a general friction coefficient. The most common case is the telegraph equation in which case

$$B = aI$$

where a is a positive constant. In this case (1.4) holds with $\gamma_0 = a$. An important case is

$$B = aS^\alpha$$

where the constants a, α satisfy

$$a > 0, \quad \alpha \in [0, 1).$$

In this case, $\gamma_0 = a^{\frac{1}{1-\alpha}}$ in (3.4). The only interesting case is when S is unbounded, which is the case here when α is positive, B is an unbounded operator. The strongly damped wave equation refers to the case when B is also unbounded. B is always assumed to be injective.

Let S, $B = F(S)$ be as before and suppose f, g are such that (3.1), (3.2) has a unique solution u. Consider the corresponding first order equation, obtained by erasing $u''(t)$ in (3.1) and replacing u by v:

$$2Bv'(t) + S^2 v(t) = 0, \quad t \geq 0, \tag{3.5}$$

with initial condition

$$v(0) = h. \tag{3.6}$$

This vector h is given by

$$h = \frac{1}{2}\chi_{(0,\gamma_0)}(S)\left\{\left(F(S)^2 - S^2\right)^{\frac{1}{2}}(F(S)f + g) + f\right\},$$

a generalization of the formula we wrote before, and we need f, g are such that $h \neq 0$. Note that (3.5) reduces to (2.3) when $f = f_1$, $g = f_2$, and $B = F(S) = \alpha I$. We will show that

$$l(t) := \frac{\|u(t) - v(t)\|}{\|v(t)\|} \longrightarrow 0, \tag{3.7}$$

as $t \longrightarrow 0$, and we will find closed subspaces E_n of H such that $E_n \subset E_{n+1}$,

$$\bigcup_{n=1}^{\infty} E_n \text{ is dense in } H,$$

and

$$l(t) \leq C_n e^{-\epsilon_n t}$$

for f, $g \in E_n$ where C_n, ϵ_n are positive constants.

The point of the theorem is that, for large times the solution of the "hyperbolic equation" (3.1) looks like the solution of the "parabolic equation" (3.5). In the telegraph equation case when $B = F(S) = aI$, (3.5) becomes

$$2av'(t) + S^2 v(t) = 0,$$

which, with (3.6), is solved by

$$v(t) = e^{-\frac{t}{2a}S^2} h.$$

In the case of strong damping, the solution of the limiting parabolic problem is

$$v(t) = e^{-\frac{t}{2}B^{-1}S^2} h.$$

Think of $S = (-\Delta)^{\frac{1}{2}}$ on $L^2(\mathbb{R}^N)$ and

$$B = aS^{\alpha} = a(-\Delta)^{\frac{\alpha}{2}},$$

$0 < \alpha < 1$. Then $B^{-1}S^2 = \frac{1}{a}(-\Delta)^{1-\frac{\alpha}{2}}$ with domain

$$D\left(B^{-1}S^2\right) = H^{2-\alpha}\left(\mathbb{R}^N\right),$$

while

$$D\left(S^2\right) = H^2\left(\mathbb{R}^N\right).$$

The solution of the heat equation

$$2B\frac{dv}{dt} + S^2 v(t) = 0, \;\; v(0) = h \text{ is}$$

$$v(t) = e^{-\frac{t}{2}B^{-1}S^2}h.$$

Here we use the standard Sobolev space notation. Thus $B^{-1}S^2$ is a pseudodifferential operator of lower order $2-\alpha$ than that of the Laplacian, unless $\alpha = 0$ in which case we have the telegraph equation.

3.2 SELFADJOINT AND NORMAL OPERATORS

Let S be a selfadjoint operator on H with spectrum $\sigma(S)$. By the Spectral Theorem there exists an L^2 space $L^2(\Lambda, \Sigma, \nu)$, and a unitary operator

$$W : H \longrightarrow L^2(\Lambda, \Sigma, \nu)$$

such that S is unitarily equivalent, via W, to the maximally defined operator of multiplication by a $\Sigma-$measurable function

$$m : \Lambda \longrightarrow \sigma(S) \subset \mathbb{R},$$

i.e.,

$$Sf = W^{-1}M_m W f,$$

for

$$f \in D(S) = \left\{W^{-1}g \in H : \ mg \in L^2(\Lambda, \Sigma, \nu)\right\}$$

and

$$(M_m g)(x) = m(x)\, g(x),$$

for $x \in \Lambda,\ g \in L^2(\Lambda, \Sigma, \nu)$.

Two selfadjoint operators $S_1,\ S_2$ *commute* if and only if the bounded operators

$$(\lambda_1 I - S_1)^{-1},\ (\lambda_2 I - S_2)^{-1}$$

commute for all $\lambda_1,\ \lambda_2 \in \mathbb{C}\backslash\mathbb{R}$ if and only if

$$e^{itS_1},\ e^{isS_2}$$

commute for all $t,\ s \in \mathbb{R}$. Similarly, two normal operators $N_1,\ N_2$ with

$$\sup \operatorname{Re} \sigma(N_j) < +\infty, \quad j = 1, 2$$

commute if and only if e^{tN_1} and e^{sN_2} commute for all $t,\ s \geq 0$; here N is normal means $N = S_1 + iS_2$, where $S_1,\ S_2$ are commuting selfadjoint operators.

The functional calculus for S selfadjoint says that for every Borel measurable function F from $\sigma(S) \subset \mathbb{R}$ to $\mathbb{C}$, $F(S)$ defined by

$$F(S) = W^{-1} M_{F(m)} W$$

is normal, and any two of this operators commute. Moreover,

$$F \longrightarrow F(S)$$

is linear and is an algebra homomorphism, thus

$$F_1(S)\, F_2(S) = (F_1 F_2)(S),$$

etc. Also, $F(S)$ is bounded on H if and only if F is bounded on $\sigma(S)$, and $F(S)$ is selfadjoint if and only if F is real valued. And for $S = S^*$, $F(S)$ is semibounded (above or below) if and only if $F(\sigma(S))$ is semibounded in $\mathbb{R}$.

In particular, for Γ a Borel set in $[0,+\infty)$, $P_\Gamma = \chi_\Gamma(S)$ is the orthogonal projection of H onto $\chi_\Gamma(S)(H)$; and

$$P_\Gamma F(S) = F(S) P_\Gamma = P_\Gamma F(S P_\Gamma)$$

is the part of $F(S)$ in Γ, and its spectrum is contained in Γ.

If F_1, F_2 are complex Borel functions on $\sigma(S)$ that are bounded above, it follows that $F_j(S)$ and $\sum_{k=1}^{n} F_k(S)$ generate (C_0) semigroups on H and

$$e^{t\sum_{k=1}^{n} F_k(S)} = \prod_{k=1}^{n} e^{tF_k(S)}, \tag{3.8}$$

and the product can be taken in any order. Finally, if $L = F(S) = L^* \geq 0$, then $[L]^{\frac{1}{2}}$ denotes the unique nonnegative square root of L.

3.3 THE FrGGR THEOREM

Consider the problem (3.1), (3.2), which we rewrite as

$$u'' + 2Bu' + S^2 u = 0, \quad t \geq 0, \tag{3.9}$$

$$u(0) = f, \quad u'(0) = g, \tag{3.10}$$

where $S = S^* \geq 0$ on H,

$$\inf \sigma(S) = 0 \notin \sigma_\rho(S), \tag{3.11}$$

$B = F(S)$ where F is a continuous function from $(0, +\infty)$ to $(0, +\infty)$ which is bounded near zero and strictly less than the identity function near infinity, in the sense that for some $\delta > 0$,

$$\lim_{x \longrightarrow +\infty} \inf ((1 - \delta) x - F(x)) \geq 0. \tag{3.12}$$

We also assume there exists $\gamma_0 > 0$ such that (3.4) holds, namely

$$\begin{cases} F(x) > x & for\ 0 < x < \gamma_0, \\ F(\gamma_0) = \gamma_0, & \\ F(x) < x & for\ x > \gamma_0, \\ \limsup_{x \longrightarrow 0^+} F(x) < +\infty. & \end{cases} \tag{3.13}$$

Let Γ be the open interval $(0, \gamma_0)$ and let

$$P_\Gamma = \chi_\Gamma(S). \tag{3.14}$$

We explain our notation for square roots of selfadjoint operators. If $m : \Omega \to \mathbb{R}$ satisfies $m(\omega) = 1$ for all $\omega \in \Omega$, then

$$I = M_m$$

is the identity on $L^2(\Omega, \Sigma, \mu)$. Let $\Gamma \in \Sigma$ be arbitrary and let $\Gamma^c = \Omega \setminus \Gamma$,

$$s_\Gamma(\omega) = \chi_\Gamma(\omega) - \chi_{\Gamma^c}(\omega).$$

Note that $s_{\Gamma_1} = s_{\Gamma_2}$ a.e. if and only if

$$\mu(\Gamma_1 \setminus \Gamma_2) + \mu(\Gamma_2 \setminus \Gamma_1) = 0.$$

Since $s_\Gamma^2 = m$ for all Γ, I has many selfadjoint square roots. But it only has one positive selfadjoint square root.

If $T = T^* \geq 0$ on H, let $[T]^{\frac{1}{2}}$ be the unique nonnegative selfadjoint square root of T. If $T = T^*$, write

$$T = T_+ - T_-$$

where

$$T_+ = \chi_\Gamma(T)\, T,$$

$$T_- = -\chi_{\Gamma^c}(T)\, T$$

and

$$\Gamma = [0, \infty).$$

Define

$$[T]^{\frac{1}{2}} = \chi_\Gamma(T)\,[T_+]^{\frac{1}{2}} + i\chi_{\Gamma^c}(T)\,[T_-]^{\frac{1}{2}}.$$

This is the uniquely defined square root T that we will use. Note that the general solution of

$$\omega'' + T\omega = 0$$

is given by

$$\omega(t) = e^{t[T]^{\frac{1}{2}}} f_1 + e^{-t[T]^{\frac{1}{2}}} f_2$$

for $f_1,\ f_2 \in \mathcal{D}\left([T]^{\frac{1}{2}}\right)$.

Theorem 3.1.(FrGGR) *Consider the damped wave equation with initial conditions which we rewrite as*

$$u'' + 2Bu' + S^2 u = 0, \quad t \geq 0,$$

$$u(0) = f, \quad u'(0) = g,$$

where $S = S^* \geq 0$ *on* H,

$$\inf \sigma(S) = 0 \notin \sigma_\rho(S),\ \gamma_0 \notin \sigma_\rho(S),$$

Let v *be a solution of the corresponding heat equation*

$$2Bv' + S^2 v = 0,$$

obtained by deleting the second derivative term. Let v *satisfy*

$$v(0) = h := \frac{1}{2}\chi_{(0,\gamma_0)}\left(S^2\right)\left(f + \left[\left(B^2 - S^2\right)\chi_{(0,\gamma_0)}\left(S^2\right)\right]^{\frac{1}{2}}(Bf + g)\right).$$

Then, for u *the solution of* (3.9), (3.10),

$$u(t) = v(t)(1 + o(t)) \tag{3.17}$$

holds as $t \longrightarrow +\infty$, *provided* $h \neq 0$. *Moreover, if* $\Gamma_n = \left[\frac{1}{n}, \gamma_0 - \frac{1}{n}\right]$ *and if* $0 \neq h \in$

$P_{\Gamma_n}(H)$ for some $n \in \mathbb{N}$ then

$$u(t) = v(t)\left(1 + o\left(e^{-\epsilon_n t}\right)\right) \tag{3.18}$$

for some $\epsilon_n > 0$.

Remark 3.2. Note that

$$P_{\Gamma_n}(H) \subset P_{\Gamma_{n+1}}(H),$$

and $\bigcup_{n=1}^{\infty} P_{\Gamma_n}(H)$ is dense in $P_{\Gamma}(H)$.

Proof. Recall that the square root of $(B^2 - S^2) P_\Gamma$ refers to the nonnegative square root. We first show that the problem (3.9) and (3.10) is wellposed by showing that it is governed by a (C_0) contraction semigroup.

We first treat the case of $B = 0$ in (3.9).

Rewrite (3.9), (3.10) as, for $U = \begin{pmatrix} Su \\ u' \end{pmatrix}$,

$$U' = \begin{pmatrix} Su' \\ u'' \end{pmatrix} = \begin{pmatrix} 0 & S \\ -S & 0 \end{pmatrix} \begin{pmatrix} Su \\ u' \end{pmatrix} = GU, \tag{3.19}$$

$$U(0) = L = \begin{pmatrix} Sf \\ g \end{pmatrix}.$$

Let K be the completion of $D(S) \oplus D(S)$ in the norm

$$\left\| \begin{pmatrix} m \\ n \end{pmatrix} \right\|_K = \left(\|m\|^2 + \|n\|^2\right)^{\frac{1}{2}},$$

where $\|\cdot\|$ is the norm on H. Then G defined by (3.19) is skewadjoint on K and thus generates a (C_0) unitary group by Stone's Theorem. We consider this as a semigroup since we are only concerned with times $t \geq 0$. Next, (3.9) can be written in K as

$$U' = (G + P)\,U$$

where

$$P = \begin{pmatrix} 0 & 0 \\ 0 & -2B \end{pmatrix}.$$

Since $B = B^* \geq 0$,

$$\begin{aligned} \left\| P \begin{pmatrix} m \\ n \end{pmatrix} \right\|_K &= \|2Bn\| \leq (1-\epsilon)\|Sn\| + C_\epsilon \|n\| \\ &\leq (1-\epsilon) \left\| P \begin{pmatrix} m \\ n \end{pmatrix} \right\|_K + C_\epsilon \left\| \begin{pmatrix} m \\ n \end{pmatrix} \right\|_K \end{aligned}$$

for some $\epsilon > 0$ and a corresponding $C_\epsilon > 0$, for all $n \in D(S) \subset D(B)$, thanks to (3.12) and the last line in (3.13). Namely, write $B = B_1 + B_2 := BP_{(0,M)} + BP_{[M,\infty)}$, where M is such that

$$x \geq F(x) + \delta x,$$

i.e.

$$F(x) \leq (1-\delta)\,x$$

for $x \geq M$ and $F(x)$ is bounded in $[0, M]$. Thus B_1 is bounded, $B_2 = B_2^* \geq 0$ and

$$\|B_2 n\| \leq (1-\delta)\|Sn\| + \|B_2\|\,\|n\|$$

for all $n \in D(S)$. Thus for $N = \begin{pmatrix} m \\ n \end{pmatrix}$,

$$\|PN\|_K \leq (1-\delta)\|GN\|_K + M\|N\|_K$$

where $\delta > 0$ and $M = \|B_2\|$. It follows that $G+P$ is $m-$dissipative and generates a (C_0) contraction semigroup on K, since P is obviously dissipative. Then (3.9), (3.10) has a unique strongly C^2 solution (resp. mild solution) if $f \in D(S^2)$, $g \in D(S)$ (resp. $f \in D(S)$, $g \in H$).

We shall express the unique solution using d'Alembert's formula. We seek a solution of the form

$$u(t) = e^{tC}h$$

where C is a Borel function of S. By (3.9), C must satisfy

$$C^2 + 2BC + S^2 = 0.$$

Formally,

$$C = C_{\pm} = -B \pm \left(B^2 - S^2\right)^{\frac{1}{2}}.$$

Selfadjoint operators have many square roots, but nonnegative selfadjoint operators have unique nonnegative square roots. Thus we uniquely define $C_{\pm}$ by

$$C_{\pm} = -B \pm (Q_0 + iQ) \tag{3.20}$$

where

$$Q_0 = \left[\left(B^2 - S^2\right)\chi_{(0,\gamma_0)}(S)\right]^{\frac{1}{2}}, \quad Q = \left[\left(S^2 - B^2\right)\chi_{[\gamma_0,+\infty)}(S)\right]^{\frac{1}{2}}. \tag{3.21}$$

Thus the solution u of (3.9), (3.10) can be written as

$$u(t) = e^{tC_+}h_+ + e^{tC_-}h_-,$$

where $C_\pm$ are defined by (3.20), (3.21). There are strong C^2 solutions (resp. mild solutions) if and only if $h_\pm \in D(S^2)$ (resp. $h_\pm \in D(S)$).

Given $f = u(0)$, $g = u'(0)$, we obtain $h_\pm$ by inverting the 2×2 system

$$f = h_+ + h_-$$

$$g = C_+h_+ + C_-h_-.$$

An elementary calculation gives

$$h_- = \frac{1}{2}\left(f - (Q_0 + iQ)^{-1}(Bf + g)\right) \tag{3.22}$$

$$h_+ = \frac{1}{2}\left(f + (Q_0 + iQ)^{-1}(Bf + g)\right). \tag{3.23}$$

Write

$$u = u_1 + u_2 + u_3$$

where

$$u_1(t) = e^{tC_+}P_{(0,\gamma_0)}h_+,$$

$$u_2(t) = e^{tC_+}P_{[\gamma_0,+\infty)}h_+,$$

$$u_3(t) = e^{tC_-}h_-.$$

First,

$$\|u_3(t)\| = \left\|e^{-itQ}e^{-tQ_0}e^{-tB}h_-\right\| \le \left\|e^{-tB}h_-\right\|$$

since e^{-itQ} is unitary and $\left\|e^{-tQ_0}\right\| \leq 1$. Next,

$$\left\|u_2\left(t\right)\right\| = \left\|e^{itQ}e^{-tB}P_{[\gamma_0,+\infty)}h_+\right\| \leq \left\|e^{-tB}h_+\right\|.$$

The next estimate is the key one.

For

$$h := P_{(0,\gamma_0)}\left(h_+\right), \tag{3.24}$$

$$\left\|u_1\left(t\right)\right\| = \left\|e^{t(-B+Q_0)}h_+\right\|.$$

We know that $h \in P_{(0,\gamma_0)}\left(H\right)$: assume

$$0 \neq h \in P_{[\delta,\gamma_0-\delta]}\left(H\right) =: H_\delta \tag{3.25}$$

for some $\delta > 0$. Let

$$Q_{0\delta} = Q_0 P_{[\delta,\gamma-\delta]}.$$

Then, since $F\left(x\right) > x$ on $\left[\delta,\gamma_0-\delta\right]$, $F\left(x\right) - x \geq \epsilon$ on $[\delta,\gamma_0-\delta]$ for some $\epsilon > 0$. Thus

$$Q_{0\delta} \geq \epsilon I.$$

Consequently

$$\left\|u_1\left(t\right)\right\| \geq e^{\epsilon t}\left\|e^{-tB}h\right\|.$$

It follows that for some constant C_0,

$$\left\|u_2\left(t\right)\right\| + \left\|u_3\left(t\right)\right\| \leq C_0 e^{-\epsilon t}\left\|u_1\left(t\right)\right\|.$$

Thus

$$u(t) = u_1(t)\left(1 + O\left(e^{-\epsilon t}\right)\right). \tag{3.26}$$

We must show that this holds with u_1 replaced by v.

The unique solution of (3.15) is

$$v(t) = e^{-\frac{t}{2}B^{-1}S^2}h.$$

Note that h as defined by (3.16) is $P_\Gamma h_+$ where h_+ is as in (3.23). To compare u_1 with v, we need Taylor's formula with integral remainder, which for $g \in C^3[0,l]$ for some $l > 0$ says that

$$g(x) = g(0) + g'(0)\,x + \frac{g''(0)}{2}x^2 + \frac{1}{2}\int_0^x (x-y)^2\, g'''(y)\,dy.$$

Applying this to

$$g(x) = 1 - (1-x)^{\frac{1}{2}}, \quad 0 < x < 1, \tag{3.27}$$

yields

$$g(L)\,f = \frac{1}{2}Lf + \frac{1}{8}L^2 f + Rf \tag{3.28}$$

where R is a bounded operator commuting with L and satisfying

$$R = R^* \geq 0.$$

Consequently

$$\begin{aligned}\|u_1(t)-v(t)\| &= \left\|e^{-tB\left(-I+B^{-1}Q_0\right)}h - e^{-\frac{t}{2}B^{-1}S^2}h\right\| \\ &= \left\|e^{-tB\left[I-\left(B^{-2}Q_0^2\right)^{\frac{1}{2}}\right]}h - e^{-\frac{t}{2}B^{-1}S^2}h\right\| \\ &= \left\|e^{-\frac{t}{2}B^{-1}S^2}\left\{e^{-\frac{t}{8}L^2}e^{-tR}-I\right\}h\right\| \qquad (3.29)\end{aligned}$$

by (3.27), (3.28) with $L = B^{-2}\left(B^2-S^2\right)P_{(0,\gamma_0)} = \left(I-B^{-2}S^2\right)P_{(0,\gamma_0)}$.

We have

$$\zeta_1 I \le R \le \zeta_2 I$$

on H_δ for some constants $0<\zeta_1<\zeta_2<+\infty$. Furthermore, we also have

$$\zeta_3 I \le L \le \zeta_4 I$$

on H_δ for some positive constants ζ_3, ζ_4. It now follows from (3.28) that

$$\|u_1(t)-v(t)\| = \left\|e^{-\frac{t}{2}B^{-1}S^2}\left(I-e^{-\frac{t}{8}L}e^{-tR}h\right)\right\|$$

and

$$\left\|e^{-\frac{t}{8}L}e^{-tR}h\right\| \le e^{-t\zeta_5}\|h\|$$

where

$$\zeta_5 = \frac{\zeta_3}{8}+\zeta_1 > 0.$$

Consequently

$$\|u(t)-v(t)\| \le \|v(t)\|\, O\left(e^{-t\zeta_5}\right).$$

Combining this inequality with (3.26) yields the desired asymptotic relation

$$\frac{\|u(t) - v(t)\|}{\|v(t)\|} = o\left(e^{-t\epsilon_\delta}\right)$$

for some $\epsilon_\delta > 0$.

Now let $0 \neq h \in P_{(0,\gamma_0)}(H)$. We must show that

$$\frac{\|u(t) - v(t)\|}{\|v(t)\|} \longrightarrow 0, \quad \text{and} \quad t \longrightarrow +\infty. \tag{3.30}$$

We proceed by contradiction. Suppose (3.30) fails to hold for some $h \neq 0$ in $P_{(0,\gamma_0)}(H)$. Then, there exists $\epsilon_1 > 0$ and $t_n \longrightarrow +\infty$ such that

$$\frac{\|u(t_n) - v(t_n)\|}{\|v(t_n)\|} \geq \epsilon_1 \tag{3.31}$$

for all $n \in \mathbb{N}$. Choose $\delta > 0$ and $\widetilde{h} \in H_\delta = P_{[\delta,\gamma_0-\delta]}(H)$ (depending on ϵ_1) such that

$$\left\|h - \widetilde{h}\right\| < \frac{\epsilon_1}{4}$$

and let $\widetilde{f}, \widetilde{g}$ be the corresponding initial data. Note that

$$P_{[\gamma_0,+\infty)}l = P_{[\gamma_0,+\infty)}\widetilde{l}$$

for $l = f, g$, and f and g are modified only on the subspace $P_\Lambda(H)$

$$\Lambda := [\delta - \delta_1, \delta + \delta_1] \cup [\gamma_0 - \delta - \delta_1, \gamma_0 - \delta + \delta_1],$$

for some $\delta_1 > 0$ which can be chosen to be arbitrarily small. In particular, given

$\epsilon_2 > 0$ we may choose $\widetilde{f}$, $\widetilde{g}$ as above and additionally satisfying

$$\left\| f - \widetilde{f} \right\| + \| g - \widetilde{g} \| < \epsilon_2,$$

$$\frac{\left\| \widetilde{h} \right\|}{\| h \|} \in [1 - \epsilon_2, 1 + \epsilon_2].$$

It follows that

$$\| u(t) - \widetilde{u}(t) \|, \quad \| v(t) - \widetilde{v}(t) \| < \frac{\epsilon_1}{4}$$

for all $t > 0$. Consequently

$$\frac{\| u(t) - v(t) \|}{\| v(t) \|} \leq \frac{\| \widetilde{u}(t) - \widetilde{v}(t) \|}{\| \widetilde{v}(t) \|} \left(\frac{1 + \epsilon_2}{1 - \epsilon_2} \right) + \frac{\epsilon_1}{4}$$

$$\leq \tau_0 e^{-\epsilon_3 t} \left(\frac{1 + \epsilon_2}{1 - \epsilon_2} \right) + \frac{\epsilon_1}{4} \longrightarrow \frac{\epsilon_1}{4}, \tag{3.32}$$

as $t \longrightarrow +\infty$, since $0 \neq \widetilde{h} \in H_\delta$, and τ_0, ϵ_3 are positive constants depending on δ. But (3.32) contradicts (3.31) for $t = t_n$ with n large enough. It follows that (3.30) holds. This completes the proof of Theorem 3.1. $\square$

Example 3.1 We recall the "Wentzell Laplacian" of Chapter 1. Let Ω be an unbounded domain in $\mathbb{R}^N$ with nonempty boundary $\partial\Omega$, such that for every $R > 0$ there exists a ball $B(x_R, R)$ in Ω. Let $\mathcal{A}(x)$ be an $N \times N$ matrix for $x \in \overline{\Omega}$ such that $\mathcal{A}(x)$ is real, Hermitian and

$$\alpha_0 |\xi|^2 \leq \mathcal{A}(x) \xi \cdot \xi \leq \alpha_1 |\xi|^2 \tag{3.33}$$

for all $x \in \overline{\Omega}$, $\xi \in \mathbb{R}^N$, where

$$0 < \alpha_0 \leq \alpha_1 < \infty$$

are constants. Similarly let $\mathcal{B}(x)$ for $x \in \partial\Omega$ be a real Hermitian $(N-1) \times (N-1)$ matrix satisfying (3.33) for all $x \in \partial\Omega$ with the same α_0, α_1. Because of Theorem 1.1, we need not make the usual smoothness assumptions on $\mathcal{A}$, $\mathcal{B}$, γ, and β. Define distributional differential operators on Ω (resp. $\partial\Omega$) by

$$Lu_1 = \nabla \cdot (\mathcal{A}(x) \nabla u_1),$$

$$L_\partial u_2 = \nabla_\tau \cdot (\mathcal{B}(x) \nabla_\tau u_2)$$

for u_1 (resp. u_2) defined on Ω (resp. $\partial\Omega$). Here ∇_τ is the tangential gradient on $\partial\Omega$. The damped wave equation we consider is

$$u_{tt} + 2Bu_t = Lu \text{ in } \Omega, \tag{3.34}$$

$$Lu + \beta\partial_\nu^{\mathcal{A}} u + \gamma u - q\beta L_\partial u = 0 \text{ on } \partial\Omega. \tag{3.35}$$

Here the conormal derivative term is

$$\partial_\nu^{\mathcal{A}} u = (\mathcal{A}\nabla u) \cdot \nu$$

at $x \in \partial\Omega$, where ν is the unit outer normal to $\partial\Omega$ at x; $\beta > 0$, $\gamma \geq 0$, γ is bounded, and $q \in [0, +\infty)$.

The problem (3.34), (3.35) can be rewritten as

$$u'' + 2F(S)\,u' + S^2 u = 0$$

$$u(0) = f, \quad u'(0) = g$$

where the Hilbert space is

$$H = L^2(\Omega, dx) \oplus L^2(\partial\Omega, dS/\beta).$$

The operator S^2 has the matrix representation

$$S^2 = \begin{pmatrix} -L & 0 \\ \beta\partial_\nu^{\mathcal{A}} & \gamma - q\beta L_\partial \end{pmatrix}.$$

It was shown earlier that $S = [S^2]^{\frac{1}{2}}$, with a suitable domain, satisfies

$$S = S^* \geq 0, \quad 0 = \inf\sigma(S), \quad 0 \neq \sigma_\rho(S).$$

Furthermore, for all $u \in D(S)$, we have $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, where $u_2 = tr(u_1)$, the trace of u_1. Then Theorem 3.1 applies to

$$u'' + 2aS^{\frac{\alpha k}{2}}u' + S^{2k}u = 0, \tag{3.36}$$

$$u(0) = f, \quad u'(0) = g, \quad k \in \mathbb{N}.$$

If S is a pseudodifferential operator of order r, then S^δ is a pseudodifferential operator of order $r\delta$. It can be a partial differential operator if $r\delta \in \mathbb{N}_0$.

Thus (3.36) can be a partial differential equation only when $\frac{\alpha k}{2} \in \mathbb{N}$. If $k = 3$ and $\alpha = \frac{2}{3}$, the corresponding parabolic problem is

$$v' + \frac{1}{2a} S^4 v = 0, \quad v(0) = h.$$

The boundary conditions associated with (3.36) are

$$Lw + \beta \partial_\nu^A w + \gamma w - q\beta L_\partial w = 0 \; on \; \partial\Omega$$

for $w = S^{2j} u$, $j = 0, 1, ..., k-1$.

The solution to a sixth order wave equation is asymptotically equal to the solution of a fourth order heat equation.

What is new here is that β, γ, $\mathcal{A}$, $\mathcal{B}$ can be "very bad" relative to the previous known results.

Example 3.2 The simplest examples of unidirectional waves in one dimension are described by the equation (for t, $x \in \mathbb{R}$)

$$u_t = cu_x + bu_{xxx} =: Mu, \tag{3.37}$$

where $(b, c) \in \mathbb{R}^2 \backslash \{(0,0)\}$. The most common case is $c \neq 0$, $b = 0$, in which case the corresponding equation for bidirectional waves is

$$\left(\frac{\partial}{\partial t} - M\right)\left(\frac{\partial}{\partial t} + M\right) u = u_{tt} - c^2 u_{xx} = 0.$$

The case of $b \neq 0$ is the Airy equation, and (3.37) is the linearization of the KdV equation

$$u_t = cu_x + bu_{xxx} + c_1 uu_x.$$

For $c = 0 \neq b$, the bidirectional version of (3.37) is

$$\left(\frac{\partial}{\partial t} - M\right)\left(\frac{\partial}{\partial t} + M\right) u = u_{tt} - b^2 u_{xxxxxx} = 0.$$

Now, let $H = L^2(\mathbb{R})$, $D = \frac{d}{dx}$ and $T = -D^2 = T^* \geq 0$. Let

$$S^2 = T^3 + a_0 T^2 + a_1 T,$$

where a_0, $a_1 \in [0, +\infty)$. Consider

$$u_{tt} - 2au_{xxt} - u_{xxxxxx} + a_0 u_{xxxx} - a_1 u_{xx} = 0,$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x).$$

In this case,

$$B = aT = F(S) = F\left(\left(T^3 + a_0 T^2 + a_1 T\right)^{\frac{1}{2}}\right).$$

For $x > 0$, we want to consider the function

$$G(x) = \frac{1}{a}\left(x^3 + a_0 x^2 + a_1 x\right),$$

so that $G(x) - x$ is negative in $(0, \gamma_0)$ and positive on (γ_0, ∞) for some $\gamma_0 > 0$. But

$$\frac{d}{dx}\left(\frac{G(x) - x}{x}\right) = \frac{d}{dx}\left(\frac{1}{a}\left(x^2 + a_0 x + (a_1 - a)\right)\right) = \frac{1}{a}(2x + a_0) > 0,$$

and $G(x) = x$ for $x \neq 0$ if and only if

$$x = \frac{1}{2}\left(-a_0 \pm \sqrt{a_0^2 - 4(a_1 - a)}\right).$$

Thus we get exactly one positive root if and only if

$$a > a_1 + \frac{a_0^2}{4},$$

which we assume. It is now elementary to check that $B = F(S)$ and F satisfies the assumptions of Theorem 3.1. In this case

$$\gamma = \frac{1}{2}\left(-a_0 \pm \sqrt{a_0^2 - 4(a_1 - a)}\right).$$

4 ASYMPTOTIC PARABOLICITY OF WAVES WITH TIME DEPENDENT DAMPING

4.1 THE SETUP

Let B, S be commuting nonnegative selfadjoint operators on a complex Hilbert space H. Assume $\mathcal{D}(S) \supset \mathcal{D}(B)$ and $\inf \sigma(S) = 0$, $0 \notin \sigma_\rho(S)$. This implies that S is injective but S^{-1} is unbounded. Additional restrictions will be placed on B, S. The interesting cases are when S is unbounded. It was shown in [9] that, under additional hypotheses, the unique mild solution u of the strongly damped wave equation

$$\frac{d^2u}{dt^2} + 2B\frac{du}{dt} + S^2u = 0, \quad t \geq 0, \tag{4.1}$$

$$u(0) = f, \quad u'(0) = g, \tag{4.2}$$

is asymptotically equal to the unique mild solution v of the corresponding "heat equation"

$$\begin{cases} 2B\frac{dv}{dt} + S^2v = 0, \quad t \geq 0, \\ v(0) = h, \end{cases} \tag{4.3}$$

for a suitable h (depending on f, g, B, S) in the sense that

$$u(t) = v(t)(1 + o(1))$$

as $t \to \infty$; that is

$$\|u(t) - v(t)\| \leq \epsilon(t)\|v(t)\|$$

where $\epsilon(t) \to 0$ as $t \to \infty$, provided that $h \neq 0$. This is the idea of asymptotic parabolicity.

In this chapter we establish an analogous result for

$$u''(t) + 2(B + C(t))\, u'(t) + S^2 u(t) = 0, \quad t \geq 0 \tag{4.4}$$

with initial conditions (4.2), where each $C(t)$ is selfadjoint, all the operators commute, and $C(t) \to 0$ as $t \to \infty$ in a suitable sense.

In case each $C(t)$ is bounded and

$$\int_0^\infty \|C(t)\|\, dt < \infty,$$

and $C(\cdot)$ is continuously differentiable as a function of time, at least in some sense, then asymptotic parabolicity holds, i.e.

$$u(t) = v(t)(1 + o(1))$$

where v satisfies (4.3) with an h depending on f, g, B, S and $C(\cdot)$, again assuming $h \neq 0$.

The presence of $C(\cdot)$ makes the construction of h much more complicated. The mutually commuting hypothesis makes the construction of h possible; the noncommuting case is much harder. The main theorem of the final chapter of my thesis is the first result of this kind, using time dependent friction. We treat the case of unbounded $C(t)$.

Hypothesis 4.1. *Let* $\{S, B,\ C(t) : t \geq 0\}$ *be a family of commuting selfad-*

joint operators on H, *with*

$$0=\inf\sigma(S),\quad 0\notin\sigma_\rho(S),$$

$$\sup\sigma(S)=\infty,$$

and there exists a unique $\gamma_0>0$ *such that* $\gamma_0\notin\sigma_\rho(S)$ *and*

$$B\chi_{(0,\gamma_0)}(S)\geq S\chi_{(0,\gamma_0)}(S),$$

$$B\chi_{(\gamma_0,\infty)}(S)\leq S\chi_{(\gamma_0,\infty)}(S).$$

The last condition holds if $B=F(S)$ where $F\in C(\mathbb{R}^+,(0,\infty))$,

$$F(x)>x\quad for\quad 0<x<\gamma_0,$$

$$F(\gamma_0)=\gamma_0,$$

$$F(x)<x\quad if\quad x>\gamma_0;$$

and

$$(1-\delta_1)x>F(x)\tag{4.5}$$

for some $\delta_1>0$ and all sufficiently large x.

An important example we have in mind is

$$B=aS^\alpha$$

where $a>0,\ 0\leq\alpha<1$ are constants. The telegraph equation corresponds to $\alpha=0$ and $0<\alpha<1$ is the usual strongly damped wave equation.

In this case,

$$\gamma_0 = a^{\frac{1}{1-\alpha}}.$$

More generally we can take

$$B = \sum_{j=1}^{m} a_j S^{\alpha_j},$$

$a_j > 0,\ \ 0 \leq \alpha_j < 1,\ \ m \in \mathbb{N}.$

For $C(t)$ we can take

$$C(t) = \sum_{j=1}^{n} c_j(t) S^{\beta_j} + C_0(t) := C_1(t) + C_0(t),$$

$0 \leq \beta_j < 1,\ \ 0 \leq c_j \in \mathcal{C}_0 \cap L^1(\mathbb{R}^+),\ \ C_0 \in (\mathcal{C}_0 \cap L^1)(\mathbb{R}^+, \mathcal{L}(H)),$ and $C_0(t)$ need not be nonnegative. This will all be made precise in Hypothesis 4.2 in Section 4.2 below.

4.2 SOLUTION OF THE DAMPED WAVE EQUATION

We recall the spectral theorem and the associated functional calculus. Given $S = S^*$ on H, then exists an L^2 space $L^2(\Omega, \Sigma, \mu)$ and a unitary operator

$$U_0 : H \to L^2(\Omega, \Sigma, \mu)$$

such that S is unitarily equivalent to a multiplication operator on $L^2 = L^2(\Omega, \Sigma, \mu)$,

$$S = U_0^{-1} M_m U_0,$$

where

$$m : \Omega \rightarrow \sigma(S) \subset \mathbb{R},$$

$$(M_m g)(\omega) = m(\omega)\, g(\omega), \quad \omega \in \Omega,$$

and $g \in \mathcal{D}(M_m)$ if and only if both g and mg are in L^2. Thus M_m is the operator of multiplication by m with maximal domain. Thus $f \in \mathcal{D}(S)$ if and only if $U_0 f \in \mathcal{D}(M_m)$, and furthermore

$$\begin{aligned} B &= U_0^{-1} M_{F(m)} U_0, \\ C(t) &= U_0^{-1} M_{G_t(m)} U_0 \end{aligned}$$

for real valued Borel functions F, G_t on $\sigma(S) \subset \mathbb{R}$.

The usual approach to (4.4) is to write it as a first order equation for

$$U(t) = \begin{pmatrix} Su(t) \\ u'(t) \end{pmatrix}, \quad t \geq 0.$$

We have

$$\begin{aligned} U' &= \begin{pmatrix} Su' \\ u'' \end{pmatrix} \\ &= \begin{pmatrix} 0 & S \\ -S & 0 \end{pmatrix} U + \begin{pmatrix} 0 & 0 \\ 0 & -\widetilde{B}(t) \end{pmatrix} U \end{aligned}$$

when $\widetilde{B}(t) = B + C(t)$. We rewrite this as

$$U' = A_1 U + A_2(t)\, U$$

on $H \oplus H$, with

$$A_1 = S \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -A_1;$$

$$A_2(t) = -\widetilde{B}(t) \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} = A_2(t)^*$$

and

$$\left[S, \widetilde{B}(t)\right] = 0.$$

But

$$[A_1, A_2(t)] \neq 0$$

since

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

Also, if E_1, E_2 are commuting semigroup generators then

$$e^{t\overline{(E_1+E_2)}} = e^{tE_1} e^{tE_2};$$

this fails when $[E_1, E_2] \neq 0$. We avoid this noncommutativity issue by working directly with (4.4) in H.

We first consider (4.4) with

$$\widetilde{B}(t) = B + C(t)$$

being independent of t. This reduces to

$$u'' + 2\widetilde{B}(0)\,u' + S^2 u = 0, \tag{4.6}$$

which is (3.1) with different notation, as in Section 3.1.

The general solution of (4.6) is given by the d'Alembert formula

$$u(t) = e^{t\Lambda_+} f_+ + e^{t\Lambda_-} f_-,$$

where $f_\pm$ are suitable vectors in $\mathcal{D}(S)$ and

$$\Lambda_\pm = -\widetilde{B}(0) \pm \left[\widetilde{B}(0)^2 - S^2\right]^{\frac{1}{2}}, \tag{4.7}$$

and we use the convention for square roots described in Section 3.3.

Similarly, we define

$$\Lambda_\pm(s) = -\widetilde{B}(s) \pm \left[\widetilde{B}(s)^2 - S^2\right]^{\frac{1}{2}}, \quad \text{for } \ s \geq 0. \tag{4.8}$$

By the spectral theorem and the functional calculus, the general solution of (4.4) is easily shown to be given by the generalized d'Alembert formula

$$u(t) = e^{\int_0^t \Lambda_+(s)ds} f_+ + e^{\int_0^t \Lambda_-(s)ds} f_- \tag{4.9}$$

for $f_{\pm} \in \mathcal{D}(S)$. This u is a strong (C^2 in time) solution if $f_{\pm} \in \mathcal{D}(S^2)$ and a mild solution otherwise. Note that u is not a sum of semigroup orbits; rather it is a sum of evolution operator orbits.

We pause to discuss the integrals of the form

$$e^{\int_a^b R(s)ds} \tag{4.10}$$

where each $R(s)$ is selfadjoint and these operators all commute. Thus there is a selfadjoint (or normal) operator T on H such that

$$R(s) = g(s, T) \tag{4.11}$$

for some Borel function

$$g : (a, b) \times \Omega_1 \to \mathbb{R} \quad (or \quad \mathbb{C})$$

where

$$T = U_1^{-1} M_{m_1} U_1,$$

U_1 is unitary from H to $L^2 = L^2(\Omega_1, \Sigma_1, \mu_1)$,

$$m_1 : \Omega_1 \to \mathbb{R}$$

is Σ_1-measurable, and for $s \in (a, b)$,

$$R(s) = U_1^{-1} M_{g(s,m_1)} U_1.$$

This explains (4.11).

We need $T \geq 0$, T is injective, and $R(s)$ has smooth enough dependence on s so that

$$\begin{aligned}\frac{d^2}{dt^2} e^{\int_a^t R(s) ds} f &= \frac{d}{dt}\left(R(t) e^{\int_a^t R(s) ds} f\right)\\ &= \left(R'(t) + R(t)^2\right) e^{\int_a^t R(s) ds} f.\end{aligned}$$

Thus

$$\widehat{u}(t) = e^{\int_0^t R(s) ds} f$$

is a C^2 solution of

$$\begin{aligned}&\widehat{u}'' + \widehat{R}(t)\widehat{u}'\\ = &\left(R'(t) + R(t)^2 + \widehat{R}(t) R(t)\right)\widehat{u},\end{aligned}$$

$$u(0) = f, \quad u'(0) = R(0) f$$

for f in a common core of

$$\{T, \ \{R(s): \ a < s < b\}\}.$$

For instance, we may assume

$$\bigcap_{a<s<b} \mathcal{D}(R(s)) \supset \mathcal{D}(T^{\alpha})$$

for some $\alpha < 1$. For example, let

$$R(s) = R_0(s) + \sum_{j=1}^{\nu} \widehat{\beta}_j(s) T^{\widehat{\alpha}_j}$$

where $0 \leq \widehat{\alpha}_j < 1$ and $\widehat{\beta}_j \in C^2(a,b)$ for each j and

$$R_0 \in C^2([a,b], \mathcal{L}(H))$$

with

$$R_0(s) = R_0(s)^*$$

for each $s \in [a,b]$.

If $a \in \mathbb{R}$ and $b = \infty$, we also assume

$$\widehat{\beta}_j \in C^2[a,\infty],$$

$$\widehat{\beta}_j(\infty) = 0, \quad R_0(\infty) = 0,$$

and

$$\int_0^\infty \left(\sum_{j=0}^{\nu} \widehat{\beta}_j(s) + \|R_0(s)\| \right) ds < \infty.$$

We write

$$\Lambda_\pm(s) = U_0^{-1} M_{\lambda_\pm(s)} U_0,$$

where

$$\lambda_\pm(s) := -(F(m) + G_s(m)) \pm \left[(F(m) + G_{j_s}(m))^2 - m^2 \right]^{\frac{1}{2}}$$

are functions on m. For $x \in \mathbb{R}$,

$$[x]^{\frac{1}{2}} = \begin{cases} \sqrt{x} \geq 0 & \text{if } x \geq 0, \\ i\sqrt{x} & \text{if } x < 0. \end{cases}$$

Let

$$\lambda_\pm^0(0) := -F(m) \pm \left[F(m)^2 - m^2 \right]^{\frac{1}{2}}.$$

Write

$$u(t) = e^{\int_0^t \Lambda_+(s)ds} f_+ + e^{\int_0^t \Lambda_-(s)ds} f_-$$
$$= e^{t\Lambda_+^0} P_+(t) f_+ + e^{t\Lambda_-^0} P_-(t) f_-$$

where

$$\Lambda_\pm^0 = U_0^{-1} M_{\lambda_\pm^0(0)} U_0,$$

$$P_\pm(t) = e^{-\int_0^t C(s)ds \pm \int_0^t Q(s)ds},$$

$$Q(t) = \left[(B + C(t))^2 - S^2\right]^{\frac{1}{2}} - \left[B^2 - S^2\right]^{\frac{1}{2}}$$

We assume

Hypothesis 4.2. *Let* $C(t) = C_1(t) + C_0(t)$; *for each* j, t,

$$C_j(t) = G_j(t, S)$$

for a jointly continuous function $(t, x) \to G_j(t, x)$ *on* $\mathbb{R}^+ \times \mathbb{R}^+$ *which is* C^2 *in* t; $C_0(\cdot), C_0'(\cdot) \in (\mathcal{C}_0 \cap L^1)(\mathbb{R}^+, \mathcal{L}(H))$, $0 \leq C_1(t), |C_1'(t)| \leq K_0\left(I + S^{1-\delta_2}\right)$ *for some* $\delta_2 > 0$; *and for each* $f \in \mathcal{D}\left(S^{1-\delta_2}\right)$, $t \geq 0$,

$$\|C_1'(t) f\| + \|C_1(t) f\| \leq a(t)\left(\|f\| + \left\|S^{1-\delta_2} f\right\|\right),$$

where $a \in \mathcal{C}_0 \cap L^1(\mathbb{R}^+)$.

Theorem 4.1. *Let Hypothesis 4.1 and Hypothesis 4.2 hold. Let* u *be the unique solution of* (4.4), (4.2). *Then there is a dense set* $D^0 \subset H \oplus H$ *such that for*

$(f, g) \in D^0$, *there is a canonically defined*

$$h = h(f, g, S, B, C(\cdot))$$

such that, for the unique solution v *of*

$$\begin{aligned} 2B\frac{dv}{dt} + S^2 v &= 0, \quad t \geq 0 \\ v(0) &= h, \end{aligned}$$

u *satisfies*

$$u(t) = v(t)(1 + o(1))$$

as $t \to \infty$, *provided* $h \neq 0$.

Proof. Let u solve (4.4), (4.2) and let w be the unique solution of (4.6) with initial data

$$w(0) = \widehat{f}, \quad w'(0) = \widehat{g}.$$

Our goal is to show that

$$u(t) = w(t)(1 + o(1))$$

as $t \to \infty$, provided $\widehat{f}$ and $\widehat{g}$ are chosen appropriately. Then the conclusion of Theorem 4.1 follows from Theorem 3.1, provided h is appropriately chosen (as a function of $\widehat{f}$, $\widehat{g}$) and $h \neq 0$.

By the calculation prior to the statement of Theorem 4.1, we know that

$$u(t) = e^{t\Lambda_+^0} P_+(t) f_+ + e^{t\Lambda_-^0} P_-(t) f_-,$$

and

$$w(t) = e^{t\Lambda_+^0} g_+ + e^{t\Lambda_-^0} g_-$$

for suitable $g_\pm$. The idea is to show that

$$\lim_{t\to\infty} P_\pm(t) f_\pm$$

exists (and we call it $g_\pm$), and so (4.12) follows from that since $\left\{e^{t\Lambda_+^0} : t \in \mathbb{R}^+\right\}$ is a (C_0) contraction semigroup. Note that $\{P_\pm(t) : t \in \mathbb{R}^+\}$ are bounded operators, but

$$\lim_{t\to\infty} \|P_\pm(t)\| = \infty$$

can happen. Thus f, g must be restricted in order that

$$\lim_{t\to\infty} P_\pm(t) f_\pm$$

exists.

Recall that

$$P_\pm(t) = e^{-\int_0^t C(s)ds} e^{\pm\int_0^t Q(s)ds}, \tag{4.13}$$

$$Q(t) = \left[(B + C(t))^2 - S^2\right]^{\frac{1}{2}} - \left[B^2 - S^2\right]^{\frac{1}{2}}. \tag{4.14}$$

By our assumptions, there exists $\gamma_0 > 0$ such that

$$B\chi_{(0,\gamma_0)}(S) \geq S\chi_{(0,\gamma_0)}(S),$$

$$B\chi_{(\gamma_0,\infty)}(S) \leq S\chi_{(\gamma_0,\infty)}(S).$$

By Hypothesis 4.2, since $a(t) \to 0$ as $t \to \infty$, we have

$$(B + C(t))^2 \chi_{(\gamma_1,\infty)}(S) \geq S\chi_{(\gamma_1,\infty)}(S)$$

for some $\gamma_1 \in [\gamma_0, \infty)$. Fix this γ_1. Define

$$H_+ = \chi_{(\gamma_1,\infty)}(S)(H),$$

$$H_- = H_+^1 = \chi_{(0,\gamma_1]}(S)(H).$$

Let $R_\pm$ be the orthogonal projection onto $H_\pm$. We summarize some properties of $P_\pm$, $Q_\pm$: $Q_\pm(t) R_+$ is skew adjoint for each $t > \tau_1$ for some $\tau_1 = \tau_1(s, \gamma_1) > 0$;

$$0 \leq e^{-\int_0^t C(s)ds} \leq e^{-\int_0^t C_0(s)ds},$$

$$0 \leq e^{-\int_0^\infty C(s)ds} \leq e^{-\int_0^\infty C_0(s)ds},$$

and $e^{-\int_0^\infty C(s)ds}$ is positive, selfadjoint and bounded;

$$e^{\pm \int_{\tau_1}^t Q(s)ds} \to e^{\pm \int_{\tau_1}^\infty Q(s)ds} \tag{4.15}$$

is unitary;

$$T_\pm := e^{\pm \int_0^{\tau_1} Q(s)ds}$$

is a normal operator commuting with S, $T_\pm$ is injective, and for a constant $K > 0$,

$$0 \leq \operatorname{Re} T_\pm \leq e^{\tau_1 K} e^{\tau_1 KS}$$

by Hypothesis 4.2.

All this follow easily, except for the convergence in (4.15). We treat this now.

$$(B + C(t))^2 - S^2 = B^2 - S^2 + 2BC(t) + C(t)^2,$$

$$2BC(t) + C(t)^2 = 2C_0(t)B + 2BC_1(t) + C(t)^2,$$

hence

$$\begin{aligned} -2(C_0(t))B &\leq 2BC(t) + C(t)^2 \\ &\leq a_2(t)\left(S^{2-\delta_3} + I\right) \end{aligned}$$

for some constant $\delta_3 > 0$ and $a_2 \in \mathcal{C}_0 \cap L^1(\mathbb{R}^+)$.

For $t > 0$,

$$(B + C(t))^2 - S^2 \leq B^2 - S^2 + a_2(t)\left(S^{2-\delta_3} + I\right) \leq 0$$

on $\chi_{(\gamma_3,\infty)}(S)(H)$ for some $\gamma_3 > 0$. Moreover,

$$B^2 - S^2 + a_2(t)\left(S^{2-\delta_3} + I\right) \leq 0$$

for $t \geq \tau_2 = \tau_2(\delta_3, S)$. Using the functional calculus associated with the spectral theorem,

$$\left[(B + C(t))^2 - S^2\right]^{\frac{1}{2}} f \to \left[B^2 - S^2\right]^{\frac{1}{2}} f$$

as $t \to \infty$ for all $f \in \mathcal{D}(S)$, the latter set being a core for all of the operators under construction.

It only remains to identify the limit

$$\lim_{t\to\infty} P_{\pm}(t)\, f_{\pm}$$

and to see precisely how this restricts $f_{\pm}$ (or, equivalently, f and g).

$$e^{-\int_0^t C(s)ds}h \to e^{-\int_0^\infty C(s)ds}h$$

as $t \to \infty$ for all $h \in H$ and $e^{-\int_0^\infty C(s)ds}$ is positive selfadjoint and bounded. Next consider

$$W_{\pm}(t) := e^{\pm \int_0^t Q(s)ds}$$

on H_- and on H_+. $W_{\pm}(t)$ is unitary on H_+ for all $t \geq 0$ and it is unitary on all of H for all $t > \tau_1$. Let $0 \leq t \leq \tau_1$.

$$\int_0^{\tau_1} Q(s)\, ds = \int_0^{\tau_1} \operatorname{Re} Q(s)\, ds + i \int_0^{\tau} \operatorname{Im} Q(s)\, ds,$$

$$W_{\pm} = \widehat{W}_{\pm} = e^{\pm \int_0^{\tau_1} \operatorname{Re} Q(s)ds}$$

where $\widehat{W}_{\pm}$ is unitary on H. Also,

$$\pm \left|\operatorname{Re} Q(s)\right| \leq K_3 \left(I + S^{1-\delta_3}\right)$$

for suitable positive constants K_3, δ_3.

Thus for all h in the dense set $\mathcal{D}\left(e^{\tau_1 S^{1-\delta_3}}\right)$,

$$e^{\pm \int_0^{\tau_1} \operatorname{Re} Q(s)dsh} \to \widetilde{T}_{\pm}h$$

and $\widetilde{T}_{\pm}$ is a positive selfadjoint operator and

$$\left\|\widetilde{T}_{\pm}h\right\| \leq K_3\tau_1\left(\left\|e^{tS^{1-\delta_3}}h\right\| + \|h\|\right).$$

Thus

$$\lim_{t\to\pm\infty} P_{\pm}(t)\, f_{\pm} = Z_{\pm}\widetilde{T}_{\pm}f_{\pm}$$

where $\widetilde{T}_{\pm}$ is positive and selfadjoint, while

$$Z_{\pm} = e^{-\int_0^{\infty} C(s)ds}W_{\pm},$$

this first factor being positive selfadjoint operator and the second factor being unitary, and all of them operators commute.

Thus for Theorem 4.1, $\{f,\ g\}$ must be restricted to a smaller domain which is a common core for all the operators we consider. By imposing rather insignificant additional hypotheses, we can arrange so that $\widetilde{T}_{\pm} = 0$, but we prefer to have the theorem valid in full generality. □

Remark: The construction of h follows from the above proof. We summarize this now.

We have

$$u(t) = e^{t\Lambda_+^0}P_+(t)\, f_+ + e^{t\Lambda_-^0}P_-(t)\, f_-,$$

where

$$P_{\pm}(t) = \exp\left(-\int_0^t (C(s) \pm Q(s))\, ds\right),$$

$$Q(s) = \left[(B + C(s))^2 - S^2\right]^{\frac{1}{2}} - \left[B^2 - S^2\right]^{\frac{1}{2}}.$$

The limit

$$g_{\pm} := P_{\pm}(\infty) f_{\pm} \tag{4.16}$$

exist, and $f_{\pm}$ are constructed from

$$u(0) = f, \quad u'(0) = g$$

by notifying that

$$u(0) = f = P_{+}(0) f_{+} + P_{-}(0) f_{-},$$

$$u'(0) = g = \left(\Lambda_{+}^{0} P_{+}(0) + P_{+}'(0)\right) f_{+} + \left(\Lambda_{-}^{0} P_{-}(0) + P_{-}'(0)\right) f_{-},$$

and it is elementary to show these two equations for $f_{\pm}$ in terms of f, g.

Next,

$$w(t) := e^{t\Lambda_{+}^{0}} g_{+} + e^{t\Lambda_{-}^{0}} g_{-} \tag{4.17}$$

(with $g_{\pm}$ given by (4.16)) satisfies a damped wave equation of the form

$$w'' + 2\widehat{B}w' + \widehat{S}^{2}w = 0,$$

and

$$w(0), \; w'(0)$$

are obtained from $g_{\pm}$ using (4.17). Moreover,

$$w(t) = u(t)(1 + o(t))$$

follows by the calculation in this chapter.

By our previous results, we can construct h in terms of $w(0)$, $w'(0)$ so that the solution v of

$$2\widehat{B}v' + \widehat{S}^2 v = 0, \quad v(0) = h$$

satisfies

$$v(t) = w(t)(1 + o(1)),$$

provided $h \neq 0$, for this case

$$v(t) = u(t)(1 + o(1)),$$

and this completes our (complicated) explanation of how h is "cannonically" constructed.

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