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Investigating and Validating Several Characteristics of the Pythagorean Triples

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INVESTIGATING AND VALIDATING SEVERAL CHARACTERISTICS OF THE PYTHAGOREAN TRIPLES

by

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DEDICATION

I dedicate this to my deserving family, who laboriously strive to achieve more with their given disabilities than many think possible. To my loving and encouraging wife, Paula, who always believed in me and gave me the confidence to reach for this educational status, which I once thought was impossible. To my children, Kira and Nathaniel, who allowed me to take the time to accomplish this dream and fought through their own learning disabilities to become successful contributors to their community. To my parents, David and Linda, who gave me life, trained me to live it properly, and gave guidance and wisdom at the appropriate times. To my parents-in-law, Skip and Ruth, who contributed toward this journey by giving me encouragement and financial assistance. To all of you, my family, I extend my deepest gratitude for your contributions which made this quest attainable.

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ABSTRACT

Lewis, Jeffrey David. MS. The University of Memphis. May 2014. Investigating and Validating Several Characteristics of the Pythagorean Triples. Major Professor: Alistair Windsor, Ph.D.

Working with the Pythagorean triples in Number Theory class, I became intrigued by several attributes within this special set of numbers. Therefore, I will be investigating and validating the following characteristics of the Pythagorean triples:

1) A Pythagorean triple consists of either one even number along with two odd numbers or three even numbers.
2) Every Pythagorean triple contains a multiple of 3, a multiple of 4 and a multiple of 5.
3) Every odd number greater than 1 is part of a Twin Pythagorean triple.
4) All natural numbers greater than 2 appear at least once in a Pythagorean triple and several numbers appear in two or more distinct triples.
5) Every prime number of the form $4k + 1$ is a hypotenuse of a Pythagorean triple.
6) The smallest value of c for which there exist four distinct solutions with a, b, and c pairwise co-prime is 1105.

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Introduction

History of Pythagoras

The Aegean Sea and the Sea of Crete, which are major subdivisions of the Mediterranean Sea, are sprinkled with numerous islands. These islands are all part of the country Greece, including Samos. Samos is off the coast of present-day Turkey in an area referred to as Ionia and was characteristic of the Greek politics and culture of that time. Born to his parents, Mnesarchus and Parthenis (Pythais), on the island of Samos during the second half of the sixth century, was a great philosopher and mathematician by the name of Pythagoras. He was also known as a scientist and religious leader who believed in the concept of transmigration, better known as reincarnation [15].

Pythagoras was a well-traveled man, who thrived on learning new concepts and incorporating new ideas into his own developing philosophy. The earliest teacher to influence Pythagoras was a philosopher named Pherecydes, who taught him in his youth. Around the age of eighteen, Pythagoras set off to study philosophy at the Ionian school in Miletus under the tutelage of Thales and Anaximander, two highly respected philosophers in Greece during that time. He was inspired significantly by these two great minds in the areas of mathematics, astronomy, geometry and cosmology. Because of a recommendation from Thales, Pythagoras then traveled to Egypt in order to continue his education and gain knowledge in applied mathematics for the next twenty-two years.

The Egyptians were known for their great tradition in mathematics, and especially for their applications in the area of geometry. The wherewithal to create massive and miraculous pyramids demonstrates their keen geometric expertise. The spiritual practices of the Egyptians also influenced Pythagoras, and he was ultimately initiated into the priesthood. After being exiled as an Egyptian priest to Babylon around 525 BC, Pythagoras studied with the magi, or Babylonian priests, for the next twelve years. It is said that the Babylonians discovered the concept of the Pythagorean Theorem over 1,000 years prior to Pythagoras himself discovering this amazing formula.

The great mathematician and philosopher returned to his homeland of Greece after being absent for nearly 40 years. Once there, Pythagoras established a school which was named the Semicircle, but distractions forced him to relocate to an area in southern Italy known as Croton. Here, Pythagoras did much of his teaching, and here he set up his secretive Pythagorean society, which has caused him to be such a mysterious figure today. His followers were sworn to secrecy, so little is known about the teachings of Pythagoras. Only a select group of approximately 500 students, the mathematikoi, were allowed in the inner circle. These were Pythagoras's most devoted disciples, and those who were unable to meet the high standards of their teacher were expelled and shunned for the rest of their life.

After a while, people began to distrust the secretiveness of the Pythagorean community, and many of the colony's rulers were followers of Pythagoras. The Crotonians became angry and rose up in opposition, wishing to expel the Pythagoreans from Croton. Cylon, a wealthy nobleman who was refused membership into the society, led an attack which forced Pythagoras to flee to Metapontium. It was here, early in the fifth century BC, that the life of Pythagoras ended. Details of his death vary from suicide to starvation.

Following the death of Pythagoras, his ideas quickly spread among many other locations. His influence can be seen in the works of Plato, Nicolaus Copernicus, Johannes Kepler, and Euclid. Much of the contents of Euclid's *Elements* can be traced back to ideas of Pythagoras and his school. Recognized as being the first to call himself a philosopher, a term he fashioned to denote "lover of wisdom," Pythagoras believed the simple expression, "All is number." This Pythagorean motto implied that all reality is mathematical and everything can be depicted by numbers, or a ratio of numbers (positive rational numbers). Pythagoras made contributions in several areas including: religion, numerology, cosmology, astronomy, music and mathematics; most importantly, in the area of geometry. Although Pythagoras is credited with classifying numbers into odd and even, prime and composite, he is most often associated with the theorem that is identified by his name.

The Pythagorean Theorem, $a^2 + b^2 = c^2$, states that the sum of the squares of the legs in a right triangle are equal to the square of the hypotenuse. Although Pythagoras was thought to be the first one to have discovered a proof of the theorem, he was not the first to utilize the rule. There is proof to support that the Egyptians applied the exact concept centuries before the followers of Pythagoras. Furthermore, a proposal written on clay tablets during the time of King Hammurabi suggests the Babylonians were aware of the idea more than 1,000 years earlier. Credit is thought to be given to Pythagoras for extensive work he and his followers did to scientifically justify the theorem.

They did not think of the rule as a simple algorithm, in which you square one number and add it to another number squared, and whose sum is equal to the square of a third number. In their minds, they saw it geometrically as squares built onto each side of the right triangle, as seen in Figure 1. In this example, the right triangle has two legs with lengths of 3 and 4 units and a hypotenuse measuring 5 units in length. A square with side lengths of 3 units would have an area of 9 $units^2$; a square with side lengths of 4 units would have an area of 16 $units^2$; and a square with side lengths of 5 units would have an area of 25 $units^2$. The sum of the areas of the squares formed using the lengths of the right triangle's legs is equal to the area of the square formed using the length of the hypotenuse.

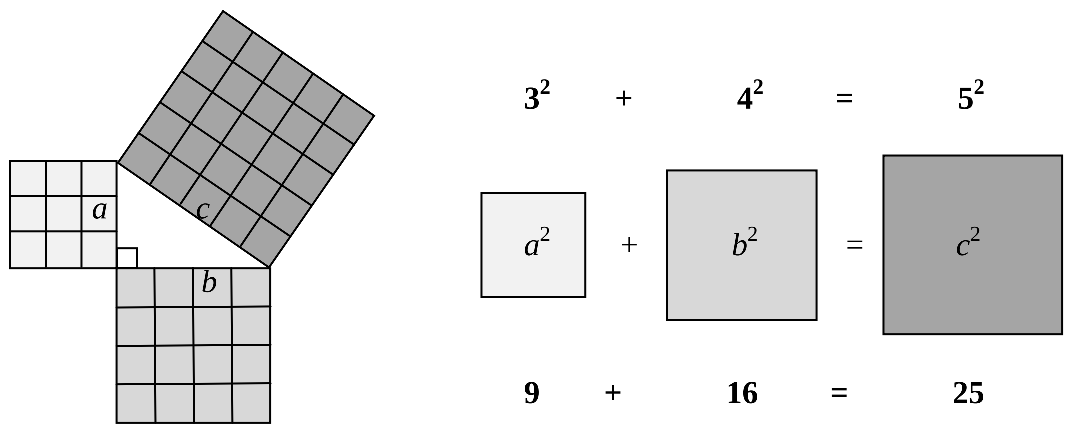

Figure 1. The area in square units for each side of a right triangle separated out and shown as the Pythagorean Theorem.

Proving the Pythagorean Theorem

There are a myriad of unique approaches to discovering the Pythagorean Theorem, ranging from cultures such as the Babylonians to the Chinese, from people such as Euclid to President Garfield, and from mathematical areas such as Algebra or Geometry to Calculus. Several of these methods will be presented hereafter, including using similar right triangles, transforming four congruent right triangles, using properties of Algebra and four congruent right triangles, showing equal areas of rectangles using triangles, and dissecting and rearranging polygonal pieces.

The first procedure to be demonstrated involves similar right triangles formed by drawing an altitude (*h*) from the vertex of the right angle to the hypotenuse of an original right triangle, as seen in Figure 2. The original triangle and the two triangles formed inside the original are all similar triangles based on a theorem from Geometry. Similar triangles are defined as having all corresponding angles congruent and all corresponding side lengths proportional. If we let x represent the shorter segment of the original hypotenuse and let y represent the longer segment, then $x + y = c$.

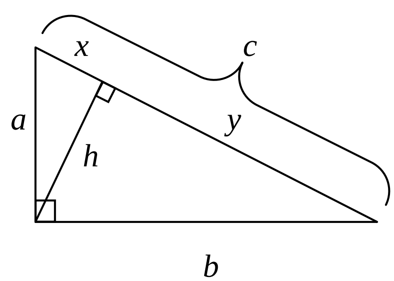

Figure 2. A right triangle with an altitude included from the right angle to the hypotenuse.

Now we can separate the triangles and see the three different sizes, the congruent angles and the corresponding sides, as seen in Figure 3. Sides x, h, and a in the small triangle correspond, respectively, to sides h, y, and b in the medium triangle and to sides a, b, and c in the largest triangle. Any ratio of two sides in one triangle will be equal to the ratio of the two corresponding sides in another triangle.

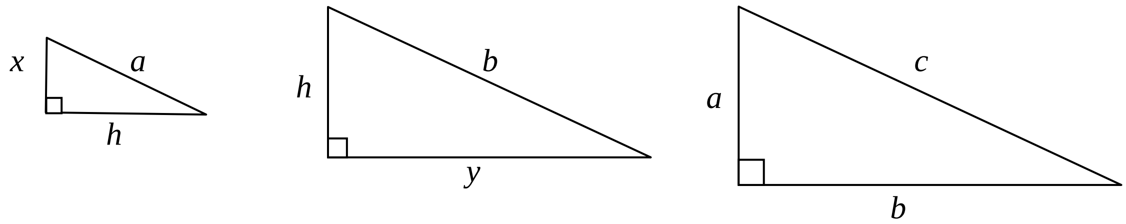

Figure 3. Three similar right triangles taken from the original right triangle in Figure 2.

Next construct a couple of the proportions formed by these three similar triangles, this can be seen in the work that follows. In the first step create a proportion using the ratio of the short leg and the hypotenuse from the smallest triangle and setting it equal the ratio of the short leg and the hypotenuse from the largest triangle. In Geometry *a* is known as the geometric mean of *x* and *c*, which we express as *a* equals the square root of the product of *x* and *c*, or a^2 equals *x* times *c*. This can also be done by finding the cross products of the proportion, where the extremes equal the means. When dividing both sides of the equation by *c* and using the symmetric property of equality to solve for the variable *x* our result is *x* equals *a* squared divided by *c*. Repeating this process in step two, we take the ratio for the long leg and the hypotenuse of the medium size triangle and set it equal to the ratio of the long leg and the hypotenuse of the largest triangle. Find the geometric mean or the product of the extremes and the product of the means, then solve for the variable *y*. We now have expressions for both *x* and *y* which, recalling, added together equal *c*. So in step three substitute the expressions for *x* and *y*, then simplify the equation by multiplying everything by *c* and we arrive at the Pythagorean Theorem, $a^2 + b^2 = c^2$.

Step 1: $\frac{x}{a} = \frac{a}{c}$ Step 2: $\frac{y}{b} = \frac{b}{c}$ (1)

$a^2 = xc$ $b^2 = yc$ (2)

$x = \frac{a^2}{c}$ $y = \frac{b^2}{c}$ (3)

Step 3: $x + y = c$ (4)

$$\frac{a^2}{c} + \frac{b^2}{c} = c \quad (5)$$

$$a^2 + b^2 = c^2 \quad (6)$$

A second way of discovering the Pythagorean Theorem comes from a Chinese proof, which generates four copies of the original triangle and uses geometric transformations to show the theorem, as seen in Figure 4. To begin, the triangles are arranged in such a way as to produce the area of a square with side lengths *c* and 4 right angles. The right angles are supplements of the two acute angles of the right triangles which are complementary. The first transformation is to translate the bottom left triangle up and to the right so that the two hypotenuses coincide. Next translate the leftmost triangle down to create a square with side lengths *a*. Then translate the bottom right triangle to the left so that the two hypotenuses coincide and create a square with side lengths *b*. The area c^2 has been transformed into the area of a^2 and b^2, therefore $a^2 + b^2 = c^2$ [16].

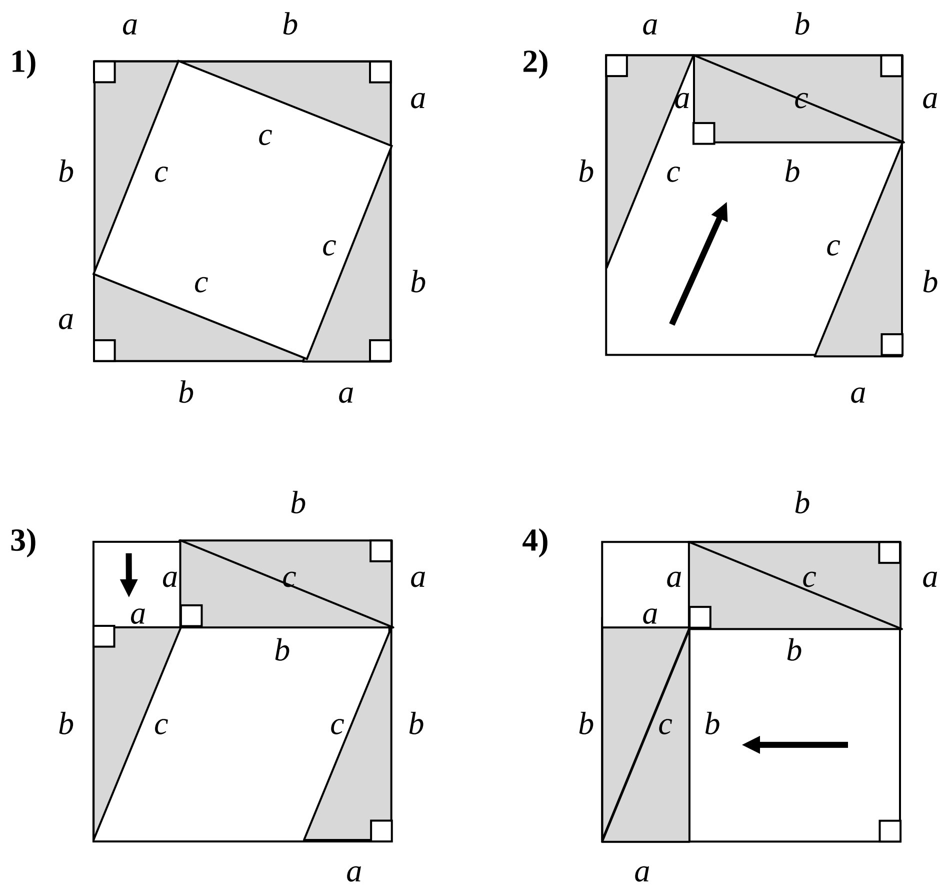

Figure 4. Transforming four congruent right triangles to prove the Pythagorean Theorem.

A proof, more in line with the Babylonian way of thinking, uses the same figure and some Algebraic properties. To find the area of the shaded triangles in figure four, first find the area of the large square and then subtract the square on the inside. To do this we add *a* and *b* to find the length of one side of the larger square. Therefore, the area of the large square is $(a + b)^2$. The area of the smaller square would be c^2. If we subtract the two areas, $(a + b)^2 - c^2$ it yields the area of the shaded portion.

An alternate way of finding the shaded area of figure four is to find the area of one triangle and multiply it by four, since all the triangles are congruent. Knowing the formula for the area of a triangle is base times height divided by 2, and if we let *a* represent the base and *b* represent the height, then we would have $\frac{a\,b}{2}$. Multiply this expression by four, $4\,(\frac{a\,b}{2})$ and we have the shaded area.

The equation in example 2 reflects two methods of finding the area of the large square. So if we square the binomial on the left and multiply the expression on the right by four we are left with *a* squared plus the product of 2, *a* and *b* plus *b* squared equals the product of 2, *a* and *b* plus *c* squared. Next subtract 2*ab* from both sides giving the Pythagorean Theorem once again.

$$(a + b)^2 = 4\left(\frac{(ab)}{2}\right) + c^2 \tag{7}$$

$$a^2 + 2ab + b^2 = 2ab + c^2 \tag{8}$$

$$a^2 + b^2 = c^2 \tag{9}$$

Another way to prove the Pythagorean Theorem appears in Book I, Proposition 47 of Euclid's Elements, written around 300 BC. It states, "In right-angled triangles the square on the side opposite of the right angle equals the sum of the squares on the sides containing the right angle." This can be seen in Figure 5. Start by drawing a perpendicular line through the right triangle from the vertex of the right angle to the hypotenuse and extend it through the square. Then, by drawing diagonal AK in the small square, we divide its area in half. If we stretch vertex A of the triangle ACK down to vertex B, we have a triangle BCK with equal area, on account of the height not changing. Rotating this triangle 90° counterclockwise produces the congruent triangle ACE, which in turn is has area congruent to the triangle XCE or half the area of rectangle CEYX. As a result, we know the area of square AHKC is congruent to the area of rectangle CEYX [18].

Going back to the original figure and remembering what we have already proven, looking at Figure 6, we now draw a diagonal in square ABFG to create the triangle AFB. The triangle CFB would have area congruent to AFB since it has the same base and height. Next rotate triangle CFB 90° clockwise and we will obtain triangle ABD, which has area congruent to triangle XBD or half the area of rectangle DBXY. This results in equal areas of square FGAB and rectangle DBXY. This means the area of the medium-sized square and the larger rectangle are congruent, as well as area of the small square and the small rectangle. Therefore the sum of the areas of the squares on the sides containing the right angle are equal to the area of the square on the side opposite the right angle, just as it is stated in Book One, Proposition 47 of Euclid's Elements, thus proving the Pythagorean Theorem once again. Likewise, this method can be executed using any type of similar figures, and is also contained in Euclid's Elements in Book Six, Proposition 31.

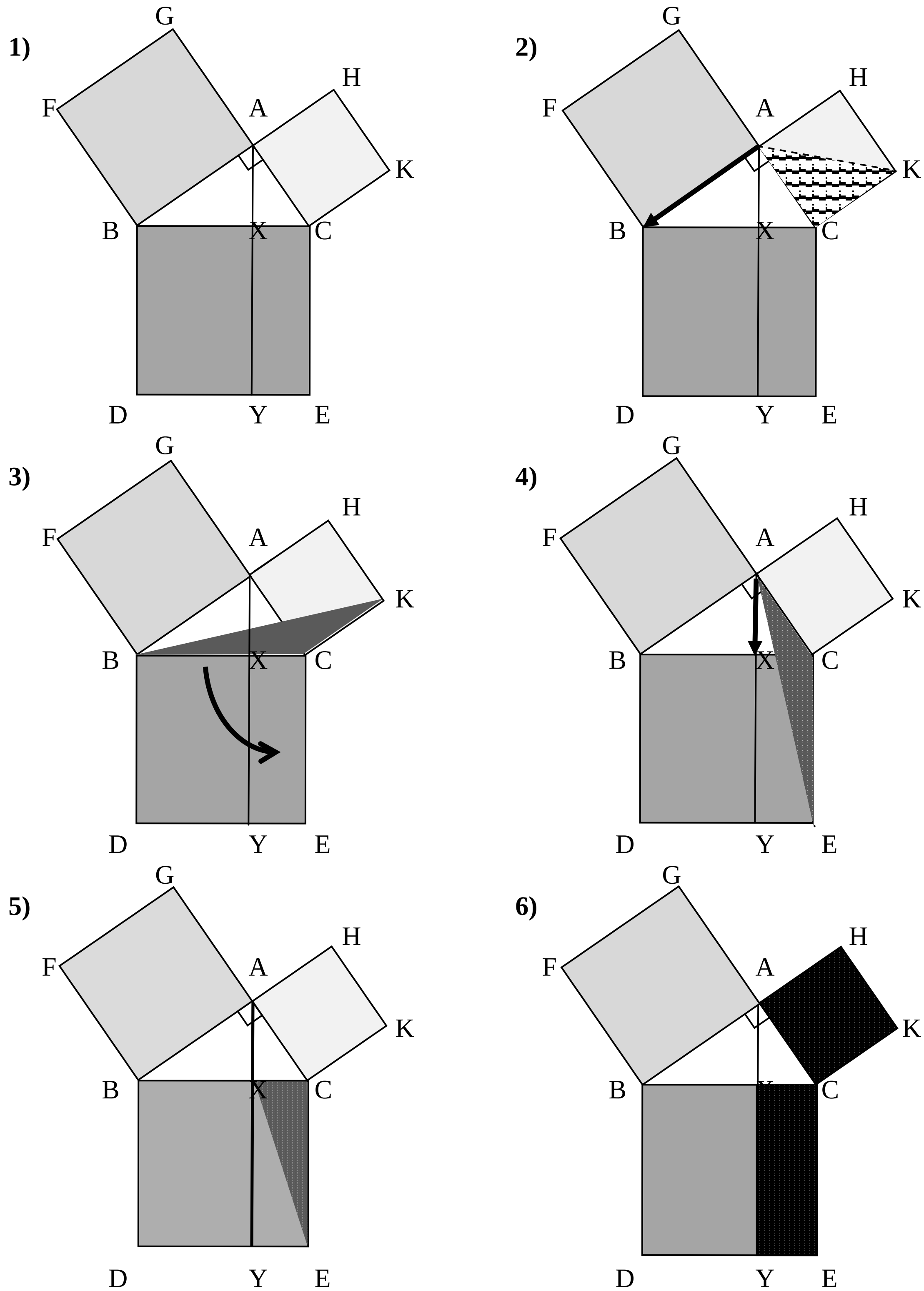

Figure 5. Transforming the area of the small square into the small rectangle.

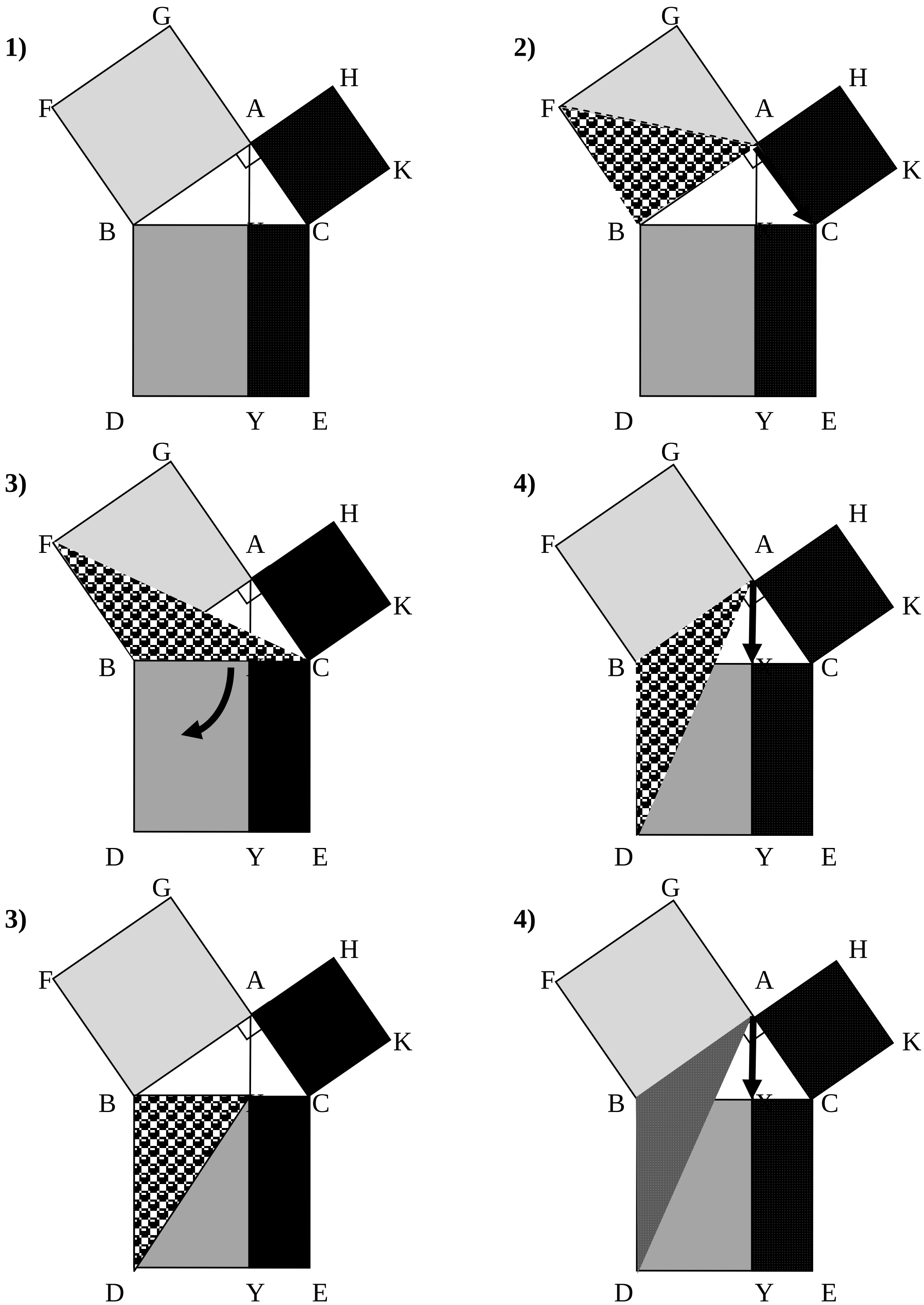

Figure 6. Transforming the area of the medium square into the large rectangle.

The final proof of the Pythagorean Theorem which we will examine involves dissecting and rearranging polygonal pieces, as seen in Figure 7. As we have done previously, start with a right triangle and squares attached to each side of the triangle with lengths corresponding to the three sides of the triangle. Next, take the smallest square, a^2, and place it inside the largest one, c^2. Extend each side of the small square in one direction until it intersects the large square, such that the length is equal to b which is the length of the medium square, and thus creating four quadrilateral areas. Label the four quadrilaterals areas w^2, x^2, y^2, and z^2, and we can now observe the area c^2 is equivalent to the sum of the areas a^2, w^2, x^2, y^2, and z^2. As we can see in the work below, by rearranging the four quadrilateral pieces and placing them into b^2 we will notice that the sum of the areas of w^2, x^2, y^2, and z^2 is equal to b^2. Thus, the Pythagorean Theorem is proven once more.

$$c^2 = a^2 + (w^2 + x^2 + y^2 + z^2) \tag{10}$$

$$b^2 = w^2 + x^2 + y^2 + z^2 \tag{11}$$

$$c^2 = a^2 + b^2 \tag{12}$$

$$a^2 + b^2 = c^2 \tag{13}$$

As previously mentioned, there are several ways to prove the Pythagorean Theorem, and we have looked at a few. There are also infinitely many triples of rational and irrational numbers that can be generated with the Pythagorean Theorem. We will reduce the multitude of numbers slightly and look solely at the positive integer solutions for the formula. These triples are known as Pythagorean Triples.

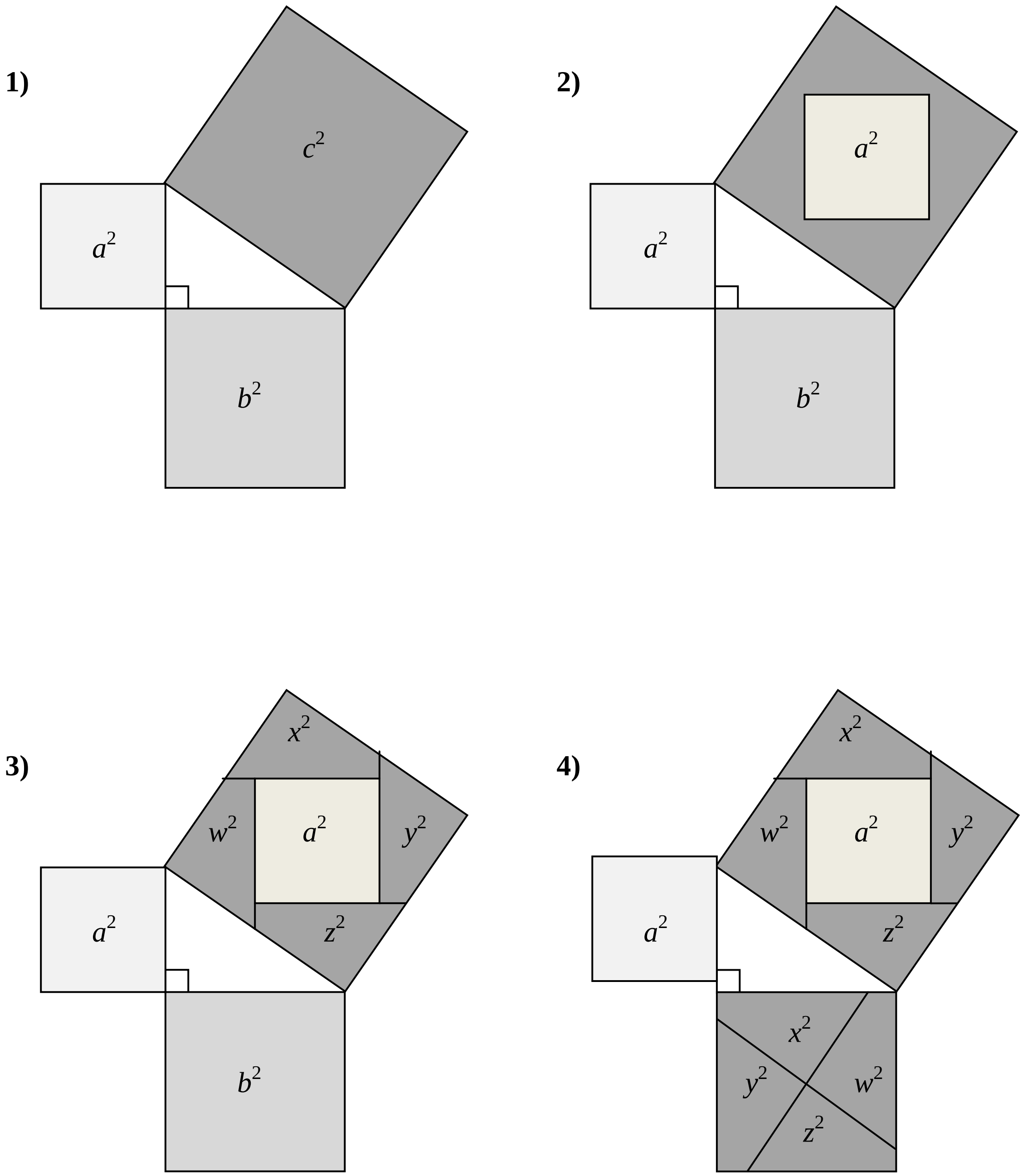

Figure 7. Proving Pythagorean Theorem using quadrilaterals.

CHAPTER 2

Literature Review

Pythagorean Triples

There is documentation of the Babylonians having an understanding of the Pythagorean Theorem long before the time of Pythagoras. A clay tablet written in characters recognized as old Babylonian style cuneiform script, known as *Plimpton 322*, contains examples of early Babylonian mathematics. *Plimpton 322*, whose dimensions are 13 cm across by 9 cm tall by 2 cm thick, is part of the G. A. Plimpton collection at Columbia University in New York. This clay fashioned document dates back to the time period between 1900 and 1600 BC; some people believe it was written about 1800 BC, possibly between 1822 and 1784 BC. The tablet has suffered some deterioration and damage including; the entire left edge missing, large indentations in the upper left corner and in the middle right side, as well as the lower right corner broken off.

Plimpton 322 possesses a table of numbers that demonstrate that the Babylonians did comprehend the concept of the Pythagorean Theorem. The table consists of four columns and fifteen rows, in which the last row is simply a column of numbers from one to fifteen and serves to count the entries. The second and third columns are designated as "*breadth*" and "*diagonal*", which we can classify as the length of the short leg and the hypotenuse of a right triangle. The first column is difficult to read with the damage to the left margin of the document, but the numbers are thought to represent the length of the long leg of a right triangle, possibly a whole number in the form of a fraction [16]. Hence the document proves the Babylonians were capable of calculating what we call, Pythagorean triples.

The Pythagorean Theorem operates for all real numbers, both rational and irrational. On the other hand, Pythagorean triples are the natural numbers or positive integers; (a, b, c), that satisfy the equation, $a^2 + b^2 = c^2$. A few of the more commonly known triples are (3, 4, 5), (5, 12, 13), (7, 24, 25) and (8, 15, 17). The first question that arises is whether there are infinitely many Pythagorean triples, and the answer to the question is an emphatic, YES! When multiplying (a, b, c) by any natural number r we can acquire infinitely many Pythagorean triples; (ra, rb, rc). For example, when you multiply the common triple, (3, 4, 5), by $r = 2, 3, 4, \ldots, n$ the resulting products yield infinitely many Pythagorean triples; (6, 8, 10), (9, 12, 15), (12, 16, 20), …, ($3n$, $4n$, $5n$). This is not particularly satisfying so we decide to call a Pythagorean triple primitive if the only number that divides all three numbers is 1. The question thus remains whether there are infinitely many primitive Pythagorean triples. We remark that if a prime divides two of the three numbers in a Pythagorean triple then it must divide the remaining number since the square of the remaining number can be written as a sum or difference of the squares of the two divisible numbers and if a prime divides the square of a number then it divides the number itself. Thus to show that a Pythagorean triple is primitive we can just show that two of the numbers have no factors in common (such numbers are called relatively prime or co-prime).

Euclid proved there are infinitely many primitive Pythagorean triples, using the observation that any odd number can be written as the difference of perfect squares of two consecutive natural numbers; this can be seen in Table 1. Since there are an infinite number of perfect squares that are odd (i.e. 9, 25, 49, 81…), then we can conclude that there are infinitely many Pythagorean triples as well. Since two consecutive numbers are always relatively prime the Pythagorean triples found this way are primitive Pythagorean triples. We will return to the discussion of odd numbers and perfect squares again as we look at the many patterns found within the Pythagorean triples.

Table 1.

n	n^2	Difference	Primitive Triple
1	1		
2	4	4 – 1 = 3	
3	9	9 – 4 = 5	
4	16	16 – 9 = 7	
5	25	25 – 16 = 9	(3, 4, 5)
6	36	36 – 25 = 11	
7	49	49 – 36 = 13	
…	…	…	
12	144	144 – 121 = 23	
13	169	169 – 144 = 25	(5, 12, 13)
…	…	…	
24	576	576 – 529 = 47	
25	625	625 – 576 = 49	(7, 24, 25)
…	…	…	

Now let's turn our attention to a pair of mathematicians who furthered the study of Pythagorean triples, namely Diophantus of Alexandria and Pierre de Fermat. Diophantus was a Greek mathematician, just like Pythagoras, who lived sometime between the years of 200 to 298 AD. Pierre de Fermat was a French mathematician, who lived during the early 1600's. Diophantus made substantial contributions to the area of algebra and has been called "the father of algebra." He was the author of a series of 13 books called *Arithmetica* [4]. Unlike Diophantus, Fermat, who is considered "the father of modern number theory," was quite mysterious and published very little of the work he accomplished. He is best known for the conjecture known as Fermat's Last Theorem, which was inspired by reading a copy of Diophantus's *Arithmetica.*

Diophantus was interested in finding only the integer solutions to polynomial equations; today we call these types of equations, Diophantine equations. Pythagorean triples are solutions to, the Diophantine equation, $a^2 + b^2 = c^2$. Fermat expanded the idea of the Pythagorean Theorem when he wrote this note in the margin of his copy of *Arithmetica.*

> It is impossible to write a cube as a sum of two cubes, a fourth as a sum of two fourth powers, and, in general, any power beyond the second as a sum of two similar powers. For this, I have discovered a truly wonderful proof, but the margin is too small to contain it.

This equation, $x^n + y^n = z^n$, that Fermat asserted has no integer solutions is also an example of a Diophantine equation. Fermat's assertion was not fully proven for another 350 years. A British mathematician, Andrew Wiles, presented a complete and correct proof of Fermat's Last Theorem in September 1994.

Focusing once again on the famous Pythagorean Theorem, $a^2 + b^2 = c^2$, let us consider the task of identifying all natural number solutions. Pythagoras, himself, set forth the following equations that yield infinitely many Pythagorean triples; where n is any random positive integer.

$a = 2n + 1, \qquad b = 2n^2 + 2n = 2n(n + 1), \qquad c = 2n^2 + 2n + 1 = 2n(n + 1) + 1$

If $n = 1$, then (a, b, c) = (3, 4, 5)

If $n = 2$, then (a, b, c) = (5, 12, 13)

If $n = 3$, then (a, b, c) = (7, 24, 25)

↓ ↓ ↓

Unfortunately, this does not include all possible Pythagorean triples and does not even include all primitive Pythagorean triples. It was Euclid, who later produced a formula for generating Pythagorean triples using two positive integers, which yields a more exhaustive list of the special set of numbers.

Allowing the variables m and n to represent two arbitrary positive integers and requiring that $m > n > 0$ we then let $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$ to form an infinite set of Pythagorean triples. It is immediately clear that (a, b, c) formed in this fashion is a Pythagorean triple since $a^2 = m^4 - 2m^2n^2 + n^4$, $b^2 = 4m^2n^2$, so $a^2 + b^2 = m^4 + 2m^2n^2 + n^4 = (m^2 + n^2)^2 = c^2$. You may check that (9, 12, 15) is not generated by Euclid's formulas since $12 = 2mn$ means either $m = 3$ and $n = 2$ or $m = 6$ and $n = 1$ but that would give either $a = 5$ and $c = 13$ or $a = 35$ and $c = 37$. Thus Euclid's formulas do not give all Pythagorean triples.

Primitive Pythagorean Triples

If a Pythagorean triple (a, b, c) does not have a common factor greater than 1 it is specified as a *primitive* Pythagorean triple, also known as a *reduced* triple. One can make an even more profound assertion saying no pair of side lengths of a Pythagorean Triangle can have a common factor other than 1, i.e. GCF $(a, b) = 1$, GCF $(b, c) = 1$, and GCF $(c, a) = 1$ where GCF denotes the greatest common factor. Furthermore, for any primitive Pythagorean triple only one of the legs can be even, either a or b; but never both a and b at the same time. If a and b are even numbers simultaneously we would have $(\text{even})^2 + (\text{even})^2 = \text{even}$, making c^2 even and this would not be a primitive since each number would be divisible by 2 and GCF $(a, b, c) \neq 1$.If a and b are odd numbers, then they can be written as $a = 2x + 1$ and $b = 2y + 1$, and once squared we have $a^2 = 4x^2 + 4x + 1 \equiv 1 \pmod 4$ and $b^2 = 4y^2 + 4y + 1 \equiv 1 \pmod 4$. Clearly c^2 is even and thus c is even and can be written as $c = 2z$. Once we square we have $c^2 = 4z^2 \equiv 0 \pmod 4$. Substituting into the Pythagorean Theorem we would have $1 + 1 \equiv 0 \pmod 4$, which is a contradiction. Therefore exactly one of a or b is odd in any primitive Pythagorean triple.

If we go back to Euclid's formulas for finding Pythagorean triples, where $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$ we see that this produces many triples that are not primitive. Clearly a necessary condition for the triple to be primitive is that GCF(m, n) = 1 since any common factor of m and n will be a common factor of a, b, and c. Observe that if both m and n are odd then both $m^2 - n^2$ and $m^2 + n^2$ are even and we will not have a primitive triple. Thus the following conditions are necessary:

1) m and n are positive integers
2) $m > n > 0$
3) GCF(m, n) = 1
4) exactly one of m and n is odd

These conditions are sufficient for a primitive triple since if a prime p is a common factor of a, b, and c then p must be a common factor of both $m^2 - n^2$ and $m^2 + n^2$. Thus p is a common factor of $(m^2 + n^2) + (m^2 - n^2) = 2m^2$ and of $(m^2 + n^2) - (m^2 - n^2) = 2n^2$. Our property 4 means that both $m^2 - n^2$ and $m^2 + n^2$ are odd and so p cannot be 2. Consequently we must have that p is a common factor of both m^2 and n^2 and since p is prime it must be a common factor of m and n. This is a contradiction with property 3. Thus any triple constructed using m and n that satisfies properties 1 – 4 above must be primitive. It turns out that in fact all primitive triples may be constructed in this way.

What follows is an elementary proof that Euclid's formulas for generating Pythagorean triples give all primitive triples. Let $a^2 + b^2 = c^2$ be a primitive Pythagorean triple. We will assume that b is the even leg. We have that $b^2 = c^2 - a^2$ by using the subtraction property of equality. Next we factor the difference of squares which gives us $b^2 = (c + a) \cdot (c - a)$. Recognizing the fact that b is the geometric mean of the factors $(c + a)$ and $(c - a)$ we can rewrite the equation as such, $\frac{(c+a)}{b} = \frac{b}{(c-a)}$. By allowing $\frac{(c+a)}{b} = \frac{m}{n}$ such that GCF $(m, n) = 1$, we have a fraction in simplest form. Therefore $\frac{b}{(c-a)} = \frac{m}{n}$ by substitution and by the multiplicative inverse property we have $\frac{(c-a)}{b} = \frac{n}{m}$. Then perform the following algebraic steps;

$$\frac{(c+a)}{b} = \frac{m}{n} \qquad \frac{(c-a)}{b} = \frac{n}{m} \tag{14}$$

$$\frac{c}{b} + \frac{a}{b} = \frac{m}{n} \qquad \text{(B)}\ \frac{c}{b} - \frac{a}{b} = \frac{n}{m} \tag{15}$$

$$\text{(A)}\ \frac{c}{b} = \frac{m}{n} - \frac{a}{b} \tag{16}$$

Substitute (A) into (B)

$$\left(\frac{m}{n} - \frac{a}{b}\right) - \frac{a}{b} = \frac{n}{m} \tag{17}$$

$$-2\,\frac{a}{b} = \frac{n}{m} - \frac{m}{n} \tag{18}$$

$$-2\,\frac{a}{b} = \frac{n^2 - m^2}{mn} \tag{19}$$

$$\text{(C)}\quad \frac{a}{b} = \frac{m^2 - n^2}{2mn} \tag{20}$$

Substitute (C) into (A)

$$\frac{c}{b} = \frac{m}{n} - \left(\frac{m^2 - n^2}{2mn}\right) \tag{21}$$

$$\frac{c}{b} = \frac{2m^2}{2mn} - \frac{m^2}{2mn} + \frac{n^2}{2mn} \tag{22}$$

$$\frac{c}{b} = \frac{2m^2 - m^2 + n^2}{2mn} \tag{23}$$

$$\frac{c}{b} = \frac{m^2 + n^2}{2mn} \tag{24}$$

Therefore $\frac{a}{b} = \frac{m^2 - n^2}{2mn}$ and $\frac{c}{b} = \frac{m^2 + n^2}{2mn}$. If we can equate denominators and numerators then we are done. Since $\frac{a}{b}$ and $\frac{c}{b}$ are in simplest form we know that $\frac{m^2-n^2}{2mn}$ and $\frac{m^2+n^2}{2mn}$ are in simplest form. Since GCF $(m, n) = 1$ the only factor that the numerators and denominators might share is 2. Again since GCF $(m, n) = 1$ the only way that $m^2 - n^2$ and $m^2 + n^2$ can be even is if both m and n are odd. Let $m = 2l + 1$ and $n = 2k + 1$. Now $m^2 + n^2 = (2l + 1)^2 + (2k + 1)^2 = (4l^2 + 4l + 1) + (4k^2 + 4k + 1) = 4l^2 + 4l + 4k^2 + 4k + 2$. If we write $\frac{m^2+n^2}{2mn}$ in lowest form we will have to divide numerator and denominator by 2 to produce $\frac{2l^2 + 2l + 2k^2 + 2k + 1}{mn}$. Now both the numerator and denominator are relatively prime and they are odd. This is a contradiction since now we can equate coefficients so $b = mn$ but we assumed at the outset that b was even. Therefore we must have that that $\frac{m^2-n^2}{2mn}$ and $\frac{m^2+n^2}{2mn}$ are in simplest form and we may equate the numerators and denominators to get $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$ as stated.

Included on the first page of the appendix is a chart showing the triples formed for $m = (2, 3, 4, 5, 6, 7, 8, 9, 10)$ and $n = (1, 2, 3, 4, 5, 6, 7, 8, 9)$. While this method does not include every possible triple, it does include all of the primitive Pythagorean triples. With a few refinements to the equations and variables we can generate all of the triples. If we let $a = k \cdot (m^2 - n^2)$, $b = k \cdot (2mn)$, and $c = k \cdot (m^2 + n^2)$, where m, n satisfy our properties 1 – 4 for primitive Pythagorean triples and k is an arbitrary positive integer, then we will be able to produce all Pythagorean triples. This is clear since every Pythagorean triple is a multiple of a primitive Pythagorean triple.

Rational points on the circle

We know that $x^2 + y^2 = 1$ is the equation for the unit circle, where (x, y) is a point on the circle. If (a, b, c) is a Pythagorean triple then we could write $\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$ where $(x, y) = \left(\frac{a}{c}, \frac{b}{c}\right)$. Thus every Pythagorean triple gives rise to a rational point on the arc of the unit circle in the first quadrant. We will show a one to one correspondence between the Pythagorean triples given by Euler's formulas with *m* and *n* relatively prime positive integers satisfying $m > n$ and the rational points on the circle. We now want to show that every rational point in the first quadrant of the circle comes from a Pythagorean triple given by Euler's formulas. The circle is one dimensional so we can find formulas $x = \varphi(t)$ and $y = \omega(t)$ that determine *x* and *y* from one number *t*. We will do this in a clever fashion so that if *t* is rational then *x* and *y* are rational and vice versa. The point $(\varphi(t), \omega(t))$ is the intersection of the circle with the line of slope *t* through (-1, 0). The equation for the line can be written as $y = t(x + 1)$, as seen in Figure 8.

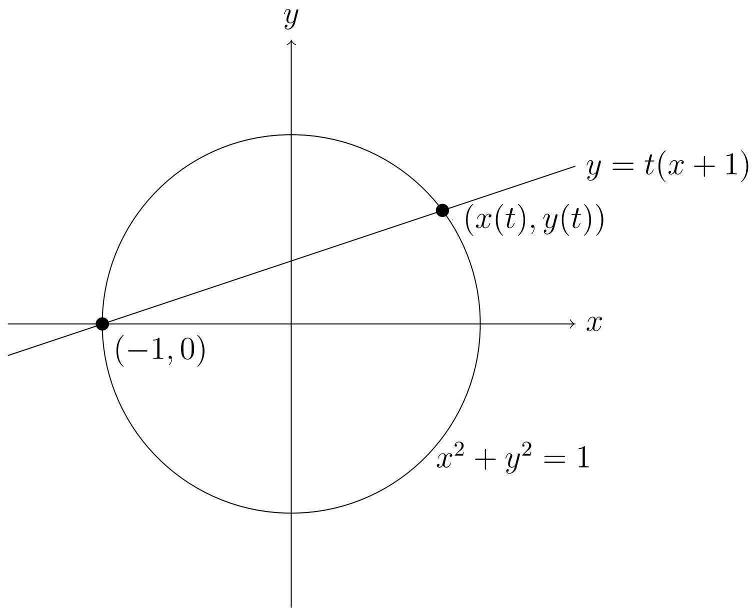

Figure 8.The Unit Circle

Geometrically we see that for $0 < t < 1$ we have *(x(t), y(t))* in the first quadrant. If *(x(t), y(t))* is rational then the slope formula immediately gives us that *t* is rational. If we substitute this expression for *y* into the equation for the unit circle we have [7],

$$x^2 + t^2 (x + 1)^2 = 1 \tag{25}$$

$$x^2 + t^2x^2 + 2\,t^2x + t^2 = 1 \tag{26}$$

$$(1 + t^2)\,x^2 + (2\,t^2)x + (t^2 - 1) = 0 \tag{27}$$

Divide through by $(1 + t^2)$ so the coefficient of x^2 is 1.

(D) $$x^2 + \frac{(2\,t^2)}{(1+t^2)}x + \frac{(t^2-1)}{(1+t^2)} = 0 \tag{28}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2} \tag{29}$$

Quadratic formula with *a=1*

(E) $x = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - c}$ (30)

$$x = -\frac{t^2}{1+t^2} \pm \sqrt{\left(\frac{t^2}{1+t^2}\right)^2 - \frac{t^2-1}{1+t^2}} \quad (31)$$

Substitute (D) into (E)

$$x = -\frac{t^2}{1+t^2} \pm \sqrt{\frac{t^4 - t^4 + 1}{(1+t^2)^2}} \quad (32)$$

$$x = \frac{-t^2 \pm 1}{1+t^2} \quad (33)$$

$$x = \frac{-t^2+1}{1+t^2} \quad or \quad x = \frac{-t^2-1}{1+t^2} \quad (34)$$

(F) $x = \frac{1-t^2}{1+t^2} \quad or \quad x = -\frac{t^2+1}{1+t^2} = -1$ (35)

We ignore the $x = -1$ intersection

(G) $y = t\,(x + 1)$ (36)

Equation of the line

$$y = t\left(\frac{1-t^2}{1+t^2} + 1\right) \quad (37)$$

Substitute (F) into (G)

$$y = t\left(\frac{2}{1+t^2}\right) \quad (38)$$

$$y = \frac{2t}{1+t^2} \quad (39)$$

From this we see that if t is rational then $x(t)$ and $y(t)$ are rational. If t is a positive proper fraction then we may write $t = \frac{n}{m}$, where $m > n > 0$ and GCF $(m, n) = 1$. Now we can substitute t to find x and y in terms of m and n.

$$x = \frac{1-t^2}{1+t^2} = \frac{1-\frac{n^2}{m^2}}{1+\frac{n^2}{m^2}} = \frac{m^2-n^2}{m^2+n^2} = \frac{a}{c} \quad (40)$$

$$y = \frac{2t}{1+t^2} = \frac{2\left(\frac{n}{m}\right)}{1+\frac{n^2}{m^2}} = \frac{2mn}{m^2+n^2} = \frac{b}{c} \qquad (41)$$

Therefore $x = \frac{a}{c}$ and $y = \frac{b}{c}$, where *a*, *b*, and *c* are given by Euler's formulas. Geometrically it is obvious that this map is one to one since distinct slopes clearly lead to distinct intersections.

We know that some of the triples that we get from Euler's formulas will not be primitive even with *m* and *n* relatively prime positive integers satisfying $m > n$. At first sight this seems to contradict our statement about the one to one correspondence since surely a Pythagorean triple should lead to the same rational point as the primitive Pythagorean triple it is associated with. This is not the case as we will demonstrate with a simple example: If we let $m = 5$ and $n = 3$ then we get $a = 16$, $b = 30$, and $c = 34$, which is not primitive, the associated primitive triple comes from $m = 4$ and $n = 1$ which gives $a = 15$, $b = 8$, and $c = 17$. The first Pythagorean triple leads to the rational point $\left(\frac{16}{34}, \frac{30}{34}\right)$, which is equivalent to $\left(\frac{8}{17}, \frac{15}{17}\right)$, while the second Pythagorean triple leads to $(\frac{15}{17}, \frac{8}{17})$.

CHAPTER 3

Analysis

Odds and Evens

When we examine the three numbers that form a Pythagorean triple we notice a pattern regarding the parity of the numbers in each set; that is whether each one is an even number or an odd number. Every Pythagorean triple contains three even numbers or two odd numbers and one even number. None of the triples are comprised of all odd numbers or two evens numbers and an odd number. We will take a look at each case and see why it is possible for some parity combinations and why it is not possible for others.

Let's start with a review of some very basic number theory. When we square an even number it yields an even number and when we square an odd number it yields an odd number. We know an even number is divisible by 2; therefore we can represent an even number by the expression $2k$. If we square this expression $(2k)^2 = 4k^2$, the result is an even number due to the factor 4 being even. Similarly, in the case of an odd number, which we will represent as $2l + 1$, when squared $(2l + 1)^2 = 4l^2 + 4l + 1$ we see the result is an odd number as well. In addition, the sum of two even numbers or two odd numbers is always even and the sum of an odd number and an even number is odd.

<u>even + even = even</u>	<u>odd + odd = even</u>	<u>odd + even = odd</u>
$2k + 2l = 2(k + l)$	$(2k + 1) + (2l + 1) = 2(k + l + 1)$	$(2k + 1) + 2l = 2(k + l) + 1$

From this we can see that it is impossible to have all three numbers be odd. So now we can focus on the other three cases.

First let us prove that the second case an odd + odd = even is not possible. When $a = 2k + 1$ and $b = 2l + 1$, such that they are odd, then their squares would be $4k^2 + 4k + 1$

and $4l^2 + 4l + 1$. Meaning $a^2 = 4k^2 + 4k + 1 \equiv 1 \pmod 4$ and $b^2 = 4l^2 + 4l + 1 \equiv 1 \pmod 4$ and their sum would be $a^2 + b^2 = (4k^2 + 4k + 1) + (4l^2 + 4l + 1) = 4k^2 + 4k + 4l^2 + 4l + 2 \equiv 2 \pmod 4$. Conversely, if $c = 2k + 2l + 2$ is even, then $c^2 = 4k^2 + 4l^2 + 8kl + 8k + 8l + 4 \equiv 0 \pmod 4$. This shows that $a^2 + b^2 \neq c^2$ and explains why we cannot have a Pythagorean triple in the form (odd, odd, even).

Next let us show that the first case is a possible Pythagorean triple. When $a = 2k$ and $b = 2l$ then their squares would be $4k^2$ and $4l^2$. Meaning $a^2 = 4k^2 \equiv 0 \pmod 4$ and $b^2 = 4l^2 \equiv 0 \pmod 4$ and their sum would be $a^2 + b^2 = (4k^2) + (4l^2) \equiv 0 \pmod 4$. Similarly, if $c = 2k + 2l$ is even, then $c^2 = 4k^2 + 4kl + 4l^2 \equiv 0 \pmod 4$. Which shows that $a^2 + b^2 = c^2$ and explains why we can have a Pythagorean triple with all even numbers.

Finally let us look at that the third case of odd + even = odd, when $a = 2k + 1$ and $b = 2l$. Meaning $a^2 = 4k^2 + 4k + 1 \equiv 1 \pmod 4$ and $b^2 = 4l^2 \equiv 0 \pmod 4$ and their sum would be $a^2 + b^2 = (4k^2 + 4k + 1) + (4l^2) = 4k^2 + 4l^2 + 4k + 1 \equiv 1 \pmod 4$. And $c = 2k + 2l + 1$ and $c^2 = 4k2 + 4l^2 + 8kl + 4k + 4l + 1 \equiv 1 \pmod 4$. Showing that $a^2 + b^2 = c^2$ and proving we can have an (odd, even, odd) Pythagorean triple.

Multiple of 3, 4, or 5

The most well known and purest of the Pythagorean triples, (3, 4, 5), is extremely integrated into each and every one of the other infinitely many triples. We can show each of these three numbers is a factor of one of the numbers in every other triple (*a*, *b, c*). One of the legs in a Pythagorean triangle, *a* or *b*, is always divisible by 3. This is also true for the divisibility of 4 in regards to the two legs. In addition, it can be verified that one of the legs or the hypotenuse will always be divisible by 5. Proof for each of these cases can be given using modular arithmetic.

First we will establish the fact that a or b, the legs of a Pythagorean triangle must be divisible by 3. We may assume that our Pythagorean triple is primitive since if it holds for primitive Pythagorean triples it must hold for all Pythagorean triples since all other Pythagorean triples are multiples of a primitive Pythagorean triple.

Since Euclid's formula finds all primitive Pythagorean triples we may take $a = m^2 - n^2$ and $b = 2mn$

$m^2 - n^2 \equiv \pmod 3$

$m^2 - n^2$

$_n\backslash^m$	**0**	**1**	**2**
0	0	1	1
1	2	0	0
2	2	0	0

$2mn \equiv \pmod 3$

$2mn$

$_n\backslash^m$	**0**	**1**	**2**
0	0	0	0
1	0	2	1
2	0	1	2

We see that in every case at least one of a or b is divisible by 3.

Next we will establish the fact that a or b, the legs of a Pythagorean triangle must be divisible by 4.

Again we may assume our Pythagorean triple is primitive and hence may be written as $a = m^2 - n^2$ and $b = 2mn$

$m^2 - n^2 \equiv \pmod 4$

$m^2 - n^2$

$_n\backslash^m$	**0**	**1**	**2**	**3**
0	0	1	0	1
1	3	0	3	0
2	0	1	0	1
3	3	0	3	0

$2mn \equiv \pmod 4$

$2mn$

$_n\backslash^m$	**0**	**1**	**2**	**3**
0	0	0	0	0
1	0	2	0	2
2	0	0	0	0
3	0	2	0	2

We see that in every case at least one of a or b is divisible by 4.

Finally we will establish the fact that a, b, or c, one of the legs or the hypotenuse of a Pythagorean triangle must be divisible by 5.

Given $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$

$m^2 - n^2 \equiv \pmod 5$

$m^2 - n^2$

$_n\backslash^m$	**0**	**1**	**2**	**3**	**4**
0	0	1	4	4	1
1	4	0	3	3	0
2	1	2	0	0	2
3	1	2	0	0	2
4	4	0	3	3	0

$2mn \equiv \pmod 5$

$2mn$

$_n\backslash^m$	**0**	**1**	**2**	**3**	**4**
0	0	0	0	0	0
1	0	3	1	4	2
2	0	1	2	3	4
3	0	4	3	2	1
4	0	2	4	1	3

$m^2 + n^2 \equiv \text{(mod 5)}$

$m^2 + n^2$

$_n\backslash^m$	**0**	**1**	**2**	**3**	**4**
0	0	1	4	4	1
1	1	2	0	0	2
2	4	0	3	3	0
3	4	0	3	3	0
4	1	2	0	0	2

Twin Pythagorean Triples

A twin Pythagorean triple is one in which two sides of the right triangle are consecutive integers. This is possible with a leg and the hypotenuse as well as with two legs, such that (a, b, c) can be written as $(a, b, b+1)$ or $(a, a+1, c)$. Here is a list of some of the twin Pythagorean triples; first with a leg and the hypotenuse (3, 4, 5), (5, 12, 13), (7, 24, 25), (9, 40, 41), (11, 60, 61), (13, 84, 85), (15, 112, 113) . . . and also with two legs (3, 4, 5), (20, 21, 29), (119, 120, 169), (696, 697, 985) . . . Additionally it can also be shown that every odd number greater than 1 is part of a twin Pythagorean triple.

Let's start by showing the case where a leg and the hypotenuse are consecutive integers. Clearly such a Pythagorean triple is primitive so we may use the Euler formulae for the Pythagorean triple. It is readily observed that a and c cannot be successive integers. That is we have $b + 1 = c$. Using the Euler formulae we have $2mn + 1 = m^2 + n^2$ where $m > n > 0$, then $1 = m^2 - 2mn + n^2 = (m - n)^2$. When you take the square root you have $m - n = \pm 1$ and solving for m we get $m = n + 1$, remembering that $m > n > 0$. So, if $n = 1$, then $m = 2$ and $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$. Therefore, $a = 2^2 - 1^2 = 3$, $b = 2(2)(1) = 4$, and $c = 2^2 + 1^2 = 5$ giving us the only Pythagorean triple (3, 4, 5) where all three sides are consecutive integers. Continuing with n = 2, then $m = 3$, $a = 3^2 - 2^2 = 5$, $b = 2(3)(2) = 12$, and $c = 3^2 + 2^2 = 13$ giving us the twin Pythagorean triple (5, 12, 13). One final example with n = 3, then $m = 4$, $a = 4^2 - 3^2 = 7$, $b = 2(4)(3) = 24$, and $c = 4^2 + 3^2 = 25$ giving us another twin Pythagorean triple with a leg and the hypotenuse being consecutive integers (7, 24, 25). It is easy to show every odd number greater than 1 is part of a twin Pythagorean triple where a leg and the hypotenuse are consecutive integers. Since we have proven $m = n + 1$ in twin Pythagorean triples where a leg and the hypotenuse are consecutive integers, we can substitute $(n + 1)$ in place of m for the odd leg, $m^2 - n^2$. This gives us $(n + 1)^2 - n^2$, which in turn becomes $2n + 1$. Therefore when $n = 1, 2, 3, 4, 5, 6 \ldots r$ the leg, $m^2 - n^2 = 3, 5, 7, 9, 11, 13 \ldots s$, which is every odd number greater than 1.

In a like manner we can show that two legs of a Pythagorean triple can be consecutive integers. This time we let $2mn \pm 1 = m^2 - n^2$. From this we get $m^2 - 2mn - n^2 = \pm 1$ or equivalently $(m - n)^2 - 2n^2 = \pm 1$. If we let $m - n = x$ and $n = y$ then we obtain $x^2 - 2n^2 = \pm 1$. This is the famous Pell equation. Luckily the solutions of the Pell equation are known.

$$x = \frac{\left(1+\sqrt{2}\right)^r + \left(1-\sqrt{2}\right)^r}{2} \qquad (42)$$

$$y = \frac{\left(1+\sqrt{2}\right)^r - \left(1-\sqrt{2}\right)^r}{2\sqrt{2}} \qquad (43)$$

If we work backward we see that $m = x + y$, $n = y$ so that $a = m^2 - n^2 = x^2 + 2xy$, $b = 2mn = 2xy + 2y^2$, and $c = m^2 + n^2 = x^2 + 2xy + 2y^2$. Now plugging in the solutions above we obtain

$$a = \frac{(-1)^r}{2} + \frac{\left(1-\sqrt{2}\right)^{1+2r} + \left(1+\sqrt{2}\right)^{1+2r}}{4} \tag{44}$$

$$b = -\frac{(-1)^r}{2} + \frac{\left(1-\sqrt{2}\right)^{1+2r} + \left(1+\sqrt{2}\right)^{1+2r}}{4} \tag{45}$$

$$c = \frac{\left(1+\sqrt{2}\right)^{1+2r} - \left(1-\sqrt{2}\right)^{1+2r}}{2\sqrt{2}} \tag{46}$$

Plugging in r from 1 to 10 we obtain the first ten triples with legs that are consecutive integers. They are:

(3, 4, 5), (21, 20, 29), (119, 120, 169), (697, 696, 985), (4059, 4060, 5741),

(23661, 23660, 33461), (137903, 137904, 195025), (803761, 803760, 1136689),

(4684659, 4684660, 6625109), (27304197, 27304196, 38613965)

There are infinitely many such triples.

Number of Appearances

Every integer can be a side in a Pythagorean triple. We can show this using m and n, where $2mn$ is even and $m^2 - n^2$ is odd. If E is an even integer and we let $n = 1$ and $m = E/2$, then $2mn = 2(E/2)(1) = E$. Likewise if F is an odd integer and we let $m = (F + 1)/2$ and $n = (F - 1)/2$, then $m^2 - n^2 = (m - n)(m + n) = 1 \cdot F = F$ [9]. So every integer can be a side in a Pythagorean triple.

There are several numbers which appear in two or more distinct Pythagorean triples. The number 12 occurs twice in primitive Pythagorean triples (5, 12, 13) & (12, 53, 37) and two more times as a multiple of (3, 4, 5) in (9, 12, 15) & (12, 16, 20). There is not much interest in triples which are multiples of (3, 4, 5) so we will focus our attention on numbers occurring multiple times in primitive Pythagorean triples. There are three cases we will need to look at including even legs, odd legs, and an odd hypotenuse. The odd legs will be differentiated as prime and non-prime.

For the first case let's look at the number 8, such that, $8 = 2mn$, where the possible integer solutions are $(m, n) = (1, 4)$ where the GCF is 1.This indicates that there is one occurrence of 8 in a primitive Pythagorean triple. (8, 15, 17) For the number 36 we see that $36 = 2mn$, where the possible solutions are $(m, n) = (3, 6), (2, 9), (1, 18)$, but one of these have a common factor greater than 1. (3, 6) = 3 Only two solutions have a GCF = 1and therefore there are only two occurrences of the number 36 as a leg in a primitive Pythagorean triple. (36, 77, 85) and (36, 323, 325) There is one exception to this rule and that is every fourth number after 2. The numbers 6, 10, 14, 18 . . . $4k + 2$ cannot be the side lengths in a primitive Pythagorean triple.

In the case of odd legs, if the number is prime, it will only appear once in the form of $m^2 - n^2$. However in the case of odd legs which are not prime the same rule applies as with the even legs. For example the number 9 has factors of $(3, 3) \neq 1$ and $(1, 9) = 1$. Since there is only one pair of factors with a GCF of 1 the number 9 only appears once in a primitive Pythagorean triple. (9, 40, 41) As we see with 15 which has two pairs of factors (3, 5) and (1, 15) it will appear twice in a primitive Pythagorean triple, (8, 15, 17) and (15, 112, 113) the number 105 actually appears 4 times (88, 105, 137), (105, 208, 233), (105, 608, 617), and (105, 5512, 5513) because it has four pair of factors with a GCF of 1. (7, 15), (3, 35), (5, 21), and (7, 15)

For the final case of an odd hypotenuse we will see that these numbers are not only odd, but also prime or at least a product of prime numbers. We will discuss this more in depth in the next two sections. We will see that it is not possible for some numbers to be a hypotenuse in any Pythagorean triples and there are some numbers that do appear in multiple primitive Pythagorean triples.

A Prime Number Hypotenuse

The numbers that can be the length of the hypotenuse in a primitive Pythagorean triple are much less numerous. This list does include all prime numbers of the form $4k + 1$. Some examples are $5 = 4(1) + 1$, $13 = 4(3) + 1$, $17 = 4(4) + 1$, $29 = 4(7) + 1$, and $37 = 4(9) + 1$. Also included as lengths of the hypotenuse for a primitive Pythagorean triple are numbers that are products of primes in the form $4k + 1$. This would be numbers like $25 = 5\cdot5$, $65 = 5\cdot13$, and $221 = 13\cdot17$. We can see that each of these numbers can also be produced using $m^2 + n^2$. $5 = 2^2 + 1^2$, $13 = 3^2 + 2^2$, $17 = 4^2 + 1^2$, $29 = 5^2 + 2^2$, $37 = 6^2 + 1^2$, and $25 = 4^2 + 3^2$, $65 = 8^2 + 1^2$, $221 = 11^2 + 10^2$

Even though the numbers 7, 11, 19, and 23 are prime numbers they are not of the form $4k + 1$, but actually in the form $4k – 1$; therefore they cannot be a hypotenuse in a primitive Pythagorean triple. Similarly, even though 9 is of the form $4k + 1$ is not a possible hypotenuse in a primitive Pythagorean triple because its prime factors are not in the form $4k + 1$. This is also true of numbers like 21, 33, 45, and 49.

Hypotenuse with Four Solutions

As we have seen every hypotenuse in a primitive Pythagorean triple is a prime number or a product of prime numbers in the form $4k + 1$. From 1 through 100 this includes 5, 13, 17, 25, 29, 37, 41, 53, 61, 65, 73, 85, 89, and 97. Most of these numbers only appear in one primitive Pythagorean triple, but 65 and 85 in the list actually appear in two different primitive Pythagorean triples. We call these pairs of Pythagorean triples siblings when they have the same hypotenuse. This double occurrence is due to the number of distinct factors in the form $4k + 1$. Notice that even though $25 = 5 \cdot 5$ has a pair of prime factors, they are the same and not distinct. Whereas $65 = 5 \cdot 13$ and $85 = 5 \cdot 17$ have two distinct prime factors. This double appearance would also include the numbers $145 = 5 \cdot 29$, $185 = 5 \cdot 37$, $205 = 5 \cdot 41$, $221 = 13 \cdot 17$ and many more.

The pattern continues in an exponential fashion such that the smallest possible number to appear in four different primitive Pythagorean triples would be 1105, since the three smallest distinct prime factors of the form $4k + 1$ are 5, 13, 17 and $1105 = 5 \cdot 13 \cdot 17$.[22] Therefore the smallest value of c for which there exist four distinct solutions with a, b, and c pairwise co-prime is 1105. If we continue the pattern the next number would appear eight times in a primitive Pythagorean triple and it would be the product of the four smallest distinct prime factors of the form $4k + 1$. That would be $5 \cdot 13 \cdot 17 \cdot 29 = 32045$. The numbers in the sequence 5, 65, 1105, 32045, 1185665, 48612265, 2576450045... are the smallest hypotenuses that harbor 1, 2, 4, 8, 16, 32, 64 ... primitive Pythagorean triples. The terms are given by multiplying successive primes that satisfy $p \equiv 1 \pmod{4}$: $5 = 5$, $65 = 5 \cdot 13$, $1105 = 5 \cdot 13 \cdot 17$, $32045 = 5 \cdot 13 \cdot 17 \cdot 29$, and $1185665 = 5 \cdot 13 \cdot 17 \cdot 29 \cdot 37$.

CHAPTER 4

Summary

Uses in the classroom

The applications of the Pythagorean Theorem, Pythagorean triples and the patterns found therein are endless. These exercises can be included as early as basic math in an elementary school classroom and continue to be used all the way through the highest levels of mathematics. The Pythagorean Theorem is much more than just finding the missing side of a right triangle. In addition it can be used to discuss number parity, prime numbers, common factors, divisibility, series and sequences, perimeter and area, mathematical history and the list goes on and on.

In this section we would like to share just a few ideas for optional methods of blending Pythagorean triples and other mathematical content. This will encompass the education levels of elementary, middle and high school. The topics mentioned previously will be incorporated along with the Pythagorean triples to introduce or sometimes expand on the subject matter. This is a way to show how to integrate an important mathematical concept like the Pythagorean Theorem along with various other mathematical concepts.

Let's start with the elementary level where we could use Pythagorean triples to discuss odd and even numbers along with the concept of the perimeter of a right triangle. If some pairs of students were given eight pieces of dry spaghetti noodles and first asked to cut them into even lengths of 6, 8, 10, 12, 14, 16, 18, and 20 units. Next they would be told to lay the 6 and 8 unit pieces on grid paper in an "L" shape. Then measure the distance needed to form a right triangle, and they would find it to be 10 units. Complete the activity by asking them to use the spaghetti pieces to find three other even numbers that could be the sides of a right triangle.

Different pairs could be doing the activity with three odd lengths and still other pairs could use one odd and one even leg. Of course the pairs using all odd lengths will not find a Pythagorean triple and the class could discuss how the perimeter is comprised of all even numbers or two odds and one even. In the end have the students take the three even numbers and divide them by two to see that they reduce to the same numbers in one of the two odd and one even Pythagorean triples.

When students start to look at common factors give them Pythagorean triples and tell them to find the GCF. Discuss the fact that some only have a common factor of one while others have larger common factors. We can introduce the idea that those Pythagorean triples only having a common factor of one are called primitive Pythagorean triples. This would also be a good time to have students see that some of the numbers in a primitive Pythagorean triple are prime, but ask them what are the possible factors of the nonprime numbers. Hopefully they will see that 3, 4, and 5 are factors of one of the numbers every time.

In middle school students could contemplate the various ways to find Pythagorean triples. This fits in well with the common core concept of understanding multiple processes to get a solution. The students could research online to find different methods of producing Pythagorean triples and make a list of them. Then the teacher would assign a group to one of the methods and the students would have to write a paper or design a presentation for the class on their procedure.

One simple way to produce Pythagorean triples is to take any odd number, like 7, and square it to get 49. Then find the two consecutive numbers that add up to 49, which would be 24 and 25. This gives you the Pythagorean triple (7, 24, 25) and can lead into a discussion of twin Pythagorean triples. We could also use the method of Pythagoras where he let $a = 2n + 1$, $b = 2n^2 + 2n = 2n(n + 1)$, and $c = 2n^2 + 2n + 1 = 2n(n + 1) + 1$. Another method for generating Pythagorean triples uses numbers from the Fibonacci sequence. We pick four consecutive numbers from the sequence F_n, F_{n+1}, F_{n+2}, and F_{n+3}, where $a = F_n \cdot F_{n+3}$, $b = 2 \cdot F_{n+1} \cdot F_{n+2}$, and $c = F_{n+1}^2 \cdot F_{n+2}^2$. Of course we would need to include Euclid's formula where $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$ when m and n are positive integers.

Finally on the high school level, we could involve literacy as part of the lesson by having students do a research paper on the many people and their proofs of the Pythagorean Theorem. The students would be assigned a group of people like the Babylonians or Chinese, or individuals like Euclid, President Garfield or Pythagoras himself. They would be required to write some background information about the person(s) assigned and describe how they proved the Pythagorean Theorem. This could count as a project for the grading period.

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APPENDIX

Calculating Pythagorean Triples

Where m and n are positive integers, such that $m > n > 0$,

and $m^2 - n^2$, $2mn$, and $m^2 + n^2$ are Pythagorean triples.

	***n* = 1**	***n* = 2**	***n* = 3**	***n* = 4**	***n* = 5**	***n* = 6**	***n* = 7**	***n* = 8**	***n* = 9**
***m* = 2**	(3, 4, 5)								
***m* = 3**	(8, 6, 10)	(5, 12, 13)							
***m* = 4**	(15, 8, 17)	(12, 16, 20)	(7, 24, 25)						
***m* = 5**	(24, 10, 26)	(21, 20, 29)	(16, 30, 34)	(9, 40, 41)					
***m* = 6**	(35, 12, 37)	(32, 24, 40)	(27, 36, 45)	(20, 48, 52)	(11, 60, 61)				
***m* = 7**	(48, 14, 50)	(45, 28, 53)	(40, 42, 58)	(33, 56, 65)	(24, 70, 74)	(13, 84, 85)			
***m* = 8**	(63, 16, 65)	(60, 32, 68)	(55, 48, 73)	(48, 64, 80)	(39, 80, 89)	(28, 96, 100)	(15, 112, 113)		
***m* = 9**	(80, 18, 82)	(77, 36, 85)	(72, 54, 90)	(65, 72, 97)	(56, 90, 106)	(45, 108, 117)	(32, 126, 130)	(17, 144, 145)	
***m* = 10**	(99, 20, 101)	(96, 40, 104)	(91, 60, 109)	(84, 80, 116)	(75, 100, 125)	(64, 120, 136)	(51, 140, 149)	(36, 160, 164)	(19, 180, 181)

Calculating Primitive Pythagorean Triples

Where m and n are positive integers, such that $(m, n) = 1$, $m > n > 0$, exactly one of m and n is even,

and $m^2 - n^2$, $2mn$, and $m^2 + n^2$ are Pythagorean triples.

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$
$m = 2$	(3, 4, 5)								
$m = 3$		(5, 12, 13)							
$m = 4$	(15, 8, 17)		(7, 24, 25)						
$m = 5$		(21, 20, 29)		(9, 40, 41)					
$m = 6$	(35, 12, 37)				(11, 60, 61)				
$m = 7$		(45, 28, 53)		(33, 56, 65)		(13, 84, 85)			
$m = 8$	(63, 16, 65)		(55, 48, 73)		(39, 80, 89)		(15, 112, 113)		
$m = 9$		(77, 36, 85)		(65, 72, 97)		(45, 108, 117)		(17, 144, 145)	
$m = 10$	(99, 20, 101)		(91, 60, 109)				(51, 140, 149)		(19, 180, 181)