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D* Sets and AIP* Sets in Z and Countable Fields

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D* SETS AND AIP* SETS IN Z AND COUNTABLE FIELDS

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ABSTRACT

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We consider the set of ultrafilters in $\mathbb{Z}$, denoted $\beta\mathbb{Z}$. An IP set in $\mathbb{Z}$ is a set that contains some infinite sequence and all of its finite sums. An IP* set is a set that meets every IP set non-trivially. An AIP* set is a set A having the property that for some set B of 0 upper Banach density, $A \cup B$ is IP*. Alternately, call a set IP rich if it is still IP upon the removal of any 0 upper Banach density set. Then a set is AIP* if and only if it intersects every IP rich set non-trivially. Some ultrafilters have the property that all of their members have positive upper Banach density. Such ultrafilters are called essential and their members are called D sets. A set that is a member of every essential idempotent ultrafilter is called a D^* set. V. Bergelson asked whether or not every D* set is AIP*. Equivalently, whether every D set is IP rich. We give a negative answer to this question. That is, we construct a set A that is an IP rich set but not a D set. We also extend our construction to countably infinite fields of finite characteristic.

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Chapter 1. Introduction

1.1 Filters and Ultrafilters

In this introductory section, we provide some general definitions and facts needed for this dissertation. Most of the literature provided in this chapter can be found in **[HS]** in a much more detailed form.

Throughout this section, S will denote an arbitrary set with the discrete topology. $\mathcal{P}(\mathcal{S})$ is the power set of S and " $\subset$ " will denote strict inclusion. ($\subseteq$ will denote non-strict inclusion.) Note that " $\subset$ " is a strict partial order on $\mathcal{P}(S)$, i.e., a relation " $\prec$ " on $\mathcal{P}(S)$ such that:

1. $\Omega \prec \Omega$ never holds, and
2. If $\Omega_1 \prec \Omega_2$ and $\Omega_2 \prec \Omega_3$, then $\Omega_1 \prec \Omega_3$.

Given a set S, a *filter* on S is a non-empty set $\Upsilon \subset \mathcal{P}(S)$ with the following properties:

(a) If $A, B \in \Upsilon$, then $A \cap B \in \Upsilon$.

(b) If $A \in \Upsilon$ and $A \subseteq B \subseteq S$, then $B \in \Upsilon$.

(c) $\emptyset \notin \Upsilon$.

Here are some examples of filters:

1. Let $A \neq \emptyset$ be a subset of $\mathbb{N}$. The set $\{B \subseteq \mathbb{N} : A \subseteq B\}$ satisfiies the filter requirement.
2. Given $a \in \mathbb{Z}$. The set $\{B \subseteq \mathbb{Z} : a \in B\}$ is also a filter. (It is actually an ultrafilter; see below.)
3. $\{A \subseteq \mathbb{N} : \mathbb{N} \backslash A \text{ is finite}\}$ is a filter.

Much of this section will be about ultrafilters, which are abstract collections of sets having certain properties. We begin with a pair of definitions.

1.1.1. Definition. A collection $\mathcal{C}$ of subsets of S is said to have the finite intersection property if for every finite subcollection $\{C_1, C_2, \cdots, C_n\}$ of $\mathcal{C}$, the intersection $C_1 \cap C_2 \cap \cdots \cap C_n$ is nonempty.

1.1.2. Definition. Let S be a non-empty set. An *ultrafilter* on S is a filter on S which is not properly contained in any other filter on S.

Non-trivial examples of ultrafilters require some version of the axiom of choice. We shall be using the axiom of choice in the form of Zorn's Lemma freely here. As a step toward its formulation, we remind readers that a chain on a strict partially ordered set $(\mathcal{A}, \prec)$ is a subset $\mathcal{B} \subseteq \mathcal{A}$ such that for every A and B in $\mathcal{B}$ for which $A \neq B$, either $A \prec B$ or $B \prec A$.

1.1.3. Zorn's Lemma. Let $\mathcal{A}$ be a non-empty set that is strictly partially ordered. If every non-empty chain in $\mathcal{A}$ has an upper bound in $\mathcal{A}$, then $\mathcal{A}$ has a maximal element.

With Zorn's Lemma in hand, we can establish that ultrafilters are plentiful.

1.1.4. Theorem. Let S be a non-empty set and let $\mathcal{V} \subseteq \mathcal{P}(S)$ be a filter on S. There exists an ultrafilter $\mathcal{U} \subseteq \mathcal{P}(\mathcal{S})$ such that $\mathcal{V} \subseteq \mathcal{U}$. Moreover, if $\mathcal{V}$ is a filter and $\mathcal{V} \subseteq \mathcal{W}$ with $\mathcal{W}$ having the finite intersection property, then there exists an ultrafilter $\mathcal{U}$ containing $\mathcal{W}$.

Proof. Let

$$T = \{\mathcal{B} \subseteq \mathcal{P}(\mathcal{S}) : \mathcal{V} \subseteq \mathcal{B} \text{ and } \mathcal{B} \text{ has the finite intersection property}\}.$$

Note that T is non-empty since $\mathcal{V} \in T$. Let T be (partially) ordered by strict inclusion (i.e., " $\subset$ "). Our strategy is to find a maximal element in T, which we will then show is an ultrafilter. We use Zorn's Lemma. Let $B \subseteq T$ be a non-empty chain in T. We need to show that B has an upper bound in T. We have

$$\mathcal{G} = (\bigcup_{\mathcal{B} \in B} \mathcal{B}) \in T.$$

To see this, we must show that $\mathcal{V} \subseteq \mathcal{G}$ and that $\mathcal{G}$ has the finite intersection property. Obviously $\mathcal{V} \subseteq \mathcal{G}$. We need to show that if $\mathcal{X}_i \in \mathcal{P}(S)$, $1 \leq i \leq n$, then $\bigcap \mathcal{X}_i \neq \emptyset$. We'll show this for $n = 2$, leaving the general case as an exercise. If $\mathcal{X}, \mathcal{Y} \in \mathcal{P}(\mathcal{S})$

and $\mathcal{X}, \mathcal{Y} \in \mathcal{G}$ then $\mathcal{X} \in \mathcal{B}_i$ and $\mathcal{Y} \in \mathcal{B}_j$ for some $B_i, B_j \in \mathcal{G}$. Since B_i, B_j belong to the chain B, either $B_i \subseteq B_j$ or $B_j \subseteq B_i$. We may assume without loss of generality that $B_i \subseteq B_j$. Than we have $\mathcal{X}, \mathcal{Y} \in B_j$, which implies $\mathcal{X} \cap \mathcal{Y} \neq \emptyset$, establishing finite intersectivity and hence that $\mathcal{G} \in T$.

Since $\mathcal{G}$ is an upper bound for the arbitrary chain B, it follows from Zorn's Lemma that T contains a maximal element. Let $\mathcal{U}$ be such a maximal element. We are now left to show that $\mathcal{U}$ is an ultrafilter. Trivially, $\emptyset \notin \mathcal{U}$ and $\mathcal{U}$ has the finite intersection property. $\mathcal{U}$ is also closed under supersets. For if not then there exists $A \in \mathcal{U}$ with $A \subset B$ and $B \notin \mathcal{U}$. Now we can show that $\mathcal{U} \cup \{B\}$ has the finite intersection property. Indeed, let $\{A_i\}_{i=1}^k$ be a finite collection of subsets of S with $A_i \in \mathcal{U}$ for all i. Then

$$\emptyset \neq (\bigcap_{i=1}^{k} A_i) \cap A \subseteq (\bigcap_{i=1}^{k} A_i) \cap B.$$

So $\mathcal{U} \cup \{B\}$ contains $\mathcal{V}$ and is closed under finite intersections, and hence belongs to T. But $\mathcal{U} \subset (\mathcal{U} \cup \{B\})$, contradicting the maximality of $\mathcal{U}$. So in fact $\mathcal{U}$ is closed under supersets.

We have shown that $\mathcal{U}$ is a filter. It remains to show that it is a maximal filter. So suppose to the contrary that there exists a filter $\mathcal{C}$ such that $\mathcal{U} \subset \mathcal{C}$. Then $\mathcal{C} \in T$, again contradicting the maximality of $\mathcal{U}$. Therefore $\mathcal{V} \subset \mathcal{U}$, then $\mathcal{U}$ is an ultrafilter.

To establish the final claim, we just note that if $\mathcal{W} \subseteq \mathcal{P}(\mathcal{S})$ has the finite intersection property and $\mathcal{V} \subseteq \mathcal{W}$, then $\mathcal{W} \subseteq \mathcal{U}$ by construction.

□

Unlike filters, the only concrete examples of ultrafilters are of a somewhat degenerate form. Namely, if $a \in S, \{A \in \mathcal{P}(S) : a \in A\}$ is easily seen to be an ultrafilter on S. This ultrafilter is called the *principal ultrafilter* defined by a. We mention here a map that will be useful later. Let $e : S \to \beta S$ be defined by the rule that $e(a)$ is the principal ultrafilter generated by a. Next, we give a useful characterization of ultrafilters.

1.1.5. Theorem. Let $\mathcal{U}$ be a filter on a set S. Then $\mathcal{U}$ is an ultrafilter if and only if for all $A \subseteq S$, either $A \in \mathcal{U}$ or $A^c \in \mathcal{U}$ (but not both).

Proof. Let $\mathcal{U}$ be an ultrafilter on S and let $A \subseteq S$. If both A and A^c belong to $\mathcal{U}$ then their intersection does as well. But their intersection is empty, so this cannot be. (This much actually follows from the mere fact that $\mathcal{U}$ is a filter.). Suppose next that neither A nor A^c belong to $\mathcal{U}$. Let $B \in \mathcal{U}$. Then $B \cap A \neq \emptyset$, for otherwise, we would have $B \subseteq A^c$, which would imply that $A^c \in \mathcal{U}$ since $\mathcal{U}$ is closed under supersets. It can be shown along the same lines that in fact $\mathcal{U} \cup \{A\}$ has the finite intersection property. Indeed, any finite intersection of elements of $\mathcal{U}$ is in $\mathcal{U}$, so every finite intersection of elements of $\mathcal{U} \cup \{A\}$ is of the form B or $B \cap A$, where $B \in \mathcal{U}$. So now by Theorem 1.1.4, it would follow that there exists an ultrafilter $\mathcal{V}$ such that $\mathcal{U} \subset \mathcal{U} \cup \{A\} \subseteq \mathcal{V}$, contradicting the fact that $\mathcal{U}$ is a maximal filter.

Conversely, supppose that for all $A \subseteq S$, either $A \in \mathcal{U}$ or $A^c \in \mathcal{U}$. If $\mathcal{U}$ is not an ultrafilter then there exists a filter $\mathcal{V}$ with $\mathcal{U} \subset \mathcal{V}$. Let $A \in \mathcal{V} \backslash \mathcal{U}$. Then by assumption $A^c \in \mathcal{U}$. But $\mathcal{U} \subset \mathcal{V}$, so $A^c \in \mathcal{V}$. Therefore we have $A \cap A^c = \emptyset \in \mathcal{V}$, a contradiction.

□

The above theorem has the following handy corollary.

1.1.6. Theorem. Let $\mathcal{U} \subset \mathcal{P}(\mathcal{S})$ be an ultrafilter on S. If for some integer k, $S = A_1 \cup \cdots \cup A_k$, then there exists $i \in \{1, \ldots k\}$ such that $A_i \in \mathcal{U}$.

Proof. Let $S = A_1 \cup \ldots \cup A_k$ and $F = \{1, \ldots, k\}$. If for any $i \in \{1, \ldots, k\}$, $A_i \in \mathcal{U}$, then we are done. So suppose $A_i \notin \mathcal{U}$ for all i. By Theorem 1.1.5, we have $A_i^c \in \mathcal{U}$ for all i. Therefore

$$\bigcap_{i=1}^{k} A_i^c = (\bigcup_{i=1}^{k} A_i)^c = S^c = \emptyset \in \mathcal{U},$$

a contradiction.

□

1.2 Topology on βS

Now let $(S, \cdot)$ be endowed with the discrete topology (i.e. every point in S is an open set), we wish to define a topology on the set of ultrafilters generated by S. We proceed to definitions and notations.

1.2.1. Definition. $\beta S = \{p : p \text{ is an ultrafilter on } S\}$.

1.2.2. Definition. Given $A \subset S$, we let $\overline{A} = \{p \in \beta S : A \in p\}$.

From this point forward ultrafilters will generally be denoted by lower case letters, as we shall think of them as points in the space βS.

Let $p \in \beta S$. A moment's thought will convince us that $A \cap B \in p$ if and only if both $A \in p$ and $B \in p$. (Here we are using the facts that p is closed under supersets and that p is closed under finite intersections.) With this in mind, we can see that

$$\begin{aligned}\overline{A} \cap \overline{B} &= \{p \in \beta S : p \in \overline{A} \text{ and } p \in \overline{B}\}\\ &= \{p \in \beta S : A \in p \text{ and } B \in p\}\\ &= \{p \in \beta S : A \cap B \in p\} = \overline{A \cap B}.\end{aligned}$$

Similary,

$$\begin{aligned}\overline{A} \cup \overline{B} &= \{p \in \beta S : p \in \overline{A} \text{ or } p \in \overline{B}\}\\ &= \{p \in \beta S : A \in p \text{ or } B \in p\}\\ &= \{p \in \beta S : A \cup B \in p\} = \overline{A \cup B}.\end{aligned}$$

Recall that a collection $\mathcal{B}$ of subsets of a set X forms a basis for a topology on X if $\bigcup \mathcal{B} = X$ and if for every $B, C \in \mathcal{B}$ and $x \in B \cap C$ there is some $D \in \mathcal{B}$ with $x \in D$ and $D \subseteq B \cap C$. It follows that the collection $\Lambda = \{\overline{A} : A \subset S\}$ serves as a basis for a topology on βS. For:

1. Given $p \in \beta S$, one can find some $A \in p$, and $\overline{A}$ contains p. Hence the sets $\overline{A}$ cover βS. And

2. If $p \in \overline{A_1} \cap \overline{A_2}$ then, by our observation above, $A_1 \cap A_2 \in p$. This implies that $p \in \overline{A_1 \cap A_2} \in \Lambda$ and $\overline{A_1 \cap A_2} \subseteq \overline{A_1} \cap \overline{A_2}$.

We endow βS with the topology generated by Λ. Observe that, for $p \in \beta S$, the family of sets $\{\overline{A} : A \in p\}$ serves as a neighborhood basis at p. Also note that

$$\beta S \backslash \overline{A} = \{p \in \beta S : A \notin p\} = \{p \in \beta S : S \backslash A \in p\} = \overline{S \backslash A}.$$

Therefore for $A \subseteq S$, $\overline{A}$ is closed as well as open.

Recall that a topological space X is a Hausdorff space if for every $x, y \in X$ with $x \neq y$, there are neighborhoods A and B of x and y, respectively, such that $A \cap B = \emptyset$. Recall as well that X is compact if and only if every family of closed subsets of X having the finite intersection property has non-empty intersection.

1.2.3. Theorem. βS is a compact Hausdorff space.

Proof. Let p, q be distinct members of βS. Since $p \neq q$, pick $A \in p \backslash q$. Then $S \backslash A \in q \backslash p$. So $\overline{A}$ and $\overline{S \backslash A}$ are neighborhoods of p and q respectively which are, moreover, disjoint. Therefore βS is Hausdorff.

To see that βS is compact, let $\mathcal{B}$ be a collection of closed subsets of βS having the finite intersection property. We must show that $\bigcap \mathcal{B} \neq \emptyset$. Suppose then, to the contrary, that $\bigcap \mathcal{B} = \emptyset$. Then for every $p \in \beta S$ one can find $B_p \in \mathcal{B}$ with $p \in B_p^c$. Since B_p^c is open, there exists some $A_p \in S$ with $p \in \overline{A_p} \subseteq B_p^c$.

Consider the collection of sets $\{A_p^c : p \in \beta S\}$. This collection has the finite intersection property. For if not, one could find $p_1, \ldots, p_k \in \beta S$ with

$$A_{p_1}^c \cap A_{p_2}^c \cap \cdots \cap A_{p_k}^c = \emptyset,$$

which would imply that

$$A_{p_1} \cup A_{p_2} \cup \cdots \cup A_{p_k} = S.$$

But this would mean that

$$\overline{A_{p_1}} \cup \overline{A_{p_2}} \cup \cdots \cup \overline{A_{p_k}} = \overline{S} = \beta S.$$

Moreover, $\overline{A}_{p_i} \subseteq B^c_{p_i}$ for $1 \leq i \leq k$, so that

$$B^c_{p_1} \cup B^c_{p_2} \cup \cdots \cup B^c_{p_k} = S$$

and

$$B_{p_1} \cap B_{p_2} \cap \cdots \cap B_{p_k} = \emptyset,$$

a contradiction.

□

1.3 Stone-Čech Compactification of $(S, \cdot)$

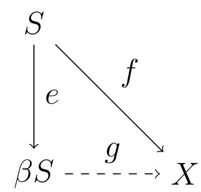
Suppose that S is countable discrete space. We now move to a formulation of the fact that, modulo a certain natural identification, the space βS is a Stone-Čech compactification of S. We begin with several definitions.

1.3.1. Definition. Let Y be a topological space and suppose that $f : S \to Y$ is a homeomorphism into, i.e., $f(S)$ is the homeomorphic image of S under f when $f(S)$ is given the relative topology induced by Y. In this case, we say that f is an embedding of S into Y.

1.3.2. Definition. A compactification of S is a pair (f, Y) where Y is a compact Hausdorff space and f is an embedding of S as a dense subset of Y. In other words, $\overline{f(S)} = Y$.

1.3.3. Definition. Let (e, Y) be a compactification of S. If for any compact space X and any continuous function $f : S \to X$, f induces a unique extension $g : Y \to X$ then we say that (e, Y) is a Stone-Čech compactification of S. (By extension here we mean that $g \circ e = f$ on S.)

The following standard theorem tells us that the map $e : S \to \beta S$ sending $s \in S$ to the principal ultrafilter defined by s is a Stone-Čech compactification of S. In particular, given any compact space X and a continuous function $f : S \to X$, there exists a unique continuous function $g : \beta S \to X$, such that the following diagram commutes:



1.3.4. Theorem. $(e, \beta S)$ is a Stone-Čech compactification of S.

Proof. [HS, Theorem 3.27].

Suppose now that the countable discrete set S is endowed with an associative operation $\cdot$. That is, let $(S, \cdot)$ be a discrete semigroup. One can use the fact that βS is the Stone-Čech compactification of S to show that there is a unique associative operation $\star$ on βS that both extends $\cdot$ (modulo identification of S with $e(S) \subset \beta S$) and having the property that for every $q \in \beta S$, the function $p \to p \star q$ is continuous. This brings us to the following definition.

1.3.5. Definition. A right topological semigroup is a topological space $(\Omega, \mathscr{T})$ endowed with an associative operation $\star$ such that for all $\omega \in \Omega$, the function $f_\omega : \Omega \to \Omega$ defined by $f_\omega(\gamma) = \gamma \cdot \omega$ is continuous. If $(\Omega, \mathscr{T})$ also compact and Hausdorff, we say that Ω is a *compact right topological semigroup.*

And the following theorem.

1.3.6. Theorem. There is a unique binary operation $\star : \beta S \times \beta S \to \beta S$ satisfying the following three conditions:

1.) For every $s, t \in S$, $e(s) \star e(t) = e(s \cdot t)$.

2.) For each $q \in \beta S$, the function $p_q : \beta S \to \beta S$ is continuous, where $p_q(p) = p \star q$.

3.) For each $s \in S$, the function $\lambda_s : \beta S \to \beta S$ is continuous, where $\lambda_s(q) = e(s) \star q$.

Proof. [HS, Theorem 4.1].

1.4 Operation on βS

We now take a few moments to collect some thoughts. Our aim for this introductory section was to formulate justification for the fact that $(\beta S, \star)$ is a compact right topological semigroup. Compactness and the Hausdorff property were established in Theorem 1.2.3. Right continuity and the extension property were established in Theorem 1.3.6. It remains to show that $(\beta S, \star)$ is in fact a semigroup. That is, we must establish the associativity of $(\beta S, \star)$. We first state the following definition:

1.4.1. Definition. Suppose that $A \subseteq S$ and $s \in S$. Denote:

1. $s^{-1}A = \{t \in S : s \cdot t \in A\}$.
2. $As^{-1} = \{t \in S : t \cdot s \in A\}$.

Since points of βS are ultrafilters and the operation $\star$ on βS is derived from rather abstract extensions of continuous functions, it is somewhat difficult, at the stage, to work with. A natural question to ask is which subsets of S are members of the ultrafilter $p \star q$. Knowing this, the operation $\star$ becomes easier to work with. The question is answered by the following theorem.

1.4.2. Theorem. Let $A \subset S$.

1. For any $s \in S$ and $q \in \beta S$, $A \in e(s) \star q$ if and only if $s^{-1}A \in q$.
2. For any $p, q \in \beta S$, $A \in p \star q$ if and only if $\{s \in S : s^{-1}A \in q\} \in p$.

Proof. 1. Suppose that $A \in e(s) \star q$. Then $\overline{A}$ is a neighborhood of $e(s) \star q$. Let $\lambda_s : \beta S \to \beta S$ be the continuous function described in Theorem 1.3.6 (3). By continuity of λ_s at q there exists a basic neighborhood $\overline{B}$ of q such that $\lambda_s(\overline{B}) \subseteq \overline{A}$. Now $q \in \overline{B}$ implies $B \in q$. Let $b \in B$. We have $\lambda_s(e(b)) = e(s) \star e(b) = e(s \cdot b)$ by Theorem 1.8 (1). So we have $e(s \cdot b) \in \overline{A}$. Hence $s \cdot b \in A$. This holds for all $b \in B$, so $sB \subseteq A$ which implies that $B \subseteq s^{-1}A$. But $B \in q$, so $s^{-1}A \in q$ as desired.

Conversely, assume $s^{-1}A \in q$ and suppose that $A \notin e(s) \star q$. Then $S \backslash A \in e(s) \star q$. By the above argument, $s^{-1}(S \backslash A) \in q$. But this is a contradiction since $s^{-1}A \cap s^{-1}(S \backslash A) = \emptyset$.

2. The argument is similar to the above. Let $A \in p{\star}q$. Then $\overline{A}$ is a neighborhood of $p \star q$. Pick a basic neighborhood $\overline{B}$ of p such that $p_q(\overline{B}) \subseteq \overline{A}$, where p_q is the map defined in Theorem 1.8 (2), namely $p_q(p) = p \star q$. So $B \in p$. We claim $B \subseteq \{s \in S : s^{-1}A \in q\}$. To see this, let $b \in B$. Then $p_q(e(b)) \in \overline{A}$, so $e(b) \star q \in \overline{A}$. Hence $A \in e(b) \star q$ and (by 1.) $b^{-1}A \in q$. Therefore $B \subseteq \{s \in S : s^{-1}A \in q\}$. Since also $B \in p$, we have

$$\{s \in S : s^{-1}A \in q\} \in p.$$

Conversely, suppose $\{s \in S : s^{-1}A \in q\} \in p$ and $A \notin p \star q$. Then $S \backslash A \in p \star q$. Again by (1), $\{s \in S : s^{-1}(S \backslash A) \in q\} \in p$. But this is a contradiction since $\{s \in S : s^{-1}A \in q\} \cap \{s \in S : s^{-1}(S \backslash A) \in q\} = \emptyset$.

□

Now that we know which subsets of S belong to $p{\star}q$, we can establish associativity of $\star$, completing the proof that $(\beta S, \star)$ is a compact right topological semigroup.

1.4.3. Theorem. The operation $\star$ on βS is associative.

Proof. Let $p, q, r \in \beta S$. We note that for any nonempty $A \subseteq S$ and $y \in S$, $y^{-1}\{s \in S : s^{-1}A \in p\} = \{x \in S : x^{-1}y^{-1}A \in p\}$. Now $A \in (p \star q) \star r$

$$\begin{aligned}
&\iff \{s \in S : s^{-1}A \in r\} \in p \star q \\
&\iff \{y \in S : y^{-1}\{s \in S : s^{-1}A \in r\} \in q\} \in p \\
&\iff \{y \in S : \{x \in S : x^{-1}y^{-1}A \in r\} \in q\} \in p \\
&\iff \{y \in S : y^{-1}A \in q \star r\} \in p \\
&\iff A \in p \star (q \star r).
\end{aligned}$$

□

1.5 Ellis' Theorem

We will discuss some connections between ultrafilters and Ramsey-type results in the next section. A theorem by Ellis, which we shall presently state, will be needed throughout.

1.5.1. Definition. Let $(S, \cdot)$ be a semigroup. An element $p \in S$ is an idempotent if and only if $p \cdot p = p$.

1.5.2. Theorem (Ellis). If $(S, \cdot)$ is a compact right topological semigroup, then S has an idempotent.

For the original proof of Ellis' Theorem, see [E]. With this theorem in mind, we remark for any semigroup $(S, \cdot)$, the set of ultrafilters on S (i.e., βS) with the induced operation $\star$ contains an idempotent.

Chapter 2. Applications and Notions of Largeness

2.1 IP sets

Knowing the properties of certain ultrafilters can provide interesting Ramsey-type results. Perhaps one of the most elegant applications of ultrafilters is the derivation of Hindman's theorem by F. Galvin and S. Glazer, which we will present later. Before we start, we want to make a fairly simple observation. We would like to remark here that all the definitions and results presented in Section 1 were in a general semigroup $(S, \cdot)$ setting. They of course apply in particular to $(\mathbb{N}, +)$. (In this thesis, we take $\mathbb{N} = \{1, 2, \ldots\}$. That is, we take it that $0 \notin \mathbb{N}$.) With that in mind, for $p, q \in \beta\mathbb{N}$, we have

$$A \in p \star q \iff \{n \in \mathbb{N} : -n + A \in q\} \in p = \{n \in \mathbb{N} : A - n \in q\} \in p.$$

When $S = \mathbb{N}$ we shall write $p + q$ instead of $p \star q$, with the understanding that the operation "$+$" is the unique associative extension to $\beta\mathbb{N}$ described in Chapter 1. Readers should however keep in mind that most of the properties regarding $(\beta\mathbb{N}, +)$ that we will discuss can be applied to $(\beta S, \cdot)$ for an arbitrary discrete semigroup $(S, \cdot)$.

We now define the central object of this thesis. Namely, IP sets.

2.1.1. Definition. Let $\langle x \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$. A finite sums set is a set of the form

$$FS(\langle x \rangle_{n=1}^{\infty}) = \{x_{i_1} + x_{i_2} + \cdots + x_{i_k} : i_1 < i_2 < \cdots < i_k; k \in \mathbb{N}\}.$$

Let $A \subset \mathbb{N}$. A is said to be an IP set if and only if there exists $\langle x_n \rangle_{n=1}^{\infty}$ in A with the property that $FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$.

These objects have analogs in general semigroups $(S, \cdot)$, with one caveat. In case $(S, \cdot)$ is non-commutative, order of products matters in the definition of "finite sums set". In general there are two choices. One can take products with increasing indices, or with decreasing indices. Since however we will deal here only with commutative semigroups, we shall not worry about this distinction.

2.1.2. Definition. Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in a commutative semigroup S with operation "$\cdot$". A finite product set is a set of the form

$$FP(\langle x_n \rangle_{n=1}^{\infty}) = \{x_{i_1} \cdots x_{i_k} : i_1 < \cdots < i_k; k \in \mathbb{N}\}$$

If $A \subset S$. A is said to be an IP set if and only if there exists $\langle x_n \rangle_{n=1}^{\infty}$ in A with the property that $FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$.

2.2 Hindman's Finite Sums Theorem

Now we state Hindman's theorem, the core result about IP sets.

2.2.1. (HFS) Theorem. If $\bigcup_{i=1}^{k} C_i$ is a finite partition of $\mathbb{N}$, then there exists $j \in \{1, 2 \ldots, k\}$ such that C_j is an IP set.

Proof. Suppose that $\mathbb{N} = \bigcup_{i=1}^{k} C_i$. Let $p \in \beta\mathbb{N}$ be any idempotent ultrafilter. We have $\bigcup_{i=1}^{k} C_i \in p$, so by Lemma 1.3 there exists $j \in \{1, 2, \ldots, k\}$ such that $C_j \in p$. For convenience, write $D = C_j$. Then $D \in p+p$, so by definition $\{n \in \mathbb{N} : D-n \in p\} \in p$. It follows that $(D \cap \{n \in \mathbb{N} : D - n \in p\}) \in p$. Let now

$$n_1 \in D \cap \{n \in \mathbb{N} : D - n \in p\} \text{ and } D_1 = (D \cap (D - n_1)) \in p.$$

By idempotence, $D_1 \in p + p$, so $\{n \in \mathbb{N} : D_1 - n \in p\} \in p$, from which it follows that $(D_1 \cap \{n \in \mathbb{N} : D_1 - n \in p\}) \in p$. Let now

$$n_2 \in D_1 \cap \{n \in \mathbb{N} : D_1 - n \in p\} \text{ and } D_2 = (D_1 \cap (D_1 - n_2)) \in p.$$

Now we note that $\{n_1, n_2, n_1 + n_2\} \subset D$. Moreover, we can iterate this process. At the next step we obtain

$$n_3 \in D_2 \cap \{n \in \mathbb{N} : D_2 - n \in p\} \text{ and } D_3 = (D_2 \cap (D_2 - n_3)) \in p.$$

Notice

$$D_2 = D \cap (D - n_1) \cap ((D - n_2) \cap (D - n_1 - n_2)).$$

Hence we have $n_3 \in D, n_1 + n_3 \in D, n_2 + n_3 \in D, n_1 + n_2 + n_3 \in D$. By continuing in this way, we get a sequence $\langle n_i \rangle_{i=1}^{\infty}$ such that $FS(\langle n_i \rangle_{i=1}^{\infty}) \subseteq D$. So D contains a finite sums set and is therefore an IP set.

□

2.2.2 Remark. We note here that the above algorithm to construct a finite sums set can be carried out in any member of any idempotent ultrafilter p. That is, if $A \in p$ and p is an idempotent ultrafilter of βS, then A is an IP set.

2.3 IP* sets

As mentioned in the very beginning of this dissertation, one of our major goals is to provide a negative answer to the question whether every D^* set is AIP^* (definitions will be stated later). In order to reach that goal, we need another important notion of largeness defined as follows.

2.3.1. Definition. Let $(S, \cdot)$ be a semigroup. A set $A \subseteq S$ is an IP* set if and only if for every IP set $B \subseteq S$, $A \cap B \neq \emptyset$.

One fairly immediate, albeit non-trivial, example of an IP* set in $\mathbb{N}$ is $2\mathbb{N} = \{2, 4, 6, \ldots\}$. For if $A \subseteq \mathbb{N}$ is an IP set, it contains an finite sums set, say with generators $\langle x_n \rangle_{n=1}^{\infty}$ (so that $FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$). One now checks that, e.g., at least one of $x_1, x_2, x_1 + x_2$ must be even.

Indeed, $k\mathbb{N}$ is IP* for any natural number k. For by the Pigeon-Hole Principle, given $k+1$ distinct generators $x_1, \ldots, x_{k+1}$, some two members of the set

$$\{x_1, x_1 + x_2, x_1 + x_2 + x_3, \ldots, x_1 + x_2 + x_3 + \cdots x_{k+1}\}$$

must belong to the same residue class modulo k. The difference of these two numbers will belong to $FS(\langle x_n \rangle_{n=1}^{k+1})$ and be divisible by k.

Next, we review a characterization and some useful properties about IP* sets in the next few lemmas. The following result (due to F. Galvin) is needed for a characterization of IP* sets. Before we start, let us denote, for a set A, $\mathscr{P}_f(A) = \{F :$

$\emptyset \neq F \subseteq A$ and F is finite$\}$.

2.3.2. Theorem. Given a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{N}$, let

$$G = \bigcap_{i=1}^{\infty} \overline{FS(\langle x_n \rangle_{n=i}^{\infty})},$$

where this closure is in $\beta\mathbb{N}$. Then G is a semigroup. In particular, there is an idempotent $p \in G$ with the property that for each $m \in \mathbb{N}$, $FS(\{x_n\}_{n=m}^{\infty}) \in p$.

Proof. We first note that the family of sets $\{FS(\langle x_n \rangle_{n=m}^{\infty}) : m \in \mathbb{N}\}$ has the finite intersection property. Since $\beta\mathbb{N}$ is compact, this implies that G is non-empty. To show that G is a semigroup, we must show that it is closed under $+$. Let $p, q \in G$. We must show that $p + q \in G$. That is, we must show that for all $m \in \mathbb{N}, A_m = FS(\langle x_n \rangle_{n=m}^{\infty}) \in p + q$.

Fix $m \in \mathbb{N}$. Let $a \in A_m$ and pick $F \in \mathscr{P}_f(\mathbb{N})$ such that $\sum_{i \in F} x_i = a$. Then for $k = \max F + 1$, we have $FS(\langle x_n \rangle_{n=k}^{\infty}) + a \subset A_m$, which implies that $FS(\langle x_n \rangle_{n=k}^{\infty}) \subset A_m - a$. Now $q \in G$, which implies that $FS(\langle x_n \rangle_{n=k}^{\infty}) \in q$. It follows that $A_m - a \in q$. Since a was arbitrarily picked from A_m, we have that $A_m - a \in q$ for all $a \in A_m$. In other words, $A_m \subseteq \{a \in \mathbb{N} : A_m - a \in q\}$. Finally, $p \in G$, so $A_m \in p$. Therefore $\{a \in \mathbb{N} : A_m - a \in q\} \in p$, i.e.$A_m \in p + q$.

The final claim follows from the fact that G is a semigroup and obviously closed, hence compact. Therefore, by Ellis' theorem, G contains an idempotent p.

□

2.3.3. Lemma. $A \subseteq S$ is IP* if and only if A is a member of every idempotent ultrafilter of βS.

Proof. Let $A \subseteq S$ be IP* and let $p \in \beta S$ be an idempotent. Suppose that $A \notin p$. By Theorem 1.2, $S \backslash A \in p$. So by Remark 2.2.2, $S \backslash A$ is an IP set, contradicting the fact that A is IP*.

Conversely, suppose that A is a member of every idempotent ultrafilter. Let B be an IP set. By Theorem 2.3.2, there exists an idempotent ultrafilter p such that

$B \in p$. Now $A \in p$ by assumption, so $A \cap B \neq \emptyset$. That is to say, A meets every IP set B and is therefore IP*.

□

2.3.4. Corollary. If $A \subseteq S$ is IP*, then for any IP set $B \subseteq S$, $A \cap B$ is an IP set.

Proof. Let B be an IP set. Let p be an idempotent ultrafilter with $B \in p$. By the previous lemma, A is also in p. Therefore, $A \cap B \in p$ and again by Remark 2.2, $A \cap B$ is an IP set.

□

That the IP* notion is a very robust notion of largness is attested to by the fact that the family of IP* sets has the finite intersection property:

2.3.5. Lemma. Suppose that A and B are IP* sets. Then $A \cap B$ is again an IP* set.

Proof. Let A, B be IP* sets and suppose to the contrary that $A \cap B$ is not an IP* set. Then we can then find an IP set C such that $(A \cap B) \cap C = \emptyset$. But write $(A \cap B) \cap C = A \cap (B \cap C)$. By the previous corollary, $B \cap C$ is an IP set, and so meets the IP* set A. This is a contradiction.

□

2.4 Essential Idempotents, D and D* sets

In this dissertation we will be marrying the notions of largeness derived from membership in idempotent ultrafilters (IP and IP*) with different, more classical notions of largeness based on asymptotic frequency (density, upper density, upper Banach density). The latter notions have a storied history, thanks in large part to a famous result of E. Szemerédi which states, in one of its incarnations, that if $E \subseteq \mathbb{N}$ has positive upper Banach density (we will give the definition of this presently), then E contains arbitrarily long arithmetic progressions.

Here is the promised definition of upper Banach density. We formulate it in $\mathbb{Z}$, since that is where we will be working in the next section. However, it is essentially no different in $\mathbb{N}$.

2.4.1. Definition. Let $A \subseteq \mathbb{Z}$. The upper Banach density of A is given by

$$d^*(A) = \limsup_{N-M\to\infty} \frac{|A \cap \{M, M+1, \ldots, N-1\}|}{N-M}.$$

Lower Banach density (denoted $d_*(A)$) is similarly defined. (i.e. with limsup replaced by liminf in the above expression.)

We mention here a useful property of upper Banach density.

2.4.2. Theorem Let $E \subset \mathbb{Z}$ and let $(k_n)_{n=1}^{\infty}$ be any increasing sequence of numbers. If for any intervals $(I_n)_{n=1}^{\infty}$ with $|I_n| = k_n$, one has

$$\frac{|I_n \cap E|}{|I_n|} \to 0,$$

then $d^*(E) = 0$.

Proof. Suppose not. Then there exist $\alpha > 0$ and intervals $(J_k)_{k=1}^{\infty}$ with $|J_k| \to \infty$ such that

$$\frac{|E \cap J_k|}{|J_k|} > \alpha \text{ for all } k.$$

Let n be so large that for any interval I_n with $|I_n| = k_n$,

$$\frac{|E \cap I_n|}{|I_n|} < \frac{\alpha}{2}.$$

Choose k such that $|J_k| > k_n$ and pick I_n such that $|I_n| = k_n$ and $\min I_n = \min J_k$. Let c be the floor of $\frac{|J_k|}{|I_n|}$, so that

$$(c+1)|I_n| > |J_k| \geq c|I_n|.$$

Now we tile the interval J_k using the disjoint intervals $\{I_n + sk_n\}_{s=0}^{c}$. With some algebraic manipulations, we obtain:

$$\begin{aligned}
\frac{|E \cap J_k|}{|J_k|} &\leq \frac{|E \cap I_n|}{|J_k|} + \frac{|E \cap \{I_n + |I_n|\}|}{|J_k|} + \cdots + \frac{|E \cap \{I_n + c|I_n|\}|}{|J_k|} \\
&= \frac{1}{c}\left(\frac{|E \cap I_n|}{\frac{|J_k|}{c}} + \frac{|E \cap \{I_n + |I_n|\}|}{\frac{|J_k|}{c}} + \cdots + \frac{|E \cap \{I_n + c|I_n|\}|}{\frac{|J_k|}{c}}\right) \\
&\leq \frac{1}{c}\left(\frac{|E \cap I_n|}{|I_n|} + \frac{|E \cap \{I_n + |I_n|\}|}{|I_n|} + \cdots + \frac{|E \cap \{I_n + c|I_n|\}|}{|I_n|}\right) \\
&< \frac{1}{c}(c+1)\frac{\alpha}{2} \\
&= \frac{\alpha}{2} + \frac{\alpha}{2c} \leq \alpha,
\end{aligned}$$

a contradiction.

□

Here now is the first "marriage" of the largeness notions based on idempotence and density.

2.4.3. Definition. An ultrafilter $p \in \beta Z$ having the property that $d^*(A) > 0$ for every $A \in p$ is said to be essential. If p is an essential idempotent and $A \in p$, we say that A is a D set. If B^c is not a D set, we say that B is a D^* set.

Remark. The notion of essential idempotents was first introduced in [**BD**]. Indeed, readers are encouraged to refer to that paper for a more proper treatment.

Of course, formulating the definition of essential idempotent does not ensure that such exist. However, they do.

2.4.4. Theorem. There are essential idempotents in $\beta\mathbb{Z}$.

Proof. Let

$$F = \{A \subset \mathbb{Z} : d^*(A^c) = 0\} = \{A \subset \mathbb{Z} : d_*(A) = 1\}.$$

We claim that F is a filter. Obviously, $\emptyset \notin F$ and F is closed under supersets. If $A, B \in F$, then $d^*(A^c) = d^*(B^c) = 0$. Hence $d^*(A^c \cup B^c) = 0$, which implies that

$$(A^c \cup B^c)^c = A \cap B \in F.$$

By Theorem 1.1.5, F is contained in some ultrafilter q. Furthermore, if $A \in q$, then $d^*(A) > 0$. For, suppose to the contrary that there exists $A \in q$ with $d^*(A) = 0$. Then $A^c \in F \subset q$, a contradiction.

So far, we have shown the existence of essential ultrafilters. Next we show the existence of essential idempotents. Recall that

$$A \in r + q \iff \{n \in \mathbb{Z} : A - n \in q\} \in r.$$

So if $r \in \beta\mathbb{Z}$ is arbitrary and q is essential, $A \in r + q$ implies $A - n \in q$ for some n. Therefore $d^*(A) = d^*(A - n) > 0$, so $r + q$ is essential. Now look at the set

$$S = \beta\mathbb{Z} + q = \{r + q : r \in \beta\mathbb{Z}\}.$$

We claim that S is a semigroup. S is clearly closed under $+$, and of course associativity is satisfied since the operation is inherited from $\beta\mathbb{Z}$. S is compact, for it is the image of the compact space $\beta\mathbb{Z}$ under the continuous map $r \to r + q$. Hence S is a compact right-topological semigroup, and so contains idempotents by Ellis' Theorem.

□

2.4.5. Theorem. A set $B \subset \mathbb{Z}$ is D^* if and only if B belongs to every essential idempotent in $\beta\mathbb{Z}$.

Proof. Let B be a D^* set and p be an essential idempotent. Suppose $B \notin p$, then $B^c \in p$. So B^c is a D set, which implies B is not a D^* set, a contradiction. Conversely, let $B \in p$ for all essential idempotent ultrafilters p. Then $B^c \notin p$ for any essential idempotent ultrafilter p. Therefore B^c is not a D set, proving B is a D^* set.

□

2.5 IP rich and AIP* sets

We now move to the second new collection of notions.

2.5.1. Definition. A set $A \subseteq \mathbb{Z}$ is called IP rich, or an AIP set, if, for every $E \subset \mathbb{Z}$ with $d^*(E) = 0$, $A \backslash E$ is an IP set. If $B^c \subset \mathbb{Z}$ is not IP rich we say that B is AIP*.

So an IP rich set is a set A that is so "rich" in finite sums sets that no set of zero upper Banach density can possibly meet them all.

Note: AIP* is supposed to stand for "almost IP*". (Accordingly, we prefer "IP rich" to "AIP".) Here is a useful characterization of these sets.

2.5.2. Theorem. Let $B \subseteq \mathbb{Z}$. Then B is AIP* if and only if $B \cup E$ is IP* for some E with $d^*(E) = 0$.

Proof. By definition, we have: $\iff$ B is AIP* if and only if B^c is not IP rich.

$\iff$ There exists E with $d^*(E) = 0$ such that $B^c \setminus E$ contains no IP set.

$\iff$ $(B^c \setminus E)^c \in p$ for all idempotent ultrafilter p.

$\iff$ $(B \cup E) \in p$ for all idempotent ultrafilter p.

$\iff$ $(B \cup E)$ is IP*.

□

Let us discuss briefly the interconnections between the two aforementioned sets of notions.

IP* sets are clearly AIP*. To see this, just note that if A is IP*, then $A \cup E$ is trivially IP* as well for some (in fact any) zero upper Banach density set E. AIP* sets, in turn, are necessarily D*. Indeed if A is AIP* then $A \cup E$ is IP* for some $E \subset \mathbb{Z}$ having zero upper Banach density. Then $A \cup E$ is a member of every idempotent $p \in \beta\mathbb{Z}$. E does not belong to any essential idempotent, however, so A must belong to all of them.

Going the other direction, not every AIP* set is IP*. To see this, let A be a 0 upper Banach density IP set. (For a cheap example, take $\{0\}$. But there are non-trivial examples as well, such as $FS(\langle 3^n \rangle_{n=1}^{\infty})$. Then $A^c \cup A = \mathbb{Z}$ (surely an IP* set), so A^c is AIP*. But A^c is certainly not IP*, as it fails to meet the IP set A.

It turns out not to be so easy to determine whether or not every D* set is AIP*. Indeed, Vitaly Bergelson recently asked us this question. In the next chapter, we give a negative answer.

Chapter 3. D* and AIP* sets in $\mathbb{Z}$

This chapter, which is the most important chapter of this dissertation, consists of three sections. Section 1 is introductory and contains a formulation of the Main Theorem to be proved. In section 2 we provide combinatorial characterizations of D sets and IP rich sets, and in section 3 we provide the example that proves the Main Theorem.

3.1 Main Theorem

In this section we will be dealing with large sets in $\mathbb{Z}$, which leads us to consideration of the space $\beta\mathbb{Z}$ of ultrafilters on $\mathbb{Z}$, endowed with the usual algebraic structure and topology introduced in the previous chapter. We take a few moments to recapitulate the main points.

$\beta\mathbb{Z}$ is given the topology generated by clopen sets $\overline{A} = \{p \in \beta\mathbb{Z} : A \in p\}$. With this topology, $\beta\mathbb{Z}$ becomes a compact Hausdorff space.

Identifying $z \in \mathbb{Z}$ with the *principal* ultrafilter $e(z) = \{A \subset \mathbb{Z} : z \in A\}$, $\beta\mathbb{Z}$ becomes a representation of the Stone-Čech compactification of $\mathbb{Z}$. It follows from this that there is a unique associative extension to $\beta\mathbb{Z}$ of the operation $+$ on $\mathbb{Z}$ having the property that for every $q \in \beta\mathbb{Z}$, the function $p \to p + q$ is continuous. The operation $+$ makes $(\beta\mathbb{Z}, +)$ into a compact right topological semigroup, and may be characterized by:

$$A \in p + q \Leftrightarrow \{x \in \mathbb{Z} : A - x \in q\} \in p.$$

By Ellis' theorem, $\beta\mathbb{Z}$ has idempotents. Actually, this is a triviality, as $e(0)$ is idempotent! But in fact we get many idempotents, as per Theorem 2.3.2. Remark 2.2.2. applies: if $p \in \beta\mathbb{Z}$ is an idempotent, then any member of p contains an IP set. A degenerate case is given by the idempotent $e(0)$: the singleton $\{0\}$, for example, which is a member of $e(0)$, is an IP set. (It contains the finite sums systems generated by the constant sequence $x_i = 0$.) Any member of a non-zero idempotent, however, contains an infinite finite sums system, and we call an IP set non-trivial if it contains an infinite finite sums system.

In this chapter we will prove the following theorem, which answers the question of V. Bergelson alluded to in the previous section.

3.1.1. [MZ] Main Theorem. There are D* subsets of $\mathbb{Z}$ that are not AIP*.

This yields the following proper containments:

$$IP^* \subsetneq AIP^* \subsetneq D^*, \text{ or } IP \supsetneq AIP \supsetneq D.$$

The proof, carried out in Part 3, proceeds via construction of an IP rich set that is not a D set. The construction relies on workable characterizations of D sets and IP rich sets. These are of independent interest, and are given in Part 2. Our equivalent condition for IP richness, which we call FS tree richness, already appears in the literature. In [HS, Theorem 20.17] it is shown to be a necessary property of D sets (making its non-sufficiency potentially interesting to a different crowd), while in [T] it is proved by elementary means to be a partition regular property. Our equivalent condition for D sets, meanwhile, is inspired by and comparable to a combinatorial characterization of central sets given in [HMS].

3.2 Tree structure characterizations of IP rich sets and D sets.

In this section we give characterizations of D sets and IP rich sets. These are modeled on an elementary characterization of so-called central sets by Hindman, Maleki and Strauss in [HMS]. We begin with several definitions.

3.2.1. Definition. Let Ω be the set of finite sequences of integers, including the empty sequence.

Sometimes, we will want to include zero in our finite sums sets.

3.2.2. Definition. $FS_0(\langle x_i \rangle) = FS(\langle x_i \rangle) \cup \{0\}$.

The following definition will be instrumental in the inductive process whereby we construct IP rich sets.

3.2.3. Definition. If $A \subseteq \mathbb{Z}$ and $f = (x_1, \dots, x_k) \in \Omega$, we say that A is IP rich over f if for every $E \subset \mathbb{Z} \backslash FS(\langle x_1, \dots, x_k \rangle)$ with $d^*(E) = 0$ there exist non-zero $x_{k+1}, x_{k+2}, \dots \in \mathbb{Z}$ such that $FS(\langle x_i \rangle_{i=1}^{\infty}) \subset A \backslash E$. A is IP rich if A is IP rich over the empty sequence $()$.

Recall that a non-trivial IP set is just an infinite IP set.

3.2.4. Definition. Let $F \subset \mathbb{Z}$ be a finite non-empty set. An IP_F set is a set $R + F$, where R is a non-trivial IP set. A set $J \subset \mathbb{Z}$ is IP_F^* if J intersects every IP_F set non-trivially.

The following lemma is a generalization of the fact that for any IP^* set B and any $n \in \mathbb{Z}$, the set $n\mathbb{Z} \cap B$ is again IP^* by Lemma 2.3.5.

3.2.5. Lemma Let $F \subset \mathbb{Z}$ be a finite set. If $J \subseteq \mathbb{Z}$ is an IP_F^* set and $n \in \mathbb{N}$ then $(n\mathbb{Z} + F) \cap J$ is IP_F^* as well.

Proof. For every non-trivial IP set R,

$$R + F \not\subseteq J^c,$$

so

$$R \not\subseteq \bigcap_{f \in F} (J^c - f).$$

Therefore

$$\{0\} \cup \bigcup_{f \in F} (J - f) = \{0\} \cup \Big(\bigcap_{f \in F} (J^c - f) \Big)^c$$

is IP^*, which implies that

$$\{0\} \cup \big(n\mathbb{Z} \cap \bigcup_{f \in F} (J - f) \big)$$

is IP*. So, for every non-trivial IP set R there exist $r \in R$, $f \in F$, $z \in \mathbb{Z}$ and $j \in J$ such that $r = nz = j - f$, so that $r + f = nz + f = j$, whence $(R+F) \cap \big((n\mathbb{Z}+F) \cap J\big) \neq \emptyset$.

□

We now move to our characterization of IP rich sets.

3.2.6. Definition. A set $A \subset \mathbb{Z}$ is FS tree rich if there is a subset $T \subseteq \Omega$ having the following properties:

I1. $() \in T$;

I2. If $f \in T$ then $d^*\big(B_f\big) > 0$, where $B_{(x_1,\ldots,x_k)} = \{x \in \mathbb{Z} : (x_1, \ldots, x_k, x) \in T\}$.

I3. If $(x_1, \ldots, x_k) \in T$ then $FS\big(\langle x_1, \ldots, x_k \rangle\big) \subset A$.

Hindman and Strauss show (cf. [HS, Theorem 20.17]) that FS tree richness is necessary for D sets. We establish now that FS tree richness is necessary (and sufficient) for IP richness.

3.2.7. [MZ] Theorem Let $A \subset \mathbb{Z}$. Then A is IP rich if and only if it is FS-tree rich.

Proof. We start with:

Claim. If A is IP rich over $(x_1, \ldots, x_k)$ then

$$B = \Big\{x \in \mathbb{Z} \backslash FS\big(\langle x_1, \ldots, x_k \rangle\big) : A \text{ is IP rich over } (x_1, \ldots, x_k, x)\Big\} \qquad (*)$$

has positive upper Banach density.

Suppose **Claim** is false. Pick recalcitrant $(x_1, \ldots, x_k)$ and put

$$F = FS_0\big(\langle x_1, \ldots, x_k \rangle\big).$$

We will construct a set $E \subset \mathbb{Z} \backslash FS\big(\langle x_1, \ldots, x_k \rangle\big)$ with $d^*(E) = 0$ such that $A^c \cup E$ is IP$^*_F$, which will yield a contradiction.

To see the contradiction, by assumption A is IP rich over $(x_1, \ldots, x_k)$. Hence there exists an IP set R' and a sequence $\langle x_n \rangle_{n=1}^{\infty}$ agreeing with $(x_1, \ldots, x_k)$ on the first k terms, such that

$$FS(\langle x_n\rangle_{n=1}^{\infty}) \subseteq R' \subseteq A\backslash E.$$

Also notice that $R'' = FS(\langle x_i\rangle_{i=k+1}^{\infty})$ is again an IP set contained in $A\backslash E$ and $R''+F \subseteq R'$. Therefore

$$R'' + F \subseteq A\backslash E.$$

But if $(A\backslash E)^c = A^c \cup E$ is IP_F^* then $A^c \cup E$ meets $R'' + F$, which is a contradiction.

Let

$$K = \big\{x \in \mathbb{Z}\backslash FS\big(\langle x_1, \ldots, x_k\rangle\big) : FS\big(\langle x_1, \ldots, x_k, x\rangle\big) \subset A\big\}.$$

Let $\prec$ be a well-order on $\mathbb{Z}$. We will construct sequences $(k_x)_{x\in K\backslash B}$ and $(E'_x)_{x\in K\backslash B}$ (of numbers tending to ∞ and sets, respectively) satisfying the following:

(a) For every $x \in K\backslash B$ and every interval I with $|I| \geq k_x$,

$$\Big|I \cap \bigcup_{y\in K\backslash B, y\prec x} E'_y\Big| \leq \frac{|I|}{|x|+1}.$$

(b) For every $x \in K\backslash B$, $d^*(E'_x) = 0$.

(c) For every $x \in K\backslash B$, $E'_x \subset k_x\mathbb{Z}$.

(d) If $x, y \in K\backslash B$ with $y \prec x$ then $k_y | k_x$.

(e) For every $x \in K\backslash B$,

$$\bigcap_{y\in F_x} \big((A\backslash E'_x) - y\big)\backslash\{0\}$$

is not IP, where $F_x = FS_0\big(\langle x_1, \ldots, x_k, x\rangle\big)$.

(f) For every $x \in K\backslash B$,

$$E'_x \subset \mathbb{Z}\backslash FS\big(\langle x_1, \ldots, x_k\rangle\big).$$

Supposing that this construction has been carried out, let

$$E = B \cup \bigcup_{x\in K\backslash B} E'_x.$$

By $(*)$ and (f),

$$E \subset \mathbb{Z}\backslash FS\big(\langle x_1, \ldots, x_k\rangle\big).$$

Also $d^*(E) = 0$. To see this, note that if I_x are intervals with $|I_x| = k_x$ then by (c) and (d) at most one member of

$$\bigcup_{y\in K\backslash B, y\not\prec x} E'_y$$

can belong to I_x, whereas by (a) and the fact that $d^*(B) = 0$ one has

$$\left| I_x \cap \bigcup_{y\in K\backslash B, y\prec x} E'_y \right| + |I_x \cap B| = |I_x| o(1),$$

so that $|I_x \cap E| = |I_x| o(1)$. Moreover, $A^c \cup E$ is IP^*_F, as desired. For if its complement $A\backslash E$ contains $R + F$ for some (non-trivial) IP set R then picking $x \in R$ and an IP set R' not having 0 as a member such that $R' + \{0, x\} \subseteq R$ one will have

$$R + F \subseteq A\backslash E,$$

which implies that

$$R \subseteq \bigcap_{y\in F} (A\backslash E) - y,$$

and

$$R' + \{0, x\} \subseteq R \subseteq \bigcap_{y\in F} (A\backslash E) - y.$$

Therefore

$$0 \notin R' \subseteq \bigcap_{y\in F} \Big(\big((A\backslash E) - y\big) - x \Big) \backslash \{0\} \subseteq \bigcap_{y\in F_x} \big((A\backslash E) - y\big) \backslash \{0\}.$$

The latter set is therefore IP, but $x \in K\backslash B$ ($x \in K$ by definition and $x \notin E \supset B$) and $E'_x \subset E$, so the potentially larger set

$$\bigcap_{y\in F_x} \big((A\backslash E'_x) - y\big) \backslash \{0\}$$

is IP as well, contradicting (e).

We have therefore shown, modulo the possibility of the construction, that $A^c \cup E$ is IP^*_F, yielding the contradiction that establishes **Claim.**

It remains to show that one can carry out the construction. Suppose $x \in K\backslash B$ and k_y, E'_y have been determined for all $y \in K\backslash B$ with $y \prec x$. Since $x \notin B$, A is not IP rich over $(x_1, \ldots, x_k, x)$, so there is a set

$$E_x \subset \mathbb{Z}\backslash FS\big(\langle x_1, \ldots, x_k, x\rangle\big)$$

with $d^*(E_x) = 0$ such that $A\backslash E_x$ contains no set of the form $FS\big(\langle x_i\rangle_{i=1}^{\infty}\big)$ with $x_{k+1} = x$ and x_i non-zero for $i \geq k+2$. In particular,

$$\bigcap_{y\in F_x} \big((A\backslash E_x) - y\big) \setminus \{0\}$$

is not IP.

It is clear that we may choose k_x in conformity with (a) and (d). Now put

$$E'_x = \Big(k_x\mathbb{Z} \cap \bigcup_{y\in F_x} (E_x - y)\Big)\backslash\{0\}.$$

Note that (b) and (c) are satisfied, and since $0 \notin E'_x$, (f) is as well provided k_x is large enough (given $x \in K\backslash B$, let $k_x > \sum_{i=1}^{k} x_i$ satisfying (d)). We now establish (e).

We know that

$$\bigcap_{y\in F_x} \big((A \setminus E_x) - y\big) \setminus \{0\}$$

is not IP, so its complement

$$\{0\} \cup \bigcup_{y\in F_x} \big((A^c \cup E_x) - y\big)$$

is IP*. Thus by 3.2.5,

$$k_x\mathbb{Z} \cap \Big(\{0\} \cup \bigcup_{y\in F_x} \big((A^c - y) \cup (E_x - y)\big)\Big)$$

is IP*, so that the potentially larger

$$\{0\} \cup \bigcup_{y\in F_x} (A^c - y) \cup \big(k_x\mathbb{Z} \cap \bigcup_{y\in F_x} (E_x - y)\big) = \{0\} \cup \bigcup_{y\in F_x} (A^c - y) \cup E'_x$$

is IP* as well. This set is however contained in

$$\{0\} \cup \bigcup_{y \in F_x} \big((A^c \cup E'_x) - y\big),$$

which is therefore IP*, implying that its complement

$$\bigcap_{y \in F_x} \big((A \setminus E'_x) - y\big) \setminus \{0\}$$

is not IP, yielding (e) and establishing **Claim**.

In light of the above claim, it is now easy to check that

$$T = \{f \in \Omega : A \text{ is IP rich over } f\}$$

satisfies I1-I3 above.

Conversely, suppose that T satisfies I1-I3 and let $E \subset \mathbb{Z}$ with $d^*(E) = 0$. We must show that $A \backslash E$ contains an IP set. Since $() \in T$, $d^*\big(\{x \in \mathbb{Z} : (x) \in T\}\big) > 0$, and for all x in this set, $x \in A$. So we may choose x_1 such that $(x_1) \in T$ and $x_1 \notin E$. Next we have

$$d^*\big(\{x \in \mathbb{Z} : (x_1, x) \in T\}\big) > 0,$$

and for every x in this set, $\{x, x+x_1\} \subset A$. Since $d^*\big(E \cup (E - x_1)\big) = 0$, we may choose x_2 such that $(x_1, x_2) \in T$ and $x_2 \notin E \cup (E - x_1)$. Note now that $FS\big(\{x_1, x_2\}\big) \subset A \backslash E$. It is clear that this process can be continued and will yield a sequence $\langle x_i \rangle_{i=1}^{\infty}$ for which $FS\big(\langle x_i \rangle_{i=1}^{\infty}\big) \subset A \setminus E$.

□

We next move to our elementary characterization of D sets. One will immediately see that it is similar to the FS-tree richness condition, but stronger, in that the intersection of the successor sets of any finite family of nodes must have positive upper Banach density.

3.2.8. [MZ] Theorem. Let $A \subseteq \mathbb{Z}$. Then A is a D set if and only if there is a subset $T \subseteq \Omega$ having the following properties:

D1. $() \in T$;

D2. If $f_1, \ldots, f_t \in T$ then $d^*\big(B_{f_1} \cap \cdots \cap B_{f_t}\big) > 0$, where

$$B_{(x_1,\ldots,x_k)} = \{x \in \mathbb{Z} : (x_1, \ldots, x_k, x) \in T\}.$$

D3. If $(x_1, \ldots, x_k) \in T$ then $FS\big(\langle x_1, \ldots, x_k\rangle\big) \subset A$.

Proof. We will be using the standard fact that if p is idempotent and $A \in p$ then $A \in p + p$, i.e., $\{m : A - m \in p\} \in p$. Let p be an essential idempotent with $A \in p$. Let

$$A_{()} = A \cap \{m : A - m \in p\}.$$

For $x \in A_{()}$, let

$$A_{(x)} = A \cap (A - x) \cap \{m : (A \cap (A - x)) - m \in p\} \in p.$$

Note that for such x, $x \in A$. Now for $y \in A_{(x)}$, let

$$A_{(x,y)} = A \cap (A - x) \cap (A - y) \cap (A - x - y)$$
$$\cap\big\{m : \big(A \cap (A - x) \cap (A - y) \cap (A - x - y)\big) - m \in p\big\} \in p.$$

Note that for such x, y, $FS\big(\langle x, y\rangle\big) \subset A$. Now for $z \in A_{(x,y)}$ one defines $A_{(x,y,z)} \in p$, etc. Continuing in this fashion, one defines p-sets $\{A_f : f \in T\}$ for some set $T \subset \Omega$. Letting

$$B_{(x_1,\ldots,x_k)} = \{x \in \mathbb{Z} : (x_1, \ldots, x_k, x) \in T\}$$

one has

$$B_{(x_1,\ldots,x_k)} = A_{(x_1,\ldots,x_k)},$$

and D1-D3 above are satisfied.

Conversely, suppose that T satisfies D1-D3. If necessary, we expand T to T' with T' satisfying the following condition.

D4. If $(x_1, \dots, x_k) \in T$ and $L_1, L_2, \dots, L_r$ are consecutive blocks of natural numbers whose union is $\{1, 2, \dots, k\}$ then, letting $y_i = \sum_{j \in L_i} x_j$, one has $(y_1, \dots, y_r) \in T'$.

We don't do this expansion in an arbitrary fashion, however, as we want to preserve D1-3. So, we just include in T' everything in T as well as every such $(y_1, \dots, y_r)$ for $(x_1, \dots, x_k)$ originally in T. Now we check that T' satisfies D1-D3.

Obviously T' satisfies D1 and D3. For D2, we first note that $B_{(x_1,\dots,x_k)} \subseteq B_{(y_1,\dots,y_r)}$ whenever $(x_1, \dots, x_k) \in T$ and $(y_1, \dots, y_r) \in T'$ with $y_i = \sum_{j \in L_i} x_j$, etc. To see this, observe that if $x \in B_{(x_1,\dots,x_k)}$ then $(x_1, \dots, x_k, x) \in T$, which implies $(y_1, \dots, y_r, x) \in T'$, which implies that $x \in B_{(y_1,\dots,y_r)}$. In other words, $B_{(x_1,\dots,x_k)} \subseteq B_{(y_1,\dots,y_r)}$. So finite intersectivity for the sets $B_{(y_1,\dots,y_r)}$ is inherited from the same property for the sets $B_{(x_1,\dots,x_k)}$. So D2 will still be satisfied by T'.

Now let

$$S = \bigcap_{\substack{f \in T', E \subset \mathbb{Z}, \\ d^*(E) = 0}} \overline{(B_f \setminus E)}.$$

As the sets $B_f \setminus E$ have the finite intersection property, S is non-empty and of course closed. Moreover if $p \in S$ and $C \in p$ then $d^*(C) > 0$, as otherwise $(B_{()} \setminus C) \in p$, a contradiction. Also $A \in p$ for all $p \in S$. We claim that S is a semigroup and thus contains idempotents; such idempotents will be essential and will contain A, and this will complete the proof.

We move to the proof that S is a semigroup. Let $p, q \in S$. We need to show that $p + q \in S$. Let $C \in p + q$ be arbitrary. It suffices to find $r \in S$ with $C \in r$. (If $p + q$ were not a member of the closed set S, one could find a basic neighborhood $\overline{C} = \{r : C \in r\}$ of $p + q$ disjoint from S.) In order to show this it is sufficient to show that

$$d^*\Big(\bigcap_{i=1}^{h} B_{f_i} \cap C\Big) > 0$$

for every $f_1, \ldots, f_h \in T'$, as then we can choose

$$r \in \bigcap_{\substack{f \in T', E \subset \mathbb{Z}, \\ d^*(E) = 0}} \overline{((B_f \cap C) \setminus E)}.$$

One has $\{x \in \mathbb{Z} : C - x \in q\} \in p$, so since $p \in S$, for every $f_1, \ldots, f_h \in T'$,

$$d^*\Big(\big\{x \in \bigcap_{i=1}^{h} B_{f_i} : C - x \in q\big\}\Big) > 0.$$

Fix $f_i = (x_1^{(i)}, \ldots, x_{k_i}^{(i)}) \in T'$, $1 \leq i \leq h$. We may choose $x \in \bigcap_{i=1}^{h} B_{f_i}$ with $C - x \in q$. Since $q \in S$,

$$d^*\big(\{n \in \bigcap_{i=1}^{h} B_{(x_1^{(i)}, \ldots, x_{k_i}^{(i)}, x)} : C - x \in p_n\}\big)$$

$$= d^*\big(\{n \in \bigcap_{i=1}^{h} B_{(x_1^{(i)}, \ldots, x_{k_i}^{(i)}, x)} : x + n \in C\}\big)$$

$$= d^*\big(\bigcap_{i=1}^{h} B_{(x_1^{(i)}, \ldots, x_{k_i}^{(i)}, x)} \cap (C - x)\big) > 0,$$

where p_n is the principal ultrafilter on n. Put another way,

$$d^*\big(\{n \in \bigcap_{i=1}^{h} B_{(x_1^{(i)}, \ldots, x_{k_i}^{(i)}, x)} : n + x \in C\}\big) > 0.$$

Observe now that (by D4)

$$n \in \bigcap_{i=1}^{h} B_{(x_1^{(i)}, \ldots, x_{k_i}^{(i)}, x)} \Rightarrow (x_1^{(i)}, \ldots, x_{k_i}^{(i)}, x, n) \in T'$$

for all i, which implies

$$(x_1^{(i)}, \ldots, x_{k_i}^{(i)}, x + n) \in T' \Rightarrow x + n \in \bigcap_{i=1}^{h} B_{f_i}$$

for all i. Therefore, we conclude

$$x + \{n \in \bigcap_{i=1}^{h} B_{(x_1^{(i)}, \ldots, x_{k_i}^{(i)}, x)} : n + x \in C\} \subset \bigcap_{i=1}^{h} B_{f_i} \cap C,$$

so

$$d^*\Big(\bigcap_{i=1}^{h} B_{f_i} \cap C\Big) > 0,$$

which completes the proof.

□

With the two workable characterizations of IP rich and D sets provided, we are ready to establish our Main Theorem, which we do in section 3.

3.3 An IP rich set that is not a D set and proof of Main Theorem.

We are now ready to construct the set that will enable us to prove our main theorem.

3.3.1. [MZ] Theorem. There exists a set $A \subset \mathbb{Z}$ such that A is IP rich and A is not a D set.

Proof. We say that a subset of $\mathbb{Z}$ is thick if it contains arbitrarily long intervals. Our first task is to construct countable pairwise disjoint family $\{S_i : i \in \mathbb{N}\}$ of thick subsets of $\mathbb{N}$. We provide one standard construction here. Let $\{I_n\}_{n=1}^{\infty}$ be consecutive intervals of length 2^n in $\mathbb{N}$ (i.e., $I_1 = \{1, 2\}, I_2 = \{3, 4, 5, 6\}, I_3 = \{7, 8, 9, 10, 11, 12, 13, 14\}$ and so on). Now we assign the collection $\{I_n\}_{n=1}^{\infty}$ to our family $\{S_i : i \in \mathbb{N}\}$ as follows.

Step 1: We assign I_1 to S_1.

Step 2: We assign I_2 to S_1 and I_3 to S_2.

Step 3: We assign I_4 to S_1, I_5 to S_2 and I_6 to S_3.

We continue in such fashion. At step n: we assign some previously unassigned collection $\{I_n\}_{n=i}^{j}$ with $j - i = n$ to S_1 to S_n. Carrying on the construction this way, we see that each S_i contains infinitely many members from the collection $\{I_n\}_{n=1}^{\infty}$ and hence contains arbitrarity long intervals. They are also pairwise disjoint since the collection $\{I_n\}_{n=1}^{\infty}$ is pairwise disjoint.

Next we will be constructing countably many sets A_f of positive upper Banach density in this proof. Each of these will be assumed to be contained in a separate member of such a family. By an n-spaced subset of some S_i we mean a set $B \subset S_i \cap [n, \infty)$ having the property that if $x \in B$ and $0 < |x - y| < n$ then $y \in S_i \setminus B$.

We are now ready to construct our set.

Step 1. Let $A_{()} \subset S_1$ be a set of odd numbers with $d^*(A_{()}) > 0$.

Step 2. Let x_1 be the least member of $A_{()}$. Choose m_1 with $2^{m_1} > x_1$ and let $A_{(x_1)}$ be a 2^{m_1+2}-spaced subset of S_2 consisting of numbers equal to $2^{m_1} \pmod{2^{m_1+1}}$ with $d^*(A_{(x_1)}) > 0$.

Step 3. Now pick the least member x_2 of $A_{()} \cup A_{(x_1)}$ not already used (i.e. not x_1). Suppose that x_2 comes from $A_{(x_1)}$. Choose $m_2 > m_1$ with $2^{m_2} > (x_1 + x_2)$ and let $A_{(x_1,x_2)}$ be a 2^{m_2+2}-spaced subset of S_3 consisting of numbers equal to 2^{m_2} (mod 2^{m_2+1}) with $d^*(A_{(x_1,x_2)}) > 0$.

Step 4. Let x_3 be the least member of $A_{()} \cup A_{(x_1)} \cup A_{(x_1,x_2)}$ not already used. Say it comes from $A_{()}$. Choose $m_3 > m_2$ with $2^{m_3} > x_1 + x_2 + x_3$ and let $A_{(x_3)}$ be a 2^{m_3+2}-spaced subset of S_4 consisting of numbers equal to $2^{m_3} \pmod{2^{m_3+1}}$ with $d^*(A_{(x_3)}) > 0$.

(...continue in this fashion...)

Step k+1. Let x_k be the least member of any of the sets constructed in previous stages that was not already used. Say it comes from a set $A_{(a_1,\ldots,a_t)}$. Choose $m_k > m_{k-1}$ with $2^{m_k} > x_1 + \cdots + x_k$ and let $A_{(a_1,\ldots,a_t,x_k)}$ be a 2^{m_k+2}-spaced subset of S_{k+1} consisting of numbers equal to $2^{m_k} \pmod{2^{m_k+1}}$ with $d^*(A_{(a_1,\ldots,a_t,x_k)}) > 0$.

We now make a series of observations:

A. $A_{(a_1,\ldots,a_t,x_k)} + FS_0\big(\langle a_1, \ldots, a_t, x_k\rangle\big) + \{0, 1, \ldots, 2^{m_k}\} \subset S_{k+1}$.

This follows from $A_{(a_1,\ldots,a_t,x_k)}$ being a 2^{m_k+2}-spaced subset of S_{k+1}. Indeed, from our construction, we have

$$2^{m_k} > a_1 + \ldots, a_t + x_k.$$

Since $A_{(a_1,\ldots,a_t,x_k)}$ is a 2^{m_k+2}-spaced subset, if $x \in A_{(a_1,\ldots,a_t,x_k)}$, then

$$(x - 2^{m_k+2}, x + 2^{m_k+2}) \subset S_{k+1}.$$

Therefore

$$A_{(a_1,\ldots,a_t,x_k)} + FS_0\big(\langle a_1,\ldots,a_t,x_k\rangle\big) + \{0,1,\ldots,2^{m_k}\} \subset S_{k+1}.$$

B. Every member of $A_{(a_1,\ldots,a_t,x_k)}$ is divisible by 2^{m_k}.

To see this, let $x \in A_{(a_1,\ldots,a_t,x_k)}$, then $x - 2^{m_k} = n2^{m_k+1}$ for some n. Hence $x = (2n+1)2^{m_k}$.

C. No member of $A_{(a_1,\ldots,a_t,x_k)} + FS_0\big(\langle a_1,\ldots,a_t,x_k\rangle\big)$ is divisible by 2^{m_k+1}.

We first note that if $x \in A_{(a_1,\ldots,a_t,x_k)}$, then x is not divisible by 2^{m_k+1}. Suppose to the contrary that $x = b2^{m_k+1} = (2b)2^{m_k}$ for some b. In **B**, we have already established that $x = (2n+1)2^{m_k}$. So we have $2b = 2n+1$, a contradiction. Therefore, no member of $A_{(a_1,\ldots,a_t,x_k)}$ is divisible by 2^{m_k+1}.

Now let's suppose $x + b$ is divisible by 2^{m_k+1} for some $x \in A_{(a_1,\ldots,a_t,x_k)}$ and $b \in FS_0\big(\langle a_1,\ldots,a_t,x_k\rangle\big)$. Then we have $x + b = c2^{m_k+1} = 2c2^{m_k}$ for some c. By **B**, $x = n2^{m_k}$ for some n. So $2c > n$ and $b = (2c-n)2^{m_k}$, which implies $b > 2^{m_k}$, (by our construction) a contradiction.

Let T be the set of $(a_1,\ldots,a_k) \in \Omega$ used as subscripts for sets $A_{(\cdot)}$ in this construction.

D. Letting

$$B_{(a_1,\ldots,a_k)} = \{x \in \mathbb{Z} : (a_1,\ldots,a_k,x) \in T\},$$

one has $B_{(a_1,\ldots,a_k)} = A_{(a_1,\ldots,a_k)}$.

Next put

$$A = \bigcup_{(a_1,\ldots,a_k)\in T} \Big(A_{(a_1,\ldots,a_k)} + FS_0\big(\langle a_1,\ldots,a_k\rangle\big)\Big) = \bigcup_{(a_1,\ldots,a_k)\in T} FS\big(\langle a_1,\ldots,a_k\rangle\big).$$

To justify the last equality above, let x be an element of the left hand side. Then for some $(a_1,\ldots,a_k) \in T$, some $b \in A_{(a_1,\ldots,a_k)}$ and some $1 \le i_1 < i_2 < \cdots < i_t \le k$

(possibly an empty sequence), $x = b + a_{i_1} + \cdots + a_{i_j}$. Now we use the fact that $(a_1, \ldots, a_k, b) \in T$, whereupon x is a member of the right-hand side.

Conversely, let x be an element of the right-hand side. So there exists $(a_1, \ldots, a_k) \in T$ and $1 \le i_1 < i_2 < \cdots < i_t \le k$ such that $x = a_{i_1} + \cdots + a_{i_t}$. We now use the fact that $(a_1, \ldots, a_{i_t - 1}) \in T$ and $a_{i_t} \in A_{(a_1, \ldots, a_{i_t - 1})}$, whereupon x is seen to be an element of the left hand side.

Then I1-I3 above are plainly satisfied, so A is IP rich. We now turn to showing that A is not a D set.

E. If $(a_1, \ldots, a_t) \in T$ and $m \in \mathbb{N}$ then

(1) $4a_i \le a_{i+1}$, $1 \le i < t$.

(2) If $a_t \equiv 2^m \pmod{2^{m+1}}$ then $a_i \not\equiv 0 \pmod{2^m}$, $1 \le i < t$.

(3) If for some $1 \le i_1 < i_2 < \cdots < i_k \le t$ one has

$$a_{i_1} + a_{i_2} + \cdots + a_{i_k} \equiv 0 \pmod{2^m}$$

then $a_{i_1} \equiv 0 \pmod{2^m}$. (Hence $a_{i_j} \equiv 0 \pmod{2^m}$, $1 \le j \le k$.)

(1) follows from the fact that $A_{(a_1, \ldots, a_j, x_k)}$ is 2^{m_k+2}-spaced with $2^{m_k} > a_1 + \cdots + a_j$.

For (2), the argument is very similar to **B**. Suppose the negation, that is, suppose that there exists i such that $a_i = n2^m$ for some n. Now by construction, $a_i \equiv 2^j \pmod{2^{j+1}}$ for some $j < m$. Let $b = m - j$. We then have $a_i = (2k+1)2^j$ for some k and

$$a_i = n2^m = n2^{j+b} = 2^b n 2^j.$$

This implies that $2^b n = 2k + 1$, a contradiction.

For (3), assume the contrary that there exists $1 \le i_1 \le \ldots \le i_k \le t$ such that

$$a_{i_1} + \cdots + a_{i_k} \equiv 0 \pmod{2^m} \text{ and } a_{i_i} \not\equiv 0 \pmod{2^m}.$$

Choose a shortest (i.e. minimum k, but note $k \ge 2$) subsequence $1 \le i_1 \le \cdots \le i_k \le t$ satisfying our assumption. Then $a_{i_k} \not\equiv 0 \pmod{2^m}$, since if not, 2^m would divide the shorter sequence $(a_{i_1} + \cdots + a_{i_{k-1}})$. Now choose r such that $a_{i_k} \equiv 2^r \pmod{2^{r+1}}$. Then $r < m$, since if not, then 2^m divides a_{i_k}, which would again yield a shorter sequence. Then

$$a_{i_1} + a_{i_2} + \cdots + a_{i_{k-1}} \equiv 0 \pmod{2^r}$$

but by (2) $a_{i_1} \not\equiv 0 \pmod{2^r}$. So this is a shorter sequence, which is a contradiction.

F. If $\langle a_i \rangle_{i=1}^{\infty}$ is a sequence having the property that $(a_1, \ldots, a_t) \in T$ for every $t \in \mathbb{N}$ then $d^*\big(FS\big(\langle a_i \rangle_{i=1}^{\infty}\big)\big) = 0$.

This follows from **E** (1). For let $t \in \mathbb{N}$ and let I be any interval of length 4^t. Since $a_{t+1} \geq 4^t$, I contains at most one member of $x + FS(\langle a_i \rangle_{i=t+1}^{\infty})$ for any $x \in FS(\langle a_i \rangle_{i=1}^{t})$. Therefore, I contains no more than 2^t members of $FS(\langle a_i \rangle_{i=1}^{\infty})$, but $|F_{2m}| = k^{2m}$.

G. For all $x, y \in A$, if there exists an IP set $R \subset \mathbb{N}$ with

$$R \cup (R + x) \cup (R + y) \subset A$$

then there exists some $(a_1, \ldots, a_k) \in T$ such that $\{x, y\} \subset FS\big(\langle a_1, \ldots, a_k \rangle\big)$.

To see this, pick m such that 2^m is greater than $\max\{x, y\}$. Since R is an IP set and $2^m\mathbb{N}$ is IP*, there exists $h \in \mathbb{N}$ such that $h2^m \in R$. So A must contain a configuration of the form

$$\{h2^m, h2^m + x, h2^m + y\}.$$

By definition of A, $h2^m$ is a member of some set

$$A_{(a_1,\ldots,a_t,x_k)} + FS_0\big(\langle a_1, \ldots, a_t, x_k \rangle\big).$$

By **C** no member of that set is divisible by 2^{m_k+1}. This implies that $m \le m_k$, so that $\max\{x,y\} < 2^{m_k}$. Then by **A**,

$$\{h2^m, h2^m + x, h2^m + y\} \subset S_{k+1},$$

which implies that in fact

$$\{h2^m, h2^m + x, h2^m + y\} \subset A \cap S_{k+1} = A_{(a_1,\ldots,a_t,x_k)} + FS_0\big(\langle a_1,\ldots,a_t,x_k\rangle\big).$$

But $A_{(a_1,\ldots,a_t,x_k)}$ is 2^{m_k+2}-spaced, $2^{m_k} > x_1 + \cdots + x_k$ and $\max\{x,y\} < 2^{m_k}$, so for some $x_j \in A_{(a_1,\ldots,a_t,x_k)}$ one actually has

$$\{h2^m, h2^m + x, h2^m + y\} \subset \{x_j\} + FS_0\big(\langle a_1,\ldots,a_t,x_k\rangle\big) \subset FS\big(\langle a_1,\ldots,a_t,x_k,x_j\rangle\big).$$

Now write

$$(x_{i_1},\ldots,x_{i_z}) = (a_1,\ldots,a_t,x_k,x_j)$$

and suppose that $x_{i_1},\ldots,x_{i_q}$ are not divisible by 2^m while $x_{i_{q+1}},\ldots,x_{i_z}$ are; this is possible by **E** (2). By **E** (3), no member of $FS(\langle x_{i_1},\ldots,x_{i_q}\rangle)$ is divisible by 2^m, so

$$h2^m \in FS\big(\langle x_{i_{q+1}},\ldots,x_{i_z}\rangle\big).$$

Now, every member of $FS\big(\langle x_{i_{q+1}},\ldots,x_{i_z}\rangle\big)$ is divisible by $2^{m_{i_q}}$. On the other hand, by **C** $x_{i_{q+1}}$ is not divisible by $2^{m_{i_q}+1}$. Therefore $m_{i_q} \ge m$, whence $\max\{x,y\} < 2^{m_{i_q}}$. Now since

$$M + x' = h2^m + x$$

for some $M \in FS\big(\langle x_{i_{q+1}},\ldots,x_{i_z}\rangle\big)$ and $x' \in FS\big(\langle x_{i_1},\ldots,x_{i_q}\rangle\big)$,

$$2^{m_{i_q}} \mid (h2^m - M) = (x' - x).$$

We have established that $x < 2^{m_{i_q}}$ and by our construction,

$$x' \le x_{i_1} + \cdots + x_{i_q} < 2^{m_{i_q}}.$$

So $x = x' \in FS\big(\langle x_{i_1}, \ldots, x_{i_q}\rangle\big)$. As a similar argument applies to y, we have

$$\{x, y\} \subset FS\big(\langle x_{i_1}, \ldots, x_{i_q}\rangle\big).$$

H. Suppose that $(x_{i_1}, \ldots, x_{i_k}), (x_{j_1}, \ldots, x_{j_t}) \in T$. If

$$\Big(FS\big(\langle x_{i_1}, \ldots, x_{i_k}\rangle\big) \setminus FS\big(\langle x_{i_1}, \ldots, x_{i_{k-1}}\rangle\big)\Big)$$

$$\cap\Big(FS\big(\langle x_{j_1}, \ldots, x_{j_t}\rangle\big) \setminus FS\big(\langle x_{j_1}, \ldots, x_{j_{t-1}}\rangle\big)\Big)$$

is non-empty then $k = t$ and $i_s = j_s$, $1 \leq s \leq t$.

Note that, by construction, $(a_1, \ldots, a_j) \in T$ is uniquely determined by a_j. (If $a_j \in S_{k+1}$ then $a_{j-1} = x_k$. Now use induction.) So by symmetry we may assume that if there is a counterexample to **H** then there is a counterexample with $x_{i_k} < x_{j_t}$. Hence, assume

$$x_{i_k} + a = x_{j_t} + b, \tag{$*$}$$

for

$$a \in FS\big(\langle x_{i_1}, \ldots, x_{i_{k-1}}\rangle\big) \text{ and } b \in FS\big(\langle x_{j_1}, \ldots, x_{j_{t-1}}\rangle\big).$$

But since $x_{i_k} \in A_{(x_{i_1}, \ldots, x_{i_{k-1}})}$, x_{i_k} has distance at least $4(x_1 + \cdots + x_{i_{k-1}})$ from any other x_i and $a < x_1 + \cdots + x_{i_{k-1}}$. It follows that $x_{i_k} + a < x_{j_t}$, contradicting $(*)$.

Suppose now that A is a D set. Then there is a tree $T' \subset \Omega$ which, together with its successor sets B'_f, satisfies D1-D3 above. By D2, for each $y, z \in B'_{()}$, $B'_{(y)} \cap B'_{(z)} \neq \emptyset$. Pick $x_1 \in B'_y \cap B'_z$. Again by D2, $B'_{(y,x_1)} \cap B'_{(z,x_1)} \neq \emptyset$. Pick $x_2 \in B'_{(y,x_1)} \cap B'_{(z,x_1)}$. Continue in such fashion, we obtain a sequence $\langle x_n\rangle_{n=1}^{\infty}$. By D3, $FS(\langle y, x_1, \ldots\rangle) \subset A$ and $FS(\langle z, x_1, \ldots\rangle) \subset A$. If we let R be the IP set $FS(\langle x_n\rangle_{n=1}^{\infty})$, then one easily sees that $R \cup (R + y) \cup (R + z) \subset A$. By **G**, then, for every $y, z \in B'_{()}$ there there exists some $(a_1, \ldots, a_k) \in T$ such that

$$\{y, z\} \subset FS\big(\langle a_1, \ldots, a_k\rangle\big).$$

Consider now the map from $B'_{()}$ to T that sends $x \in B'_{()}$ to the unique (by **H**) $\pi(x) = (a_1, \ldots a_k) \in T$ having the property that $x = a_k + y$ for some $y \in FS_0\big(\langle a_1, \ldots, a_{k-1}\rangle\big)$. What **G** tells us is that for every $y, z \in B'_{()}$, either $\pi(y)$ is an initial segment of $\pi(z)$ or vice-versa. Since for any fixed y there can be only finitely many $z \in B'_{()}$ such that $\pi(z)$ is an initial segment of $\pi(y)$, the length of $\pi(y)$ as y ranges over the infinite set $B'_{()}$ is unbounded and there exists at least one infinite sequence $(a_1, a_2, \ldots)$ in the closure of $\pi(B'_{()})$ (topology of pointwise convergence). So $\pi(y)$ is an initial segment of $(a_1, a_2, \ldots)$ for every $y \in B'_{()}$ (otherwise we could find $z \in B'_{()}$ such that neither of $\pi(y), \pi(z)$ was an initial segment of the other). Therefore $B'_{()} \subset FS\big(\langle a_i\rangle_{i=1}^{\infty}\big)$, and since by **E** (1) $4a_i \leq a_{i+1}$ for every i, one has $d^*\big(FS\big(\langle a_i\rangle_{i=1}^{\infty}\big)\big) = 0$ by **F**, contradicting $d^*(B'_{()}) > 0$.

□

Recall now our Main Theorem, which we are finally ready to prove.

3.1.1. [MZ] Main Theorem. There are D* subsets of $\mathbb{Z}$ that are not AIP*.

Proof of Main Theorem. Let A be the set constructed in the previous theorem. Then $\mathbb{Z} \setminus A$ is D* but not AIP*.

□

Chapter 4. Extension to Countable Fields

4.1 Commentary

In this chapter, we show that the construction given in section 3 for a D* set that is not AIP* in $\mathbb{Z}$ can be generalized to any countable field with positive characteristic. This provides a valuable commentary on a recent result obtained by V. Bergelson and D. Robertson [**BR**], as we explain:

In [**MW**], R. McCutcheon and A. Windsor show that in a countable field F with finite characteristic k, if $\{T_g\}_{g\in F}$ is a measure preserving action of a probability space $(X, \mathscr{A}, \mu)$ and $A \in \mathscr{A}$ with $\mu(A) > 0$, then for any polynomial $p(x) \in F[x]$ with $p(0) = 0$ and $\epsilon > 0$, the set

$$B = \{g \in F : \mu(A \cap T_{p(g)}^{-1} A) > \mu(A)^2 - \epsilon\}$$

is D^*. With the implicit hope that there are sets $A \subset F$ that are D* but not AIP* in mind, Bergelson and Robertson show that B is actually AIP*, ostensively strengthening the result given in [**MW**]. We shall state the result given in [**BR**] here and then show by construction that it does in fact provide a strengthening of [**MW**].

In truth, what we are stating is an early version of their result, which was eventually superceded by their result that B is in fact AIP$^*_r$ for any r–although we do not define this here, it should be noted, in fairness to Bergelson and Robertson, that there is an easy proof that the family of AIP$^*_r$ sets is strictly smaller than the family of D* sets; on the other hand, when r is fixed it does not have the finite intersection property.

4.2 IP ring

Let $\mathcal{F}$ denote the family of all finite subsets of $\mathbb{N}$. We recall the following definitions.

4.2.1. Definition. Let $\{\alpha_i\}_{i=1}^{\infty}$ be a sequence of subsets in $\mathcal{F}$. We write

$$FU\big((\alpha_i)_{i=1}^{\infty}\big) = \big\{\alpha_{i_1} \cup \alpha_{i_2} \cup \cdots \cup \alpha_{i_k} : i_1 < i_2 < \ldots < i_k; k \in \mathbb{N}\big\}.$$

4.2.2. Definition. For $\alpha, \beta \in \mathcal{F}$, we write $\alpha < \beta$ if $i < j$ for every $i \in \alpha$ and every $j \in \beta$. If $\{\alpha_i\}_{i=1}^{\infty} \subset \mathcal{F}$ with $\alpha_1 < \alpha_2 < \cdots$, then the sub family

$$\mathcal{F}^{(1)} = \big\{ \cup_{i \in \beta} \alpha_i : \beta \in \mathcal{F} \big\} = FU\big((\alpha_i)_{i=1}^{\infty}\big)$$

is call an IP-ring.

4.2.3. Definition. Let F be a field and let $(X, \mathscr{A}, \mu)$ be a probability space. For $g \in F$, the map $T_g : X \to X$ is called a measure preserving action if it satisfies:

1). $T_{g_1}(T_{g_2}(x)) = T_{g_1+g_2}(x)$ for all $g_1, g_2 \in F$ and $x \in X$.

2). If e is the additive identity of F, then $T_e(x) = x$ for all $x \in X$.

3). For all $A \in \mathscr{A}, \mu(A) = \mu(T_g^{-1}(A))$ for all $g \in F$.

Now we will prove:

4.2.4. Theorem. (weak version of [**BR**], Theorem 1.2]). Let F be a countable field with positive characteristic k and let $\{T_g\}_{g \in F}$ be a measure preserving action of a probability space $(X, \mathscr{A}, \mu)$. Let $\epsilon > 0$ be given. If $A \in \mathscr{A}$ with $\mu(A) > 0$ and $p(x) \in F[x]$ with $p(0) = 0$, then the set

$$\{g \in F : \mu(A \cap T_{p(g)}^{-1} A) > \mu(A)^2 - \epsilon\}$$

is AIP*.

Before we start, we go over some preliminaries required in the proof.

Let G be an abelian group. A sequence of finite subsets $\langle A_n \rangle_{n=1}^{\infty}$ in G is said to be a Følner sequence if for each $g \in G$,

$$\lim_{n \to \infty} \frac{|\ gA_n \bigtriangleup A_n\ |}{|\ A_n\ |} = 0.$$

Let $B \subset G$. For every Følner sequence $\langle A_n \rangle_{n=1}^{\infty}$ one has notions of density and upper density, namely

$$d_{\langle A_n \rangle}(B) = \lim_{n \to \infty} \frac{|\ A_n \cap B\ |}{|\ A_n\ |}$$

if the limit exists, and in any case

$$\bar{d}_{\langle A_n \rangle}(B) = \limsup_{n\to\infty} \frac{\mid A_n \cap B \mid}{\mid A_n \mid}.$$

We let $d^*(B)$ to be the supremum of $\bar{d}_{\langle A_n \rangle}(B)$ running over all Følner sequences (A_n) and call it the upper Banach density of B.

4.2.5. Definition. Let (Ω, d) be a metric space and $\{x_g\}_{g\in G} \subseteq \Omega$ be a sequence indexed by a countable abelian group G. We write

$$\text{D-lim}_{g\in G} x_g = x$$

if for all $\epsilon > 0$, $d^*(\{g : |x_g - x| > \epsilon\}) = 0$.

4.2.6. Definition. Let (Ω, d) be a metric space and suppose $\mathcal{F}^{(1)}$ is an IP-ring. If $f : \mathcal{F}^{(1)} \to \Omega$, we write

$$\underset{\alpha\in\mathcal{F}^{(1)}}{\text{IP-lim}} \quad f(\alpha) = x$$

if for all $\epsilon > 0$ there exists $\alpha_0 \in \mathcal{F}^{(1)}$ such that $d(f(\alpha), x) < \epsilon$ for all $\alpha \in \mathcal{F}^{(1)}$ with $\alpha > \alpha_0$.

4.2.7. Theorem. Let (Ω, d) be a compact metric space. Let $\mathcal{F}^{(1)}$ be an IP-ring and $g : \mathcal{F}^{(1)} \to \Omega$. There exists $x \in \Omega$ and an IP-ring $\mathcal{F}^{(2)} \subset \mathcal{F}^{(1)}$ such that

$$\underset{\alpha\in\mathcal{F}^{(2)}}{\text{IP-lim}} \quad g(\alpha) = x.$$

Proof. Let $g : \mathcal{F}^{(1)} \to \Omega$.

Step 1: For any $x \in \Omega$, define $B_x^{(1)} = \{y \in \Omega : |x - y| < 1\}$. $\{B_x^{(1)} : x \in \Omega\}$ is clearly an open cover for Ω. By compactness, there exsts a finite subcover, say

$$\{B_{x_i}^{(1)}\}_{i=1}^{k_1}$$

for some $k_1 \in \mathbb{N}$. Notice that the collection of subsets

$$\{B_{x_i}^{(1)}\}_{i=1}^{k_1}$$

induces a finite partition of

$$\mathcal{F}^{(1)} = \cup_{i=1}^{k_1} C_i$$

by defining $\alpha \in C_j$ if and only if $j = \min\{j : g(\alpha) \in B_{x_j}^{(1)}\}$. By Hindman's theorem, there exists an IP-subring $\mathcal{F}^{(2)} \subset \mathcal{F}^{(1)}$ such that for all $\alpha \in \mathcal{F}^{(2)}$, $\alpha \in C_{d_1}$ for some $d_1 \in \{1, \dots, k_1\}$ or equivalently,

$$g(\alpha) \in B_{x_{d_1}}^{(1)}$$

for all $\alpha \in \mathcal{F}^2$.

Step 2: For any $x \in \overline{B_{x_{d_1}}^{(1)}}$, define

$$B_x^{(\frac{1}{2})} = \big\{y \in \overline{B_{x_{d_1}}^{(1)}} : |x - y| < \frac{1}{2}\big\}.$$

Now

$$\big\{B_x^{(\frac{1}{2})} : x \in \overline{B_{x_{d_1}}^{(1)}}\big\}$$

is clearly an open cover of $\overline{B_{x_{d_1}}^{(1)}}$. Let

$$\big\{B_{x_i}^{(\frac{1}{2})}\big\}_{i=1}^{k_2}$$

be a finite subcover for $\overline{B_{x_{d_1}}^{(1)}}$ for some $k_2 \in \mathbb{N}$. Using the same argument as in Step 1, let

$$\mathcal{F}^{(2)} = \cup_{i=1}^{k_2} C_i$$

be partitioned by $\alpha \in C_j$ if and only if $j = \min\{j : g(\alpha) \in B_{x_j}^{(\frac{1}{2})}\}\}$. Again by Hindman's theorem, there exists an IP-subring $\mathcal{F}^{(3)} \subset \mathcal{F}^{(2)}$ such that for all $\alpha \in \mathcal{F}^{(3)}, \alpha \in C_{d_2}$ for some $d_2 \in \{1, \dots, k_2\}$ or equivalently,

$$g(\alpha) \in B_{x_{d_2}}^{(\frac{1}{2})}$$

for all $\alpha \in \mathcal{F}^{(3)}$.

We continue in such fashion:

Step n: We find an IP-subring $\mathcal{F}^{(n)} \subset \mathcal{F}^{(n-1)}$ with the property that

$$g(\alpha) \in B_{x_{d_n}}^{(\frac{1}{n})}$$

for all $\alpha \in \mathcal{F}^{(n)}$. We note here that

$$A = \bigcap_{n=1}^{\infty} \overline{B_{x_{d_n}}^{(\frac{1}{n})}} \neq \emptyset$$

since A is the intersection of a nested sequence of closed sets in a compact space.

For $i \in \mathbb{N}$, we let $(\alpha_{i,j})_{j=1}^{\infty}$ be the set of generators of $\mathcal{F}^{(i)}$. Here we put these generators in matrix form.

$$\begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \dots \\ \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \dots \\ \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} & \dots \\ . & . & . & \dots \\ . & . & . & \dots \end{pmatrix}$$

Let $\mathcal{F}^{(*)}$ be the IP-ring generated by $(\alpha_{i,j})_{i=j}$ (the diagonal) from the above matrix. Pick $y \in A$ and we have

$$\text{IP-}\lim_{\alpha \in \mathcal{F}^*} g(\alpha) = y.$$

To see this, let $\epsilon > 0$ be given and pick $n \in \mathbb{N}$ satisfying $\frac{1}{n} < \frac{\epsilon}{2}$. Then

$$B_{x_{d_n}}^{(\frac{1}{n})} \subset B_y^{(\epsilon)}.$$

It follow that for every $\alpha \in \mathcal{F}^{(*)}$ with $\alpha > \alpha_{n,n}$,

$$\alpha \in \mathcal{F}^{(n)} \text{ implies } g(\alpha) \in B_{x_{d_n}}^{(\frac{1}{n})} \text{ implies } d(y, g(\alpha)) < \epsilon.$$

□

4.3 VIP system

Recall that our immediate goal is to establish a recurrence property for families of measure preserving transformations $\{T_{p(g)} : g \in F\}$. In order to do this, we need to rely on the "polynomial" properties of subfamilies $\{T_{p(g(\alpha))} : \alpha \in \mathcal{F}\}$, where $g : \mathcal{F} \to F$ is an IP-system. This leads to the following definition.

4.3.1. Definition. Let $(G, +)$ be an abelian group and $\mathcal{F}^{(1)}$ be an IP-ring. A function $g : \mathcal{F}^{(1)} \to G$ is a VIP system of degree at most d if for any disjoint $\alpha_0, \alpha_1, \ldots, \alpha_d \in \mathcal{F}^{(1)}$,

$$\sum_{\alpha \in FU(\alpha_0,\alpha_1,\ldots,\alpha_d)} (-1)^{|\alpha|} g(\alpha) = e.$$

The most typical example of a VIP system is $g(\alpha) = p(n(\alpha))$, where n is an IP system. (In, say, any commutative ring R, where $p(x) \in R[x]$ has zero constant term.) There are other examples, however. Indeed, if $(d_{ij})_{i,j=1}^{\infty}$ is any infinite matrix from R then $g(\alpha) = \sum_{i,j\in\alpha} d_{ij}$ defines a VIP system of degree at most 2.

The utility of the VIP system notion for us is summed up in the following lemma.

4.3.2. Lemma. Let $(L, \cdot)$ be a compact, commutative topological group with identity e. (In particular, the map $L \times L \to L$ defined by $(x, y) \to xy$ is continuous.) If $\mathcal{F}^{(1)}$ is an IP ring and $x : \mathcal{F}^{(1)} \to L$ is a VIP system of degree at most d for some $d < \infty$ then there exists an IP-subring $\mathcal{F}^2 \subseteq \mathcal{F}^1$ such that

$$\underset{\alpha\in\mathcal{F}^{(2)}}{\text{IP-lim}} \quad x(\alpha) = e.$$

Proof. Using Hindman's theorem we pass to a subring $\mathcal{F}^{(2)} \subseteq \mathcal{F}^{(1)}$ such that

$$\underset{\alpha\in\mathcal{F}^{(2)}}{\text{IP-lim}} \quad x(\alpha) = h$$

exists. Now choose M sufficiently large, so that $x(\alpha) \approx h$ whenever min $\alpha \geq M$ for all $\alpha \in \mathcal{F}^{(2)}$. Choose pairwise disjoint $\alpha_0, \ldots, \alpha_d \in \mathcal{F}^{(2)}$ satisfying min $\alpha_i \geq M$. By definition, one has

$$e = \sum_{\alpha \in FU(\alpha_0,\dots,\alpha_d)} (-1)^{|\alpha|} x(\alpha) \approx -2^d h + (2^d - 1)h = -h$$

so $h = e$.

□

4.4 Koopman-von Neumann Decomposition Theorem.

We will need an polynomial ergodic splitting theorem for field actions due to Bergelson, Leibman and McCutcheon (proved using a theorem of Larick) that is, in turn, an extension of the so-called Koopman-von Neumann Decomposition Theorem, a classical result that we shall formulate first:

Let $L^2(X, \mathscr{A}, \mu)$ (denoted $L^2(X)$ for convenience) be the usual Hilbert space (of real valued functions) with inner product defined as:

$$\langle f, g \rangle = \int fg \, d\mu.$$

4.4.1. Theorem. (See, e.g., [**BS**] Theorem 4.10.5.) Let G be a countable abelian group, $(X, \mathscr{A}, \mu)$ be a probability space and let $\{T_g\}_{g \in G}$ be a measure preserving action of X. Put

$$H_c^G = \{f \in L^2(X) : \{T_g f : g \in G\} \text{ is precompact }\}$$

and

$$H_{wm}^G = \{f \in L^2(X) : \text{D-lim}_g \langle T_g f, h \rangle = 0, \text{ for all } h \in L^2(X)\}.$$

Then $L^2(X) = H_c^G \bigoplus H_{wm}^G$.

Here now is the Bergelson-Leibman-McCutcheon splitting theorem alluded to above.

4.4.2. Theorem. (See [**BLM**, Theorem 3.17].) Let F be a countable field of finite characteristic k, let $\{T_g : g \in F\}$ be a measure preserving G-action on a probability space $(X, \mathcal{A}, \mu)$, and let $p(x) \in F[x]$ be a polynomial having zero constant term, i.e., $p(0) = 0$. Let now:

$$\mathcal{H}^c = \{f \in L^2(X) : \{T_{p(g)}f : g \in F\} \text{ is precompact}\}$$

and

$$\mathcal{H}^{wm} = \{f \in L^2(X) : \text{D-lim}_{g \in F}\langle T_{p(g)}f, h\rangle = 0 \text{ for all } h \in L^2(X)\}.$$

Letting $G = \text{span}(\text{Range } p)$, one has

(a) $\mathcal{H}^c = H_c^G$, and

(b) $\mathcal{H}^{wm} = H_{wm}^G$.

In particular, it follows that

(c) $L^2(X) = \mathcal{H}^c \bigoplus \mathcal{H}^{wm}$.

Next we proceed to prove Theorem 4.2.4.

Step 1. By Theorem 4.4.2. we may write $L^2(X) = \mathcal{H}^c \bigoplus \mathcal{H}^{wm}$, where these subspaces are defined as in the theorem.

Let $A \in \mathscr{A}$ and write $1_A = h_c + h_{wm}$ with $h_c \in \mathcal{H}^c$ and $h_{wm} \in \mathcal{H}^{wm}$. Put

$$L = \overline{\{T_g h_c : g \in F\}},$$

where closure is in the norm topology. Now we wish to put a group structure on L, so as to transform it into a commutative, compact topological group.

For $l_1, l_2 \in L$, pick $\{g_i^{(1)}\}_{i=1}^{\infty}$ with $T_{g_i^{(1)}}h_c \to l_1$ and $\{g_i^{(2)}\}_{i=1}^{\infty}$ with $T_{g_i^{(2)}}h_c \to l_2$. Define now

$$l_1 \cdot l_2 = \lim_{i\to\infty} T_{g_i^{(1)}+g_i^{(2)}}h_c.$$

We can check simultaneously that the above limit exists and that the operation is well defined. To wit, pick potentially novel sequences $\{k_i^{(1)}\}_{i=1}^{\infty}$ with $T_{k_i^{(1)}}h_c \to l_1$

and $\{k_i^{(2)}\}_{i=1}^{\infty}$ with $T_{k_i^{(2)}}h_c \to l_2$. By application of the simple trick

$$\|a-b\| = \|a-c+c-b\| \leq \|a-c\| + \|c-b\|$$

(a routine consequence of the triangle inequality) multiple times, we get:

$$\|T_{k_i^{(1)}+k_i^{(2)}}h_c - T_{g_i^{(1)}+g_i^{(2)}}h_c\| \leq \|T_{k_i^{(1)}+k_i^{(2)}}h_c - T_{k_i^{(1)}}l_2\| + \|T_{k_i^{(1)}}l_2 - T_{k_i^{(1)}}T_{g_i^{(2)}}h_c\|$$

$$+\|T_{g_i^{(2)}}T_{k_i^{(1)}}h_c - T_{g_i^{(2)}}l_1\| + \|T_{g_i^{(2)}}l_1 - T_{g_i^{(1)}}T_{g_i^{(2)}}h_c\|$$

$$\leq ||T_{k_i^{(2)}}h_c - l_2|| + ||l_2 - T_{g_i^{(2)}}h_c|| + ||T_{k_i^{(1)}}h_c - l_1|| + ||l_1 - T_{g_i^{(1)}}h_c|| \to 0$$

as $i \to \infty$. Hence $l_1 \cdot l_2$ is well defined.

To check commutativity, we note that

$$l_1 \cdot l_2 = \lim_{i\to\infty} T_{g_i^{(1)}+g_i^{(2)}}h_c = \lim_{i\to\infty} T_{g_i^{(2)}+g_i^{(1)}}h_c = l_2 \cdot l_1.$$

Lastly, we mention the following lemma useful to check the continuity of " $\cdot$ ".

4.4.3. Lemma Given $l_1 \in L$, let $l_1^{(i)}$ be a sequence converging to l_1. Then for any $\epsilon > 0$ and sequences $j^{(i)}, k^{(i)}$, we have

$$\|j^{(i)} \cdot l_1^{(i)} - k^{(i)} \cdot l_1^{(i)}\| \leq \|j^{(i)} - k^{(i)}\| + \epsilon$$

for all sufficiently large i.

Proof. For all i, choose three sequences $\{g_t^{(i)}\}, \{f_t^{(i)}\}$ and $\{d_t^{(i)}\}$ with $T_{g_t^{(i)}}h_c \to j^{(i)}, T_{f_t^{(i)}}h_c \to k^{(i)}$ and $T_{d_t^{(i)}}h_c \to l_1^{(i)}$. By definition, for t sufficiently large

$$\|j^{(i)} \cdot l_1^{(i)} - T_{g_t^{(i)}+d_t^{(i)}}h_c\| \approx 0 \text{ and } \|k^{(i)} \cdot l_1^{(i)} - T_{f_t^{(i)}+d_t^{(i)}}h_c\| \approx 0.$$

Therefore,

$$\|j^{(i)} \cdot l_1^{(i)} - k^{(i)} \cdot l_1^{(i)}\| \approx \|T_{g_t^{(i)}+d_t^{(i)}}h_c - T_{f_t^{(i)}+d_t^{(i)}}h_c\| = \|T_{g_t^{(i)}}h_c - T_{f_t^{(i)}}h_c\|.$$

But

$$\|T_{g_t^{(i)}}h_c - T_{f_t^{(i)}}h_c\| \to \|j^{(i)} - k^{(i)}\|$$

as $t \to \infty$.

$\square$

We proceed to show the continuty of "$\cdot$". Let $l_1, l_2 \in L$ with sequences $l_1^{(i)} \to l_1$ and $l_2^{(i)} \to l_2$. It suffices to show that

$$\|l_1 \cdot l_2 - l_1^{(i)} \cdot l_2^{(i)}\| \to 0$$

as $i \to \infty$. By the triangle inequality,

$$\|l_1 \cdot l_2 - l_1^{(i)} \cdot l_2^{(i)}\| \le \|l_1 \cdot l_2 - l_1 \cdot l_2^{(i)}\| + \|l_1 \cdot l_2^{(i)} - l_1^{(i)} \cdot l_2^{(i)}\|.$$

Apply lemma 4.4.3. to the first term of the above inquality on the right (let $j^{(i)}$ and $k^{(i)} = l_1$ for all i), we have

$$\|l_1 \cdot l_2 - l_1 \cdot l_2^{(i)}\| \le \|l_1 - l_1\| + \epsilon = \epsilon.$$

Again apply lemma 4.4.3. to the second term on the right (let $j^{(i)} = l_1$ and $k^{(i)} = l_1^{(i)}$), we get

$$\|l_1 \cdot l_2^{(i)} - l_1^{(i)} \cdot l_2^{(i)}\| \approx \|j^{(i)} - k^{(i)}\| + \epsilon = \|l_1 - l_1^{(i)}\| + \epsilon.$$

Hence, $\|l_1 \cdot l_2 - l_1^{(i)} \cdot l_2^{(i)}\| \le 2\epsilon$.

We conclude that L is a compact commutative topological group. Note, moreover, that the map $g \to T_g h_c$ defines a group homomorphism $F \to L$.

Step 2. Recall that we wish to find an IP* set E and a 0 upper Banach density set D such that for all $g \in E \setminus D$,

$$\mu(A \cap T_{p(g)}^{-1} A) > \mu(A)^2 - \epsilon.$$

Let

$$E = \{g : \langle h_c, T_{p(g)} h_c \rangle \geq \mu(A)^2 - \frac{\epsilon}{2}\}.$$

We claim that E is IP*. To this end, let $\{g_\alpha\}_{\alpha \in \mathcal{F}} \subseteq F$ be an IP system; that is, $g_{\alpha \cup \beta} = g_\alpha + g_\beta$ whenever $\alpha \cap \beta = \emptyset$. We must show that there exists $\alpha \in \mathcal{F}$ such that $g_\alpha \in E$. Let

$$x(\alpha) = x_\alpha = T_{p(g_\alpha)} h_c \in X \text{ for } \alpha \in \mathcal{F}.$$

Then, using the fact that $g \to T_g h_c$ is a homomorphism, $\{x_\alpha\}_{\alpha \in \mathcal{F}}$ is a VIP-system of degree at most $d = \deg\ p(x)$ in L, i.e.,

$$\sum_{\alpha \in FU(\alpha_0, \ldots, \alpha_d)} (-1)^{|\alpha|} x_\alpha = e$$

whenever $\alpha_0, \ldots, \alpha_d$ are pairwise disjoint members of $\mathcal{F}$.

By the above lemma, then, one has

$$\text{IP-}\lim_{\alpha \in \mathcal{F}^{(1)}} \quad x_\alpha = e$$

in the topology of L (which comes from the norm topology on $L^2(X)$. Here $e = h_c$ is the identity element of L. It follows that

$$\text{IP-}\lim_{\alpha \in \mathcal{F}^{(1)}} \quad \langle h_c, T_{p(g_\alpha)} h_c \rangle = \|h_c\|^2 \geq \mu(A)^2.$$

The last inequality follows from the fact that the constants are in $\mathcal{H}^c$, and h_c is the orthogonal projection of 1_A onto $\mathcal{H}^c$. So in fact one may write $h_c = \mu(A) + m$ for some function m orthogonal to the constants.

Step 3. Recall that $h_{wm} \in \mathcal{H}^{wm}$. It therefore follows that

$$D = \{g \in F : |\langle h_{wm}, T_{p(g)} h_{wm} \rangle| \geq \frac{\epsilon}{2}\}$$

satisfies $d^*(D) = 0$.

If $g \in E \setminus D$ then

$$|\langle h_{wm}, T_{p(g)} h_{wm} \rangle| \leq \frac{\epsilon}{2}$$

and

$$\langle h_c, T_{p(g)} h_c \rangle \geq \mu(A)^2 - \frac{\epsilon}{2}.$$

So

$$\mu(A \cap T_{p(g)}^{-1} A) = \int 1_A T_{p(g)} 1_A d\mu = \langle 1_A, T_{p(g)} 1_A \rangle = \langle (h_c + h_{wm}), T_{p(g)}(h_c + h_{wm}) \rangle$$

$$= \langle h_c, T_{p(g)} h_c \rangle + \langle h_c, T_{p(g)} h_{wm} \rangle + \langle h_{wm}, T_{p(g)} h_c \rangle + \langle h_{wm}, T_{p(g)} h_{wm} \rangle$$

$$\geq \mu(A) - \frac{\epsilon}{2} + 0 + 0 - \frac{\epsilon}{2} = \mu(A)^2 - \epsilon.$$

This concludes the proof of Theorem 4.2.4.

□

4.5 An extension of our previous construction

In order to know that Theorem 4.2.4. marks an advance on the result, due to McCutcheon and Windsor, that the returns set in question is D*, one needs to know that, in a finite characteristic field F as in $\mathbb{Z}$, not every D* set is AIP*. In the rest of this section, we will establish this fact.

Fix a countable field F having finite characteristic k. We do not need to consider the multiplicative structure of F, as the notions of IP, IP-rich, D-set etc. are based solely on the additive structure. The structure of F as an additive group is that of a direct sum of countably many copies of $\mathbb{Z}_k$. Let $\{g_1, g_2, \ldots\}$ be a set of generators of this group. For $n \in \mathbb{N}$, let F_n be the group generated by $\{g_1, \ldots, g_n\}$ and let G_n be the group generated by $\{g_{n+1}, g_{n+2}, \ldots\}$. Write

$$x \equiv y \ (\bmod G_n) \text{ if } x - y \in G_n.$$

Largeness notions defined earlier for $\mathbb{Z}$ have obvious analogs in F.

4.5.1. Definitions. We denote by βF the set of all ultrafilters on F. A sequence $\{\Phi_n\}$ of finite subsets of F is a Følner sequence if for every $g \in F$ one has $\frac{|(g+\Phi_n)\cap\Phi_n|}{|\Phi_n|} \to 1$ as $n \to \infty$. If $E \subseteq F$, we define $d^*(E)$ to be the supremum of $\limsup_n \frac{|E\cap\Phi_n|}{|\Phi_n|}$ over all

Følner sequences $\{\Phi_n\}$ for F. An ultrafilter $p \in \beta F$ is said to be essential if $d^*(E) > 0$ for every $E \in p$. A D set is a subset $E \subseteq F$ belonging to some essential idempotent, and a D* set is a set that intersects every D set non-trivially. Equivalently, a set is D* if it belongs to every essential idempotent. A set $E \subseteq F$ is IP if it contains the finite sums of some infinite sequence, and is IP* if it meets every IP set. $E \subseteq F$ is IP rich if $E \setminus C$ is IP for every $C \subset F$ with $d^*(C) = 0$. A set is AIP* if it meets every IP rich set. Equivalently, a set is AIP* if there is some set $C \subset F$ with $d^*(C) = 0$ such that $E \cup C$ is IP*.

We move now to the particular notions germane to our constructions. Let Ω be the family of all finite sequences taken from F, including the empty sequence $()$.

4.5.2. Definitions.

(i) A set $A \subseteq F$ is *FS tree rich* if there is a subset $T \subseteq \Omega$ having the following properties:

I1. $() \in T$;

I2. If $f \in T$ then $d^*\big(B_f\big) > 0$, where $B_{(x_1,\ldots,x_k)} = \{x \in F : (x_1, \ldots, x_k, x) \in T\}$.

I3. If $(x_1, \ldots, x_k) \in T$ then $FS\big(\langle x_1, \ldots, x_k\rangle\big) \subset A$.

(ii) A set $A \subseteq F$ is *intersectively FS tree rich* if there is a subset $T \subseteq \Omega$ having the following properties:

D1. $() \in T$;

D2. If $f_1, \ldots, f_t \in T$ then $d^*\big(B_{f_1} \cap \cdots \cap B_{f_t}\big) > 0$, where

$$B_{(x_1,\ldots,x_k)} = \{x \in F : (x_1, \ldots, x_k, x) \in T\}.$$

D3. If $(x_1, \ldots, x_k) \in T$ then $FS\big(\langle x_1, \ldots, x_k\rangle\big) \subset A$.

In the previous section we established a pair of equivalences for $\mathbb{Z}$. First we showed that FS tree richness is equivalent to IP richness, then we showed that the D sets are precisely the intersectively FS tree rich sets. Although these equivalences are surely valid in F, we shall only undertake what we need of these tasks for our present purpose. Namely, that FS tree richness is sufficient for IP richness and that intersective FS tree richness is necessary for D sets.

Here is the first of these tasks.

4.5.3. Theorem. Let $A \subseteq F$. If A is FS-tree rich then it is IP rich.

Proof. Suppose that T satisfies I1-I3 and let $E \subset F$ with $d^*(E) = 0$. We must show that $A \setminus E$ is IP. Since $() \in T$,

$$d^*\big(\{x \in F : (x) \in T\}\big) > 0,$$

and for all x in this set, $x \in A$. So we may choose x_1 such that $(x_1) \in T$ and $x_1 \notin E$. Next we have

$$d^*\big(\{x \in F : (x_1, x) \in T\}\big) > 0,$$

and for every x in this set, $\{x, x+x_1\} \subset A$. Since $d^*\big(E \cup (E - x_1)\big) = 0$, we may choose x_2 such that $(x_1, x_2) \in T$ and $x_2 \notin E \cup (E - x_1)$. Note now that $FS\big(\{x_1, x_2\}\big) \subset A \setminus E$. It is clear that this process can be continued and will yield a sequence $\langle x_i \rangle_{i=1}^{\infty}$ for which

$$FS\big(\langle x_i \rangle_{i=1}^{\infty}\big) \subset A \setminus E.$$

□

And here is the second.

4.5.4. Theorem. Let $A \subseteq F$. If A is a D set then it is intersectively FS tree rich.

Proof. We will be using the standard fact that if p is idempotent and $A \in p$ then $A \in p + p$, i.e. $\{m : A - m \in p\} \in p$. Let p be an essential idempotent with $A \in p$. Let

$$A_{()} = A \cap \{m : A - m \in p\}.$$

For $x \in A_{()}$, let

$$A_{(x)} = A \cap (A - x) \cap \{m : (A \cap (A - x)) - m \in p\} \in p.$$

Note that for such x, $x \in A$. Now for $y \in A_{(x)}$, let

$$A_{(x,y)} = A \cap (A - x) \cap (A - y) \cap (A - x - y)$$

$$\cap \{m : \big(A \cap (A - x) \cap (A - y) \cap (A - x - y)\big) - m \in p\} \in p.$$

Note that for such x, y, $FS\big(\langle x, y\rangle\big) \subset A$. Now for $z \in A_{(x,y)}$ one defines $A_{(x,y,z)} \in p$, etc. Continuing in this fashion, one defines p-sets $\{A_f : f \in T\}$ for some set $T \subset \Omega$. Letting

$$B_{(x_1,\ldots,x_k)} = \{x \in F : (x_1, \ldots, x_k, x) \in T\}$$

one has $B_{(x_1,\ldots,x_k)} = A_{(x_1,\ldots,x_k)}$, and D1-D3 above are satisfied.

□

We now come to the main result of this section.

4.5.5. Theorem. There exists a set $A \subset F$ such that A is IP rich and A is not a D set.

Proof. A subset of F is *thick* if it contains shifts of F_n for every n. We first show that a collection of pairwise disjoint thick subsets (call these $\{S_i : i \in \mathbb{N}\}$) of F can be constructed. Recall that $\{g_1, g_2, \ldots\}$ is a set of generators for F. Also note that the collection$\{F_n + g_{n+1}\}_{n=1}^{\infty}$ are pairwise disjoint subsets of F.

Step 1: We start our construction by assigning $\{F_1 + g_2\}$ to S_1.

Step 2: We assign $\{F_2 + g_3\}$ to S_1 and $\{F_3 + g_4\}$ to S_2

Step 3: We assign $\{F_4 + g_4\}$ to S_1, $\{F_5 + g_6\}$ to S_2 and $\{F_6 + g_7\}$ to S_3.

We continue in such fashion. At step n, we assign some previously unassigned collection $\{F_n + g_{n+1}\}_{n=i}^{j}$ with $j - i = n$ to the collection $\{S_i\}_{i=1}^{n}$. Carrying out the construction this way, we see that each member of the disjoint collection $\{S_i : i \in \mathbb{N}\}$ contains a shifted copy of F_n for all $n \in \mathbb{N}$.

So let $\{S_i : i \in \mathbb{N}\}$ be a pairwise disjoint collection of thick subsets of F. We move on to our main construction. By an n-sparse subset of F we mean a set $B \subset F \setminus F_n$ whose members come from distinct cosets of F_n.

It suffices to construct A such that A is FS tree rich but not intersectively FS tree rich. Let $A_{()} \subset S_1$ be a set of group elements equal to $g_1 \pmod{G_1}$ with $d^*(A_{()}) > 0$. Order F by

$$\sum_i a_i g_i < \sum_j b_j g_j \text{ if and only if } \sum_i a_i k^i < \sum_j b_j k^j,$$

where $0 \leq a_i, b_j < k$. Let x_1 be the least member of $A_{()}$. Choose $m_1 \geq 3$ with $x_1 \in F_{m_1 - 1}$ and let

$$A_{(x_1)} \subset \bigcap_{g \in F_{m_1+1}} (S_2 - g)$$

consist of group elements equal to $g_{m_1} \pmod{G_{m_1}}$ with $d^*(A_{(x_1)}) > 0$.

Now pick the least member x_2 of $A_{()} \cup A_{(x_1)}$ not already used (i.e. not x_1). Suppose that x_2 comes from $A_{(x_1)}$. Choose $m_2 \geq m_1 + 2$ with $x_2 \in F_{m_2 - 1}$ and let

$$A_{(x_1, x_2)} \subset \bigcap_{g \in F_{m_2+1}} (S_3 - g)$$

consist of group elements equal to $g_{m_2} \pmod{G_{m_2}}$ with $d^*(A_{(x_1,x_2)}) > 0$. Continue in this fashion; at the stage where we are ready to create a set within S_{k+1}, we let x_k be the least member of any of the sets constructed in previous stages that was not already used. Say it comes from a set $A_{(a_1,\ldots,a_t)}$. Choose $m_k > m_{k-1}$ with $x_k \in F_{m_k - 1}$ and let

$$A_{(a_1,\ldots,a_t,x_k)} \subset \bigcap_{g \in F_{m_k}} (S_{k+1} - g)$$

consist of group elements equal to $g_{m_k} \pmod{G_{m_k}}$ with $d^*(A_{(a_1,\ldots,a_t,x_k)}) > 0$.

We note that:

A. $A_{(a_1,\ldots,a_t,x_k)} + FS_0\big(\langle a_1, \ldots, a_t, x_k \rangle\big) + F_{m_k+1} \subset S_{k+1}$.

B. $A_{(a_1,\ldots,a_t,x_k)} \subset G_{m_k - 1}$.

C. No member of $A_{(a_1,\ldots,a_t,x_k)} + FS_0\big(\langle a_1, \ldots, a_t, x_k \rangle\big)$ belongs to G_{m_k}.

Let T be the set of $(a_1, \ldots, a_k) \in \Omega$ used as subscripts for sets $A_{(\cdot)}$ in this construction.

D. Letting

$$B_{(a_1,\ldots,a_k)} = \{x \in F : (a_1, \ldots, a_k, x) \in T\},$$

one has $B_{(a_1,\ldots,a_k)} = A_{(a_1,\ldots,a_k)}$.

Next put

$$A = \bigcup_{(a_1,\ldots,a_k)\in T} \Big(A_{(a_1,\ldots,a_k)} + FS_0\big(\langle a_1, \ldots, a_k\rangle\big)\Big) = \bigcup_{(a_1,\ldots,a_k)\in T} FS\big(\langle a_1, \ldots, a_k\rangle\big).$$

Then I1-I3 above are plainly satisfied, so A is FS tree rich. We now turn to showing that A is not intersectively FS tree rich.

E. If $(a_1, \ldots, a_t) \in T$ and $m \in \mathbb{N}$ then

(1) $a_t \in G_{2t-2}$.

(2) If $a_t \equiv g_m \pmod{G_m}$ then $a_i \not\equiv 0 \pmod{G_{m-1}}$, $1 \le i < t$. (In fact, $a_i \in F_{m-1}$.)

(3) If for some $1 \le i_1 < i_2 < \cdots < i_k \le t$ one has $a_{i_1} + a_{i_2} + \cdots + a_{i_k} \in G_m$ then $a_{i_1} \in G_m$. (Hence $a_{i_j} \in G_m$), $1 \le j \le k$.)

(1) follows from the fact that $m_{k+1} \ge m_k + 2$.

For (2), note that for some $i_1 < i_2 < \cdots < i_t$, $(a_1, \ldots, a_t) = (x_{i_1}, \ldots, x_{i_t})$. Note that $x_{i_t} \in A_{(x_{i_1},\ldots,x_{i_{t-1}})}$ and $x_{i_t} \equiv g_{m_{i_{t-1}}} \pmod{G_{m_{i_{t-1}}+1}}$. This implies that $m = m_{i_{t-1}}$. Now use the fact that the sequence m_j increases with j.

For (3), assume the negation and choose a shortest (i.e. minimum k, but note $k \ge 2$) counterexample. Then obviously $a_{i_k} \notin G_m$. Choose r such that $a_{i_k} \equiv g_r$ (mod G_r). Then

$$a_{i_1} + a_{i_2} + \cdots + a_{i_{k-1}} \in G_{r-1}$$

but $a_{i_1} \notin G_{r-1}$ (again, since m_j increases with j). So this is a shorter counterexample, which is a contradiction.

F. If $\langle a_i\rangle_{i=1}^{\infty}$ is a sequence having the property that $(a_1, \ldots, a_t) \in T$ for every $t \in \mathbb{N}$ then $d^*\big(FS\big(\langle a_i\rangle_{i=1}^{\infty}\big)\big) = 0$.

This follows from **E** (1). For let $m \in \mathbb{N}$ and let I be any shift of F_{2m}. Since $a_i \in G_{2m}$, $i > m$, I contains at most one member of $x + FS(\langle a_i \rangle_{i=m+1}^{\infty})$ for any $x \in FS(\langle a_i \rangle_{i=1}^{m})$. Therefore, I contains no more than 2^m members of $FS(\langle a_i \rangle_{i=1}^{\infty})$.

G. For all $x, y \in A$, if there exists an infinite IP set $R \subset F$ with $R \cup (R+x) \cup (R+y) \subset A$ then there exists some $(a_1, \ldots, a_k) \in T$ such that

$$\{x, y\} \subset FS\big(\langle a_1, \ldots, a_k \rangle\big).$$

To see this, pick m such that $x, y \in F_m$. A must contain a configuration of the form $\{g, g+x, g+y\}$ for some $g \in G_m$. By definition of A, g is a member of some set $A_{(a_1,\ldots,a_t,x_k)} + FS_0\big(\langle a_1, \ldots, a_t, x_k \rangle\big)$. By **C** no member of that set belongs to G_{m_k}. This implies that $m < m_k$, so that $x, y \in F_{m_k - 1}$. Then by **A**, $\{g, g+x, g+y\} \subset S_{k+1}$, which implies that in fact

$$\{g, g+x, g+y\} \subset A \cap S_{k+1} = A_{(a_1,\ldots,a_t,x_k)} + FS_0\big(\langle a_1, \ldots, a_t, x_k \rangle\big).$$

But $A_{(a_1,\ldots,a_t,x_k)} \subset G_{m_k - 1}$ and

$$\{x, y\} \cup FS_0\big(\langle a_1, \ldots, a_t, x_k \rangle\big) \subset F_{m_k - 1},$$

so for some $x_j \in A_{(a_1,\ldots,a_t,x_k)}$ one actually has

$$\{g, g+x, g+y\} \subset \{x_j\} + FS_0\big(\langle a_1, \ldots, a_t, x_k \rangle\big) \subset FS\big(\langle a_1, \ldots, a_t, x_k, x_j \rangle\big).$$

Now write

$$(x_{i_1}, \ldots, x_{i_z}) = (a_1, \ldots, a_t, x_k, x_j)$$

and suppose that $x_{i_1}, \ldots, x_{i_q}$ are not members of G_m while $x_{i_{q+1}}, \ldots, x_{i_z}$ are; this is possible by **E** (2). By **E** (3), no member of $FS(\langle x_{i_1}, \ldots, x_{i_q} \rangle)$ belongs to G_m, so $g \in FS\big(\langle x_{i_{q+1}}, \ldots, x_{i_z} \rangle\big)$. Now, every member of $FS\big(\langle x_{i_{q+1}}, \ldots, x_{i_z} \rangle\big)$ belongs to $G_{m_{i_q} - 1}$. On the other hand, by **C** $x_{i_{q+1}}$ does not belong to $G_{m_{i_q}}$. Therefore $m_{i_q} > m$, whence $\{x,y\} \subset F_{m_{i_q}-1}$. It's also the case (by stipulation; see the construction) that

$$x_{i_1} + \cdots + x_{i_q} \in F_{m_{i_q}-1}.$$

$M + x' = g + x$ for some $M \in FS\big(\langle x_{i_{q+1}}, \ldots, x_{i_z}\rangle\big)$ and $x' \in FS\big(\langle x_{i_1}, \ldots, x_{i_q}\rangle\big)$, so one has $x' - x = M - g \in G_{m_{i_q}-1}$. It follows that $x = x' \in FS\big(\langle x_{i_1}, \ldots, x_{i_q}\rangle\big)$. As a similar argument applies to y, we have $\{x,y\} \subset FS\big(\langle x_{i_1}, \ldots, x_{i_q}\rangle\big)$.

H. Suppose that $(x_{i_1}, \ldots, x_{i_k}), (x_{j_1}, \ldots, x_{j_t}) \in T$. If

$$\Big(x_{i_k} + FS_0\big(\langle x_{i_1}, \ldots, x_{i_{k-1}}\rangle\big)\Big) \cap \Big(x_{j_t} + FS_0\big(\langle x_{j_1}, \ldots, x_{j_{t-1}}\rangle\big)\Big)$$

is non-empty then $k = t$ and $i_s = j_s$, $1 \le s \le t$.

Note that, by construction, $(a_1, \ldots, a_j) \in T$ is uniquely determined by a_j. (If $a_j \in S_{k+1}$ then $a_{j-1} = x_k$. Now use induction.) So if there is a counterexample to **H** then it must satisfy also $x_{i_k} \neq x_{j_t}$. We now consider two cases.

First case: $(x_{i_1}, \ldots, x_{i_{k-1}}) = (x_{j_1}, \ldots, x_{j_{t-1}}) = (a_1, \ldots, a_r)$. If this case, for $m = m_{i_{k-1}-1}$, x_{i_k} and x_{j_t} are distinct elements of $A_{(a_1,\ldots,a_r)}$, which is contained in G_m. But $FS\big(\langle a_1, \ldots, a_r\rangle\big) \subset F_m$. Therefore

$$x_{i_k} + FS_0\big(\langle x_{i_1}, \ldots, x_{i_{k-1}}\rangle\big) = x_{i_k} + FS_0\big(\langle a_1, \ldots, a_r\rangle\big)$$

and

$$x_{j_t} + FS_0\big(\langle x_{j_1}, \ldots, x_{j_{t-1}}\rangle\big) = x_{j_t} + FS_0\big(\langle a_1, \ldots, a_r\rangle\big)$$

belong to distinct cosets of F_m, and therefore cannot meet. This is a contradiction.

Second case: $(x_{i_1}, \ldots, x_{i_{k-1}}) \neq (x_{j_1}, \ldots, x_{j_{t-1}})$. In this case x_{i_k} and x_{j_t} are members of different sets $A_{(a_1,\ldots,a_j)}$, so that

$$x_{i_k} + FS_0\big(\langle x_{i_1}, \ldots, x_{i_{k-1}}\rangle\big)$$

and

$$x_{j_t} + FS_0\big(\langle x_{j_1}, \ldots, x_{j_{t-1}}\rangle\big)$$

belong to different sets S_k, again yielding a contradiction.

Suppose now that A is a D set. Then there is a tree $T' \subset \Omega$ which, together with its successor sets B'_f, satisfies D1-D3 above. In particular, for each $y, z \in B'_{()}$ there is some IP set $R \subset \mathbb{N}$ with

$$R \cup (R+y) \cup (R+z) \subset A.$$

By **G**, then, for every $y, z \in B'_{()}$ there there exists some $(a_1, \ldots, a_k) \in T$ such that $\{y, z\} \subset FS(\langle a_1, \ldots, a_k \rangle)$.

Consider now the map from $B'_{()}$ to T that sends $x \in B'_{()}$ to the unique (by **H**) $\pi(x) = (a_1, \ldots a_k) \in T$ having the property that $x = a_k + y$ for some $y \in FS_0(\langle a_1, \ldots, a_{k-1} \rangle)$. What **G** tells us is that for every $y, z \in B'_{()}$, either $\pi(y)$ is an initial segment of $\pi(z)$ or vice-versa. Since for any fixed y there can be only finitely many $z \in B'_{()}$ such that $\pi(z)$ is an initial segment of $\pi(y)$, the length of $\pi(y)$ as y ranges over the infinite set $B'_{()}$ is unbounded and there exists at least one infinite sequence $(a_1, a_2, \ldots)$ in the closure of $\pi(B'_{()})$ (topology of pointwise convergence). So $\pi(y)$ is an initial segment of $(a_1, a_2, \ldots)$ for every $y \in B'_{()}$ (otherwise we could find $z \in B'_{()}$ such that neither of $\pi(y), \pi(z)$ was an initial segment of the other). Therefore $B'_{()} \subset FS(\langle a_i \rangle_{i=1}^{\infty})$, so by **F** $d^*(B'_{()}) = 0$.

□

We close with the obvious corollary to the previous construction.

4.5.6. Theorem. There are D* subsets of F that are not AIP*.

Proof. Let A be the set constructed in the previous theorem. Then $F \setminus A$ is D* but not AIP*.

□

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