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Some Geometrical Properties of Orlicz-Lorentz Spaces and their Kothe Duals

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SOME GEOMETRICAL PROPERTIES OF ORLICZ-LORENTZ SPACES AND THEIR KÖTHE DUALS

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ABSTRACT

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Orlicz-Lorentz function and sequence spaces $\Lambda_{\varphi,w}$ and $\lambda_{\varphi,w}$, which appeared as interpolation spaces between Lorentz spaces and L^∞ via Calderón's method of interpolation, are an important class of Köthe spaces. The recent discovery of their explicit Köthe duals, namely $\mathcal{M}_{\varphi,w}$ for functions and $\mathfrak{m}_{\varphi,w}$ for sequences, has provided more opportunities to study geometrical properties of Orlicz-Lorentz spaces.

First, we present basic properties of the spaces $\mathcal{M}_{\varphi,w}$ and $\mathfrak{m}_{\varphi,w}$. To this end, the relationship between level functions and the modular $P_{\varphi,w}$ is mentioned. We provide explicit formulas for the Orlicz norm in these spaces. In addition, we characterize separability of the spaces $\mathcal{M}_{\varphi,w}$ and $\mathfrak{m}_{\varphi,w}$. Consequently, we obtain a characterization of these spaces containing isomorphic and isometric copies of c_0 and l^∞.

Second, we consider the M-ideal properties in Orlicz-Lorentz function and sequence spaces. The main results in this part are explicit formulas for the norm of bounded linear functionals on these spaces equipped with the Luxemburg norm or the Orlicz norm. Subsequently, the order-continuous subspaces are M-ideals if the spaces are equipped with the Luxemburg norm, while this is not the case with the Orlicz norm. As an application, we provide a characterization of norm-attaining functionals on Orlicz-Lorentz function and sequence spaces. We also show the existence of unique norm-preserving extensions of integral functionals for the function space. In addition, M-embeddedness of Orlicz-Lorentz spaces is studied via the geometric version of the M-ideal property described by the intersection of balls.

Third, we study the u-ideal properties, which are more general than the M-ideal properties, of Orlicz-Lorentz spaces. We provide a characterization of strict u-ideals

in biduals for Orlicz-Lorentz spaces with the existence of an isomorphic copy of l^1 through the Matuszewska-Orilcz indices.

Finally, we provide a characterization of Orlicz-Lorentz function and sequence spaces with the local diameter two property, the diameter two property, and the Radon-Nikodým property in terms of the Δ_2 condition. From our M-embeddedness result, we can obtain a sufficient condition for the strong diameter two property of Orlicz-Lorentz function and sequence spaces. In addition, we provide a sufficient condition for Orlicz-Lorentz spaces having the weak* strong diameter two property.

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CHAPTER 1

INTRODUCTION

Orlicz-Lorentz function and sequence spaces, which are interpolation spaces between Lorentz spaces and L^∞ by Calderón's method, are an important class of Köthe spaces. The basic properties of these spaces were investigated by Kamińska in [24]. Also, many geometric properties of the spaces, for instance, rotundity [50, 51, 53] and the existence of isomorphic copies of l_p [29, 30], have been studied by other researchers. In 2014, the explicit isometric duals of Orlicz-Lorentz spaces were discovered by Kamińska, Leśnik, and Raynaud [27], and this has provided more opportunities to study the geometry of Orlicz-Lorentz spaces in a more general setting.

It is well known that the isometric dual of an Orlicz space L_φ is another Orlicz space L_{φ_*}, where φ_* is the complementary function of φ. However, the isometric dual $\mathcal{M}_{\varphi_*,w}$ (resp. $\mathfrak{m}_{\varphi_*,w}$) of an Orlicz-Lorentz space $\Lambda_{\varphi,w}$ (resp. $\lambda_{\varphi,w}$) is not an Orlicz-Lorentz space because the space is defined by the different modular $P_{\varphi_*,w}$ (resp. $p_{\varphi_*,w}$) employing the Hardy-Littlewood inequalities [31].

Furthermore, the relationship between the modular $P_{\varphi,w}$ as well as the modular $p_{\varphi,w}$ and Halperin's level functions in [27] makes investigation of the space $\mathcal{M}_{\varphi,w}$ and $\mathfrak{m}_{\varphi,w}$ highly nontrivial. Recently, a paper by Foralewski, Leśnik, and Maligranda [16] provided a very useful relationship between Halperin's level functions and another version of level functions developed by Sinnamon [46, 47, 48, 49], and this relationship has allowed us to study properties of the modulars $P_{\varphi,w}$ and $p_{\varphi,w}$ in depth.

This dissertation consists of five chapters. In Chapter 1, notations that are used throughout the dissertation as well as some basic facts from the theory of Banach function lattices are given. We provide the definitions of distribution functions as well as decreasing rearrangement and their properties which are used extensively in this dissertation (e.g. Lemma 1.1 and Lemma 1.3). In Section 1.1, we introduce Orlicz-Lorentz function and sequence spaces, $\Lambda_{\varphi,w}$ and $\lambda_{\varphi,w}$. Also, their standard norms, namely the Luxemburg norm and the Orlicz norm, are provided. We show that if two Orlicz functions are equivalent, then the Orlicz-Lorentz spaces defined by these functions coincide as sets and have equivalent norms (Lemma 1.4). The Δ_2 condition plays a significant role in the theory of Orlicz spaces, so we provide three versions of the Δ_2 condition and their equivalent statements (Lemma 1.5).

The spaces $\mathcal{M}_{\varphi,w}$ and $\mathfrak{m}_{\varphi,w}$ as well as the Luxemburg norm and the Orlicz norm for these spaces are introduced in Section 1.2. The difference between the modular $P_{\varphi,w}$ and the modular $\rho_{\varphi,w}$ that defines Orlicz-Lorentz function spaces comes from the fact that the modular $P_{\varphi,w}$ is given by means of Hardy-Littlewood inequality or by means of level functions. The similar fact is true for the modular $p_{\varphi,w}$ comparing to the modular $\alpha_{\varphi,w}$ that defines Orlicz-Lorentz sequence spaces. We recall a known fact that the space $\mathcal{M}^0_{\varphi_*,w}$ and $\mathfrak{m}^0_{\varphi_*,w}$ (resp. $\mathcal{M}_{\varphi_*,w}$ and $\mathfrak{m}_{\varphi_*,w}$) are the Köthe duals of Orlicz-Lorentz spaces $\Lambda_{\varphi,w}$ and $\lambda_{\varphi,w}$ (resp. $\Lambda^0_{\varphi,w}$ and $\lambda^0_{\varphi,w}$) [27].

Since the modulars $P_{\varphi,w}$ and $p_{\varphi,w}$ have a deep connection with Halperin's level functions, we provide some facts about Halperin's level functions through another version of level functions given by Sinnamon in Section 1.3. By some basic properties proven for Sinnamon's level functions and the relationship between Halperin's level functions and Sinnamon level functions provided in [16], we show that the analogous properties are also true for Halperin's level functions (Proposition 1.16). The similar results hold for the sequence version of level functions (Proposition 1.20).

In Chapter 2, we consider basic properties of the spaces $\mathcal{M}_{\varphi,w}$ and $\mathfrak{m}_{\varphi,w}$. Section 2.1 starts with showing the equivalence of the Luxemburg norm and the Orlicz norm on the spaces $\mathcal{M}_{\varphi,w}$ and $\mathfrak{m}_{\varphi,w}$ through the theory of Banach function lattices (Theorem 2.5.(1)). To this end, we present equivalent formulas for the

Orlicz norm. In addition, we provide a characterization of the Orlicz norm for $\mathcal{M}_{\varphi,w}$ achieving the infimum (Theorem 2.5.(3)). We also show this for the space $\mathfrak{m}_{\varphi,w}$.

To provide a characterization of separability of the spaces $\mathcal{M}_{\varphi,w}$ and $\mathfrak{m}_{\varphi,w}$ in Section 2.2, we prove that $(\mathcal{M}_{\varphi,w})_a = (\mathcal{M}_{\varphi,w})_b$ by computing the fundamental function of the space $\mathcal{M}_{\varphi,w}$ (Proposition 2.8). The analogous result for the space $\mathfrak{m}_{\varphi,w}$ holds immediately from the theory of Banach function lattices. We provide the explicit description of the order-continuous subspaces $(\mathcal{M}_{\varphi,w})_a$ and $(\mathfrak{m}_{\varphi,w})_a$ (Theorem 2.10). Furthermore, a characterization of the order-continuous $\mathcal{M}_{\varphi,w}$ space is given by the appropriate Δ_2 condition (Theorem 2.13). It is also shown for the space $\mathfrak{m}_{\varphi,w}$ (Theorem 2.14). By the general theory of Banach function lattices, we show that $\mathcal{M}_{\varphi,w}$ and $\mathfrak{m}_{\varphi,w}$ are separable if and only if φ satisfies the appropriate Δ_2 condition (Corollary 2.16). As a consequence, a characterization of the space $\mathcal{M}_{\varphi,w}$ not having isomorphic and isometric copies of c_0 and l^∞ (Theorem 2.19) is given. The analogous result is true for $\mathfrak{m}_{\varphi,w}$.

Chapter 3 consists of the M-ideal properties in Orlicz-Lorentz spaces. All results are shown for Orlicz-Lorentz function and sequence spaces except for Section 3.4, which are only shown for the function spaces. The results in this chapter are similar to those in Orlicz spaces, but the techniques to approach these problems are different due to dealing with decreasing rearrangement and level functions. The norm of a singular functional on Orlicz-Lorentz spaces is computed, and it shows that the formula is the same regardless of the norms equipped on the spaces (Theorem 3.5). The norm of a bounded linear functional on Orlicz-Lorentz spaces $\Lambda_{\varphi,w}$ and $\Lambda^0_{\varphi,w}$ are computed explicitly (Theorem 3.6 and Theorem 3.10). For the case of $\Lambda^0_{\varphi,w}$, more careful analysis is needed due to dealing with level functions (Lemma 3.9). These results show that the order-continuous subspaces of Orlicz-Lorentz spaces equipped with the Luxemburg norm are M-ideals, while this is not true for the spaces equipped with the Orlicz norm. In spite of the order-continuous subspace $(\Lambda^0_{\varphi,w})_a$ not being an M-ideal, a bounded linear functional on the order-continuous subspace $(\Lambda^0_{\varphi,w})_a$ attaining its norm on the unit sphere of $(\Lambda^0_{\varphi,w})_a$ has a unique norm-preserving extension to $\Lambda^0_{\varphi,w}$ (Theorem 3.21). Also, we provide a characterization of $\Lambda^0_{\varphi,w}$ for which every bounded linear functional has a unique norm-preserving extension from the order-continuous subspace to the whole space (Theorem 3.22).

M-embeddedness of Orlicz-Lorentz spaces (Theorem 3.17 and Theorem 3.18) is also considered, which has a connection with the strong diameter two property of Orlicz-Lorentz spaces (Corollary 5.13). Another application of the M-ideal properties is characterizing norm-attaining functionals (Theorem 3.23 and Theorem 3.25), which is connected to describing smooth points on the unit ball.

The u-ideal properties, known to be more general than the M-ideal properties, of Orlicz-Lorentz spaces are investigated in Chapter 4. A particular interest lies on strict u-ideals in biduals. From the general theory, it is known that a Banach space X is a strict u-ideal in its bidual if and only if X does not have an isomorphic copy of l^1. A characterization of the order-continuous subspaces of Orlicz-Lorentz spaces not containing an isomorphic copy of l^1 has been given in terms of Matuszewska-Orlicz indices [29], so we provide a characterization of strict u-ideals in biduals for Orlicz-Lorentz spaces in terms of these indices (Corollary 4.29, Corollary 4.30, and Corollary 4.31).

Chapter 5 is about diameter two properties and the Radon-Nikodým property of Orlicz-Lorentz spaces. The geometrical interpretation of the Radon-Nikodým property states that spaces with the property have slices with arbitrarily small diameter. Banach spaces with diameter two properties cannot have such slices. This shows that diameter two properties are in the opposite spectrum of the Radon-Nikodým property. We prove that Orlicz-Lorentz function and sequence spaces have the diameter two property and the local diameter two property

(Theorem 5.7 and Theorem 5.9) if and only if φ does not satisfy the appropriate Δ_2 condition. In addition, we show that Orlicz-Lorentz function and sequence spaces have the Radon-Nikodym property if and only if φ satisfies the appropriate Δ_2 condition (Corollary 5.8 and Corollary 5.10). Also, the dual versions of diameter two properties called weak* diameter two properties are considered in relation to octahedral norms [21]. We show that if φ does not satisfy the appropriate Δ_2 condition but φ_* does, then Orlicz-Lorentz function and sequence spaces equipped with the Luxemburg norm satisfy the weak* strong diameter two property. The last result is strengthened by dropping the condition on φ_*. We directly prove that Orlicz-Lorentz function spaces equipped with the Luxemburg norm satisfy the weak* strong diameter two property if φ does not satisfy the appropriate Δ_2 condition (Theorem 5.19).

Now, we introduce our notations that will be used throughout the dissertation and provide definitions and some basic facts from the theory of Banach function lattices. For general information on Banach function lattices, we refer to [8, 34, 38, 54].

The set I stands for either $[0,\gamma)$, where $0 < \gamma \leq \infty$, or $\mathbb{N}$. The set of all Lebesgue measurable functions $f : I \to \mathbb{R}$ is denoted by $L^0 = L^0(I)$, where $I = [0,\gamma)$ equipped with the Lebesgue measure m. For the case of $I = \mathbb{N}$ equipped with the counting measure $|\cdot|$, the set $l^0 = L^0(\mathbb{N})$ contains all real-valued sequences $x = (x(i))$. A Banach space $(X, \|\cdot\|)$ over I is called a Banach function lattice if $X \subset L^0(I)$ and if for $x \in L^0(I)$ and $y \in X$ with $0 \leq x \leq y$, we have $x \in X$ and $0 \leq \|x\| \leq \|y\|$. If $I = [0,\gamma)$, we call X a Banach function space, and if $I = \mathbb{N}$, a Banach sequence space.

A Banach function lattice $(X, \|\cdot\|)$ is said to have the Fatou property provided that for every sequence $(x_n) \subset X$ with $\sup_n \|x_n\| < \infty$ and $x_n \uparrow x \in L^0$ a.e., $x \in X$ and $\|x_n\| \uparrow \|x\|$. We say that $x \in X$ is order-continuous if every sequence (x_n) with

$0 \leq x_n \leq |x|$ with $x_n \downarrow 0$ a.e. implies $\|x_n\| \downarrow 0$. Denote by X_a the set of all order-continuous elements in X. It is known that X_a is a closed subspace of X ([8], Theorem 1.3.8). We also denote by X_b the closure of the set of all simple functions with supports of finite measure. For a sequence space X, X_b contains all real-valued sequences of finite support. From the general theory of Banach function lattices, we always have $X_a \subset X_b \subset X$ ([8], Theorem 1.3.11).

The Köthe dual space of a Banach function lattice X, denoted by X', is a subset of $L^0(I)$ such that every $y \in X'$ satisfies $\|y\|_{X'} = \sup\{\int_I xy : \|x\|_X \leq 1\} < \infty$. It is well known that X' with the norm $\|\cdot\|_{X'}$ is also a Banach function lattice. The space X has the Fatou property if and only if $X = X''$ ([54], pg 470, Theorem 1). We denote by X^* the set of all bounded linear functionals on X.

A linear functional $H \in X^*$ is said to be regular if there is an element $h \in X'$ which gives an integral representation $H(x) = \int_I hx$ for all $x \in X$. We denote by X_r^* the collection of all regular functionals on X. If the space X has the Fatou property and $X_a = X_b$, then $(X_a)^*$ is isometrically isomorphic to X' ([8], Theorem 2.5.5). In this case $X^* = (X_a)^* \oplus X_a^{\perp}$ is isometrically isomorphic to $X' \oplus X_s^*$, where $X_s^* = X_a^{\perp}$ is the set of singular functionals which coincides with the set of $S \in X^*$ such that $S(x) = 0$ for every $x \in X_a$. Consequently, any $F \in X^*$ has a unique decomposition $F = H + S$, where $H \in X_r^*$ and $S \in X_s^* = X_a^{\perp}$ ([54], pg 467, Theorem 2).

For convenience, we write $\{|x| > \lambda\} = \{t \in I : |x(t)| > \lambda\}$, where $\lambda > 0$. The distribution function of $x \in X$, denoted by d_x, is defined by $d_x(\lambda) = \mu\{|x| > \lambda\}$, where $\mu = m$ is the Lebesgue measure on $I = [0, \gamma)$, $0 < \gamma \leq \infty$, and the counting measure on $I = \mathbb{N}$ and $\lambda > 0$. The decreasing rearrangement of a function $f \in X$ on $I = [0, \gamma)$, $\gamma \leq \infty$ is denoted by f^* and defined as $f^*(t) = \inf\{\lambda > 0 : d_f(\lambda) \leq t\}$ where $t \in I$. For a sequence $x = (x(i))$, we define $x^* = (x^*(i))$, where $x^*(i) = \inf\{\lambda > 0 : d_x(\lambda) < i\}$ for $i \in \mathbb{N}$. For $x, y \in L^0$, x and y are equimeasurable if $d_x(\lambda) = d_y(\lambda)$ for every $\lambda > 0$, and we simply denote this by $x \sim y$. Note that x

and x^* are equimeasurable. It is well known that for a measurable subset $E \subset I$ with $\mu E = t$, $\int_E h \leq \int_0^t h^*$ ([8], Lemma 2.2.1). When $\int_0^t f^* \leq \int_0^t g^*$ for every $t \in I$, we say that f is submajorized by g in the sense of Hardy-Littlewood and denote this by $f \prec g$.

A well known result by Hardy will be used extensively in this dissertation.

Lemma 1.1. (Hardy's Lemma) [8, Proposition 3.6] Let f_1 and f_2 be non-negative Lebesgue measurable functions on $[0, \gamma)$, $0 < \gamma \leq \infty$, and suppose $\int_0^t f_1(s)ds \leq \int_0^t f_2(s)ds$ for all $t \in [0, \gamma)$. Let g be any non-negative decreasing function on $[0, \gamma)$. Then $\int_0^\gamma f_1(s)g(s)ds \leq \int_0^\gamma f_2(s)g(s)ds$. The analogous result for sequences also holds.

Also, we have the following lemmas which will be useful.

Lemma 1.2. [34, pg 67] Let $f \in L^0$ such that $d_{|f|}(\lambda) < \infty$ for all $\lambda > 0$ and (f_n) be a sequence of measurable functions such that $|f_n| \leq |f|$ a.e. for every $n \in \mathbb{N}$. If $f_n \to f$ a.e. then $(f - f_n)^* \to 0$ a.e. The same result remains true for sequences.

Lemma 1.3. [35, Lemma 2.5] For any $f \in L^0$, a decreasing function $0 \leq g$ on I and a measurable set $A \subset I$, we have

$$\int_0^\gamma (f\chi_A)^* g \leq \int_0^{mA} f^* g.$$

Proof. Since $\int_0^t f^* = \sup_{m(E)=t} \int_E |f|$ for any $f \in L^0$ ([34], pg 64), for $t \in [0, \gamma)$, we see that $\int_0^t (f\chi_A)^* = \sup_{m(E)=t} \int_E |f\chi_A| = \sup_{m(E)=t} \int_{A\cap E} |f|$. Also, by the fact that $\int_E f \leq \int_0^{mE} f^*$, we get

$$\sup_{m(E)=t} \int_{A\cap E} |f| \leq \sup_{m(E)=t} \int_0^{m(A\cap E)} f^* = \sup_{m(E)=t} \int_0^\gamma f^* \chi_{[0, m(A\cap E))}.$$

Moreover,

$$\sup_{mE=t}\int_0^\gamma f^*\chi_{[0,m(A\cap E))} \le \sup_{mE=t}\int_0^\gamma x^*\chi_{[0,\min\{mA,mE\})} = \sup_{mE=t}\int_0^\gamma x^*\chi_{[0,mA)}\chi_{[0,mE)}.$$

Since $\sup_{mE=t}\int_0^\gamma f^*\chi_{[0,mA)}\chi_{[0,mE)} = \sup_{mE=t}\int_0^{mE} f^*\chi_{[0,mA)} = \int_0^t f^*\chi_{[0,mA)}$, we have $\int_0^t (f\chi_A)^* \le \int_0^t f^*\chi_{[0,mA)}$. Hence, by Lemma 1.1, $\int_0^\gamma (f\chi_A)^*g \le \int_0^\gamma f^*\chi_{[0,mA)}g$. $\square$

We say that a Banach function lattice $(X, \|\cdot\|)$ is rearrangement invariant if $x \in X$ and $y \in L^0$ with $x \sim y$, we have $y \in X$ and $\|x\| = \|y\|$. The fundamental function of a rearrangement invariant Banach function space X, denoted by ϕ_X, is defined as $\phi_X(t) = \|\chi_{[0,t)}\|_X$ for $t \in I$.

1.1 Orlicz-Lorentz spaces $\Lambda_{\varphi,w}$ and $\lambda_{\varphi,w}$

In this section, we introduce Orlicz-Lorentz function and sequence spaces. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be an Orlicz function if φ is convex, $\varphi(0) = 0$ and $\varphi(t) > 0$ for $t > 0$. Furthermore, we say that φ is an Orlicz N-function if $\lim_{t\to 0^+} \frac{\varphi(t)}{t} = 0$ and $\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty$. The complementary function of φ is denoted by φ_* and defined as $\varphi_*(s) = \sup\{st - \varphi(t) : t \ge 0\}$. It is well known that φ is an Orlicz N-function if and only if φ_* is an Orlicz N-function. The right derivatives of φ and φ_* are denoted by p and q, respectively. The right derivatives p and q are non-negative, right-continuous, increasing functions on $\mathbb{R}_+$ [11, 33]. If φ and φ_* are Orlicz N-functions, $p(0) = \lim_{t\to 0+} p(t) = 0$ and $q(0) = \lim_{t\to 0+} q(t) = 0$ and $\lim_{t\to\infty} p(t) = \lim_{t\to\infty} q(t) = \infty$. For φ and φ_*, Young's inequality holds, that is, $ts \le \varphi(t) + \varphi_*(s)$ for every $t, s \in \mathbb{R}^+$. The equality is achieved when $t = q(s)$ or $s = p(v)$ [11, 45].

A weight function $w : I = [0,\gamma) \to (0,\gamma)$, $\gamma \le \infty$, is a decreasing, locally integrable function. In this dissertation, $W(a,b) = \int_a^b w$. It $a = 0$, we simply denote $W(0,t) = W(t)$. Notice that $W(t) < \infty$ for every $t \in I$. We always assume

$W(\infty) = \infty$ throughout the whole dissertation.

To define an Orlicz-Lorentz space on $[0, \gamma)$, we use the modular

$$\rho_{\varphi,w}(f) = \int_0^\gamma \varphi(f^*(t))w(t)dt = \int_I \varphi(f^*)w,$$

for $f \in L^0$. By the convexity of φ, we see that the modular $\rho_{\varphi,w}$ is convex. Also, for $f, g \in L^0$, if $|f| \wedge |g| = 0$, we have $\rho_{\varphi,w}(f + g) \leq \rho_{\varphi,w}(f) + \rho_{\varphi,w}(g)$ [24]. In this case, we say that the modular $\rho_{\varphi,w}$ is orthogonally subadditive. An Orlicz-Lorentz function space $\Lambda_{\varphi,w}$ is the set of all measurable functions $f \in L^0$ such that $\rho_{\varphi,w}(\lambda f) < \infty$ for some $\lambda > 0$. We introduce two norms on this space. The Luxemburg norm is defined by

$$\|f\| = \|f\|_{\Lambda_{\varphi,w}} = \inf \left\{\epsilon > 0 : \rho_{\varphi,w}(f/\epsilon) \leq 1\right\},$$

and the Orlicz norm is defined by

$$\|f\|^0 = \|f\|^0_{\Lambda_{\varphi,w}} = \sup \left\{ \int_I f^* g^* w : \rho_{\varphi_*,w}(g) \leq 1 \right\}.$$

This norm can be also computed by [50, 53]

$$\|f\|^0 = \inf_{k>0} \frac{1}{k}(1 + \rho_{\varphi,w}(kf)).$$

It is well known that $\|f\| \leq \|f\|^0 \leq 2\|f\|$ [50, 53]. In this dissertation, $\Lambda_{\varphi,w}$ stands for the Orlicz-Lorentz spaces equipped with the Luxemburg norm and $\Lambda^0_{\varphi,w}$ for the Orlicz-Lorentz spaces equipped with the Orlicz norm. Orlicz-Lorentz function spaces are rearrangement invariant Banach function lattices. Also, we have $(\Lambda_{\varphi,w})_a = (\Lambda_{\varphi,w})_b = \{f \in \Lambda_{\varphi,w} : \rho_{\varphi,w}(kf) < \infty \ \text{ for all } \ k > 0\}$ [24].

Now, we introduce the sequence analogue of Orlicz-Lorentz spaces. Let

$w = (w(i))$ be a positive, decreasing real sequence. Denote by $W(n) = \sum_{i=1}^{n} w(i)$ and assume $W(\infty) = \infty$. The modular $\alpha_{\varphi,w}$ is defined by

$$\alpha_{\varphi,w}(x) = \sum_{i=1}^{\infty} \varphi(x^*(i))w(i).$$

An Orlicz-Lorentz sequence space, denoted by $\lambda_{\varphi,w}$, consists of all real sequences $x \in l^0$ satisfying $\alpha_{\varphi,w}(\eta x) < \infty$ for some $\eta > 0$. The Luxemburg norm and the Orlicz norm on the sequence spaces are defined in a similar way to the function spaces by substituting $\rho_{\varphi,w}$ with $\alpha_{\varphi,w}$. We denote by $\lambda_{\varphi,w}$ the Orlicz-Lorentz sequence space with the Luxemburg norm and $\lambda^0_{\varphi,w}$ the sequence space with the Orlicz norm. It is known that $(\lambda_{\varphi,w})_a = (\lambda_{\varphi,w})_b = \{x \in \lambda_{\varphi,w} : \alpha_{\varphi,w}(\eta x) < \infty \text{ for all } \eta > 0\}$ [30].

We say an Orlicz function ψ is equivalent to φ over all $\mathbb{R}_+$ (resp. near ∞; near zero), if there exists $c \geq 1$ (resp. $c \geq 1$ and $t_0 \geq 0$; $c \geq 1$ and $t_0 > 0$ with $\psi(t_0) > 0$) such that for every $t > 0$ (resp. $t \geq t_0$; $0 < t \leq t_0$), $\varphi(t/c) \leq \psi(t) \leq \varphi(ct)$, and we denote it by $\varphi \sim_a \psi$ (resp. $\varphi \sim_\infty \psi$; $\varphi \sim_0 \psi$)

Lemma 1.4. If $\varphi \sim_i \psi$, $i = a, \infty$, then $\Lambda_{\varphi,w} = \Lambda_{\psi,w}$ as sets and the norms $\|\cdot\|_{\Lambda_{\varphi,w}}$ and $\|\cdot\|_{\Lambda_{\psi,w}}$ are equivalent. Similarly, if $\varphi \sim_0 \psi$, then $\lambda_{\varphi,w} = \lambda_{\psi,w}$ as sets and the norms $\|\cdot\|_{\lambda_{\varphi,w}}$ and $\|\cdot\|_{\lambda_{\psi,w}}$ are equivalent.

Proof. Since the proof for the case $i = a$ is straightforward and proving for $i = 0$ is similar to the case $i = \infty$, we only show this for $i = \infty$. In this case, $\gamma < \infty$.

Suppose that $\varphi \sim_\infty \psi$. Then, there exist $c \geq 1$ and $t_0 \geq 0$ such that $\varphi(t/c) \leq \psi(t) \leq \varphi(ct)$ for every $t \geq t_0$. If $f \in \Lambda_{\varphi,w}$, $\rho_{\varphi,w}(\lambda f) < \infty$ for some $\lambda > 0$. Consider the set $\{f^* > t_0/\lambda\} = [0, s]$, $s \in I$. We see that $\rho_{\psi,w}\left(\frac{\lambda}{c} f\right) = \int_0^s \psi\left(\frac{\lambda f^*}{c}\right) w + \int_s^\gamma \psi\left(\frac{\lambda f^*}{c}\right) w$. Since $\lambda f^* > t_0$ on $[0, s]$ and $\lambda f^* \leq t_0$ on

(s,γ), we have

$$\begin{aligned}\int_0^s \psi\left(\frac{\lambda f^*}{c}\right) w + \int_s^\gamma \psi\left(\frac{\lambda f^*}{c}\right) w &\leq \int_0^s \varphi(\lambda f^*)w + \psi\left(\frac{t_0}{c}\right)(W(\gamma)-W(s)) \\ &\leq \rho_{\varphi,w}(\lambda f) + \psi\left(\frac{t_0}{c}\right)(W(\gamma)-W(s)).\end{aligned}$$

Since $W(\gamma)<\infty$, $\psi\left(\frac{t_0}{c}\right)(W(\gamma)-W(s))<\infty$. Hence, we obtain

$$\rho_{\psi,w}\left(\frac{\lambda}{c}f\right) \leq \rho_{\varphi,w}(\lambda f) + \psi\left(\frac{t_0}{c}\right)(W(\gamma)-W(s)) < \infty,$$

and this shows that $f\in\Lambda_{\psi,w}$, in other words, $\Lambda_{\varphi,w}\subset\Lambda_{\psi,w}$. Proving the opposite inclusion uses the similar argument by replacing φ with ψ and vice versa. Thus, we have $\Lambda_{\varphi,w}=\Lambda_{\psi,w}$.

Now, we show the equivalence of the norms. Let $f\in\Lambda_{\varphi,w}$. Without loss of generality, $\|f\|_{\Lambda_{\varphi,w}}=1$. Then, $\rho_{\varphi,w}(f)\leq 1$. Consider the set $\{f^*>t_0\}=[0,s]$, where $s\in I$. Since $f^*>t_0$ on $[0,s]$ and $f^*\leq t_0$ on (s,γ),

$$\rho_{\psi,w}\left(\frac{f}{c}\right) = \int_0^s \psi\left(\frac{f^*}{c}\right) w + \int_s^\gamma \psi\left(\frac{f^*}{c}\right) w \leq \int_0^s \varphi(f^*)w + \psi\left(\frac{t_0}{c}\right)(W(\gamma)-W(s)).$$

From the fact that $\varphi\sim_\infty\psi$, $c\geq 1$ and $\rho_{\varphi,w}(f)\leq 1$,

$$\int_0^s \varphi(f^*)w + \psi\left(\frac{t_0}{c}\right)(W(\gamma)-W(s)) \leq 1 + \psi\left(\frac{t_0}{c}\right)(W(\gamma)-W(s)),$$

so $\rho_{\psi,w}\left(\frac{f}{c}\right)\leq 1+\psi\left(\frac{t_0}{c}\right)(W(\gamma)-W(s))$. Let $M_1=1+\psi\left(\frac{t_0}{c}\right)(W(\gamma)-W(s))\geq 1$. By dividing both side by M_1 and the convexity of ψ, we get $\rho_{\psi,w}\left(\frac{f}{cM_1}\right)\leq 1$, which implies that $\|f\|_{\Lambda_{\psi,w}}\leq cM_1=cM_1\|f\|_{\Lambda_{\varphi,w}}$. By replacing φ with ψ and vice versa, we can show that there exists $M_2>0$ such that $M_2\|f\|_{\Lambda_{\varphi,w}}\leq\|f\|_{\Lambda_{\psi,w}}$ by the similar argument. Therefore, the norms $\|\cdot\|_{\Lambda_{\varphi,w}}$ and $\|\cdot\|_{\Lambda_{\psi,w}}$ are equivalent. □

In the theory of Orlicz spaces, a growth condition on an Orlicz function φ plays an important role. An Orlicz function φ is said to satisfy the Δ_2 (resp. Δ_2^∞; Δ_2^0) condition if there exists $K > 0$ (resp. $K > 0$ and $u_0 \geq 0$; $K > 0$ and $u_0 > 0$) such that $\varphi(2u) \leq K\varphi(u)$ for all $u \geq 0$ (resp. $u \geq u_0$; $0 \leq u \leq u_0$). In this dissertation, the appropriate Δ_2 condition means Δ_2 when $\gamma = \infty$, Δ_2^∞ when $\gamma < \infty$, and Δ_2^0 when we consider sequences.

The following lemma provides equivalent descriptions of Δ_2, Δ_2^∞ and Δ_2^0 conditions which will be useful.

Lemma 1.5. [11, Theorem 1.13] An Orlicz function φ satisfies the Δ_2 (resp., Δ_2^∞; Δ_2^0) condition if and only if there exist $l > 1$ and $K > 1$ (resp., $l > 1, K > 1$, $u_0 \geq 0$; $l > 1, K > 1, u_0 > 0$) such that $\varphi(lu) \leq K\varphi(u)$ for all $u \geq 0$ (resp., $u \geq u_0$; $0 \leq u \leq u_0$).

1.2 The spaces $\mathcal{M}_{\varphi,w}$ and $\mathfrak{m}_{\varphi,w}$

Let φ be an Orlicz N-function and $w : I \to (0, \infty)$ be a weight function. In order to introduce the space $\mathcal{M}_{\varphi,w}$, for $f \in L^0$, we define the modular $P_{\varphi,w}(f)$ by

$$P_{\varphi,w}(f) = \inf\left\{\int_I \varphi\left(\frac{|f|}{v}\right) v : v \prec w, v \geq 0\right\} = \inf\left\{\int_I \varphi\left(\frac{f^*}{v}\right) v : v \prec w, v \downarrow\right\}.$$

This modular appeared first in a paper by Kamińska and Raynaud in 2014 [31]. In the same paper, they provided the second inequality. Similar to the modular $\rho_{\varphi,w}$, the modular $P_{\varphi,w}$ is also convex [31]. The space $\mathcal{M}_{\varphi,w}$ is defined by

$$\mathcal{M}_{\varphi,w} = \{f \in L^0 : P_{\varphi,w}(\lambda f) < \infty \text{ for some } \lambda > 0\}.$$

The difference between the modular $P_{\varphi,w}$ and the modular $\rho_{\varphi,w}$ that is orthogonally subadditive [24] comes from the fact that the modular $P_{\varphi,w}$ is orthogonally superadditive, that is, for $f, g \in L^0$ with $|f| \wedge |g| = 0$,

$P_{\varphi,w}(f+g) \geq P_{\varphi,w}(f) + P_{\varphi,w}(g)$ [31]. In this respect, we cannot use the same techniques from Orlicz-Lorentz spaces to analyze the spaces $\mathcal{M}_{\varphi,w}$ and $\mathfrak{m}_{\varphi,w}$.

The Luxemburg norm for $\mathcal{M}_{\varphi,w}$ is given by

$$\|f\|_{\mathcal{M}_{\varphi,w}} = \inf\left\{\epsilon > 0 : P_{\varphi,w}\left(\frac{f}{\epsilon}\right) \leq 1\right\},$$

and the Orlicz norm for $\mathcal{M}_{\varphi,w}$ is given by

$$\|f\|^0_{\mathcal{M}_{\varphi,w}} = \inf_{k>0}\left(\frac{1}{k}(1 + P_{\varphi,w}(kf))\right).$$

It is shown in [31] that the space $\mathcal{M}_{\varphi,w}$ is a rearrangement invariant Banach function lattice satisfying the Fatou property. The spaces $\mathcal{M}_{\varphi,w}$ and $\mathcal{M}^0_{\varphi,w}$ represent the space $\mathcal{M}_{\varphi,w}$ equipped with the Luxemburg norm and the Orlicz norm, respectively.

Also, the sequence analogue of the modular $P_{\varphi,w}$ is denoted by $p_{\varphi,w}$, and it is given by

$$p_{\varphi,w}(x) = \inf\left\{\sum_{i=1}^{\infty}\varphi\left(\frac{|f(i)|}{v(i)}\right)v(i) : v \prec w, v \geq 0\right\} = \inf\left\{\sum_{i=1}^{\infty}\varphi\left(\frac{f^*(i)}{v(i)}\right)v(i) : v \prec w, v\downarrow\right\},$$

and the sequence space $\mathfrak{m}_{\varphi,w}$ is defined by

$$\mathfrak{m}_{\varphi,w} = \{f \in l^0 : p_{\varphi,w}(\eta x) < \infty \text{ for some } \eta > 0\}.$$

Similar to the function spaces, $\mathfrak{m}_{\varphi,w}$ and $\mathfrak{m}^0_{\varphi,w}$ represent the space $\mathfrak{m}_{\varphi,w}$ equipped with the Luxemburg norm and the Orlicz norm defined by the modular $p_{\varphi,w}$, respectively. The following two theorems show that, in fact, the space $\mathcal{M}^0_{\varphi_*,w}$ and $\mathfrak{m}^0_{\varphi_*,w}$(resp. $\mathcal{M}_{\varphi_*,w}$ and $\mathfrak{m}_{\varphi_*,w}$) are the Köthe dual of $\Lambda_{\varphi,w}$ and $\lambda_{\varphi,w}$ (resp. $\Lambda^0_{\varphi,w}$ and $\lambda^0_{\varphi,w}$) .

Theorem 1.6. [27, Theorem 2.2] Let w be a positive decreasing weight and φ be an Orlicz N-function. Then the Köthe dual space to an Orlicz-Lorentz function space $\Lambda_{\varphi,w}$ (resp. $\Lambda^0_{\varphi,w}$) is expressed as

$$(\Lambda_{\varphi,w})' = \mathcal{M}^0_{\varphi_*,w} \quad (\text{resp. } (\Lambda^0_{\varphi,w})' = \mathcal{M}_{\varphi_*,w})$$

with equality of norms. Moreover any $F \in (\Lambda_{\varphi,w})^*$ (resp. $F \in (\Lambda^0_{\varphi,w})^*$) is uniquely represented as $F = H + S$, where H is a regular functional such that for some $h \in \mathcal{M}^0_{\varphi_*,w}$ (resp. $h \in \mathcal{M}_{\varphi_*,w}$) we have

$$H(x) = \int_I fh, \qquad f \in \Lambda_{\varphi,w} \quad (\text{resp. } f \in \Lambda^0_{\varphi,w}),$$

with $\|H\| = \|h\|^0_{\mathcal{M}_{\varphi_*,w}}$ (resp. $\|H\| = \|h\|_{\mathcal{M}_{\varphi_*,w}}$), and S is a singular functional such that

$$S(f) = 0 \quad \text{for all} \qquad f \in (\Lambda_{\varphi,w})_a \quad (\text{resp. } f \in (\Lambda^0_{\varphi,w})_a).$$

Theorem 1.7. [27, Theorem 5.2] Let w be a positive decreasing weight and φ be an Orlicz N-function. Then the Köthe dual space to an Orlicz-Lorentz sequence space $\lambda_{\varphi,w}$ (resp. $\lambda^0_{\varphi,w}$) is expressed as

$$(\lambda_{\varphi,w})' = \mathfrak{m}^0_{\varphi_*,w} \quad (\text{resp. } (\lambda^0_{\varphi,w})' = \mathfrak{m}_{\varphi_*,w})$$

with equality of norms. Moreover any $F \in (\lambda_{\varphi,w})^*$ (resp. $F \in (\lambda^0_{\varphi,w})^*$) is uniquely represented as $F = H + S$, where H is a regular functional such that for some $h \in \mathfrak{m}^0_{\varphi_*,w}$ (resp. $h \in \mathfrak{m}_{\varphi_*,w}$) we have

$$H(x) = \sum_{i=1}^{\infty} x(i)h(i), \qquad x \in \lambda_{\varphi,w} \quad (\text{resp. } x \in \lambda^0_{\varphi,w}),$$

with $\|H\| = \|h\|^0_{\mathfrak{m}_{\varphi_*,w}}$ (resp. $\|H\| = \|h\|_{\mathfrak{m}_{\varphi_*,w}}$), and S is a singular functional such that

$$S(x) = 0 \quad \text{for all} \qquad x \in (\lambda_{\varphi,w})_a \quad (\text{resp. } x \in (\lambda^0_{\varphi,w})_a).$$

1.3 Level functions

In this section, we provide some facts on level functions which have a deep connection with the modular $P_{\varphi,w}$. The first appearance of level functions is from a paper by Halperin in 1953 to compute the dual norm of Lorentz spaces [22]. Also, level functions were defined independently by Lorentz and Sinnamon [40, 46, 47], and Lorentz also computed the dual norm of Lorentz spaces [40]. In this dissertation, we mean level functions by Halperin's level functions. We denote by $F(a,b) = \int_a^b f$ the integral of a function f over an interval (a,b). In particular, if $a = 0$, we write $F(b) = F(0,b) = \int_0^b f$.

Definition 1.8. For $f \in L^0$, a weight function w, and $0 \leq a < b < \infty$, an interval (a,b) is a level interval of f with respect to w if

$$\frac{F(a,t)}{W(a,t)} \leq \frac{F(a,b)}{W(a,b)} \quad \text{for all} \quad a < t < b.$$

When $b = \infty$, an interval (a,∞) is a level interval of f with respect to w if

$$\frac{F(a,t)}{W(a,t)} \leq \frac{F(a,\infty)}{W(a,\infty)} = \limsup_{t\to\infty} \frac{F(a,t)}{W(a,t)} \quad \text{for all} \quad a < t < \infty.$$

A level interval (a,b) is said to be maximal if (a,b) is not contained in any other level intervals.

There are countably many maximal level intervals of f, and each maximal level intervals are disjoint [22, 27].

Definition 1.9. The level function of a non-negative function f with respect to a weight function w, denoted by f^0, is defined by

$$f^0(t) = \begin{cases} \frac{F(a_j,b_j)}{W(a_j,b_j)} w(t), & t \in (a_j, b_j) \text{ for some } j, \\ f(t), & t \notin \cup_j (a_j, b_j), \end{cases}$$

where $\{(a_j, b_j)\}_j$ is the set of maximal level intervals of f with respect to w.

The following fact will be useful later.

Theorem 1.10. [22, Theorem 3.6] Let w be a positive, locally integrable, decreasing function on $\mathbb{R}_+$. A function h is a level function with respect to w if and only if the function $\frac{h}{w}$ is a decreasing, finite, non-negative function on $\mathbb{R}_+$.

The formula for the modular $P_{\varphi,w}$ by using level functions is given as the following.

Theorem 1.11. [27, Theorem 4.7] Let φ be an Orlicz N-function and w be a positive decreasing weight function. Then for any $f \in \mathcal{M}_{\varphi,w}$,
$P_{\varphi,w}(f) = \int_I \varphi\left(\frac{(f^*)^0}{w}\right) w$.

We also provide a technical result on convergence of level functions.

Theorem 1.12. [16, Lemma 6.5] Let φ be an Orlicz function. If $0 \le f_n \le f = f^* \in \mathcal{M}_{\varphi,w}$ such that $f_n = f_n^*$ for every $n \in \mathbb{N}$, $m(\operatorname{supp} f_n) < \infty$, and $f_n \to 0$ a.e., then $f_n^0 \to 0$. a.e. The same result holds for the sequence case.

Halperin proved that $f \prec g$ implies $f^0 \prec g^0$ in his paper ([22], Theorem 3.7 (i)). In fact, the stronger result can be proven, namely $f \le g$ a.e. implies $f^0 \le g^0$ a.e. This result was shown in more general setting by Sinnamon [48, 49], and the proof turned out to be nontrivial. A recent paper by Foralewski, Leśnik, and Maligranda [16] provides a relationship between Halperin's level functions and Sinnamon's level functions. We note that Sinnamon's level functions were originally investigated with respect to a general Borel measure μ over $\mathbb{R}$. For our purpose, we consider the measure μ such that $d\mu = wdm$ over $\mathbb{R}_+$, where w is a positive, decreasing weight function with $W(\infty) = \int_0^\infty w = \infty$. To introduce Sinnamon's level functions in this setting, we start with the concept of w-concave functions.

Definition 1.13. Let w be a decreasing, positive, locally integrable function on $\mathbb{R}_+$ with $W(\infty) = \infty$. A function F is w-affine if $F(t) = cW(t) + d$, where $c, d \in \mathbb{R}_+$ and $W(t) = \int_0^t w$. A function G is said to be w-concave if there is a w-affine function F such that for every $a, b \in \mathbb{R}_+$ with $F(a) = G(a)$ and $F(b) = G(b)$, we have
$F(t) \leq G(t)$ for all $a \leq t \leq b$.

Definition 1.14. Let w be a decreasing, positive, locally integrable function on $\mathbb{R}_+$ with $W(\infty) = \infty$ and f be a non-negative, locally integrable function defined on $\mathbb{R}_+$ such that the least w-concave majorant of F, denoted by F^b, exists. The Sinnamon's level function of f, denoted by f^s, is defined by $F^b(t) = \int_0^t f^s w$, where $t \in \mathbb{R}_+$.

The relationship between f^0 and f^s is given by ([16], pg 64)

$$\frac{f^0}{w} = \left(\frac{f}{w}\right)^s. \tag{1.1}$$

We provide some properties of Sinnamon's level funtions. For the proofs of these properties, we refer to [46, 47].

Theorem 1.15. [48, 49] Let μ be a regular Borel measure on $\mathbb{R}$. For each non-negative function f, there is a corresponding decreasing function f^s such that if f, g are non-negative functions with $f \leq g$ μ-a.e., we have $f^s \leq g^s$ μ-a.e., and if (f_n) is a sequence of non-negative functions such that $f_n \uparrow f$ μ-a.e., we have $f_n^s \uparrow f^s$ μ-a.e.

Based on this result and (1.1), the analogous result holds for Halperin's level functions. Notice that $\mu A = \int_A w = 0$ if and only if $mA = 0$ since the function w is positive. Hence in our case, $f \leq g$ μ-a.e. if and only if $f \leq g$ a.e. and $f_n \uparrow f$ μ-a.e. if and only if $f_n \uparrow f$ a.e.

Proposition 1.16. Let f, g be non-negative, locally integrable functions on $\mathbb{R}_+$ such that $f \leq g$ a.e. Then, $f^0 \leq g^0$ a.e. Moreover, if (f_n) is a sequence of non-negative locally integrable functions on $\mathbb{R}_+$ with $f_n \uparrow f$ a.e., then, $f_n^0 \uparrow f^0$ a.e.

Proof. In our case, $d\mu = wdm$. Let f, g be non-negative, locally integrable functions on $\mathbb{R}_+$ such that $f \leq g$ a.e. Then, $\frac{f}{w} \leq \frac{g}{w}$ a.e. By Theorem 1.15 and the remark above $\left(\frac{f}{w}\right)^s \leq \left(\frac{g}{w}\right)^s$ a.e. From (1.1), we have $\frac{f^0}{w} \leq \frac{g^0}{w}$ a.e. and so $f^0 \leq g^0$ a.e. To show the second statement, let (f_n) be a sequence of non-negative locally integrable functions on $\mathbb{R}_+$ with $f_n \uparrow f$ a.e. Then, $\frac{f_n}{w} \uparrow \frac{f}{w}$ a.e. By Theorem 1.15, we obtain $\left(\frac{f_n}{w}\right)^s \uparrow \left(\frac{f}{w}\right)^s$ a.e. and so $f_n^0 \uparrow f^0$ a.e. □

Now, we define level sequences for $(x(i)) = x \in l^0$ with respect to a decreasing positive weight sequence $w = (w(i))$. Assume $W(\infty) = \sum_{i=1}^{\infty} w(i) = \infty$. We will use the isometric embedding technique given in [27]. Let $\bar{x} = \sum_{i=1}^{\infty} x(i)\chi_{[i-1,i)}$ and $\bar{w} = \sum_{i=1}^{\infty} w(i)\chi_{[i-1,i)}$. Define an operator $U : x = (x(i)) \mapsto \bar{x}$ such that $U(x) = \bar{x}$. This operator is a linear isometry on $\lambda_{\varphi,w}$, which image is a closed subspace of $\Lambda_{\varphi,\bar{w}}$. Indeed, for $x, y \in \lambda_{\varphi,w}$ and $a, b \in \mathbb{R}$, we have
$U(ax + by) = \sum_{i=1}^{\infty}(ax(i) + by(i))\chi_{[i-1,i)} = aU(x) + bU(y)$. Furthermore,

$$\rho_{\varphi,\bar{w}}(\bar{x}) = \int_0^{\infty} \varphi\left(\sum_{i=1}^{\infty} x^*(i)\chi_{[i-1,i)}\right) \sum_{i=1}^{\infty} w(i)\chi_{[i-1,i)} = \sum_{i=1}^{\infty} \varphi(x^*(i))w(i) = \alpha_{\varphi,w}(x),$$

Hence,

$$\|x\|_{\lambda_{\varphi,w}} = \inf\{\eta > 0 : \alpha_{\varphi,w}(\eta x) \leq 1\} = \inf\{\eta > 0 : \rho_{\varphi,\bar{w}}(\eta \bar{x}) \leq 1\} = \|\bar{x}\|_{\Lambda_{\varphi,\bar{w}}}.$$

The similar result holds for the Köthe dual spaces of Orlicz-Lorentz spaces.

Theorem 1.17. [27, Lemma 5.1] Let $x = (x(i)) \in \mathfrak{m}_{\varphi,w}$. Then we have $\|x\|_{\mathfrak{m}^0_{\varphi,w}} = \|\bar{x}\|_{\mathcal{M}^0_{\varphi,\bar{w}}}$ and $\|x\|_{\mathfrak{m}_{\varphi,w}} = \|\bar{x}\|_{\mathcal{M}_{\varphi,\bar{w}}}$. Consequently, the mapping U is a linear isometry from $\mathfrak{m}_{\varphi,w}$ to $\mathcal{M}_{\varphi,\bar{w}}$.

Let $\bar{h} = \sum_{i=1}^{\infty} h(i)\chi_{[i-1,i)}$ and $\bar{w} = \sum_{i=1}^{\infty} w(i)\chi_{[i-1,i)}$. The level function of $\bar{h}$ with respect to $\bar{w}$ is

$$\bar{h}^0(t) = \begin{cases} (\bar{H}(a_j, b_j)/W(a_j, b_j))w(t), & t \in (a_j, b_j) \text{ for some } j, \\ \bar{h}(t), & t \notin \cup_j (a_j, b_j), \end{cases}$$

where (a_j, b_j) is a maximal level interval. If $t \in [i-1, i) \not\subset \cup_j (a_j, b_j)$ for some i, we see that $\bar{h}^0(t) = \bar{h}(i)$ for all $t \in [i-1, i)$. Observe that if $t \in (a_j, a_j+1) \subset (a_j, b_j)$, then $\bar{w}(t) = w(a_j+1)$ and the value of $\bar{h}^0(t)$ is

$$\bar{h}^0(t) = \frac{\int_{a_j}^{b_j} \sum_{k=1}^{\infty} h(k)\chi_{[k-1,k)}}{\int_{a_j}^{b_j} \sum_{k=1}^{\infty} w(k)\chi_{[k-1,k)}} \bar{w}(t) = \frac{h(a_j+1) + h(a_j+2) + \cdots + h(b_j)}{w(a_j+1) + w(a_j+2) + \cdots + w(b_j)} w(a_j+1).$$

Hence for any $t \in (a_j, b_j)$,

$$\bar{h}^0(t) = \sum_{i=a_j+1}^{b_j} \left(\frac{\sum_{k=a_j+1}^{b_j} h(k)}{\sum_{k=a_j+1}^{b_j} w(k)} \right) w(i)\chi_{[i-1,i)}(t). \tag{1.2}$$

Based on our observation, we define a level interval for sequences and a level sequence.

Definition 1.18. For $0 \le a < b < \infty$ and a sequence of real numbers $h = (h(i))$, an interval $(a, b] = \{a+1, a+2, ..., b\}$ is said to be a level interval of $h = (h(i))$ with respect to $w = (w(i))$ if

$$\frac{\sum_{i=a+1}^{n} h(i)}{\sum_{i=a+1}^{n} w(i)} = \frac{H(a,n)}{W(a,n)} \le \frac{H(a,b)}{W(a,b)} \quad \text{for all} \quad a+1 \le n \le b.$$

Similar to the function case, when $n = \infty$, an interval $(a, \infty) = \{a+1, a+2, \cdots\}$ is said to be a level interval with respect to w if

$$\frac{H(a,n)}{W(a,n)} \le \frac{H(a,\infty)}{W(a,\infty)} = \limsup_{n\to\infty} \frac{H(a,n)}{W(a,n)} \quad \text{for all} \quad a+1 \le n < \infty.$$

A level interval $(a, b]$ with respect to w is said to be maximal if there is no level interval containing $(a, b]$.

Definition 1.19. The level sequence of $h = (h(i))$ with respect to $w = (w(i))$ is defined by

$$h^0(i) = \begin{cases} \frac{H(a_j,b_j)}{W(a_j,b_j)} w(i), & i \in (a_j, b_j] \text{ for some } j, \\ h(i), & i \notin \cup_j (a_j, b_j], \end{cases}$$

where $\{(a_j, b_j]\}_j$ is the set of maximal level intervals of the sequence h with respect to w.

By comparison with (1.2), we see that $\bar{h}^0(t) = h^0(i)$ for every $t \in [i-1, i)$ for all $i \in \mathbb{N}$, and so $U(h^0) = \bar{h}^0$.

From this, we have the following consequences.

Proposition 1.20. Let x, y be real-valued sequences such that $x(i) \le y(i)$ for all $i \in \mathbb{N}$. Then, $x^0 \le y^0$. Moreover, if (x_n) is a sequence of real-valued sequences x_n such that $x_n \uparrow x$, then $x_n^0 \uparrow x^0$.

Proof. Let x, y be sequences such that $x(i) \le y(i)$ for all $i \in \mathbb{N}$. Then from the mapping U, we have $\bar{x}, \bar{y}$ corresponding to x, y such that $\bar{x} \le \bar{y}$ a.e. By Proposition 1.16, we have $\bar{x}^0 \le \bar{y}^0$ a.e. Hence $x^0(i) \le y^0(i)$ for all $i \in \mathbb{N}$, and this shows that $x^0 \le y^0$.

To show the second statement, let (x_n) be a sequence of x_n such that $x_n \uparrow x$. Then we have $\bar{x}_n \uparrow \bar{x}$ a.e. and so $\bar{x}_n^0 \uparrow \bar{x}^0$ a.e. from Proposition 1.16. Hence $x_n^0(i) \uparrow x^0(i)$ for every $i \in \mathbb{N}$, and this shows that $x_n^0 \uparrow x^0$. □

Sometimes, it is better to fix the function f^* to compute $P_{\varphi,w}(f)$ instead of the weight function w [27].

Definition 1.21. For a non-negative locally integrable function f and a positive decreasing weight function w, the inverse level function of f with respect to w, denoted by w^f, is defined by

$$w^f(t) = \begin{cases} \frac{W(a_j,b_j)}{F(a_j,b_j)} f(t), & t \in (a_j, b_j) \text{ for some } j, \\ w(t), & t \notin \cup_j (a_j, b_j), \end{cases}$$

where $\{(a_j, b_j)\}_j$ is the set of maximal level intervals of f with respect to w.

The inverse level function w^{f^*} is submajorized by w, i.e. $w^{f^*} \prec w$ (Remark 4.4, [27]). From the definition of w^{f^*}, we can show that

$$P_{\varphi,w}(f) = \int_I \varphi\left(\frac{(f^*)^0}{w}\right) w = \int_I \varphi\left(\frac{f^*}{w^{f^*}}\right) w^{f^*}. \tag{1.3}$$

CHAPTER 2

SOME BASIC PROPERTIES OF THE KÖTHE DUAL OF ORLICZ-LORENTZ SPACES

We begin with the relationship between $f \in \mathcal{M}_{\varphi,w}$ and its distribution function $d_f(\lambda)$.

Proposition 2.1. Let φ be an Orlicz function. If $f \in \mathcal{M}_{\varphi,w}$, then the distribution function $d_f(\lambda) < \infty$ for all $\lambda > 0$. The similar result for the sequence space $\mathfrak{m}_{\varphi,w}$ holds.

Proof. Let $f \in \mathcal{M}_{\varphi,w}$. Then $P_{\varphi,w}(\lambda f) < \infty$ for some $\lambda > 0$. Let $c > 0$ be such that $f^*(t) > c$ for some $t \in I$ and $a = m\{f^* > c\}$ where $0 < a \leq \infty$. Notice that $c\chi_{(0,a)} < f^*$ a.e. Then by Proposition 1.16, $\frac{(c\chi_{(0,a)})^0}{w} < \frac{(f^*)^0}{w}$ a.e. From the fact that $P_{\varphi,w}(\chi_{(0,a)}) = \varphi\left(\frac{a}{W(a)}\right) W(a)$,

$$\varphi\left(\frac{\lambda ca}{W(a)}\right) W(a) = \int_I \varphi\left(\frac{\lambda c(\chi_{(0,a)})^0}{w}\right) w < \int_I \varphi\left(\frac{\lambda(f^*)^0}{w}\right) w = P_{\varphi,w}(\lambda f) < \infty. \tag{2.1}$$

If $m\{f^* > c\} = a = \infty$, then $\lim_{a\to\infty} \frac{a}{W(a)} \leq M$, where $0 < M \leq \infty$, and $\lim_{a\to\infty} W(a) = \infty$. So the left-hand side of (2.1) goes to infinity, which is a contradiction. Thus, we must have $m\{f^* > c\} = a < \infty$. Since $c > 0$ is chosen arbitrarily, the proof is finished. □

2.1 The Orlicz norm in $\mathcal{M}_{\varphi,w}$ and $\mathfrak{m}_{\varphi,w}$

In this section, we always assume φ to be an Orlicz N-function. From Theorem 1.6, the Orlicz norm for $\mathcal{M}_{\varphi,w}$ can be defined by

$$\|f\|^0_{\mathcal{M}_{\varphi,w}} = \|f\|_{(\Lambda_{\varphi_*,w})'} = \sup\left\{\int_I fg : \|g\|_{\Lambda_{\varphi_*,w}} \leq 1\right\} = \sup\left\{\int_I fg : \rho_{\varphi_*,w}(g) \leq 1\right\}.$$

Hence, we have

$$\|f\|^0_{\mathcal{M}_{\varphi,w}} = \sup\left\{\int_I fg : \rho_{\varphi_*,w}(g) \le 1\right\} = \inf_{k>0}\left(\frac{1}{k}(1 + P_{\varphi,w}(kf))\right).$$

As before, $F(a,b) = \int_a^b f$ and $F(b) = \int_0^b f$. First, we define the interval $\bar{K}(f) = [\bar{k}^*, \bar{k}^{**}]$, where

$$\bar{k}^* = \bar{k}^*(f) = \inf\left\{k > 0 : \rho_{\varphi_*,w}\left(p\left(\frac{k(f^*)^0}{w}\right)\right) \ge 1\right\},$$

and

$$\bar{k}^{**} = \bar{k}^{**}(f) = \sup\left\{k > 0 : \rho_{\varphi_*,w}\left(p\left(\frac{k(f^*)^0}{w}\right)\right) \le 1\right\}.$$

In general, $0 \le \bar{k}^* \le \bar{k}^{**} \le \infty$. Observe that $\bar{k}^{**} < \infty$ if φ is an Orlicz N-function. Indeed, assume to the contrary that $\bar{k}^{**} = \infty$. Then for every $k > 0$, $\rho_{\varphi_*,w}\left(p\left(\frac{k(f^*)^0}{w}\right)\right) \le 1$. Let $\tilde{f}$ be a simple function with support of finite measure such that $\tilde{f} \le f^*$ a.e. Let $\tilde{F}(a) = \int_0^a \tilde{f}$. By Proposition 1.16, $p\left(\frac{k\tilde{f}^0}{w}\right) \le p\left(\frac{k(f^*)^0}{w}\right)$ a.e. Hence

$$\varphi\left(p\left(\frac{k\tilde{F}(a_1)}{W(a_1)}\right)\right)W(a_1) \le \int_I \varphi\left(p\left(\frac{k\tilde{f}^0}{w}\right)\right)w \le \int_I \varphi\left(p\left(\frac{k(f^*)^0}{w}\right)\right)w \le 1,$$

where $(0, a_1)$ is the first maximal level interval of $\tilde{f}$. Moreover,

$$\varphi\left(p\left(\frac{k\tilde{F}(a_1)}{W(a_1)}\right)\right)\Big/ p\left(\frac{k\tilde{F}(a_1)}{W(a_1)}\right) \le 1\Big/\left(p\left(\frac{k\tilde{F}(a_1)}{W(a_1)}\right)W(a_1)\right).$$

Since φ is an N-function, $\lim_{t\to 0+}\frac{\varphi(t)}{t} = 0$ and $\lim_{t\to\infty}\frac{\varphi(t)}{t} = \infty$. So the left-hand side goes to infinity as $k \to \infty$, and the right-hand side goes to 0 as $k \to \infty$, which is a contradiction. Therefore $k^{**} < \infty$ when φ is an Orlicz N-function.

We also define $\bar{\theta}(f)$ for $f \in \mathcal{M}_{\varphi,w}$ by

$$\bar{\theta} = \bar{\theta}(f) = \inf\left\{\lambda > 0 : P_{\varphi,w}\left(\frac{f}{\lambda}\right) < \infty\right\}.$$

In order to prove Theorem 2.5, the following lemma related to the Luxemburg norm $\|\cdot\|_{\mathcal{M}_{\varphi,w}}$ will be useful.

Lemma 2.2. If $\|f\|_{\mathcal{M}_{\varphi,w}} \leq 1$, then $P_{\varphi,w}(f) \leq \|f\|_{\mathcal{M}_{\varphi,w}}$.

Proof. Let $f \in \mathcal{M}_{\varphi,w}$. From the definition of the Luxemburg norm, there exists a sequence of real numbers $(\lambda_n) \downarrow \|f\|_{\mathcal{M}_{\varphi,w}}$ such that $P_{\varphi,w}\left(\frac{f}{\lambda_n}\right) \leq 1$. Then $\frac{f^*}{\lambda_n} \uparrow \frac{f^*}{\|f\|_{\mathcal{M}_{\varphi,w}}}$ a.e. Furthermore, $\frac{(f^*)^0}{\lambda_n} \uparrow \frac{(f^*)^0}{\|f\|_{\mathcal{M}_{\varphi,w}}}$ a.e. from Proposition 1.16. So $\varphi\left(\frac{(f^*)^0}{\lambda_n}\right) w \uparrow \varphi\left(\frac{(f^*)^0}{\|f\|_{\mathcal{M}_{\varphi,w}}}\right) w$ a.e. By the monotone convergence theorem,

$$P_{\varphi,w}\left(\frac{f}{\|f\|_{\mathcal{M}_{\varphi,w}}}\right) = \int_I \varphi\left(\frac{(f^*)^0}{\|f\|_{\mathcal{M}_{\varphi,w}}}\right) w = \lim_{n\to\infty} \int_I \varphi\left(\frac{(f^*)^0}{\lambda_n}\right) w \leq 1.$$

Hence by the convexity of φ, we obtain $\frac{P_{\varphi,w}(f)}{\|f\|_{\mathcal{M}_{\varphi,w}}} \leq P_{\varphi,w}\left(\frac{f}{\|f\|_{\mathcal{M}_{\varphi,w}}}\right) \leq 1$. $\square$

As a consequence of Proposition 2.1, we provide the following facts which will be ingredients for proving Theorem 2.5.(3) and Theorem 2.10.

Proposition 2.3. Let $f \in L^0$ be a bounded function with support of finite measure. Then $P_{\varphi,w}(\lambda f) < \infty$ for every $\lambda > 0$.

Proof. Let $f \in L^0$ be a bounded function with the support of finite measure and $\lambda > 0$. Then there exists $c > 0$ such that $|f| \leq c$ a.e. Hence by the property of decreasing rearrangement, $f^* \leq c$ a.e. on the interval $(0, m \operatorname{supp} f)$ where $m \operatorname{supp} f < \infty$. By Proposition 1.16, we have $\varphi\left(\frac{\lambda (f^*)^0}{w}\right) w \leq \varphi\left(\frac{\lambda c (\chi_{(0,m \operatorname{supp} f)})^0}{w}\right) w$ a.e. We see that

$$\int_I \varphi\left(\frac{\lambda (f^*)^0}{w}\right) w \leq \int_I \varphi\left(\frac{\lambda c (\chi_{(0,m \operatorname{supp} f)})^0}{w}\right) w = \varphi\left(\frac{\lambda c (m \operatorname{supp} f)}{W(m \operatorname{supp} f)}\right) W(m \operatorname{supp} f) < \infty.$$

Therefore, $P_{\varphi,w}(\lambda f) < \infty$ for every $\lambda > 0$, and we finish the proof. $\square$

Lemma 2.4. Let $f \in \mathcal{M}_{\varphi,w}$ and $k > 0$. If $f_n = f\chi_{\{\frac{1}{n}<|f|\leq n\}}$, then
$\rho_{\varphi_*,w}\left(p\left(\frac{k(f_n^*)^0}{w}\right)\right) < \infty$.

Proof. Let $f \in \mathcal{M}_{\varphi,w}$ and define $f_n = f\chi_{\{\frac{1}{n}<|f|\leq n\}}$. Since f is equimeasurable to f^*, by Proposition 2.1, $m\{\frac{1}{n} < f^* \leq n\} \leq m\{f^* > \frac{1}{n}\} < \infty$ for all $n \in \mathbb{N}$. Let
$c = m\{\frac{1}{n} < f^* \leq n\}$. Hence the function f_n is a bounded function with support of finite measure. Furthermore, notice that $f_n^* \leq n\chi_{(0,c)}$ a.e., so $(f_n^*)^0 \leq (n\chi_{(0,c)})^0$ a.e. by Proposition 1.16. Since p is an increasing function, for $k > 0$, we have
$p\left(\frac{k(f_n^*)^0}{w}\right) \leq p\left(\frac{k(n\chi_{(0,c)})^0}{w}\right)$ a.e. Hence

$$\rho_{\varphi_*,w}\left(p\left(\frac{k(f_n^*)^0}{w}\right)\right) = \int_I \varphi_*\left(p\left(\frac{k(f_n^*)^0}{w}\right)\right)w \leq \int_I \varphi_*\left(p\left(\frac{k(n\chi_{(0,c)})^0}{w}\right)\right)w.$$

Moreover,

$$\int_I \varphi_*\left(p\left(\frac{k(n\chi_{(0,c)})^0}{w}\right)\right)w = \int_I \varphi_*\left(p\left(kn\frac{c}{W(c)}\chi_{(0,c)}\right)\right)w = \varphi_*(M)W(c),$$

where $M = p\left(kn\frac{c}{W(c)}\right)$. Since $W(c) < \infty$, we have $\rho_{\varphi_*,w}\left(p\left(\frac{k(f_n^*)^0}{w}\right)\right) < \infty$. $\square$

Now, we provide some results on the Orlicz norm for $\mathcal{M}_{\varphi,w}$ that are similar to those in Orlicz-Lorentz spaces [50, 51].

Theorem 2.5.

(1) $\|f\|_{\mathcal{M}_{\varphi,w}} \leq \|f\|^0_{\mathcal{M}_{\varphi,w}} \leq 2\|f\|_{\mathcal{M}_{\varphi,w}}$.

(2) If there exists $k > 0$ such that $\rho_{\varphi_*,w}\left(p\left(\frac{k(f^*)^0}{w}\right)\right) = 1$, then
$\|f\|^0_{\mathcal{M}_{\varphi,w}} = \int_I f^* p\left(\frac{k(f^*)^0}{w}\right) = \frac{1}{k}(1 + P_{\varphi,w}(kf))$.

(3) $k \in \bar{K}(f)$ if and only if $\|f\|^0_{\mathcal{M}_{\varphi,w}} = \frac{1}{k}(1 + P_{\varphi,w}(kf))$.

The analogous statements occur in the sequence space $\mathfrak{m}^0_{\varphi,w}$ when the modular $P_{\varphi,w}$ is replaced by the modular $p_{\varphi,w}$.

Proof. (1) For $g \in \Lambda_{\varphi_*,w}$, we have $\|g\|_{\Lambda_{\varphi_*,w}} \le \|g\|^0_{\Lambda_{\varphi_*,w}} \le 2\|g\|_{\Lambda_{\varphi_*,w}}$ [50]. Hence

$$\|f\|_{\mathcal{M}_{\varphi,w}} = \sup\left\{\int_I |fg| : \|g\|^0_{\Lambda_{\varphi_*,w}} \le 1\right\} \le \sup\left\{\int_I |fg| : \|g\|_{\Lambda_{\varphi_*,w}} \le 1\right\} = \|f\|^0_{\mathcal{M}_{\varphi,w}}.$$

Notice that $\left\|\frac{f}{\|f\|_{\mathcal{M}_{\varphi,w}}}\right\|_{\mathcal{M}_{\varphi,w}} = 1$ for $f \neq 0$. So by Lemma 2.2, we see that $P_{\varphi,w}\left(\frac{f}{\|f\|_{\mathcal{M}_{\varphi,w}}}\right) \le 1$. Then we have

$$\left\|\frac{f}{\|f\|_{\mathcal{M}_{\varphi,w}}}\right\|^0_{\mathcal{M}_{\varphi,w}} \le 1 + P_{\varphi,w}\left(\frac{f}{\|f\|_{\mathcal{M}_{\varphi,w}}}\right) \le 2.$$

Therefore, $\|f\|_{\mathcal{M}_{\varphi,w}} \le \|f\|^0_{\mathcal{M}_{\varphi,w}} \le 2\|f\|_{\mathcal{M}_{\varphi,w}}$.

(2) We will use the inverse level functions to show our claim. Denote the inverse level function of f^* by w^{f^*}. Assume $k > 0$ such that $\rho_{\varphi_*,w}\left(p\left(\frac{k(f^*)^0}{w}\right)\right) = 1$. From Definition 1.21, we have $\frac{f^*}{w^{f^*}} = \frac{(f^*)^0}{w}$. Moreover,

$$\int_I \varphi_*\left(p\left(\frac{kf^*}{w^{f^*}}\right)\right) w^{f^*} = \int_I \varphi_*\left(p\left(\frac{k(f^*)^0}{w}\right)\right) w = \rho_{\varphi_*,w}\left(p\left(\frac{k(f^*)^0}{w}\right)\right) = 1. \quad (2.2)$$

Observe that

$$\int_I f^* p\left(\frac{k(f^*)^0}{w}\right) = \int_I f^* p\left(\frac{kf^*}{w^{f^*}}\right) = \frac{1}{k}\int_I \frac{kf^*}{w^{f^*}} p\left(\frac{kf^*}{w^{f^*}}\right) w^{f^*}.$$

From Young's equality,

$$\frac{1}{k}\int_I \frac{kf^*}{w^{f^*}} p\left(\frac{kf^*}{w^{f^*}}\right) w^{f^*} = \frac{1}{k}\left(\int_I \varphi\left(\frac{kf^*}{w^{f^*}}\right) w^{f^*} + \int_I \varphi_*\left(p\left(\frac{kf^*}{w^{f^*}}\right)\right) w^{f^*}\right), \quad (2.3)$$

and by (1.3) and (2.2), we obtain

$$\int_I f^* p\left(\frac{k(f^*)^0}{w}\right) = \frac{1}{k}(1 + P_{\varphi,w}(kf)). \tag{2.4}$$

From the definition of the Orlicz norm for $\mathcal{M}_{\varphi,w}$ and from the fact that $\rho_{\varphi_*,w}\left(p\left(\frac{k(f^*)^0}{w}\right)\right) = 1$, we have $\|f\|^0_{\mathcal{M}_{\varphi,w}} \geq \int_I f^* p\left(\frac{k(f^*)^0}{w}\right)$. To show the converse inequality, since $\mathcal{M}_{\varphi,w}$ is rearrangement invariant, we have

$$\|f\|^0_{\mathcal{M}_{\varphi,w}} = \sup\left\{\int_I f^* g^* : \rho_{\varphi_*,w}(g) \leq 1\right\} = \frac{1}{k}\sup\left\{\int_I k f^* g^* : \rho_{\varphi_*,w}(g) \leq 1\right\}.$$

By Young's inequality and the fact that $w^{f^*} \prec w$ (Remark 4.4, [27]),

$$\int_I k f^* g^* = \int_I \frac{k f^* g^*}{w^{f^*}} w^{f^*} \leq \int_I \varphi\left(\frac{k f^*}{w^{f^*}}\right) w^{f^*} + \int_I \varphi_*(g^*) w^{f^*} \leq P_{\varphi,w}(kf) + \rho_{\varphi_*,w}(g).$$

Hence for $g \in \Lambda_{\varphi_*,w}$ such that $\rho_{\varphi,w}(g) \leq 1$, by (2.4),

$$\|f\|^0_{\mathcal{M}_{\varphi,w}} = \frac{1}{k}\sup\left\{\int_I k f^* g^* : \rho_{\varphi_*,w}(g) \leq 1\right\} \leq \frac{1}{k}(P_{\varphi,w}(kf) + 1)) = \int_I f^* p\left(\frac{k(f^*)^0}{w}\right),$$

and this proves our claim.

(3) For $f^* = f \in \mathcal{M}^0_{\varphi,w}$, define a function $T(k) = \frac{1}{k}(1 + P_{\varphi,w}(kf))$. Let $\bar{\theta} = \bar{\theta}(f)$. The function $T(k)$ is continuous on the interval $(0, 1/\bar{\theta})$. We first want to show that $\bar{k}^{**} < 1/\bar{\theta}$. Notice that for every $k < \bar{k}^{**}$, $\rho_{\varphi_*,w}\left(p\left(\frac{kf^0}{w}\right)\right) \leq 1$. In view of (2.3), $\int_I kfp\left(\frac{kf^0}{w}\right) = P_{\varphi,w}(kf) + \rho_{\varphi_*,w}\left(p\left(\frac{kf^0}{w}\right)\right)$. Hence

$$T(l) = \frac{1}{k}(1 + P_{\varphi,w}(kf)) \leq \frac{1}{k}\left(1 + \int_I kfp\left(\frac{kf^0}{w}\right)\right) \leq \frac{1}{k}(1 + \|kf\|^0_{\mathcal{M}_{\varphi,w}}),$$

and this shows that $T(k) \leq \frac{1}{k} + \|f\|^0_{\mathcal{M}_{\varphi,w}}$. Let (k_n) be a sequence of real numbers such that $k_n \uparrow \bar{k}^{**}$. By Fatou's lemma,

$$T(\bar{k}^{**}) \leq \liminf_n (\frac{1}{k_n} + \|f\|^0_{\mathcal{M}_{\varphi,w}}) \leq \liminf_n \frac{1}{k_n} + \|f\|^0_{\mathcal{M}_{\varphi,w}} = 1/\bar{k}^{**} + \|f\|^0_{\mathcal{M}_{\varphi,w}}.$$

Since φ is an Orlicz N-function, $\bar{k}^{**} < \infty$. Hence

$$P_{\varphi,w}(\bar{k}^{**} f) \leq \|\bar{k}^{**} f\|^0_{\mathcal{M}_{\varphi,w}} < \infty.$$

Thus, we get $\bar{k}^{**} < 1/\bar{\theta}$.

Now, let $k_1, k_2 \in (0, 1/\bar{\theta})$ such that $k_1 > k_2$. Define $f_n = f\chi_{\{\frac{1}{n} < |f| \leq n\}}$. From Proposition 2.1, the function f_n is a bounded function with support of finite measure. Then by Young's inequality,

$$\begin{aligned}\int_I k_1 f_n^* p\left(\frac{k_2 (f_n^*)^0}{w}\right) = \int_I \frac{k_1 f_n^*}{w^{f_n^*}} p\left(\frac{k_2 f_n^*}{w^{f_n^*}}\right) w^{f_n^*} &\leq \int_I \varphi\left(\frac{k_1 f_n^*}{w^{f_n^*}}\right) w^{f_n^*} + \int_I \varphi_*\left(p\left(\frac{k_2 f_n^*}{w^{f_n^*}}\right)\right) w^{f_n^*} \\ &= P_{\varphi,w}(k_1 f_n) + \rho_{\varphi_*,w}\left(p\left(\frac{k_2 (f_n^*)^0}{w}\right)\right).\end{aligned}$$

Since $\rho_{\varphi_*,w}\left(p\left(\frac{k_2 (f_n^*)^0}{w}\right)\right) < \infty$ by Lemma 2.4,

$$P_{\varphi,w}(k_1 f_n) \geq \int_I k_1 f_n^* p\left(\frac{k_2 (f_n^*)^0}{w}\right) - \rho_{\varphi_*,w}\left(p\left(\frac{k_2 (f_n^*)^0}{w}\right)\right). \tag{2.5}$$

In addition,

$$\begin{aligned}\int_I k_2 f_n^* p\left(\frac{k_2 (f_n^*)^0}{w}\right) = \int_I \frac{k_2 f_n^*}{w^{f_n^*}} p\left(\frac{k_2 f_n^*}{w^{f_n^*}}\right) w^{f_n^*} &= \int_I \varphi\left(\frac{k_2 f_n^*}{w^{f_n^*}}\right) w^{f_n^*} + \int_I \varphi_*\left(p\left(\frac{k_2 f_n^*}{w^{f_n^*}}\right)\right) w^{f_n^*} \\ &= P_{\varphi,w}(k_2 f_n) + \rho_{\varphi_*,w}\left(p\left(\frac{k_2 (f_n^*)^0}{w}\right)\right),\end{aligned}$$

by Young's equality. In view of Lemma 2.4, $\rho_{\varphi_*,w}\left(p\left(\frac{k_2 (f_n^*)^0}{w}\right)\right) < \infty$. So,

$$P_{\varphi,w}(k_2 f_n) = \int_I k_2 f_n^* p\left(\frac{k_2 f_n^0}{w}\right) - \rho_{\varphi_*,w}\left(p\left(\frac{k_2 f_n^0}{w}\right)\right). \tag{2.6}$$

Observe that

$$\frac{1}{k_1}(1+P_{\varphi,w}(k_1f_n))-\frac{1}{k_2}(1+P_{\varphi,w}(k_2f_n))=\frac{k_2-k_1}{k_1k_2}+\frac{1}{k_1}P_{\varphi,w}(k_1f_n)-\frac{1}{k_2}P_{\varphi,w}(k_2f_n).$$

By adding and subtracting by $\frac{1}{k_1}P_{\varphi,w}(k_2f_n)$,

$$\begin{aligned}&\frac{k_2-k_1}{k_1k_2}+\frac{1}{k_1}P_{\varphi,w}(k_1f_n)-\frac{1}{k_2}P_{\varphi,w}(k_2f_n)\\&\qquad=\frac{k_1-k_2}{k_1k_2}\left(-1+\frac{k_2}{k_1-k_2}\left(P_{\varphi,w}(k_1f_n)-P_{\varphi,w}(k_2f_n)\right)-P_{\varphi,w}(k_2f_n)\right).\end{aligned}$$

From (2.5) and (2.6),

$$\begin{aligned}&\frac{k_1-k_2}{k_1k_2}\left(-1+\frac{k_2}{k_1-k_2}(P_{\varphi,w}(k_1f_n)-P_{\varphi,w}(k_2f_n))-P_{\varphi,w}(k_2f_n)\right)\\&\qquad\geq\frac{k_1-k_2}{k_1k_2}\left(-1+\frac{k_2}{k_1-k_2}\left(\int_I(k_1-k_2)f_np\left(\frac{k_2f_n^0}{w}\right)\right)-P_{\varphi,w}(k_2f_n)\right)\\&\qquad\qquad=\frac{k_1-k_2}{k_1k_2}\left(\rho_{\varphi_*,w}\left(p\left(\frac{k_2f_n^0}{w}\right)\right)-1\right).\end{aligned}$$

Hence, we have

$$\frac{1}{k_1}(1+P_{\varphi,w}(k_1f_n))-\frac{1}{k_2}(1+P_{\varphi,w}(k_2f_n))\geq\frac{k_1-k_2}{k_1k_2}\left(\rho_{\varphi_*,w}\left(p\left(\frac{k_2f_n^0}{w}\right)\right)-1\right).$$

Since $f_n\uparrow f$ a.e. $f_n^0\uparrow f^0$ a.e. by Proposition 1.16. By the monotone convergence theorem, for $k_1>k_2$,

$$T(k_1)-T(k_2)\geq\frac{k_1-k_2}{k_1k_2}\left(\rho_{\varphi_*,w}\left(p\left(\frac{k_2f^0}{w}\right)\right)-1\right).\tag{2.7}$$

By the same argument, if $k_1<k_2$, we can also show that

$$T(k_1)-T(k_2)\geq\frac{k_1-k_2}{k_1k_2}\left(\rho_{\varphi_*,w}\left(p\left(\frac{k_1f^0}{w}\right)\right)-1\right).\tag{2.8}$$

For $0 < k_1 < k_2 < \bar{k}^*$, $\frac{k_1-k_2}{k_1k_2} < 0$ and $\rho_{\varphi_*,w}\left(p\left(\frac{k_1f^0}{w}\right)\right) < 1$. So from (2.8), $T(k)$ is decreasing over $(0,\bar{k}^*)$. Moreover, since $\frac{k_1-k_2}{k_1k_2} > 0$ and $\rho_{\varphi_*,w}\left(p\left(\frac{k_2f^0}{w}\right)\right) > 1$ for $\bar{k}^{**} < k_1 < k_2 < 1/\bar{\theta}$, $T(k)$ is increasing over $(\bar{k}^{**}, 1/\bar{\theta})$ by (2.7). Notice from the definitions of $\bar{k}^*$ and $\bar{k}^{**}$ that $\rho_{\varphi_*,w}\left(p\left(\frac{kf^0}{w}\right)\right) = 1$ for any $k \in (\bar{k}^*, \bar{k}^{**})$. In this case,

$$\|f\|^0_{\mathcal{M}_{\varphi,w}} = \inf_{l>0} T(l) = \frac{1}{k}(1 + P_{\varphi,w}(kf)).$$

by Theorem 2.5.(2). Observe that $T(l) > T(\bar{k}^*)$ for every $l < \bar{k}^*$ and $T(l) > T(\bar{k}^{**})$ for every $l > \bar{k}^{**}$. Since $T(l)$ is continuous on the interval $(0, 1/\bar{\theta})$ and $\bar{k}^*, \bar{k}^{**} \in (0, 1/\bar{\theta})$, $\|f\|^0_{\mathcal{M}_{\varphi,w}} = \inf_{l>0} T(l) = T(\bar{k}^*) = T(\bar{k}^{**})$. Hence, if $k \in \bar{K}(f) = [\bar{k}^*, \bar{k}^{**}]$, then $\|f^0\|_{\mathcal{M}_{\varphi,w}} = \frac{1}{k}(1 + P_{\varphi,w}(kf))$.

To show the converse, define $T(k)$ and $\bar{\theta}$ as before. Recall the fact that $T(k)$ is continuous on the interval $(0, 1/\bar{\theta})$. Let $k_0 > 0$ be such that $\|f\|^0_{\mathcal{M}_{\varphi,w}} = T(k_0) = \inf_{k>0} T(k)$. If $k_0 \in (0, \bar{k}^*)$, $T(k_0) > T(\bar{k}^*)$ because $T(k)$ is decreasing on the interval $(0, \bar{k}^*)$ by (2.8). Also, $T(k_0) > T(\bar{k}^{**})$ for $k_0 \in (\bar{k}^{**}, 1/\bar{\theta})$ since $T(k)$ is increasing on the interval $(\bar{k}^{**}, 1/\bar{\theta})$ by (2.7). Hence $T(k_0) = \|f\|^0_{\mathcal{M}_{\varphi,w}}$ only when $k_0 \in \bar{K}(f)$.

For the sequence case, the proof is similar to the function case, so we omit the proof. □

2.2 Separability of the space $\mathcal{M}_{\varphi,w}$ and $\mathfrak{m}_{\varphi,w}$

In this section, φ is assumed to be an Orlicz function. Since the space $\mathcal{M}_{\varphi,w}$ is a rearrangement invariant Banach function space, we first need to show the relationship between $(\mathcal{M}_{\varphi,w})_a$ and $(\mathcal{M}_{\varphi,w})_b$. We have the following results from the general theory of Banach function lattices which will be useful.

Theorem 2.6. [8, Theorem 2.5.5] Let (R, μ) be a totally σ-finite nonatomic measure space and let X be an arbitrary rearrangement invariant space over (R, μ).

The following are equivalent:

(1) $\lim_{t\to 0+} \phi_X(t) = 0$;

(2) $X_a = X_b$.

Theorem 2.7. [8, Theorem 2.5.4] Let (R, μ) be completely atomic, consisting of countably many atoms of equal measure. If X is any rearrangement invariant space over (R, μ), then $X_a = X_b$ and X_b is separable.

In view of Theorem 2.6 and Theorem 2.7, we have the following consequence for $\mathcal{M}_{\varphi,w}$.

Proposition 2.8. [35, Proposition 2.2] The fundamental function $\phi_{\mathcal{M}}$ of the space $(\mathcal{M}_{\varphi,w}, \|\cdot\|_{\mathcal{M}})$ is expressed as

$$\phi_{\mathcal{M}}(t) = \frac{t}{W(t)} \Big/ \varphi^{-1}\left(\frac{1}{W(t)}\right), \quad t \in (0, \gamma).$$

Consequently, $(\mathcal{M}_{\varphi,w})_a = (\mathcal{M}_{\varphi,w})_b$.

Proof. In order to compute the fundamental function $\phi_{\mathcal{M}}$, we will use level functions. Let $x = \chi_{(0,a)}$, $0 < a < \gamma$. We have that the interval $(0, a)$ is a level interval with respect to w, that is a is a maximal number such that $\int_0^s x/W(s) \leq a/W(a)$ for all $s \in (0, a)$ [27]. Indeed for $s \in (0, a)$, $(\int_0^s x)/W(s) = s/W(s) \leq a/W(a)$ since w is decreasing, and a is maximal since if $s > a$ then $(\int_0^s x)/W(s) = a/W(s) < a/W(a)$. Then the level function

$$x^0(s) = \frac{aw(s)}{W(a)}\chi_{[0,a)}. \tag{2.9}$$

Therefore,

$$P_{\varphi,w}(x) = \int_0^a \varphi\left(\frac{a}{W(a)}\right) w(s)ds = \varphi\left(\frac{a}{W(a)}\right) W(a). \tag{2.10}$$

By the definition of the Luxemburg norm for $\mathcal{M}_{\varphi,w}$,

$$\|\chi_{(0,a)}\|_{\mathcal{M}_{\varphi,w}} = \inf\left\{\lambda > 0 : \varphi\left(\frac{a}{\lambda W(a)}\right) W(a) \le 1\right\}.$$

we see that $\varphi\left(\frac{a}{\lambda W(a)}\right) W(a) \le 1$ if and only if $\lambda \ge \frac{a}{W(a)} \Big/ \varphi^{-1}\left(\frac{1}{W(a)}\right)$. Hence,

$$\|\chi_{(0,a)}\|_{\mathcal{M}_{\varphi,w}} = \frac{a}{W(a)} \Big/ \varphi^{-1}\left(\frac{1}{W(a)}\right).$$

The function $a \to a/W(a)$ is increasing, so $\lim_{a\to 0} a/W(a) = L$, where $0 \le L < \infty$. Moreover $\lim_{a\to 0} \varphi^{-1}(1/W(a)) = \infty$. Hence $\lim_{a\to 0+} \|\chi_{(0,a)}\|_{\mathcal{M}} = 0$. In view of Theorem 2.6, we have $(\mathcal{M}_{\varphi,w})_a = (\mathcal{M}_{\varphi,w})_b$. □

The next property is related to the norm convergence and the modular convergence of the space $\mathcal{M}_{\varphi,w}$.

Lemma 2.9. Let (f_n) be a sequence of measurable functions in $\mathcal{M}_{\varphi,w}$. Then, $\lim_{n\to\infty} \|f_n\|_{\mathcal{M}_{\varphi,w}} = 0$ if and only if $\lim_{n\to\infty} P_{\varphi,w}(\lambda f_n) = 0$ for every $\lambda > 0$. The same result holds for the sequence space $\mathfrak{m}_{\varphi,w}$.

Proof. Suppose that $\lim_{n\to\infty} \|f_n\|_{\mathcal{M}_{\varphi,w}} = 0$. Then for every $c > 1$ and $\lambda > 0$, there exists $N \in \mathbb{N}$ such that for every $n \ge N$, $\|c\lambda f_n\|_{\mathcal{M}_{\varphi,w}} \le 1$. This implies that $P_{\varphi,w}(c\lambda f_n) \le 1$ for every $n \ge N$. By the convexity of φ, we see that $cP_{\varphi,w}(\lambda f_n) \le P_{\varphi,w}(c\lambda f_n) \le 1$, so $P_{\varphi,w}(\lambda f_n) \le \frac{1}{c}$ for every $n \ge N$. Since $c > 1$ is arbitrary, we obtain $\lim_{n\to\infty} P_{\varphi,w}(\lambda f_n) = 0$ for every $\lambda > 0$.

To show the converse, suppose that for every $\lambda > 0$, $\lim_{n\to\infty} P_{\varphi,w}(\lambda f_n) = 0$. Then there exists N such that for every $n \ge N(\lambda)$, $P_{\varphi,w}(\lambda f_n) \le 1$. This implies that $\|f_n\|_{\mathcal{M}_{\varphi,w}} \le \frac{1}{\lambda}$ for every $n \ge N(\lambda)$. Since $\lambda > 0$ is arbitrary, we have $\lim_{n\to\infty} \|f_n\|_{\mathcal{M}_{\varphi,w}} = 0$. The sequence case can be proven by the similar argument, so we omit the proof. □

Theorem 2.10. $(\mathcal{M}_{\varphi,w})_a = (\mathcal{M}_{\varphi,w})_b = \{f \in L^0 : P_{\varphi,w}(\lambda f) < \infty \text{ for all } \lambda > 0\}$. For sequence spaces, $(\mathfrak{m}_{\varphi,w})_a = (\mathfrak{m}_{\varphi,w})_b = \{x \in l^0 : p_{\varphi,w}(\lambda x) < \infty \text{ for all } \lambda > 0\}$.

Proof. From Proposition 2.8, we have $(\mathcal{M}_{\varphi,w})_a = (\mathcal{M}_{\varphi,w})_b$. Let $f \in (\mathcal{M}_{\varphi,w})_b$. Then there exists a sequence of simple functions with supports of finite measure $f_n \uparrow f$ such that $0 \le f_n \le |f|$ for every $n \in \mathbb{N}$ and $\|f - f_n\|_{\mathcal{M}_{\varphi,w}} \to 0$. By Lemma 2.9, $P_{\varphi,w}(2\lambda(f - f_n)) \to 0$ for every $\lambda > 0$. Hence for every $\epsilon > 0$, we can choose $n \in \mathbb{N}$ such that $P_{\varphi,w}(2\lambda(f - f_n)) < \epsilon$. By the convexity of $P_{\varphi,w}$ and Proposition 2.3,

$$\begin{aligned} P_{\varphi,w}(\lambda f) = P_{\varphi,w}\left(\frac{2\lambda f_n + 2\lambda(f - f_n)}{2}\right) &\le \frac{1}{2}P_{\varphi,w}(2\lambda f_n) + \frac{1}{2}P_{\varphi,w}(2\lambda(f - f_n)) \\ &< \frac{1}{2}P_{\varphi,w}(2\lambda f_n) + \epsilon < \infty. \end{aligned}$$

Thus, $f \in \{f \in L^0 : P_{\varphi,w}(\lambda f) < \infty \text{ for all } \lambda > 0\}$.

Now, suppose $f \in L^0$ such that $P_{\varphi,w}(\lambda f) < \infty$ for every $\lambda > 0$. Define $f_n = f\chi_{\{\frac{1}{n} < |f| < n\}}$. From Proposition 2.1, $m\{\frac{1}{n} < |f| < n\} \le m\{|f| > \frac{1}{n}\} < \infty$, so $f_n \in (\mathcal{M}_{\varphi,w})_b$ for all $n \in \mathbb{N}$. Notice that $f_n \uparrow f$ a.e and $|f - f_n| \le |f|$ a.e. Since $d_f(\lambda) < \infty$ for every $\lambda > 0$, $(f - f_n)^* \to 0$ a.e. by Theorem 1.2. Also $(f - f_n)^* \le f^*$. From Theorem 1.12 and Proposition 1.16, $((f - f_n)^*)^0 \to 0$ a.e. and $((f - f_n)^*)^0 \le (f^*)^0$ a.e. Hence for every $\lambda > 0$, $\varphi\left(\frac{\lambda((f-f_n)^*)^0}{w}\right) w \to 0$ a.e. and $\varphi\left(\frac{\lambda((f-f_n)^*)^0}{w}\right) w \le \varphi\left(\frac{\lambda((f-f_n)^*)^0}{w}\right) w$ a.e. By the Lebesgue dominated convergence theorem, we have $P_{\varphi,w}(\lambda(f - f_n)) = \int_I \varphi\left(\frac{\lambda((f-f_n)^*)^0}{w}\right) w \to 0$. Thus, $\|f - f_n\|_{\mathcal{M}_{\varphi,w}} \to 0$ by Proposition 2.9, and this shows that $f \in (\mathcal{M}_{\varphi,w})_b = (\mathcal{M}_{\varphi,w})_a$.

For the sequence spaces, $(\mathfrak{m}_{\varphi,w})_a = (\mathfrak{m}_{\varphi,w})_b$ from Theorem 2.7. By replacing $P_{\varphi,w}$ with $p_{\varphi,w}$, we can prove the second equality by the similar argument from the function space case. □

To prove Theorem 2.13, we need the following results.

Theorem 2.11. If φ does not the satisfy the appropriate Δ_2 condition, there exists a sequence of non-negative functions $f_k \in \mathcal{M}_{\varphi,w}$ with pairwise disjoint supports such that $\|f_k\|_{\mathcal{M}_{\varphi,w}} = 1$ and $\|\sum_{k=1}^{\infty} f_k\|_{\mathcal{M}_{\varphi,w}} = 1$ for every $k \in \mathbb{N}$. The similar result holds for the sequence spaces $\mathfrak{m}_{\varphi,w}$.

Proof. Since the proof for the infinite Lebesgue measure space is similar to the finite Lebesgue measure space case, we only consider when $\gamma < \infty$. In this case, we have $\int_0^\gamma w = W(\gamma) < \infty$. Assume that φ does not satisfy the Δ_2^∞ condition. Then by Lemma 1.5, there exists an increasing sequence of real numbers (u_n) with $u_n \uparrow \infty$ such that

$$\varphi\left(\left(1+\frac{1}{n}\right)u_n\right) > 2^n\varphi(u_n) \quad \text{for every} \quad n \in \mathbb{N}. \tag{2.11}$$

Choose u_1 that satisfies $\frac{1}{\varphi(u_1)} < W(\gamma)$. Since $\sum_{n=1}^{\infty} \frac{1}{2^n\varphi(u_n)} \leq \frac{1}{\varphi(u_1)} < W(\gamma)$, there exists $t_0 \in (0,\gamma)$ such that $W(t_0) = \sum_{n=1}^{\infty} \frac{1}{2^n\varphi(u_n)}$. Hence we can choose $t_n \downarrow 0$ such that $\frac{1}{2^n\varphi(u_n)} = \int_{t_n}^{t_{n-1}} w$ for every $n \in \mathbb{N}$. Let (t_{n_i}) be a subsequence of (t_n) such that $\sum_{k=0}^{\infty} t_{n_i} < \gamma$, where $t_{n_0} = t_0$. For $k \in \mathbb{N}$, define

$$f_k = \sum_{n=n_k+1}^{\infty} u_n w \chi_{[t_n+\sum_{i=0}^{k-1} t_{n_i},\, t_{n-1}+\sum_{i=0}^{k-1} t_{n_i})}.$$

We see that $f_k^* = \sum_{n=n_k+1}^{\infty} u_n w \chi_{[t_n,t_{n-1})}$. In view of Theorem 1.10, $\frac{f_k^*}{w} = \sum_{n=n_k+1}^{\infty} u_n \chi_{[t_n,t_{n-1})}$ is a decreasing function, so $f_k^* = (f_k^*)^0$. Hence

$$P_{\varphi,w}(f_k) = \int_I \varphi\left(\frac{(f_k^*)^0}{w}\right) w = \sum_{n=n_k+1}^{\infty} \varphi(u_n) \int_{t_n}^{t_{n-1}} w = \frac{1}{2^{n_k}} \leq \frac{1}{2^k}.$$

Now, let $s > 1$. Then there exists j_0 such that for all $n > j_0$, $1 + \frac{1}{n} < s$. Let $N = \max\{j_0, n_k+1\}$. For $n \geq N$, by (2.11), we have

$$P_{\varphi,w}(sf_k) = \sum_{n=n_k+1}^{\infty} \varphi(su_n) \int_{t_n}^{t_{n-1}} w \geq \sum_{n=N}^{\infty} \varphi\left(\left(1+\frac{1}{n}\right)u_n\right) \int_{t_n}^{t_{n-1}} w > \sum_{n=N}^{\infty} \frac{2^n\varphi(u_n)}{2^n\varphi(u_n)} = \infty.$$

Hence $\|f_k\|_{\mathcal{M}_{\varphi,w}} = 1$.

To show $\|\sum_{k=1}^{\infty} f_k\|_{\mathcal{M}_{\varphi,w}} = 1$, observe that $1 = \|f_k\|_{\mathcal{M}_{\varphi,w}} \leq \|\sum_{k=1}^{\infty} f_k\|_{\mathcal{M}_{\varphi,w}}$. By the orthogonal superadditivity of $P_{\varphi,w}$, we have $P_{\varphi,w}(\sum_{k=1}^{\infty} f_k) \geq \sum_{k=1}^{\infty} P_{\varphi,w}(f_k)$. Let $r > 1$. Since $P_{\varphi,w}(rf_k) = \infty$ for all k, we have

$$P_{\varphi,w}\left(r\sum_{k=1}^{\infty} f_k\right) \geq \sum_{k=1}^{\infty} P_{\varphi,w}(rf_k) \geq P_{\varphi,w}(rf_k) = \infty.$$

Thus, $\| \sum_{k=1}^{\infty} f_k\|_{\mathcal{M}_{\varphi,w}} = 1$.

We can show the analogous result for the space $\mathfrak{m}_{\varphi,w}$ by using the similar method from the proof of Theorem 8 in [10]. In the proof, the sequence of $y_k \in \lambda_{\varphi,w}$ such that y_k's are pairwise disjoint, $\|y_k\| = 1$ for every $k \in \mathbb{N}$, and $\|\sum_{k=1}^{\infty} y_k\| = 1$, is constructed. By the fact that $(x^*w)^0 = x^*w$ for $x \in l^0$, we can also construct a sequence of z_k, where $z_k = y_k w$ for every $k \in \mathbb{N}$, such that z_k's are pairwise disjoint, $\|z_k\|_{\mathfrak{m}_{\varphi,w}} = 1$ for every $k \in \mathbb{N}$, and $\|\sum_{k=1}^{\infty} z_k\|_{\mathfrak{m}_{\varphi,w}} = 1$ in a similar way. □

From the proof of Theorem 2.11, we have the following consequence.

Corollary 2.12. If φ does not satisfy the appropriate Δ_2 condtion, there exists $f \in \mathcal{M}_{\varphi,w}$ such that $P_{\varphi,w}(f) \leq 1$ but $P_{\varphi,w}(rf) = \infty$ for $r > 1$. The similar result holds for the space $\mathfrak{m}_{\varphi,w}$.

Now, we are ready to prove the main result of this section.

Theorem 2.13. The following are equivalent:

(1) $\mathcal{M}_{\varphi,w} = \{f \in L^0 : P_{\varphi,w}(\lambda f) < \infty \text{ for all } \lambda > 0\}$;

(2) φ satisfies the Δ_2 condition when $\gamma = \infty$ or the Δ_2^{∞} condition when $\gamma < \infty$.

Proof. Suppose that φ does not satisfy the appropriate Δ_2 condition. Then by Corollary 2.12, there exists $f \in \mathcal{M}_{\varphi,w}$ such that $P_{\varphi,w}(f) \leq 1$ and $P_{\varphi,w}(2f) = \infty$. Hence $f \notin \{f \in L^0 : P_{\varphi,w}(\lambda f) < \infty \text{ for all } \lambda > 0\}$, and this shows (1) $\Rightarrow$ (2).
To prove (2) $\Rightarrow$ (1), we only show for φ satisfying the Δ_2^{∞} condition since the infinite measure space case is similar to the finite measure space case. So $I = [0, \gamma)$

where $\gamma < \infty$ and there exist $K > 0$ and $u_0 \geq 0$ such that $\varphi(2u) \leq K\varphi(u)$ for every $u \geq u_0$. Also, $W(t) = \int_0^t w < \infty$ for all $t \in I$. Let $f \in \mathcal{M}_{\varphi,w}$. Then $P_{\varphi,w}(\eta f) < \infty$ for some $\eta > 0$. We first show that $P_{\varphi,w}(2\eta f) < \infty$. Define a set $A = \left\{\frac{(f^*)^0}{w} \geq \frac{u_0}{\eta}\right\}$. In view of Theorem 1.10, $A = (0, c) \subset I$ because $\frac{(f^*)^0}{w}$ is decreasing. From the fact that φ satisfying the Δ_2^∞ condition and $\frac{(f^*)^0}{w} < \frac{u_0}{\eta}$ on (c, γ), we have

$$\begin{aligned} P_{\varphi,w}(2\eta f) = \int_I \varphi\left(\frac{2\eta(f^*)^0}{w}\right) w &= \int_I \varphi\left(\frac{2\eta(f^*)^0}{w}\right) w + \int_c^\gamma \varphi\left(\frac{2\eta(f^*)^0}{w}\right) w \\ &\leq K\int_0^c \varphi\left(\frac{\eta(f^*)^0}{w}\right) w + \varphi(2u_0)(W(\gamma) - W(c)) \\ &\leq KP_{\varphi,w}(\eta f) + \varphi(2u_0)(W(\gamma) - W(c)). \end{aligned}$$

Since $W(\gamma) < \infty$, $\varphi(2u_0)(W(\gamma) - W(c)) < \infty$. In addition, $P_{\varphi,w}(\eta f) < \infty$, so we showed $P_{\varphi,w}(2\eta f) < \infty$.

If $P_{\varphi,w}(2\eta f) < \infty$, we also have $P_{\varphi,w}(2^n\eta f) < \infty$ by the similar argument. Let $\lambda \in \mathbb{R}+$. Then there exists $n \in \mathbb{N}$ such that $\lambda < 2^n\eta$. Hence by the monotonicity of $P_{\varphi,w}$, we have $P_{\varphi,w}(\lambda f) \leq P_{\varphi,w}(2^n\eta f) < \infty$. Therefore, $P_{\varphi,w}(\lambda f) < \infty$ for every $\lambda > 0$. □

Theorem 2.14. The following are equivalent:

(1) $\mathfrak{m}_{\varphi,w} = \{x \in l^0 : p_{\varphi,w}(\lambda x) < \infty \text{ for all } \lambda > 0\}$;

(2) $\varphi \in \Delta_2^0$.

Here is a general result related to separability of a Banach function lattice.

Theorem 2.15. [8, Corollary 1.5.6] A Banach function lattice X is separable if and only if it has order-continuous norm and its underlying measure is separable.

Hence we have the following consequence of Theorem 2.13 and Theorem 2.14.

Corollary 2.16. The function space $\mathcal{M}_{\varphi,w}$ is separable if and only if φ satisfies the appropriate Δ_2 condition. The sequence space $\mathfrak{m}_{\varphi,w}$ is separable if and only if φ satisfies the Δ_2^0 condition.

Proof. From Corollary 2.16, φ satisfies the appropriate Δ_2 condition if and only if $\mathcal{M}_{\varphi,w} = \{f \in L^0 : P_{\varphi,w}(\lambda f) < \infty \text{ for all } \lambda > 0\}$. In view of Theorem 2.10, $\mathcal{M}_{\varphi,w} = (\mathcal{M}_{\varphi,w})_a = (\mathcal{M}_{\varphi,w})_b$, so the space $\mathcal{M}_{\varphi,w}$ is order-continuous. Since the Lebesgue measure m is separable, by Theorem 2.15, the space $\mathcal{M}_{\varphi,w}$ is separable.

For the space $\mathfrak{m}_{\varphi,w}$, from Theorem 2.14, φ satisfies the Δ_2^0 condition if and only if $\mathfrak{m}_{\varphi,w} = \{x \in l^0 : p_{\varphi,w}(\lambda x) < \infty \text{ for all } \lambda > 0\}$. By Theorem 2.7 and Theorem 2.10, $\mathfrak{m}_{\varphi,w} = (\mathfrak{m}_{\varphi,w})_a = (\mathfrak{m}_{\varphi,w})_b$ is separable. □

2.3 Isomorphic and isometric copies of c_0 and l^∞ in $\mathcal{M}_{\varphi,w}$ and $\mathfrak{m}_{\varphi,w}$

In this section, assume that φ is an Orlicz function. We provide the following facts which are useful for our main result.

Definition 2.17. A Banach function lattice X is said to be a KB-space if it is order-continuous with the Fatou property.

Theorem 2.18. [6, Theorem 4.60] A Banach lattice X does not have an isomorphic copy of c_0 if and only if X is a KB-space.

Now we are ready to present the main result of this section.

Theorem 2.19. The following are equivalent:

(1) φ satisfies the Δ_2 condition when $\gamma = \infty$ or φ satisfies the Δ_2^∞ condition when $\gamma < \infty$ (resp. φ satisfies the Δ_2^0 condition);

(2) $\mathcal{M}_{\varphi,w}$ (resp. $\mathfrak{m}_{\varphi,w}$) does not have an isometric copy of l^∞;

(3) $\mathcal{M}_{\varphi,w}$ (resp. $\mathfrak{m}_{\varphi,w}$) does not have an isometric copy of c_0;

(4) $\mathcal{M}_{\varphi,w}$ (resp. $\mathfrak{m}_{\varphi,w}$) does not have an isomorphic copy of c_0;

(5) $\mathcal{M}_{\varphi,w}$ (resp. $\mathfrak{m}_{\varphi,w}$) is a KB-space;

(6) $\mathcal{M}_{\varphi,w}$ (resp. $\mathfrak{m}_{\varphi,w}$) is separable.

Proof. (4) $\implies$ (3) $\implies$ (2) is clear. Since a KB-space is order-continuous and the Lebesgue measure m is separable, in view of Theorem 2.15, (5) $\iff$ (6). We have (4) $\iff$ (5) by Theorem 2.18. (1) $\iff$ (6) was shown in Corollary 2.16.

We only have to show (2) $\implies$ (1). Let φ be an Orlicz function which does not satisfy the appropriate Δ_2 condition. In view of Theorem 2.11, let (f_k) be a sequence of non-negative functions in $\mathcal{M}_{\varphi,w}$ with pairwise disjoint supports such that $\|f_k\|_{\mathcal{M}_{\varphi,w}} = 1$ and $\|\sum_{k=1}^{\infty} f_k\|_{\mathcal{M}_{\varphi,w}} = 1$ for every $k \in \mathbb{N}$. Define an operator $T : l^{\infty} \to \mathcal{M}_{\varphi,w}$ such that $Tx = \sum_{k=1}^{\infty} |x(k)| f_k$. Then we get

$$\|Tx\|_{\mathcal{M}_{\varphi,w}} = \left\| \sum_{k=1}^{\infty} |x(k)| f_k \right\|_{\mathcal{M}_{\varphi,w}} \leq \|x\|_{\infty} \left\| \sum_{k=1}^{\infty} f_k \right\|_{\mathcal{M}_{\varphi,w}} = \|x\|_{\infty}.$$

Observe that for any $0 < \lambda < 1$, there exists $|x(k_0)|$ such that $\frac{|x(k_0)|}{\lambda\|x\|_{\infty}} > 1$. Hence we have

$$P_{\varphi,w}\left(\frac{\sum_{k=1}^{\infty} |x(k)| f_k}{\lambda \|x\|_{\infty}} \right) > P_{\varphi,w}\left(\frac{|x(k_0)| f_{k_0}}{\lambda \|x\|_{\infty}} \right) = \infty,$$

which shows that $\|Tx\|_{\mathcal{M}_{\varphi,w}} = \|x\|_{\infty}$. Thus, the space $\mathcal{M}_{\varphi,w}$ contains an isometric copy of l^{∞} if φ does not satisfy the appropriate Δ_2 condition. The proof is finished.

To show (2) $\implies$ (1) for the space $\mathfrak{m}_{\varphi,w}$, the similar argument from the function case is applied in view of Theorem 2.11. □

CHAPTER 3

M-IDEAL PROPERTIES IN ORLICZ-LORENTZ SPACES

In this chapter, the M-ideal properties and M-embeddedness in Orlicz-Lorentz spaces and their applications are investigated. The materials in Section 3.1 and Section 3.2 are submitted for a publication [26].

Definition 3.1. Let Y be a closed subspace of a Banach space X. We say Y is an M-ideal in X if the range of a bounded projection $P : X^* \to X^*$ satisfying

$$\|x^*\| = \|Px^*\| + \|x^* - Px^*\| \quad \text{for all} \quad x^* \in X^*$$

is $Y^\perp$. When Y is an M-ideal in its bidual X^{**}, we say Y is M-embedded.

The general overview of M-ideals and their applications are given in [23]. There are many examples of spaces that are M-ideals and M-embedded. For instance, c_0 is an M-ideal in l^∞ [23]. It is shown that the order-continuous subspace of $L^1 + L^\infty$ with the standard norm is not an M-ideal, but there is an equivalent norm which makes this subspace an M-ideal [25]. In the case of Orlicz spaces, the order-continuous subspaces of Orlicz spaces equipped with the Luxemburg norm are M-ideals [7, 43], but this is not true when the spaces are equipped with the Orlicz norm and the Orlicz function φ does not satisfy the appropriate Δ_2 condition [12]. Also, under certain growth conditions on an Orlicz function φ and its complementary function φ_*, the order continuous subspaces of Orlicz function and sequence spaces equipped with the Luxemburg norm are M-embedded. Throughout this chapter, we always assume that φ is an Orlicz N-function.

First, we observe that for $f \in \Lambda_{\varphi,w}$, the distribution function $d_f(\lambda)$ is always finite. To show this, let $f \in \Lambda_{\varphi,w}$. Then there exists $\eta > 0$ such that $\rho_{\varphi,w}(\eta f) < \infty$.

We see that for every $\lambda > 0$,

$$\varphi(\eta\lambda)\int_0^{m\{f^*>\lambda\}} w \le \int_I \varphi(\eta_0 f^*)w = \rho_{\varphi,w}(\eta f) < \infty.$$

Since f and f^* are equimeasurable, with the assumption of $W(\infty) = \infty$, we obtain $d_f(\lambda) = d_{f^*}(\lambda) = m\{f^* > \lambda\} < \infty$ for every $\lambda > 0$.

Define $K(f) = [k^*, k^{**}]$, where $k^* = k^*(f) = \inf\{k > 0 : \rho_{\varphi_*,w}(p(kf)) \ge 1\}$ and $k^{**} = \sup\{k > 0 : \rho_{\varphi_*,w}(p(kf)) \le 1\}$. In general, $0 \le k^* \le k^{**} \le \infty$. When φ is an Orlicz N-function, $k^{**} < \infty$. Indeed, assume to the contrary that $k^{**} = \infty$. Then $\rho_{\varphi_*,w}((p(kf)) \le 1$ for all $k > 0$, so there exists a sequence of non-negative real numbers $(k_n) \uparrow \infty$ such that $\rho_{\varphi_*,w}((p(k_n f)) \le 1$. Consider a set $\{f^* > 1\}$. We have $t_0 = m\{f^* > 1\} < \infty$ from the fact that $d_f(\lambda) < \infty$ for all $\lambda > 0$. Hence

$$\varphi_*(p(k_n))W(t_0) = \int_0^{t_0} \varphi_*(p(k_n))w = \int_0^{m\{f^*>1\}} \varphi_*(p(k_n))w \le \int_I \varphi_*(p(k_n f^*)w \le 1,$$

which shows that $\frac{\varphi(p(k_n))}{p(k_n)} \le \frac{1}{W(t_0)p(k_n)}$. As k_n goes to ∞, the left-hand side also goes to ∞ by the definition of an Orlicz N-function. But the right-hand side goes to 0 since p is an increasing function and $p(\infty) = \infty$, which is a contradiction. We can also define k^*, k^{**}, and $K(x)$ for $x \in \lambda_{\varphi,w}$ by replacing $\rho_{\varphi,w}$ with $\alpha_{\varphi,w}$.

Theorem 3.2 ([50], pg 133)**.**

(1) If there exists $k > 0$ such that $\rho_{\varphi_*,w}(p(kf)) = 1$, then $\|f\|^0 = \int_0^\gamma f^* p(kf^*) = \frac{1}{k}(1 + \rho_{\varphi,w}(kf))$.

(2) For any $f \in \Lambda^0_{\varphi,w}$, $\|f\|^0 = \inf_{k>0} \frac{1}{k}(1 + \rho_{\varphi,w}(kf))$.

(3) $k \in K(f)$ if and only if $\|f\|^0 = \frac{1}{k}(1 + \rho_{\varphi,w}(kf))$.

The analogous statements occur in Orlicz-Lorentz sequence space when the modular $\rho_{\varphi,w}$ is replaced by the modular $\alpha_{\varphi,w}$.

3.1 Singular linear functionals on Orlicz-Lorentz spaces

We compute the norm of a singular linear functional on Orlicz-Lorentz spaces. For $f \in L^0$, define a function $\theta = \theta(f) = \inf\{\lambda > 0 : \rho_{\varphi,w}(f/\lambda) < \infty\}$. If $f \in \Lambda_{\varphi,w}$, then $\theta(f) < \infty$. Since $(\Lambda_{\varphi,w})_a = (\Lambda_{\varphi,w})_b = \{f \in \Lambda_{\varphi,w} : \rho_{\varphi,w}(\lambda f) < \infty \text{ for all } \lambda > 0\}$ [24], $\theta(f) = 0$ for $f \in (\Lambda_{\varphi,w})_a$. The function θ can be also defined for the sequence space l^0 by replacing f and $\rho_{\varphi,w}$ with $x \in l^0$ and $\alpha_{\varphi,w}$, and the analogous properties of θ hold for Orlicz-Lorentz sequence spaces. The results in this section are similar to ones in Orlicz spaces [11], but the proofs require different techniques, for instance, being involved with decreasing rearrangement.

The following fact will be useful in this section.

Proposition 3.3. [8, Proposition 1.3.6] A function f in a Banach function space X has order-continuous norm if and only if the following condition holds: whenever f_n, $(n = 1, 2, ...)$ and g are μ-measurable functions satisfying $|f_n| \le |f|$ for all n and $f_n \to g$ μ-a.e., then $\|f_n - g\|_X \to 0$.

First, we need the formula for θ on both Orlicz-Lorentz function and sequence spaces.

Theorem 3.4. For any $f \in \Lambda_{\varphi,w}$, $\lim_n \|f - f_n\| = \lim_n \|f - f_n\|^0 = \theta(f)$, for $f_n = f\chi_{\{\frac{1}{n} \le |f| \le n\}}$. For any $x = (x(i)) \in \lambda_{\varphi,w}$, $\lim_n \|x - x_n\| = \lim_n \|x - x_n\|^0 = \theta(x)$ for $x_n = x\chi_{\{1,2,\ldots,n\}}$.

Proof. Let $f \in (\Lambda_{\varphi,w})_a$. Then $\theta(f) = 0$. Define $f_n = f\chi_{\{\frac{1}{n} \le |f| \le n\}}$. Since $d_f(\lambda) < \infty$ for all $\lambda > 0$, we have $m\{\frac{1}{n} \le |f| \le n\} \le m\{|f| \ge \frac{1}{n}\} < \infty$. Hence $f_n \in (\Lambda_{\varphi,w})_b$. Notice that $f_n \uparrow f$ a.e. and $|f_n| \le |f|$ for all n. So $\|f - f_n\| \to 0$ by Proposition 3.3. Moreover, by the equivalence of $\|\cdot\|$ and $\|\cdot\|^0$ [50, 53], we also get $\|f - f_n\|^0 \to 0$.

Now, consider $f \in \Lambda_{\varphi,w} \setminus (\Lambda_{\varphi,w})_a$ and f_n as above. In this case, $\theta(f) > 0$. Notice that $d_f(\lambda) < \infty$ for all $\lambda > 0$, $|f - f_n| \to 0$ a.e., and $|f - f_n| \ge |f - f_{n+1}|$ for every

$n \in \mathbb{N}$. By the property of decreasing rearrangement, we have $(f - f_n)^* \geq (f - f_{n+1})^*$, and so $\rho_{\varphi,w}\left(\frac{f-f_{n+1}}{\|f-f_n\|}\right) \leq \rho_{\varphi,w}\left(\frac{f-f_n}{\|f-f_n\|}\right) \leq 1$. Hence, $\|f - f_n\|$ is monotonically decreasing. Moreover, for $g \in B_{\Lambda^0_{\varphi,w}}$, we have $\int_I (f - f_n)^* g^* w \geq \int_I (f - f_{n+1})^* g^* w$ for every $n \in \mathbb{N}$. So by the definition of the Orlicz norm, $\|f - f_n\|^0 \geq \|f - f_{n+1}\|^0$ for every $n \in \mathbb{N}$, which means that $\|f - f_n\|^0$ is also monotonically decreasing. Thus, the limits for both $\|f - f_n\|$ and $\|f - f_n\|^0$ exist. In addition, we have $(f - f_n)^* \to 0$ from Lemma 1.2.

Letting $\epsilon \in (0, \theta)$ we have $\rho_{\varphi,w}\left(\frac{f}{\theta-\epsilon}\right) = \infty$. By the orthogonal subadditivity of $\rho_{\varphi,w}$, $\infty = \rho_{\varphi,w}\left(\frac{f}{\theta-\epsilon}\right) \leq \rho_{\varphi,w}\left(\frac{f_n}{\theta-\epsilon}\right) + \rho_{\varphi,w}\left(\frac{f-f_n}{\theta-\epsilon}\right)$. Clearly, the functions f_n are bounded with supports of finite measure. This implies that $\rho_{\varphi,w}\left(\frac{f_n}{\theta-\epsilon}\right) < \infty$. Hence $\|f - f_n\| \geq \theta - \epsilon$ because $\rho_{\varphi,w}\left(\frac{f-f_n}{\theta-\epsilon}\right) = \infty$.

On the other hand for $\epsilon > 0$, we have $\rho_{\varphi,w}\left(\frac{f}{\theta+\epsilon}\right) < \infty$ by the definition of $\theta(f)$. From the fact that $(f - f_n)^* \to 0$, we get $\lim_{n\to\infty} \rho_{\varphi,w}\left(\frac{f-f_n}{\theta+\epsilon}\right) = 0$ by the Lebesgue dominated convergence theorem. Hence, in view of Theorem 3.2.(2),

$$\|f - f_n\|^0 \leq (\theta + \epsilon)\left(1 + \rho_{\varphi,w}\left(\frac{f - f_n}{\theta + \epsilon}\right)\right) \to (\theta + \epsilon),$$

as $n \to \infty$. Since $\|f\| \leq \|f\|^0$, we finally get

$$\theta - \epsilon \leq \|f - f_n\| \leq \|f - f_n\|^0 \leq \theta + \epsilon$$

for sufficiently large n and arbitrary $\epsilon > 0$, and the proof is complete in the function case. The proof in the sequence case is similar, so we skip it. □

Now, we compute the norm of a singular functional S on Orlicz-Lorentz function spaces.

Theorem 3.5. For any singular functional S on $\Lambda_{\varphi,w}$ or $\Lambda^0_{\varphi,w}$, $\|S\| = \|S\|_{(\Lambda_{\varphi,w})^*} = \|S\|_{(\Lambda^0_{\varphi,w})^*} = \sup\{S(f) : \rho_{\varphi,w}(f) < \infty\} = \sup\{\frac{S(f)}{\theta(f)} : f \in \Lambda_{\varphi,w} \setminus (\Lambda_{\varphi,w})_a\}$.

The analogous formulas hold for Orlicz-Lorentz sequence spaces $\lambda_{\varphi,w}$ and $\lambda^0_{\varphi,w}$.

Proof. For $f \in \Lambda_{\varphi,w} \setminus (\Lambda_{\varphi,w})_a$, let $f_n = f\chi_{\{\frac{1}{n} \le |f| \le n\}}$. From the fact that $f_n \in (\Lambda^0_{\varphi,w})_a$, we have $S(f_n) = 0$, so $S(f) = S(f - f_n)$. Hence $S(f) \le \|S\|_{(\Lambda^0_{\varphi,w})^*}\|f - f_n\|^0$. Since $\|f - f_n\|^0 \to \theta(f)$ as $n \to \infty$ by Theorem 3.4, $S(f) \le \|S\|_{(\Lambda^0_{\varphi,w})^*}\theta(f)$. Hence we obtain $\frac{S(f)}{\theta(f)} \le \|S\|_{(\Lambda^0_{\varphi,w})^*}$.

If $\rho_{\varphi,w}(f) < \infty$, the function $\varphi(f^*)w$ is integrable. Moreover, $\varphi((f - f_n)^*)w \le \varphi(f^*)w$ and $\varphi((f - f_n)^*)w \to 0$ a.e. Hence $\rho_{\varphi,w}(f - f_n) \to 0$ by the Lebesgue dominated convergence theorem. This implies that for sufficiently large $n \in \mathbb{N}$, $\rho_{\varphi,w}(f - f_n) \le 1$, and so $\|f - f_n\| \le 1$. By Theorem 3.4, $\theta(f) = \lim_{n\to\infty} \|f - f_n\| \le 1$. Since $S(f) = 0$ for all $f \in (\Lambda_{\varphi,w})_a$, we have $\sup\{S(f) : \rho_{\varphi,w}(f) < \infty\} = \sup\{S(f) : f \in \Lambda_{\varphi,w} \setminus (\Lambda_{\varphi,w})_a,\ \rho_{\varphi,w}(f) < \infty\}$. Notice that $S(f) \le \frac{S(f)}{\theta(f)}$ because $\theta(f) \le 1$ and $\|S\|_{(\Lambda^0_{\varphi,w})^*} \le \|S\|_{(\Lambda_{\varphi,w})^*}$ because $\|\cdot\| \le \|\cdot\|^0$ [50, 53]. Since $\|f\| \le 1$ if and only if $\rho_{\varphi,w}(f) \le 1$, we obtain

$$\begin{aligned}
\|S\|_{(\Lambda^0_{\varphi,w})^*} \le \|S\|_{(\Lambda_{\varphi,w})^*} &= \sup\{S(f) : \rho_{\varphi,w}(f) \le 1\} \\
&\le \sup\{S(f) : \rho_{\varphi,w}(f) < \infty\} \\
&\le \sup\left\{\frac{S(f)}{\theta(f)} : f \in \Lambda_{\varphi,w} \setminus (\Lambda_{\varphi,w})_a,\ \ \rho_{\varphi,w}(f) < \infty\right\} \\
&\le \sup\left\{\frac{S(f)}{\theta(f)} : f \in \Lambda_{\varphi,w} \setminus (\Lambda_{\varphi,w})_a\right\} \\
&\le \|S\|_{(\Lambda^0_{\varphi,w})^*}.
\end{aligned}$$

□

In view of Theorem 3.5, we denote by $\|S\|$ the norm of a singular functional without specifying the norms on Orlicz-Lorentz spaces.

3.2 Norm of a bounded linear functionals on Orlicz-Lorentz spaces

Let $\|F\|^0$ stand for the norm of a bounded linear functional $F \in (\Lambda_{\varphi,w})^*$. Similarly, the norm of a bounded linear functional $F \in (\Lambda^0_{\varphi,w})^*$ will be denoted by $\|F\|$. Now, we compute the value of a bounded linear functional on $\Lambda_{\varphi,w}$. Consequently, this shows that $(\Lambda_{\varphi,w})_a$ is an M-ideal in $\Lambda_{\varphi,w}$.

Theorem 3.6. Let F be a bounded linear functional on $\Lambda_{\varphi,w}$. Then $F = H + S$, where $H(f) = \int_I fh$ for some $h \in \mathcal{M}^0_{\varphi_*,w}$, $\|H\|^0 = \|h\|^0_{\mathcal{M}_{\varphi_*,w}}$, $S(f) = 0$ for all $f \in (\Lambda_{\varphi,w})_a$, and

$$\|F\|^0 = \|h\|^0_{\mathcal{M}_{\varphi_*,w}} + \|S\|.$$

Proof. By Theorem 1.6, we have $F = H + S$ uniquely, where $H(f) = \int_I hf$ for some $h \in \mathcal{M}^0_{\varphi_*,w}$ with $\|H\|^0 = \|h\|^0_{\mathcal{M}_{\varphi_*,w}}$, and $S(f) = 0$ for all $f \in (\Lambda_{\varphi,w})_a$. Observe from Theorem 3.5 that the norm of the singular functional $\|S\|$ is the same under either the Luxemburg norm or the Orlicz norm.

Clearly $\|F\|^0 = \|H + S\|^0 \leq \|H\|^0 + \|S\| = \|h\|^0_{\mathcal{M}_{\varphi_*,w}} + \|S\|$. Now we show the opposite inequality.

Let $\epsilon > 0$ be arbitrary. From the definition of $\|h\|^0_{\mathcal{M}_{\varphi_*,w}}$ and $\|S\|$, choose $f, g \in B_{\Lambda_{\varphi,w}}$ such that

$$\|h\|^0_{\mathcal{M}_{\varphi_*,w}} - \epsilon < \int_I hf \text{ and } \|S\| - \epsilon < S(g). \tag{3.1}$$

Furthermore, we can assume that f is bounded. Indeed, let $z \in S_{\Lambda_{\varphi,w}}$ such that $\|h\|^0_{\mathcal{M}_{\varphi_*,w}} - \frac{\epsilon}{2} < \int_I |hz|$. Let z_n be a non-negative bounded function with support of finite measure defined on $[0, n)$ such that $z_n \uparrow |z|$ a.e. Then $\int_I |h||z| = \lim_{n\to\infty} \int_I |h|z_n$ by the monotone convergence theorem, so for all $\epsilon > 0$, there exists z_{n_0} such that $\int_I |hz| - \frac{\epsilon}{2} \leq \int_I |h| z_{n_0}$. Hence

$$\|h\|^0_{\mathcal{M}_{\varphi_*,w}} - \frac{\epsilon}{2} - \frac{\epsilon}{2} \leq \int_I |hz| - \frac{\epsilon}{2} \leq \int_I |h| z_{n_0}.$$

Let $f = (\operatorname{sign} h) z_{n_0}$. Thus, f is a bounded function with support of finite measure such that $f \in B_{\Lambda_{\varphi,w}}$ and $\|h\|^0_{\mathcal{M}_{\varphi_*,w}} - \epsilon < \int_I hf$.

Since H is a bounded linear functional on $\Lambda_{\varphi,w}$, hf is integrable. So there exists $\delta > 0$ such that for every measurable subset $E \subset I$ with $mE < \delta$, we have

$$\int_E |hf| < \epsilon. \tag{3.2}$$

Now, we show that there exist $n \in \mathbb{N}$ and a measurable subset $E \subset I$ such that $mE < \delta$,

$$\int_E |hg| < \epsilon, \quad \int_0^{mE} \varphi(g^*)w < \frac{\epsilon}{2}, \quad \int_I \varphi((g\chi_{[n,\gamma)})^*)w < \frac{\epsilon}{2}, \quad \text{and} \quad \int_n^{\gamma} |hg| < \epsilon. \tag{3.3}$$

Indeed, let $E_n = \{g^* > n\} = [0, t_n)$ and define $g_n^* = g^* \chi_{[0,t_n)}$. We see that $g_n^* \leq g^*$ and $g_n^* \downarrow 0$ a.e., so by the Lebesgue dominated convergence theorem, $\lim_{n\to\infty} \int_I \varphi(g_n^*)w = 0$. Hence for any $\epsilon > 0$, there exists N_1 such that for every $n \geq N_1$,

$$\int_I \varphi(g_n^*)w = \int_I \varphi(g^* \chi_{[0,t_n)})w = \int_0^{t_n} \varphi(g^*)w = \int_0^{mE_n} \varphi(g^*)w < \frac{\epsilon}{2}. \tag{3.4}$$

Also $E_{n+1} \subset E_n$ for all $n \in \mathbb{N}$ and $m(\cap E_n) = m\{g^* = \infty\} = 0$. By continuity of measure, $0 = m(\cap E_n) = \lim_{n\to\infty} m\{g^* > n\}$.

Since g is equimeasurable to g^*, $\lim_{n\to\infty} m\{g^* > n\} = \lim_{n\to\infty} m\{|g| > n\} = 0$. The function hg is integrable, so $\lim_{n\to\infty} \int_{\{|g|>n\}} |hg| = 0$. Hence there exists N_2 such that $\int_{\{|g|>n\}} |hg| < \epsilon$ for $n \geq N_2$. Since $\rho_{\varphi,w}(g) < \infty$, we choose sufficiently large $n \geq N = \max\{N_1, N_2\}$ satisfying $mE_n = m\{|g| > n\} < \delta$, $\operatorname{supp} f \cap [n, \gamma) = \emptyset$, $\int_I \varphi((g\chi_{[n,\gamma)})^*)w < \frac{\epsilon}{2}$ and $\int_{[n,\gamma)} |hg| < \epsilon$. By letting $E = \{|g| > n\}$ for such n, we found $n \in \mathbb{N}$ and a measurable subset $E \subset I$ satisfying (3.3). Note that $\operatorname{supp} f \subset [0, n)$ from the construction.

Now, define

$$u(t) = \begin{cases} f(t), & t \in G_1 = \operatorname{supp} f \setminus E \\ g(t), & t \in G_2 = E \cup [n, \gamma) \\ 0, & \text{otherwise.} \end{cases}$$

By the orthogonal subadditivity of $\rho_{\varphi,w}$,

$$\begin{aligned} \rho_{\varphi,w}(u) = \int_I \varphi(f\chi_{G_1} + g\chi_{G_2})^* w &\leq \int_I \varphi((f\chi_{G_1})^*)w + \int_I \varphi((g\chi_{G_2})^*)w \\ &\leq \int_0^{mG_1} \varphi(f^*)w + \int_I \varphi(g\chi_E + g\chi_{[n,\gamma)\setminus E})^* w \end{aligned}$$

The sets E and $[n, \gamma) \setminus E$ are disjoint, so by applying the orthogonal subadditivity again, we have $\int_I \varphi(g\chi_E + g\chi_{[n,\gamma)\setminus E})^* w \leq \int_I \varphi(g\chi_E)^* w + \int_I \varphi(g\chi_{[n,\gamma)\setminus E})^* w$. Also, $\int_I \varphi(g\chi_{[n,\gamma)\setminus E})^* w \leq \int_I \varphi(g\chi_{[n,\gamma)})^* w$ and $\int_I \varphi(g\chi_E)^* w \leq \int_0^{mE} \varphi(g^*)w$ by Lemma 1.3. From the fact that $\rho_{\varphi,w}(g) \leq 1$ and (3.3)

$$\rho_{\varphi,w}(u) \leq \int_0^{mG_1} \varphi(f^*)w + \int_I \varphi(g\chi_E)^* w + \int_I \varphi(g\chi_{[n,\gamma)})^* w < 1 + \epsilon,$$

and this implies that $\rho_{\varphi,w}(\frac{u}{1+\epsilon}) \leq 1$. Thus, $\|\frac{u}{1+\epsilon}\| \leq 1$. Notice that $S(f\chi_{G_1}) = 0$ because $f \in (\Lambda_{\varphi,w})_a$. In addition, since $mG_1 = m(\operatorname{supp} f \setminus E) \leq m([0, n) \setminus E) < \infty$ and g is bounded on G_1, we have $g\chi_{G_1} \in (\Lambda_{\varphi,w})_a$, so $S(g\chi_{G_1}) = 0$. Hence $S(g) = S(g\chi_{G_1}) + S(g\chi_{G_2}) = S(g\chi_{G_2})$. Moreover, from (3.3), we have $\left|\int_{E\setminus[n,\gamma)} hg\right| \leq \int_{E\setminus[n,\gamma)} |hg| \leq \int_E |hg| < \epsilon$.

By the fact that $\rho_{\varphi,w}\left(\frac{u}{1+\epsilon}\right) \leq 1$, $(1+\epsilon)\|F\| \geq (1+\epsilon)F\left(\frac{u}{1+\epsilon}\right) = F(u)$. Therefore, it follows that

$$\begin{aligned}
(1+\epsilon)\|F\|^0 \ge F(u) &= F(f\chi_{G_1}+g\chi_{G_2}) \\
&= \int_I h(f\chi_{G_1}+g\chi_{G_2}) + S((f\chi_{G_1}+g\chi_{G_2})) \\
&= \int_I hf\chi_{G_1} + \int_I hg\chi_{G_2} + S(f\chi_{G_1}) + S(g\chi_{G_2}) \\
&= \int_{\operatorname{supp} f\setminus E} hf + \int_{E\cup[n,\gamma)} hg + S(g\chi_{G_2}) \\
&= \int_I hf - \int_E hf + \int_{E\setminus[n,\gamma)} hg + \int_{[n,\gamma)} hg + S(g) \\
&> \|h\|^0_{\mathcal{M}_{\varphi*,w}} - 2\epsilon - 2\epsilon + S(g) \ \text{(by (3.2) and (3.3))} \\
&> \|h\|^0_{\mathcal{M}_{\varphi*,w}} - 2\epsilon - 2\epsilon + \|S\| - \epsilon \ \text{(by (3.1))} \\
&= \|h\|^0_{\mathcal{M}_{\varphi*,w}} + \|S\| - 5\epsilon.
\end{aligned}$$

As $\epsilon \to 0$, the proof is done. □

The proof for the sequence version is similar to the function case, so we skip it.

Theorem 3.7. Let F be a bounded linear functional on $\lambda_{\varphi,w}$. Then $F = H + S$, where $H(x) = \sum_{i=1}^{\infty} x(i)y(i)$, $\|H\|^0 = \|y\|^0_{\mathfrak{m}_{\varphi*,w}}$, S is a singular functional vanishing on $(\lambda_{\varphi,w})_a$ and

$$\|F\|^0_{(\lambda_{\varphi,w})^*} = \|y\|^0_{\mathfrak{m}_{\varphi*,w}} + \|S\|.$$

As a consequence of Theorems 3.6 and 3.7 we obtain the following result.

Corollary 3.8. If φ does not satisfy the appropriate Δ_2 condition then the order-continuous subspaces $(\Lambda_{\varphi,w})_a$ and $(\lambda_{\varphi,w})_a$ are non-trivial M-ideals of $\Lambda_{\varphi,w}$ and $\lambda_{\varphi,w}$, respectively.

Now, we compute the norm of a bounded linear functional when Orlicz-Lorentz function and sequence spaces are equipped with the Orlicz norm. Recall from

[27, 31] that for φ and $h \in L^0$ we have

$$P_{\varphi,w}(h) = \inf\left\{\int_I \varphi\left(\frac{h^*}{v}\right) v : v \prec w, v \downarrow\right\} = \int_I \varphi\left(\frac{(h^*)^0}{w}\right) w. \tag{3.5}$$

The similar formula also holds for any sequence $h \in \ell^0$. Hence we have

$$p_{\varphi,w}(h) = \inf\left\{\sum_{i=1}^{\infty} \varphi\left(\frac{h^*(i)}{v(i)}\right) v(i) : v \prec w, v \downarrow\right\} = \sum_{i=1}^{\infty} \varphi\left(\frac{(h^*)^0(i)}{w(i)}\right) w(i).$$

Consider a decreasing simple function $h^* = \sum_{i=1}^{n} a_i \chi_{(t_{i-1},t_i)}$ where $a_1 > a_2 > \cdots > a_n > 0$ and $t_0 = 0$. Let $H^*(a,b) = \int_a^b h^*$. By Algorithm A provided in [27], the maximal level intervals of h^* are of the form $(t_{i_j}, t_{i_{j+1}})$ where $(t_{i_j})_{j=0}^{l-1}$ is a subsequence of $(t_i)_{i=1}^{n}$ with $0 = t_0 = t_{i_0} < t_{i_1} < \ldots < t_{i_l} = t_n < \infty$. Then, we have

$$\frac{(h^*)^0}{w} = \sum_{j=0}^{l-1} \frac{H^*(t_{i_j}, t_{i_{j+1}})}{W(t_{i_j}, t_{i_{j+1}})} \chi_{(t_{i_j}, t_{i_{j+1}})}. \tag{3.6}$$

Observe that the sequence $\left(\frac{H^*(t_{i_j}, t_{i_{j+1}})}{W(t_{i_j}, t_{i_{j+1}}}\right)_{j=0}^{l-1}$ is decreasing since $\frac{(h^*)^0}{w}$ is decreasing from Theorem 1.10. Furthermore, we obtain

$$P_{\varphi,w}(h) = \int_I \varphi\left(\sum_{j=0}^{l-1} \frac{H^*(t_{i_j}, t_{i_{j+1}})}{W(t_{i_j}, t_{i_{j+1}})} \chi_{(t_{i_j}, t_{i_{j+1}})}\right) w = \sum_{j=0}^{l-1} \varphi\left(\frac{H^*(t_{i_j}, t_{i_{j+1}})}{W(t_{i_j}, t_{i_{j+1}})}\right) \cdot W(t_{i_j}, t_{i_{j+1}}).$$

The next lemma is a key ingredient for computation of the norm of a bounded linear functional on $\Lambda^0_{\varphi,w}$ or $\lambda^0_{\varphi,w}$.

Lemma 3.9. Let $h \in L^0$ be a non-negative simple function with support of finite measure. Then, there exists a non-negative simple function v such that

$$P_{\varphi_*,w}(h) = \int_I \varphi_*\left(\frac{h}{v}\right) v \ \text{ and } \ \int_I \varphi\left(q\left(\frac{h}{v}\right)\right) v = \int_I \varphi\left(q\left(\frac{h}{v}\right)^*\right) w.$$

The similar formula holds for modular $p_{\varphi_*,w}(x)$ for any $x \in \ell^0$.

Proof. Let $h=\sum_{i=1}^n a_i\chi_{A_i}$ with $a_1>a_2>\cdots>a_n>0$ and $\{A_i\}_{i=1}^n$ be a family of disjoint measurable subsets of I with finite measure. Since h and h^* are equimeasurable, $mA_i=t_i-t_{i-1}$ for $i=1,\dots,n$.

It is well known in [22] and [27] that each (t_{i-1},t_i) is a level interval of h^*, contained in at most one maximal level interval $(t_{i_j},t_{i_{j+1}})$ for some $0\le j\le l-1$ [27]. So, for every j, we can see

$$m(t_{i_j},t_{i_{j+1}})=m(\cup_{i_j<i\le i_{j+1}}(t_{i-1},t_i))=m(\cup_{i_j<i\le i_{j+1}}A_i),$$

and this implies

$$H^*(t_{i_j},t_{i_{j+1}})=\int_{t_{i_j}}^{t_{i_{j+1}}}h^*=\sum_{i=i_j+1}^{i_{j+1}}\int_{t_{i-1}}^{t_i}a_i=\sum_{i=i_j+1}^{i_{j+1}}a_i(t_i-t_{i-1})=\sum_{i_j<i\le i_{j+1}}a_imA_i. \quad (3.7)$$

By (3.6),

$$\frac{(h^*)^0}{w}=\sum_{j=0}^{l-1}\sum_{i_j<i\le i_{j+1}}\frac{H^*(t_{i_j},t_{i_{j+1}})}{W(t_{i_j},t_{i_{j+1}})}\chi_{(t_{i-1},t_i)}=\left(\sum_{j=0}^{l-1}\sum_{i_j<i\le i_{j+1}}\frac{H^*(t_{i_j},t_{i_{j+1}})}{W(t_{i_j},t_{i_{j+1}})}\chi_{A_i}\right)^*.$$

Hence, by right-continuity of q,

$$q\left(\frac{(h^*)^0}{w}\right)=q\left(\sum_{j=0}^{l-1}\sum_{i_j<i\le i_{j+1}}\frac{H^*(t_{i_j},t_{i_{j+1}})}{W(t_{i_j},t_{i_{j+1}})}\chi_{A_i}\right)^*.$$

Let $v=\sum_{j=0}^{l-1}\sum_{i_j<i\le i_{j+1}}\frac{W(t_{i_j},t_{i_{j+1}})}{H^*(t_{i_j},t_{i_{j+1}})}a_i\chi_{A_i}$. Then, $q\left(\frac{(h^*)^0}{w}\right)=q\left(\frac{h}{v}\right)^*$. The functions h and v have the same supports, so the quotient h/v is set to be zero outside of the

supports of h and v. Now, we compute $\int_I \varphi_* \left(\frac{h}{v}\right) v$ and $\int_I \varphi\left(q\left(\frac{h}{v}\right)\right) v$.

$$\begin{aligned}\int_I \varphi_*\left(\frac{h}{v}\right) v &= \sum_{j=0}^{l-1} \sum_{i_j < i \leq i_{j+1}} \int_I \varphi_*\left(\frac{H^*(t_{i_j}, t_{i_{j+1}})}{W(t_{i_j}, t_{i_{j+1}})}\right) \cdot \frac{W(t_{i_j}, t_{i_{j+1}})}{H^*(t_{i_j}, t_{i_{j+1}})} a_i \chi_{A_i} \\ &= \sum_{j=0}^{l-1} \varphi_*\left(\frac{H^*(t_{i_j}, t_{i_{j+1}})}{W(t_{i_j}, t_{i_{j+1}})}\right) \cdot \frac{W(t_{i_j}, t_{i_{j+1}})}{H^*(t_{i_j}, t_{i_{j+1}})} \sum_{i_j < i \leq i_{j+1}} a_i \cdot mA_i \\ &= \sum_{j=0}^{l-1} \varphi_*\left(\frac{H^*(t_{i_j}, t_{i_{j+1}})}{W(t_{i_j}, t_{i_{j+1}})}\right) \cdot W(t_{i_j}, t_{i_{j+1}}) \text{ (by (3.7))} \\ &= P_{\varphi_*,w}(h).\end{aligned}$$

and

$$\begin{aligned}\int_I \varphi\left(q\left(\frac{h}{v}\right)\right) v &= \sum_{j=0}^{l-1} \sum_{i_j < i \leq i_{j+1}} \int_I \varphi\left(q\left(\frac{H^*(t_{i_j}, t_{i_{j+1}})}{W(t_{i_j}, t_{i_{j+1}})}\right)\right) \cdot \frac{W(t_{i_j}, t_{i_{j+1}})}{H^*(t_{i_j}, t_{i_{j+1}})} a_i \chi_{A_i} \\ &= \sum_{j=0}^{l-1} \varphi\left(q\left(\frac{H^*(t_{i_j}, t_{i_{j+1}})}{W(t_{i_j}, t_{i_{j+1}})}\right)\right) \cdot \frac{W(t_{i_j}, t_{i_{j+1}})}{H^*(t_{i_j}, t_{i_{j+1}})} \sum_{i_j < i \leq i_{j+1}} a_i \cdot mA_i \\ &= \sum_{j=0}^{l-1} \int_I \varphi\left(q\left(\frac{H^*(t_{i_j}, t_{i_{j+1}})}{W(t_{i_j}, t_{i_{j+1}})}\right)\right) \cdot w\chi_{(t_{i_j}, t_{i_{j+1}})} \text{ (by (3.7))} \\ &= \int_I \varphi\left(q\left(\sum_{j=0}^{l-1} \frac{H^*(t_{i_j}, t_{i_{j+1}})}{W(t_{i_j}, t_{i_{j+1}})} \chi_{(t_{i_j}, t_{i_{j+1}})}\right)\right) w \\ &= \int_I \varphi\left(q\left(\frac{(h^*)^0}{w}\right)\right) w = \int_I \varphi\left(q\left(\frac{h}{v}\right)^*\right) w.\end{aligned}$$

□

We are ready to compute the norm of a bounded linear functional on $\Lambda^0_{\varphi,w}$.

Theorem 3.10. Let F be a bounded linear functional on $\Lambda^0_{\varphi,w}$. Then $F = H + S$, where $H(f) = \int_I fh$ for some $h \in \mathcal{M}_{\varphi_*,w}$, $\|H\| = \|h\|_{\mathcal{M}_{\varphi_*,w}}$, $S(f) = 0$ for all $f \in (\Lambda_{\varphi,w})_a$, and $\|F\| = \inf\{\lambda > 0 : P_{\varphi_*,w}(\frac{h}{\lambda}) + \frac{1}{\lambda}\|S\| \leq 1\}$.

Proof. Similar to Theorem 3.6, we have $F = H + S$, where $H(f) = \int_I hf$ for some $h \in \mathcal{M}_{\varphi_*,w}$ with $\|H\| = \|h\|_{\mathcal{M}_{\varphi_*,w}}$ and $S(f) = 0$ for all $f \in (\Lambda^0_{\varphi,w})_a$ in view of

Theorem 1.6. Thus, we only need to show the formula for $\|F\|$. Without loss of generality, assume $\|F\| = 1$. Let $f \in S_{\Lambda^0_{\varphi,w}}$. Since $h \in \mathcal{M}_{\varphi_*,w}$, we have $P_{\varphi_*,w}(\frac{h}{\lambda}) < 1$ for some $\lambda > 0$. So we can choose $\lambda > 0$ such that $P_{\varphi_*,w}(\frac{h}{\lambda}) + \frac{1}{\lambda}\|S\| \le 1$. Let $k \in K(f)$. By Theorem 3.2.(3), $1 = \|f\|^0 = \frac{1}{k}(1 + \rho_{\varphi,w}(kf))$, and this implies that $\rho_{\varphi,w}(kf) < \infty$. For every $v \prec w$, $v \downarrow$, we have

$$\frac{1}{\lambda}F(kf) = \frac{1}{\lambda}\left(\int_I khf + S(kf)\right) \le \frac{1}{\lambda}\left(\int_I kh^* f^* + S(kf)\right) = \int_I \frac{kh^* f^* v}{\lambda v} + \frac{1}{\lambda}S(kf).$$

By Young's inequality,

$$\int_I \frac{kh^* f^* v}{\lambda v} + \frac{1}{\lambda}S(kf) \le \int_I \varphi(kf^*)v + \int_I \varphi_*\left(\frac{h^*}{\lambda v}\right) v + \frac{1}{\lambda}S(kf).$$

from (3.5) and from 1.1, we get

$$\frac{1}{\lambda}F(kf) \le \rho_{\varphi,w}(kf) + P_{\varphi_*,w}\left(\frac{h}{\lambda}\right) + \frac{1}{\lambda}S(kf)).$$

Furthermore, $S(kf) \le \|S\|$ because $\rho_{\varphi,w}(kf) < \infty$. Hence

$$\frac{1}{\lambda}F(kf) \le \rho_{\varphi,w}(kf) + P_{\varphi_*,w}\left(\frac{h}{\lambda}\right) + \frac{1}{\lambda}\|S\| \le 1 + \rho_{\varphi,w}(kf),$$

which implies that $F(f) \le \lambda \cdot \frac{1}{k}(1 + \rho_{\varphi,w}(kf)) \le \lambda\|f\|^0 = \lambda$. Since f and λ are arbitrary, we showed that $\|F\| \le \inf\{\lambda > 0 : P_{\varphi_*,w}(\frac{h}{\lambda}) + \frac{1}{\lambda}\|S\| \le 1\}$.

Now, suppose that

$$1 = \|F\| < \inf\{\lambda > 0 : P_{\varphi_*,w}\left(\frac{h}{\lambda}\right) + \frac{1}{\lambda}\|S\| \le 1\}.$$

Then there exists $\delta > 0$ such that

$$P_{\varphi_*,w}(h) + \|S\| > 1 + 3\delta.$$

From Theorem 3.5, $\|S\| = \sup\{S(f) : \rho_{\varphi,w}(f) < \infty\}$. So there exists $f \in \Lambda^0_{\varphi,w}$ such that $\rho_{\varphi,w}(f) < \infty$ and $\|S\| < S(f) + \delta$. This implies that

$$P_{\varphi_*,w}(h) + S(f) + \delta > P_{\varphi_*,w}(h) + \|S\| > 1 + 3\delta,$$

and so

$$P_{\varphi_*,w}(h) + S(f) > 1 + 2\delta.$$

Without loss of generality, let $h \geq 0$. Let $(h_n)_{n=1}^{\infty}$ be a sequence of simple functions with support of finite measure such that $h_n \uparrow h$. By Lemma 4.6 in [31], we get $P_{\varphi_*,w}(h_n) \uparrow P_{\varphi_*,w}(h)$. Hence there exists a non-negative simple function h_0 with $m(\operatorname{supp} h_0) < \infty$ such that $0 \leq h_0 \leq h$ a.e. and

$$P_{\varphi_*,w}(h) < P_{\varphi_*,w}(h_0) + \delta.$$

This implies that

$$P_{\varphi_*,w}(h_0) + S(f) > P_{\varphi_*,w}(h) + S(f) - \delta > 1 + 2\delta - \delta = 1 + \delta.$$

Now, consider a function $f_n = f\chi_{\{\frac{1}{n} \leq |f| \leq n\}}$. The function $|f - f_n| \downarrow 0$ a.e. Hence $(f - f_n)^* \to 0$ a.e. It follows that $\rho_{\varphi,w}(f - f_n) \downarrow 0$ by the Lebesgue dominated convergence theorem. Since H is a bounded linear functional on $\Lambda^0_{\varphi,w}$, $\int_I |f - f_n|h \leq \int_I |f|h < \|H\|\|f\|^0 < \infty$, and so $\int_I |f - f_n|h \to 0$. Then for $\delta > 0$, there exists N_0 such that for $n \geq N_0$,

$$\rho_{\varphi,w}(f - f_n) \leq 1 \quad \text{and} \quad \int_I |f - f_n|h < \frac{\delta}{8}.$$

Let $g_1 = f - f_n$ for some $n \geq N_0$. The function f_n is bounded with support of finite measure because $m \operatorname{supp} f_n \leq m\{|f| > \frac{1}{n}\} < \infty$. So we have

$S(f) = S(g_1) + S(f_n) = S(g_1)$ and

$$\rho_{\varphi,w}(g_1) \le 1, \quad \int_I |g_1| h < \frac{\delta}{8}, \quad \text{and} \quad P_{\varphi_*,w}(h_0) + S(g_1) > 1 + \delta. \tag{3.8}$$

Let v be the non-negative simple function constructed in Lemma 3.9 for h_0. By Young's equality,

$$\int_I q\left(\frac{h_0}{v}\right) h_0 = \int_I q\left(\frac{h_0}{v}\right) \frac{h_0}{v} v = \int_I \varphi\left(q\left(\frac{h_0}{v}\right)\right) v + \int_I \varphi_* \left(\frac{h_0}{v}\right) v.$$

Let $g_2 = q\left(\frac{h_0}{v}\right)$. It is a simple function with support of finite measure, so $g_2 \in (\Lambda^0_{\varphi,w})_a$. In view of Lemma 3.9,

$$P_{\varphi_*,w}(h_0) = \int_I \varphi_* \left(\frac{h_0}{v}\right) v = \int_I q\left(\frac{h_0}{v}\right) h_0 - \int_I \varphi\left(q\left(\frac{h_0}{v}\right)\right) v = \int_I g_2 h_0 - \int_I \varphi(g_2^*) w. \tag{3.9}$$

Since H is a bounded linear functional on $\Lambda^0_{\varphi,w}$, the function $g_2 h$ is integrable. So for $\delta > 0$, there exist $\eta > 0$, $n \in \mathbb{N}$ and $E \subset I$ such that $mE < \eta$,

$$\int_0^{mE} \varphi(g_1^*) w < \frac{\delta}{4}, \quad \int_E |g_2 h| < \frac{\delta}{2}, \quad \text{and} \quad \rho_{\varphi,w}(g_1 \chi_{[n,\gamma)}) = \int_I \varphi((g_1 \chi_{[n,\gamma)})^*) w < \frac{\delta}{8}. \tag{3.10}$$

Indeed, let $E_n = \{g_1^* > n\} = [0, t_n)$. We see that $g_1^* \chi_{E_n} \le g_1^*$ for all n and $g_1^* \chi_{E_n} \to 0$ a.e. By the Lebesgue dominated convergence theorem, for $\delta > 0$, there exists N_1 such that for all $n \ge N_1$,

$$\int_I \varphi(g_1^* \chi_{E_n}) w = \int_0^{mE_n} \varphi(g_1^*) w < \frac{\delta}{4}.$$

Since g_1 and g_1^* are equimeasurable, we have $m\{|g_1| > n\} = m\{g_1^* > n\} = mE_n$ for all n. Choose $n > N_1$ such that $mE_n < \eta$, $\operatorname{supp} h_0 \cap [n, \gamma) = \emptyset$, and $\rho_{\varphi,w}(g_1 \chi_{[n,\gamma)}) = \int_I \varphi((g_1 \chi_{[n,\gamma)})^*) w < \frac{\delta}{8}$. Finally, by letting $\{|g_1| > n\} = E$ for such n, we obtain $n \in \mathbb{N}$ and a measurable subset $E \subset I$ satisfying (3.10). Note that

$\operatorname{supp} h_0 \subset [0, n)$.

Now, we define

$$\bar{u}(t) = \begin{cases} g_2(t), & t \in A_1 = \operatorname{supp} h_0 \setminus E \\ g_1(t), & t \in A_2 = E \cup [n, \gamma) \\ 0, & \text{Otherwise.} \end{cases}$$

The function g_1 is bounded on the set A_2^c, which is a subset of $[0, n)$. So $g_1 \chi_{A_2^c} \in (\Lambda^0_{\varphi,w})_a$. Hence $S(g_1) = S(g_1 \chi_{A_2})$. In addition, g_2 is a simple function with support of finite measure, $S(g_2 \chi_{A_1}) = 0$. By the orthogonal subadditivity of $\rho_{\varphi,w}$, we get

$$\rho_{\varphi,w}(\bar{u}) \leq \rho_{\varphi,w}(g_2 \chi_{A_1}) + \rho_{\varphi,w}(g_1 \chi_{A_2}) \leq \rho_{\varphi,w}(g_2 \chi_{A_1}) + \rho_{\varphi,w}(g_1 \chi_E) + \rho_{\varphi,w}(g_1 \chi_{[n,\gamma)}),$$

and by (3.10), we have

$$\rho_{\varphi,w}(\bar{u}) < \rho_{\varphi,w}(g_2 \chi_{A_1}) + \rho_{\varphi,w}(g_1 \chi_E) + \frac{\delta}{8}.$$

Hence,

$$\int_I \bar{u}h + S(\bar{u}) - \rho_{\varphi,w}(\bar{u}) \geq \int_{A_1} g_2 h + \int_{A_2} g_1 h + S(g_1) - \rho_{\varphi,w}(g_2 \chi_{A_1}) - \rho_{\varphi,w}(g_1 \chi_E) - \frac{\delta}{8}. \quad (3.11)$$

Since $g_2 \geq 0$ and $h \geq h_0 \geq 0$, we have

$$\int_{A_1} g_2 h \geq \int_{A_1} g_2 h_0 = \int_{I \setminus E} g_2 h_0.$$

Also, in view of (5.8) and (5.10)

$$\int_{A_2} |g_1 h| < \int_I |g_1 h| < \frac{\delta}{8} \quad \text{and} \quad \int_E g_2 h_0 \leq \int_E g_2 h < \frac{\delta}{2}.$$

Then the inequality (3.11) becomes

$$\int_I h\bar{u} + S(\bar{u}) - \rho_{\varphi,w}(\bar{u}) \geq \int_{I\setminus E} g_2 h_0 - \frac{\delta}{4} + S(g_1) - \rho_{\varphi,w}(g_2\chi_{A_1}) - \rho_{\varphi,w}(g_1\chi_E).$$

Hence we obtain

$$\begin{aligned}
\int_I \bar{u}h + S(\bar{u}) - \rho_{\varphi,w}(\bar{u}) &\geq \int_{I\setminus E} g_2 h_0 - \frac{\delta}{4} + S(g_1) - \rho_{\varphi,w}(g_2\chi_{A_1}) - \rho_{\varphi,w}(g_1\chi_E) \\
&\geq \int_I g_2 h_0 - \int_E g_2 h_0 - \frac{\delta}{4} + S(g_1) - \rho_{\varphi,w}(g_2) - \rho_{\varphi,w}(g_1\chi_E) \\
&\geq \int_I g_2 h_0 - \int_E g_2 h_0 - \frac{\delta}{4} + S(g_1) - \rho_{\varphi,w}(g_2) - \frac{\delta}{4} \text{ by (5.10)} \\
&= P_{\varphi_*,w}(h_0) - \int_E g_2 h_0 + S(g_1) - \frac{\delta}{2} \text{ by (5.9)} \\
&\geq P_{\varphi_*,w}(h_0) - \frac{\delta}{2} + S(g_1) - \frac{\delta}{2} \\
&> 1 + \delta - \delta = 1. \text{ by (5.8)}
\end{aligned}$$

Finally, this implies that

$$1 = \|F\| \geq F\left(\frac{\bar{u}}{\|\bar{u}\|^0}\right) = \frac{H(\bar{u}) + S(\bar{u})}{\|\bar{u}\|^0} = \frac{\int_I \bar{u}h + S(\bar{u})}{\|\bar{u}\|^0} > \frac{1 + \rho_{\varphi,w}(\bar{u})}{\|\bar{u}\|^0} > 1,$$

which leads to a contradiction. □

The next result is the sequence analogue of Theorem 3.10.

Theorem 3.11. Let F be a bounded linear functional on $\lambda^0_{\varphi,w}$. Then $F = H + S$, where $H(x) = \sum_{i=1}^{\infty} x(i)y(i)$, $\|H\| = \|y\|_{\mathfrak{m}_{\varphi_*,w}}$, S is a singular functional vanishing on $(\lambda_{\varphi,w})_a$ and

$$\|F\| = \inf\{\eta > 0 : p_{\varphi_*,w}(\frac{h}{\eta}) + \frac{1}{\eta}\|S\| \leq 1\}.$$

Contrary to Corollary 3.8 about M-ideals in the Orlicz-Lorentz spaces equipped with the Luxemburg norm, we conclude this section by showing that $(\Lambda^0_{\varphi,w})_a$ and $(\lambda^0_{\varphi,w})_a$ are not M-ideals in $\Lambda^0_{\varphi,w}$ and $\lambda^0_{\varphi,w}$ respectively, if the Orlicz N-function φ

does not satisfy the appropriate Δ_2 condition.

Corollary 3.12. If φ does not satisfy the appropriate Δ_2 condition, then the order-continuous subspaces $(\Lambda^0_{\varphi,w})_a$ or $(\lambda^0_{\varphi,w})_a$ are not M-ideals in $\Lambda^0_{\varphi,w}$ or $\lambda^0_{\varphi,w}$, respectively.

Proof. We give a proof only for the function spaces. Let φ be an Orlicz N-function, which does not satisfy the appropriate Δ_2 condition. Then $(\Lambda^0_{\varphi,w})_a$ is a proper subspace of $\Lambda^0_{\varphi,w}$, and in view of Theorem 3.10 there exists $S \in (\Lambda^0_{\varphi,w})^*$ such that $S \neq 0$. So, choose $S \in (\Lambda^0_{\varphi,w})^*$ such that $0 < \|S\| < 1$. We show that there exist $u > 0$ and $0 < t_0 < \gamma$ such that $h = uw\chi_{(0,t_0)}$ and $\|h\|_{\mathcal{M}_{\varphi_*,w}} + \|S\| = 1$. Indeed choose u satisfying $\varphi_*(u) > 1/W(\gamma)$, where $1/W(\infty) = 0$. Then $\frac{1}{\varphi_*(u/(1-\|S\|))} < W(\gamma)$. Since W is continuous on $(0,\gamma)$, there exists $0 < t_0 < \gamma$ such that $W(t_0) = \frac{1}{\varphi_*(u/(1-\|S\|))}$. Let $h = uw\chi_{(0,t_0)}$ for such u and t_0. Clearly h is a decreasing function. Furthermore, the interval $(0,t_0)$ is the maximal level interval of h since $R(0,t) = \frac{uW(t)}{W(t)} = \frac{uW(t_0)}{W(t_0)} = R(0,t_0) = u$ for all $0 < t < t_0$, and $R(0,t_0) < R(0,t)$ for $\gamma > t > t_0$. Hence $\frac{h^0}{w} = u\chi_{(0,t_0)}$, so $P_{\varphi_*,w}(h) = \int_I \varphi_*\left(\frac{h^0}{w}\right) w = \varphi_*(u)W(t_0)$. It follows that

$$\|h\|_{\mathcal{M}_{\varphi_*,w}} = \inf\left\{\epsilon > 0 : P_{\varphi_*,w}\left(\frac{h}{\epsilon}\right) \leq 1\right\} = \inf\left\{\epsilon > 0 : \varphi_*\left(\frac{u}{\epsilon}\right) \leq \frac{1}{W(t_0)}\right\}$$
$$= \inf\left\{\epsilon > 0 : \varphi_*\left(\frac{u}{\epsilon}\right) \leq \varphi_*\left(\frac{u}{1-\|S\|}\right)\right\} = \inf\{\epsilon > 0 : \epsilon \geq 1 - \|S\|\} = 1 - \|S\|.$$

Thus, we have $\|h\|_{\mathcal{M}_{\varphi_*,w}} + \|S\| = 1$, which implies that $P_{\varphi_*,w}(\frac{h}{1-\|S\|}) \leq 1$. Now since φ is an Orlicz N-function, φ_* is also an Orlicz N-function, and so φ_* is not identical to a linear function ku for any $k > 0$. Hence for all $u > 0$, $\lambda > 1$ we have $\varphi_*(\lambda u) > \lambda\varphi_*(u)$. Therefore, by the fact that $\frac{1}{1-\|S\|} > 1$,

$$1 \geq P_{\varphi_*,w}\left(\frac{h}{1-\|S\|}\right) = \varphi_*\left(\frac{u}{1-\|S\|}\right) W(t_0) > \frac{1}{1-\|S\|} P_{\varphi_*,w}(h),$$

and this shows that

$$P_{\varphi_*,w}(h) < 1 - \|S\| = \|h\|_{\mathcal{M}_{\varphi_*,w}}. \tag{3.12}$$

On the other hand if we assume that $(\Lambda^0_{\varphi,w})_a$ is an M-ideal of $\Lambda^0_{\varphi,w}$, then $1 = \|H + S\| = \|h\|_{\mathcal{M}_{\varphi_*,w}} + \|S\| \geq P_{\varphi_*,w}(h) + \|S\|$. It follows that $P_{\varphi_*,w}(h) + \|S\| = 1$. Indeed, suppose that $P_{\varphi_*,w}(h) + \|S\| < 1$. Define $g(\lambda) = P_{\varphi_*,w}(\lambda h) + \lambda\|S\|$ for $\lambda > 0$. The function g is convex, $g(0) = 0$, and $\lim_{\lambda\to\infty} g(\lambda) = \infty$. Since $g(1) = P_{\varphi_*,w}(h) + \|S\| < 1$, there exists $\frac{1}{\lambda_0} > 1$ such that $P_{\varphi_*,w}\left(\frac{h}{\lambda_0}\right) + \frac{1}{\lambda_0}\|S\| = 1$. But then, from Theorem 3.10, we have $1 = \|H + S\| = \inf\{\lambda > 0 : P_{\varphi_*,w}(\frac{h}{\lambda}) + \frac{1}{\lambda}\|S\| \leq 1\} > 1$, which is a contradiction.

However $P_{\varphi_*,w}(h) + \|S\| = 1$ contradicts (3.12), and this completes the proof. □

3.3 M-embedded Orlicz-Lorentz spaces

In this section, we investigate M-embeddedness of Orlicz-Lorentz spaces.

Definition 3.13. A Banach space X is said to be M-embedded if X is an M-ideal in its bidual X^{**}.

The description of an M-ideal in Definition 3.1 was given in terms of bounded linear functionals. Also, there is a geometric description of M-ideal properties by using the intersection of balls in X [23].We have the following facts which will be useful.

Lemma 3.14 ([52], Lemma 1.1)**.** Suppose that there is a dense subset $\mathcal{D}$ of B_X such that for every $x^{**} \in B_{X^{**}}$, $x \in \mathcal{D}$, and $\epsilon > 0$, there exists $y \in X$ such that $\|x^{**} \pm x - y\| \leq 1 + \epsilon$. Then, X is an M-ideal in its bidual X^{**}.

Theorem 3.15 ([40], Theorem 2.6)**.** Assume that a Banach space X is an M-ideal in its bidual X^{**}. If Y is a separable closed subspace of X, then Y^* is separable.

The following result from theory of Banach lattices will be helpful.

Theorem 3.16. [8, Corollary 1.4.3] The dual X^* of a Banach function lattice X is isometrically isomorphic to the Köthe dual space X' if and only if X is order-continuous.

For the case of Orlicz spaces equipped with the Luxemburg norm, it is shown that the order-continuous subspace $(L_\varphi)_a$ is M-embedded if φ does not satisfy the appropriate Δ_2 condition while φ_* does [52]. Now, we show that this is also true for $(\Lambda_{\varphi,w})_a$ and $(\lambda_{\varphi,w})_a$.

Theorem 3.17.

(1) If both φ and φ_* satisfy the appropriate Δ_2 condition, then the order-continuous subspace $(\Lambda_{\varphi,w})_a$ is an M-ideal in its bidual $((\Lambda_{\varphi,w})_a)^{**} \simeq \Lambda_{\varphi,w}$.

(2) If both φ and φ_* does not satisfy the appropriate Δ_2 condition, then the order-continuous subspace $(\Lambda_{\varphi,w})_a$ is not an M-ideal in its bidual.

(3) If φ does not satisfy the appropriate Δ_2 condition while φ_* does, then the order-continuous subspace $(\Lambda_{\varphi,w})_a$ is an M-ideal in its bidual $((\Lambda_{\varphi,w})_a)^{**} \simeq \Lambda_{\varphi,w}$.

Proof. (1) Since φ satisfies the appropriate Δ_2 condition, $\Lambda_{\varphi,w} = (\Lambda_{\varphi,w})_a$ [24]. In view of Theorem 1.6 and Theorem 3.16, $((\Lambda_{\varphi,w})_a)^* \simeq (\Lambda_{\varphi,w})' = \mathcal{M}^0_{\varphi_*,w}$. Moreover, φ_* satisfies the appropriate Δ_2 condition, so $\mathcal{M}^0_{\varphi_*,w} = (\mathcal{M}^0_{\varphi_*,w})_a$ by Corollary 2.16. Since $\Lambda_{\varphi,w}$ has the Fatou property and $((\mathcal{M}^0_{\varphi,w})_a)^* \simeq (\mathcal{M}^0_{\varphi,w})'$ from Theorem 3.16, we have $((\Lambda_{\varphi,w})_a)^{**} \simeq ((\Lambda_{\varphi,w})_a)'' = (((\Lambda_{\varphi,w})_a)')' = (\mathcal{M}^0_{\varphi_*,w})' = \Lambda_{\varphi,w} = (\Lambda_{\varphi,w})_a$. Hence the space $(\Lambda_{\varphi,w})_a$ is reflexive. Therefore, $(\Lambda_{\varphi,w})_a$ is an M-ideal in its bidual trivially. (2) Suppose that both φ and φ_* does not satisfy the appropriate Δ_2 condition. Assume to the contrary that $(\Lambda_{\varphi,w})_a$ is an M-ideal in its bidual $((\Lambda_{\varphi,w})_a)^{**}$. In view of Theorem 3.15, the dual of $(\Lambda_{\varphi,w})_a$ has to be separable. From Theorem 3.16 and Theorem 1.6, we see that $((\Lambda_{\varphi,w})_a)^* \simeq (\Lambda_{\varphi,w})' = \mathcal{M}^0_{\varphi_*,w}$, but the space $\mathcal{M}^0_{\varphi_*,w}$ is not separable by Corollary 2.16. Thus, $(\Lambda_{\varphi,w})_a$ is not an M-ideal in its bidual $((\Lambda_{\varphi,w})_a)^{**}$.

(3) Suppose that φ does not satisfy the appropriate Δ_2 condition, but φ_* does. Notice that $((\Lambda_{\varphi,w})_a)^* \simeq (\Lambda_{\varphi,w})' = \mathcal{M}^0_{\varphi_*,w}$ from Theorem 1.6 and Theorem 3.16. By Corollary 2.16, $\mathcal{M}^0_{\varphi_*,w} = (\mathcal{M}^0_{\varphi_*,w})_a$, so $(\mathcal{M}^0_{\varphi_*,w})^* = ((\mathcal{M}^0_{\varphi_*,w})_a)^* \simeq (\mathcal{M}^0_{\varphi_*,w})' = \Lambda_{\varphi,w}$. Hence $((\Lambda_{\varphi,w})_a)^{**} \simeq \Lambda_{\varphi,w}$.

Let $f \in B_{(\Lambda_{\varphi,w})_a}$ and $g \in \mathcal{D} = B_{(\Lambda_{\varphi,w})_b}$, and $\epsilon > 0$ be arbitrary. Notice that $\rho_{\varphi,w}(g) \le 1$. Define $\bar{g} = \frac{1}{1+\epsilon} g$. From the convexity of φ, observe that $\rho_{\varphi,w}\left(\frac{g-\bar{g}}{\epsilon}\right) = \int_I \varphi\left(\frac{1}{1+\epsilon} g^*\right) w \le \frac{1}{1+\epsilon}\rho_{\varphi,w}(g) < 1$. This shows that $\|g-\bar{g}\| \le \epsilon$. Furthermore, we see that $\rho_{\varphi,w}(\bar{g}) \le \frac{1}{1+\epsilon}\rho_{\varphi,w}(g) < 1$, so there exists $\delta > 0$ such that $\rho_{\varphi,w}(\bar{g}) = 1 - \delta$.

Now, for $f \in (\Lambda_{\varphi,w})_a$, let $f_n = f\chi_{\{\frac{1}{n} \le |f| \le n\}}$. By the fact that the distribution function $d_f(\lambda) < \infty$ for every $\lambda > 0$, since $|f - f_n| \downarrow 0$ a.e. and $|f - f_n| \le |f|$, we have $(f-f_n)^* \to 0$ by Theorem 1.2. This implies that $\varphi((f-f_n)^*)w \to 0$. In addition, $\rho_{\varphi,w}(f) \le 1$. Hence $\lim_{n\to\infty} \int_I \varphi((f-f_n)^*)w = 0$ by the Lebesgue dominated convergence theorem. So there exists $n_0 \in \mathbb{N}$ such that $\int_I \varphi((f-f_{n_0})^*)w < \delta$. Define $h = f\chi_{\{\frac{1}{n_0} \le |f| \le n_0\} \cup \operatorname{supp} \bar{g}}$. Notice that $h \in (\Lambda_{\varphi,w})_a$ since h is bounded and $m(\{\frac{1}{n_0} \le |f| \le n_0\} \cup \operatorname{supp} \bar{g}) < \infty$. Also,

$$\int_I \varphi((f \pm \bar{g} - h)^*)w = \int_I \varphi((f\chi_{(\{|f| < \frac{1}{n_0}\} \cup \{|f| > n_0\}) \cap \operatorname{supp} \bar{g}^c} \pm \bar{g})^*)w.$$

The functions $f\chi_{(\{|f| < \frac{1}{n_0}\} \cup \{|f| > n_0\}) \cap \operatorname{supp} \bar{g}^c}$ and $\bar{g}$ have disjoint supports, so by the orthogonal subadditivity of $\rho_{\varphi,w}$, we have

$$\int_I \varphi((f\chi_{(\{|f|<\frac{1}{n_0}\}\cup\{|f|>n_0\})\cap \operatorname{supp}\bar{g}^c} \pm \bar{g})^*)w \leq \int_I \varphi((f\chi_{(\{|f|<\frac{1}{n_0}\}\cup\{|f|>n_0\})\cap \operatorname{supp}\bar{g}^c})^*)w + \int_I \varphi(\bar{g}^*)w.$$

Since $(\{|f| < \frac{1}{n_0}\} \cup \{|f| > n_0\}) \cap \operatorname{supp}\bar{g}^c \subset \{|f| < \frac{1}{n_0}\} \cup \{|f| > n_0\}$,

$$\int_I \varphi((f\chi_{(\{|f|<\frac{1}{n_0}\}\cup\{|f|>n_0\})\cap \operatorname{supp}\bar{g}^c})^*)w \leq \int_I \varphi((f - f_{n_0})^*)w < \delta. \tag{3.13}$$

Hence by the fact that $\int_I \varphi(\bar{g}^*)w = \rho_{\varphi,w}(\bar{g}) = 1 - \delta$ and by (3.13), we obtain

$$\int_I \varphi((f \pm \bar{g} - h)^*)w \leq \int_I \varphi((f\chi_{(\{|f|<\frac{1}{n_0}\}\cup\{|f|>n_0\})\cap \operatorname{supp}\bar{g}^c})^*)w + \int_I \varphi(\bar{g}^*)w < \delta + (1-\delta) = 1,$$

so $\|f \pm \bar{g} - h\| \leq 1$. Since $\|g - \bar{g}\| \leq \epsilon$, $\|f \pm g - h\| \leq \|f \pm \bar{g} - h\| + \|g - \bar{g}\| \leq 1 + \epsilon$. In view of Lemma 3.14, The proof is finished. □

We also prove the sequence analogue of Theorem 3.17.

Theorem 3.18.

(1) If both φ and φ_* satisfy the Δ_2^0 condition, then the order-continuous subspace $(\lambda_{\varphi,w})_a$ is an M-ideal in its bidual $((\lambda_{\varphi,w})_a)^{**} \simeq \lambda_{\varphi,w}$.

(2) If both φ and φ_* do not satisfy the Δ_2^0 condition, then the order-continuous subspace $(\lambda_{\varphi,w})_a$ is not an M-ideal in its bidual $((\lambda_{\varphi,w})_a)^{**}$.

(3) If φ does not satisfy the Δ_2^0 condition while φ_* does, then the order-continuous subspace $(\lambda_{\varphi,w})_a$ is an M-ideal in its bidual $((\lambda_{\varphi,w})_a)^{**} \simeq \lambda_{\varphi,w}$.

Proof. The proofs for (1) and (2) are similar to Theorem 3.17, so we only prove (3). Suppose that φ does not satisfy the Δ_2^0 condition but φ_* does. Notice that $((\lambda_{\varphi,w})_a)^* \simeq (\lambda_{\varphi,w})' = \mathfrak{m}^0_{\varphi_*,w}$ from Theorem 1.6 and Theorem 3.16. By Corollary

2.16, $\mathfrak{m}^0_{\varphi*,w} = (\mathfrak{m}^0_{\varphi*,w})_a$, so $(\mathfrak{m}^0_{\varphi*,w})^* = ((\mathfrak{m}^0_{\varphi*,w})_a)^* \simeq (\mathfrak{m}^0_{\varphi*,w})' = \lambda_{\varphi,w}$. Hence $((\lambda_{\varphi,w})_a)^{**} \simeq \lambda_{\varphi,w}$.

Let $a = (a(i)) \in B_{(\Lambda_{\varphi,w})_a}$, $x = (x(i)) \in \mathcal{D} = B_{(\Lambda_{\varphi,w})_b}$ and $\epsilon > 0$ be arbitrary. Assume that there exists $N_1 \in \mathbb{N}$ such that $x(i) = 0$ for $i > N_1$ for some $N_1 \in \mathbb{N}$. Now, we define $\bar{x} = (\bar{x}(i))$ where $\bar{x}(i) = \frac{x(i)}{1+\epsilon}$, $i \in \mathbb{N}$. Then by the convexity of φ, we have

$$\begin{aligned}\alpha_{\varphi,w}\left(\frac{x-\bar{x}}{\epsilon}\right) = \sum_{i=1}^{\infty}\varphi\left(\left(\left(\frac{x}{1+\epsilon}\right)\chi_{\{1,2,\ldots,N_1\}}\right)^*(i)\right)w(i) &\leq \sum_{i=1}^{N_1}\varphi\left(\frac{1}{1+\epsilon}x^*(i)\right)w(i)\\ &\leq \frac{1}{1+\epsilon}\alpha_{\varphi,w}(x) < 1,\end{aligned}$$

so $\|x-\bar{x}\| \leq \epsilon$. Since the modular $\rho_{\varphi,w}$ is convex,

$$\alpha_{\varphi,w}(\bar{x}) = \alpha_{\varphi,w}\left(\frac{x}{1+\epsilon}\right) \leq \frac{1}{1+\epsilon}\alpha_{\varphi,w}(x) < 1.$$

Hence there exists $\delta > 0$ such that $\alpha_{\varphi,w}(\bar{x}) = 1-\delta$.

Now, let $a_k = a\chi_{\{1,2,\ldots k\}}$. The distribution function $d_a(\lambda) < \infty$ for every $\lambda > 0$, so we have $(a-a_k)^* \to 0$ because $|a-a_k| \downarrow 0$ by Theorem 1.2. Hence from the fact that $\alpha_{\varphi,w}(a) \leq 1$, we can choose $N_2 > N_1$ such that $\sum_{i=1}^{\infty}\varphi((a\chi_{\{N_2,N_2+1,\ldots\}})^*(i))w(i) < \delta$ by the Lebesgue dominated convergence theorem. Define $y = (y(i))$ where $y(i) = a(i)$ for $i < N_2$ and $y(i) = 0$ otherwise. By the orthogonal subadditivity of $\alpha_{\varphi,w}$, we obtain

$$\begin{aligned}\sum_{i=1}^{\infty}\varphi\left((a\pm\bar{x}-y)^*(i)\right)w(i) &= \sum_{n=1}^{\infty}\varphi((a\chi_{\{N_2,N_2+1,\ldots\}} \pm \frac{x}{1+\epsilon}\chi_{\{1,2,\ldots,N_1\}})^*(i))w(i)\\ &\leq \sum_{n=1}^{\infty}\varphi\left(\left(\frac{x}{1+\epsilon}\chi_{\{1,2,\ldots,N_1\}}\right)^*(i)\right)w(i) + \sum_{n=1}^{\infty}\varphi((a\chi_{\{N_2,N_2+1,\ldots\}})^*(i))w(i).\end{aligned}$$

Hence $\sum_{i=1}^{\infty}\varphi((a\pm\bar{x}-y)^*(i))w(i) < 1-\delta+\delta = 1$, so $\|a\pm\bar{x}-y\| \leq 1$. From the fact that $\|x-\bar{x}\| \leq \epsilon$, we get $\|a\pm x-y\| \leq \|a\pm\bar{x}-y\| + \|x-\bar{x}\| \leq 1+\epsilon$. The proof is finished. □

3.4 On a unique norm-preserving extension of a bounded linear functional on Orlicz-Lorentz spaces

In view of Proposition 1.12 in [23], since $(\Lambda_{\varphi,w})_a$ is an M-ideal in $\Lambda_{\varphi,w}$, every integral functional $H \in (\Lambda_{\varphi,w})_a^*$ has a unique norm-preserving extension to $\Lambda_{\varphi,w}$. We consider the case when the space is equipped with the Orlicz norm.

The following Lemma is a corollary of Theorem 3.23.

Lemma 3.19. A regular linear functional $H \in (\Lambda^0_{\varphi,w})^*_r$ generated by a function $h \in (\Lambda^0_{\varphi,w})' = \mathcal{M}_{\varphi,w}$ attains its norm at $f \in S_{(\Lambda^0_{\varphi,w})_a}$ if and only if for some $k \in K(f)$, $P_{\varphi_*,w}(h/\|h\|_{\mathcal{M}_{\varphi,w}}) = 1$ and $\int_I \frac{khf}{\|h\|_{\mathcal{M}_{\varphi_*,w}}} = \rho_{\varphi,w}(kf) + P_{\varphi_*,w}(\frac{h}{\|h\|_{\mathcal{M}_{\varphi_*,w}}})$.

Now, we are ready to present the main results of this section.

Proposition 3.20. If H is a bounded linear functional on $(\Lambda^0_{\varphi,w})_a$, then it has a norm-preserving extension to the whole space $\Lambda^0_{\varphi,w}$.

Proof. Let H be a bounded linear functional on $(\Lambda^0_{\varphi,w})_a$. Then, there exists $h \in \mathcal{M}_{\varphi_*,w}$ such that $H(f) = \int hf$ for $f \in (\Lambda^0_{\varphi,w})_a$. Without loss of generality, assume $h \geq 0$. Define an extension of H to $\Lambda^0_{\varphi,w}$, denoted by $\tilde{H}$, such that $\tilde{H}(f) = \int_I hf$ for $f \in \Lambda^0_{\varphi,w}$. $|\tilde{H}(f)| = |\int_I hf| \leq \|h\|_{\mathcal{M}_{\varphi,w}} \|f\|^0$ by Hölder's inequality, so $\tilde{H}$ is bounded.

From the definition of $\|\tilde{H}\|$, for every $\epsilon > 0$, we can choose $f \in B_{\Lambda^0_{\varphi,w}}$ such that $\|\tilde{H}\| - \frac{\epsilon}{2} < \int_I |hf|$. Define $f_n = |f|\chi_{\{\frac{1}{n} \leq |f| \leq n\}}$. Due to the fact that $d_f(\lambda) < \infty$ for all $\lambda > 0$, $f_n \in (\Lambda^0_{\varphi,w})_a$. Since $f_n \uparrow |f|$ a.e., we have $\lim_{n\to\infty} \int_I hf_n = \int_I h|f|$ by the monotone convergence theorem. Then for all $\epsilon > 0$, there exists N such that for every $n \geq N$, $\int_I |hf| < \int_I |hf_n| + \frac{\epsilon}{2}$, so $\|\tilde{H}\| < \int_I hf_n + \epsilon$. Since $f_n \in B_{(\Lambda^0_{\varphi,w})_a}$, we see that $\|\tilde{H}\| \leq \|H\|$. Also from the definition of the norm of a bounded linear functional, it is easy to see that $\|H\| \leq \|\tilde{H}\|$. Thus, $\|H\| = \|\tilde{H}\|$. □

Theorem 3.21. If H is a bounded linear functional on $(\Lambda^0_{\varphi,w})_a$ which attains its norm on $S_{(\Lambda^0_{\varphi,w})_a}$, then H has a unique norm-preserving extension to $\Lambda^0_{\varphi,w}$. Similarly, this is also true for the sequence spaces $\lambda^0_{\varphi,w}$.

Proof. The existence of a norm-preserving extension of a bounded linear functional H on $(\Lambda^0_{\varphi,w})_a$ to the whole space $\Lambda^0_{\varphi,w}$, denoted by $\tilde{H}$, is shown in Proposition 3.20. Now, we show that this extension is unique among regular functionals. Indeed, suppose that we have another norm-preserving extension of H, say $\tilde{G}$. For $f \in (\Lambda^0_{\varphi,w})_a$, we have $0 = H(f) - H(f) = (\tilde{H} - \tilde{G})(f)$. Since $\tilde{H}$ and $\tilde{G}$ are regular functionals, the only possibility is when $\tilde{H} = \tilde{G}$.

Moreover, we show that $H + S$ is not a norm-preserving extension if $S \neq 0$. Without loss of generality, assume $\|H\| = \|h\|_{\mathcal{M}_{\varphi,w}} = 1$. Since H is attains its norm on $S_{(\Lambda^0_{\varphi,w})_a}$, there exists $f \in S_{(\Lambda^0_{\varphi,w})_a}$ such that $H(f) = \int_I hf = \|h\|_{\mathcal{M}_{\varphi,w}}$. In view of Lemma 3.19, we have $P_{\varphi_*,w}(h) = 1$. Define $g(\lambda) = P_{\varphi_*,w}(\frac{h}{\lambda}) + \frac{1}{\lambda}\|S\|$, $\lambda > 0$. Note that the function $g(\lambda)$ is decreasing and continuous on the interval $(1, \infty)$. We see that $g(1) = P_{\varphi_*,w}(h) + \|S\| = 1 + \|S\| > 1$. Hence there exists $\lambda_0 > 1$ such that $P_{\varphi_*,w}(\frac{h}{\lambda_0}) + \frac{1}{\lambda_0}\|S\| > 1$. But then, in view of Theorem 3.10, $\|H + S\| \geq \lambda_0 > 1 = \|H\|$. Thus, if $S \neq 0$, $H + S$ is not norm-preserving, so $\tilde{H}$ is the only norm-preserving extension of H to $\Lambda^0_{\varphi,w}$. ☐

Theorem 3.22. The following are equivalent:

(1) Every bounded linear functional H on $(\Lambda^0_{\varphi,w})_a$ has a unique norm-preserving extension to $\Lambda^0_{\varphi,w}$;

(2) φ or φ_* satisfies the appropriate Δ_2 condition.

Proof. (1) $\Longrightarrow$ (2) Suppose neither φ nor φ_* satisfies appropriate Δ_2 condition. In this case, we have a bounded linear functional $F = H + S$ on $\Lambda^0_{\varphi,w}$, where $S \neq 0$. Moreover, in view of Theorem 2.11, there exists $h \in \mathcal{M}_{\varphi_*,w}$ such that $\|H\| = \|h\|_{\mathcal{M}_{\varphi_*,w}} = 1$ but $P_{\varphi_*,w}(h) < 1$. Choose $S \neq 0$ such that $\|S\| = 1 - P_{\varphi_*,w}(h)$.

Then we obtain $P_{\varphi_*,w}(h) + \|S\| = 1$. Define $f(\lambda) = P_{\varphi_*,w}(\frac{h}{\lambda}) + \frac{1}{\lambda}\|S\|$, $\lambda > 0$. The function $f(\lambda)$ is strictly decreasing, continuous on $[1, \infty)$, and $f(1) = 1$. In view of Theorem 3.10, observe that $1 \geq f(\|H + S\|) \geq f(1) = f(\|H\|) = 1$. Hence we have $\|H + S\| = \|H\|$. Let $\tilde{H}$ be an extension of a bounded linear functional H on $(\Lambda^0_{\varphi,w})_a$ to the whole space, as given in Proposition 3.20. Then $\|H + S\| = \|H\| = \|\tilde{H}\|$. However, $H + S \neq \tilde{H}$ because $S \neq 0$, so we have two distinct norm-preserving extensions of H in this case.

(2) $\Longrightarrow$ (1) If φ satisfies the appropriate Δ_2 condition, we have $(\Lambda^0_{\varphi,w})_a = \Lambda^0_{\varphi,w}$, so there is nothing to prove [24]. Hence we only consider when φ does not satisfy appropriate Δ_2 condition but φ_* does. Assume H be a bounded linear functional on $(\Lambda^0_{\varphi,w})_a$ such that $\|H\| = \|h\|_{\mathcal{M}_{\varphi_*,w}} = 1$. Let $\tilde{H}$ be an extension of H to $\Lambda^0_{\varphi,w}$ as given in Proposition 3.20. From such an extension, we also have $\|\tilde{H}\| = \|h\|_{\mathcal{M}_{\varphi_*,w}} = 1$. Now, we show that $P_{\varphi_*,w}(h) = 1$. Assume for the contrary that $P_{\varphi_*,w}(h) < 1$. Define a function $g(\lambda) = P_{\varphi_*,w}(\frac{h}{\lambda})$. In view of Theorem 2.13, the function g is continuous on $(0, \infty)$. Note that $g(\lambda)$ is a strictly decreasing function. Then there exists $\lambda_0 < 1$ such that $P_{\varphi_*,w}(\frac{h}{\lambda_0}) = 1$, which is a contradiction to the fact that $\|h\|_{\mathcal{M}_{\varphi,w}} = 1$.

If $S \neq 0$, we have $1 < P_{\varphi_*,w}(h) + \|S\| = 1 + \|S\|$. The function $f(\lambda) = P_{\varphi_*,w}\left(\frac{h}{\lambda}\right) + \frac{1}{\lambda}\|S\|$ is continuous on $(0, \infty)$ by Theorem 2.13 and by the fact that φ_* satisfies the appropriate Δ_2 condition and $f(1) > 1$. Hence there exists $\lambda_0 > 1$ such that $P_{\varphi_*,w}\left(\frac{h}{\lambda_0}\right) + \frac{1}{\lambda_0}\|S\| > 1$. But then, we have $\|H + S\| \geq \lambda_0 > 1 = \|H\|$. Thus, $\tilde{H}$ is the only norm-preserving extension of H to $\Lambda^0_{\varphi,w}$. □

3.5 Support functionals on Orlicz-Lorentz spaces

In this section, we characterize support functionals on Orlicz-Lorentz function and sequence spaces. Let X be a Banach space. A bounded linear functional $F \in X^*$ is said to be norm-attainable at $f \in S_X$ if $\|F\| = |F(f)|$. If a bounded functional F is norm-attainable, then the bounded linear functional $\frac{F}{\|F\|}$ is called a support functional at $f \in S_X$. Support functionals on Orlicz spaces equipped with both the Luxemburg norm and the Orlicz norm and were investigated in [13]. We extend this result to Orlicz-Lorentz spaces.

Theorem 3.23. Let $F = H + S$ be a bounded linear functional on $\Lambda^0_{\varphi,w}$, where $H(f) = \int_I fh$ for some $h \in \mathcal{M}_{\varphi_*,w}$ and $S(f) = 0$ at $f \in (\Lambda^0_{\varphi,w})_a$. Then F is norm-attainable at $f \in S_{\Lambda^0_{\varphi,w}}$ if and only if for some $k \in K(f)$,

(1) $P_{\varphi_*,w}(\frac{h}{\|F\|}) + \frac{\|S\|}{\|F\|} = 1$,

(2) $\|S\| = S(kf)$, and

(3) $\int_I \frac{khf}{\|F\|} = \rho_{\varphi,w}(kf) + P_{\varphi_*,w}(\frac{h}{\|F\|})$.

are satisfied.

Proof. Let $f \in S_{\Lambda^0_{\varphi,w}}$ satisfying $\|F\| = F(f)$ and $k \in K(f)$. Then we have $\rho_{\varphi,w}(kf) < \infty$. Indeed, notice that $1 = \|f\|^0 = \frac{1}{k}(1 + \rho_{\varphi,w}(kf))$ for $k \in K(f)$ from Theorem 3.2. Hence $\rho_{\varphi,w}(kf) = k - 1 < k^{**} < \infty$. Now, for every $v \prec w$, $v\downarrow$ and $h \in \mathcal{M}_{\varphi_*,w}$, Young's inequality gives us

$$\begin{aligned} 1 = \frac{F(f)}{\|F\|} = \frac{H(f)}{\|F\|} + \frac{S(f)}{\|F\|} &\leq \frac{1}{k}\left(\int_I \frac{kh^*f^*}{v\|F\|}v + \frac{S(kf)}{\|F\|}\right) \\ &\leq \frac{1}{k}\left(\int_I \varphi(kf^*)v + \int_I \varphi_*\left(\frac{h^*}{v\|F\|}\right)v + \frac{S(kf)}{\|F\|}\right). \end{aligned}$$

From Lemma 1.1, we have $\int_I \varphi(kf^*)v \leq \int_I \varphi(kf^*)w = \rho_{\varphi,w}(kf)$. Moreover, due to the fact that $v \prec w$ where $v\downarrow$,

$$1 \leq \frac{1}{k}\left(\rho_{\varphi,w}(kf) + P_{\varphi_*,w}\left(\frac{h}{\|F\|}\right) + \frac{S(kf)}{\|F\|}\right).$$

Since $\rho_{\varphi,w}(kf) < \infty$, $S(kf) \leq \|S\|$ by Theorem 3.5. In view of Theorem 3.10,

$$P_{\varphi_*,w}\left(\frac{h}{\|F\|}\right) + \frac{\|S\|}{\|F\|} \leq 1. \tag{3.14}$$

Indeed, let (λ_n) be a sequence of real numbers such that $(\lambda_n) \downarrow \|F\|$ and $P_{\varphi_*,w}(\frac{h}{\lambda_n}) + \frac{\|S\|}{\lambda_n} \leq 1$ for every $n \in \mathbb{N}$. Let $g(k) = P_{\varphi_*,w}(kh) + k\|S\|$ for $k > 0$. The function $g(k)$ is increasing and continuous on the interval $(0, 1/\bar{\theta})$, where $\bar{\theta} = \bar{\theta}(h) = \inf\left\{\lambda > 0 : P_{\varphi_*,w}\left(\frac{h}{\lambda}\right) < \infty\right\}$. Notice that $P_{\varphi_*,w}(\frac{h}{\lambda_n}) + \frac{\|S\|}{\lambda_n} \leq 1$ and $P_{\varphi_*,w}(\frac{h}{\lambda_n}) + \frac{\|S\|}{\lambda_n} \leq P_{\varphi_*,w}(\frac{h}{\lambda_{n+1}}) + \frac{\|S\|}{\lambda_{n+1}}$ for all $n \in \mathbb{N}$. Hence

$$\lim_{n\to\infty} P_{\varphi_*,w}\left(\frac{h}{\lambda_n}\right) + \frac{\|S\|}{\|F\|} = \lim_{n\to\infty}\left(P_{\varphi_*,w}(\frac{h}{\lambda_n}) + \frac{\|S\|}{\lambda_n}\right) \leq 1.$$

Since $\frac{1}{\lambda_n} \uparrow \frac{1}{\|F\|}$, $\varphi_*\left(\frac{(h^*)^0}{\lambda_n w}\right) w \uparrow \varphi_*\left(\frac{(h^*)^0}{\|F\| w}\right) w$ a.e. By the Monotone convergence theorem,

$$\lim_{n\to\infty} P_{\varphi_*,w}\left(\frac{h}{\lambda_n}\right) = \lim_{n\to\infty}\int_I \varphi_*\left(\frac{(h^*)^0}{\lambda_n w}\right) w = \int_I \varphi_*\left(\frac{(h^*)^0}{\|F\| w}\right) w = P_{\varphi_*,w}\left(\frac{(h^*)^0}{\|F\|}\right).$$

From (3.14), we obtain

$$\begin{aligned}\frac{1}{k}\left(\rho_{\varphi,w}(kf) + P_{\varphi_*,w}\left(\frac{h}{\|F\|}\right) + \frac{S(kf)}{\|F\|}\right) &\leq \frac{1}{k}\left(\rho_{\varphi,w}(kf) + P_{\varphi_*,w}\left(\frac{h}{\|F\|}\right) + \frac{\|S\|}{\|F\|}\right) \\ &\leq \frac{1}{k}(\rho_{\varphi,w}(kf) + 1).\end{aligned}$$

Notice that $1 = \|f\|^0 = \frac{1}{k}(\rho_{\varphi,w}(kf) + 1)$ from Theorem 3.2. Hence

$$\begin{aligned}1 \leq \frac{1}{k}\left(\rho_{\varphi,w}(kf) + P_{\varphi_*,w}\left(\frac{h}{\|F\|}\right) + \frac{S(kf)}{\|F\|}\right) &\leq \frac{1}{k}\left(\rho_{\varphi,w}(kf) + P_{\varphi_*,w}\left(\frac{h}{\|F\|}\right) + \frac{\|S\|}{\|F\|}\right) \\ &\leq \frac{1}{k}(\rho_{\varphi,w}(kf) + 1) = \|f\|^0 = 1.\end{aligned}$$

Therefore, (1), (2) and (3) are satisfied.

To show the converse, let $f \in S_{\Lambda^0_{\varphi,w}}$ and $k \in K(f)$. Suppose that the conditions

(1), (2) and (3) are satisfied. Then $1 = \|f\|^0 = \frac{1}{k}(1 + \rho_{\varphi,w}(kf))$ by Theorem 3.2. Let $F = H + S$ be a bounded linear functional on $\Lambda^0_{\varphi,w}$ where $H(f) = \int_I fh$ for some $h \in \mathcal{M}_{\varphi_*,w}$ and $S(f) = 0$ for every $f \in (\Lambda^0_{\varphi,w})_a$. Hence

$$1 = \frac{1}{k}(1+\rho_{\varphi,w}(kf)) \overset{(1)}{=} \frac{1}{k}\left(\rho_{\varphi,w}(kf) + P_{\varphi_*,w}\left(\frac{h}{\|F\|}\right) + \frac{\|S\|}{\|F\|}\right) \overset{(2),(3)}{=} \frac{1}{k}\left(\int_I \frac{khf}{\|F\|} + \frac{S(kf)}{\|F\|}\right).$$

So $1 = \frac{1}{k}\left(\int_I \frac{khf}{\|F\|} + \frac{S(kf)}{\|F\|}\right) = \frac{H(f)}{\|F\|} + \frac{S(f)}{\|F\|} = \frac{F(f)}{\|F\|}$. The proof is finished. □

By the similar argument, we have the sequence analogue of Theorem 3.23.

Theorem 3.24. Let $F = H + S$ be a bounded linear functional on $\lambda^0_{\varphi,w}$, where $H(f) = \sum_{i=1}^{\infty} x(i)h(i)$ for some $h \in \mathfrak{m}_{\varphi_*,w}$ and $S(f) = 0$ at $f \in (\lambda^0_{\varphi,w})_a$. Then F is norm-attainable at $f \in S_{\lambda^0_{\varphi,w}}$ if and only if for some $k \in K(x)$,

(1) $p_{\varphi_*,w}(\frac{h}{\|F\|}) + \frac{\|S\|}{\|F\|} = 1$,

(2) $\|S\| = S(kx)$, and

(3) $\sum_{i=1}^{\infty} \frac{kh(i)x(i)}{\|F\|} = \alpha_{\varphi,w}(kx) + p_{\varphi_*,w}(\frac{h}{\|F\|})$.

are satisfied.

We also provide the characterization of support functionals at $f \in S_{\Lambda_{\varphi,w}}$.

Theorem 3.25. Let $F = H + S$ be a bounded linear functional on $\Lambda_{\varphi,w}$, where $H(f) = \int_I fh$ for some $h \in \mathcal{M}^0_{\varphi_*,w}$ and $S(f) = 0$ at $f \in (\Lambda_{\varphi,w})_a$. Then F is norm-attainable at $f \in S_{\Lambda_{\varphi,w}}$ if and only if for some $k \in \bar{K}(h)$,

(1) $\rho_{\varphi,w}(f) = 1$,

(2) $\|S\| = S(f)$, and

(3) $\int_I khf = \rho_{\varphi,w}(f) + P_{\varphi_*,w}(kh)$.

Proof. Let $f \in S_{\Lambda_{\varphi,w}}$ and $k \in \bar{K}(h)$. Suppose that $\|F\|^0 = F(f) = H(f) + S(f)$. Then by Theorem 1.6, $H(f) + S(f) = \int_I fh + S(f)$ for some $h \in \mathcal{M}^0_{\varphi,w}$. By Young's inequality, for every $v \prec w, v\downarrow$,

$$\int_I hf + S(f) = \frac{1}{k}\int_I khf + S(f) \leq \frac{1}{k}\left(\int_I \frac{kh^* f^*}{v} v\right) + S(f)$$
$$\leq \frac{1}{k}\left(\int_I \varphi_*\left(\frac{kh^*}{v}\right) v + \int_I \varphi(f^*)v\right) + S(f).$$

From Lemma 1.1, we have $\int_I \varphi(f^*)v \leq \int_I \varphi(f^*)w = \rho_{\varphi,w}(f) \leq 1$. Since this is true for all $v \prec w, v\downarrow$, by the definition of the modular $P_{\varphi_*,w}(h)$, we obtain

$$\begin{aligned}\int_I hf + S(f) = \frac{1}{k}\int_I khf + S(f) &\leq \frac{1}{k}(\rho_{\varphi,w}(f) + P_{\varphi_*,w}(kh)) + S(f)\\ &\leq \frac{1}{k}(1 + P_{\varphi_*,w}(kh)) + S(f).\end{aligned}$$

Due to the fact that $S(f) \leq \|S\|$ and $k \in \bar{K}(h)$, in view of Theorem 2.5.(3), $\frac{1}{k}(1 + P_{\varphi_*,w}(kh)) + S(f) \leq \|h\|^0_{\mathcal{M}_{\varphi_*,w}} + \|S\|$. From Theorem 3.6 $\|h\|^0_{\mathcal{M}_{\varphi_*,w}} + \|S\| = \|F\|^0$. To summarize the proof we showed the following inequalities.

$$\begin{aligned}\|F\|^0 = F(f) = \frac{1}{k}\int_I khf + S(f) &\leq \frac{1}{k}(\rho_{\varphi,w}(f) + P_{\varphi_*,w}(kh)) + S(f)\\ &\leq \frac{1}{k}(1 + P_{\varphi_*,w}(kh)) + S(f)\\ &\leq \|h\|^0_{\mathcal{M}_{\varphi,w}} + \|S\| = \|F\|^0.\end{aligned}$$

Thus, (1), (2) and (3) are satisfied.

To show the converse, suppose that (1),(2) and (3) are satisfied. Let $F = H + S$ be a bounded linear functional on $\Lambda^0_{\varphi,w}$ where $H(f) = \int_I fh$ for some $h \in \mathcal{M}_{\varphi_*,w}$ and $S(f) = 0$ for every $f \in (\Lambda^0_{\varphi,w})_a$. For $k \in \bar{K}(h)$, $\|h\|^0_{\mathcal{M}_{\varphi_*,w}} = \frac{1}{k}(1 + P_{\varphi_*,w}(kh))$ by

Theorem 2.5. Let $f \in S_{\Lambda_{\varphi,w}}$. From Theorem 3.6, $\|F\|^0 = \|h\|_{\mathcal{M}^0_{\varphi_*,w}} + \|S\|$. Hence

$$\begin{aligned}\|F\|^0 = \|h\|^0_{\mathcal{M}_{\varphi_*,w}} + \|S\| \overset{(2)}{=} \frac{1}{k}(1 + P_{\varphi_*,w}(kh)) + S(f) &\overset{(1)}{=} \frac{1}{k}(\rho_{\varphi,w}(f) + P_{\varphi_*,w}(kh)) + S(f) \\ &\overset{(3)}{=} \frac{1}{k}\int_I khf + S(f) \\ &= H(f) + S(f) = F(f).\end{aligned}$$

Therefore, $F \in (\Lambda_{\varphi,w})^*$ is norm-attainable at $f \in S_{\Lambda_{\varphi,w}}$. □

By the similar argument, we can show the sequence analogue of Theorem 3.25.

Theorem 3.26. Let $F = H + S$ be a bounded linear functional on $\lambda_{\varphi,w}$, where $H(f) = \sum_{i=1}^{\infty} x(i)h(i)$ for some $h \in \mathfrak{m}^0_{\varphi_*,w}$ and $S(f) = 0$ at $f \in (\lambda_{\varphi,w})_a$. Then F is norm-attainable at $x \in S_{\lambda_{\varphi,w}}$ if and only if for some $k \in \bar{K}(h)$,

(1) $\alpha_{\varphi,w}(x) = 1$,

(2) $\|S\| = S(x)$, and

(3) $\sum_{i=1}^{\infty} kh(i)x(i) = \alpha_{\varphi,w}(x) + p_{\varphi_*,w}(kh)$.

CHAPTER 4

U-IDEAL PROPERTIES IN ORLICZ-LORENTZ SPACES

The concept of a u-ideal was introduced in a paper [19] by Godefroy, Kalton, and Saphar in 1993. First, we provide the definition of a u-summand and a u-ideal.

Definition 4.1. A closed subspace $Y \subset X$ is said to be a u-summand if there exists a subspace $Z \subset X$ such that $Y \oplus Z = X$ and $\|y + z\| = \|y - z\|$ for all $y \in Y$ and $z \in Z$.

Definition 4.2. A closed subspace $Y \subset X$ is a u-ideal in X if there exists a projection $P : X^* \to X^*$ such that $X^* = P(X^*) \oplus \ker P$, $\ker P = (I - P)(X^*) = Y^{\perp}$, and $Y^{\perp}$ is a u-summand.

Remark 4.3. Observe that if Y is an u-ideal in X, $\|I - 2P\| = 1$ where P is a projection. Indeed, since $Y^{\perp}$ is a u-summand, for every $x^* \in X^*$,

$$\|(I - 2P)x^*\|_{X^*} = \|(I - P)x^* - Px^*\|_{X^*} = \|(I - P)x^* + Px^*\|_{X^*} = \|x^*\|_{X^*}.$$

Hence $\|I - 2P\| = 1$.

Remark 4.4. If Y is an M-ideal in X, there exists a projection $P : X^* \to X^*$ such that $X^* = P(X^*) \oplus Y^{\perp}$ and $\|x^*\|_{X^*} = \|Px^*\|_{X^*} + \|(I - P)x^*\|_{X^*}$. Let $y \in P(X^*)$ and $z \in Y^{\perp}$. Then

$$\|y - z\| = \|y\| + \| - z\| = \|y\| + \|z\| = \|y + z\|.$$

Hence Y is a u-ideal in X.

We will need the following concept to define a strict u-ideal.

Definition 4.5. A subspace V of X^* is said to be norming if for every $x \in X$, $\|x\| = \sup\{|x^*x| : \|x^*\|_{X^*} \leq 1, x^* \in V\}$.

Definition 4.6. A subspace $Y \subset X$ is a strict u-ideal in X if Y is a u-ideal in X, and $P(X^*)$ is a norming subspace of X^*.

Proposition 4.7. [38, Proposition 1.b.18]. If X is a Banach function lattice with the Fatou property, then the Köthe dual space X' is a norming subspace of X^*

To show when the order-continuous spaces $(\Lambda_{\varphi,w})_a$ and $(\Lambda^0_{\varphi,w})_a$ are strict u-ideal in their biduals $(\Lambda_{\varphi,w})_a^{**}$ and $(\lambda_{\varphi,w})_a^{**}$, we need the lower and upper Matuszewska-Orlicz indices.

Definition 4.8. [28] Let φ be an Orlicz function. The lower Matuszewska-Orlicz index for $\mathbb{R}_+$ (resp. for small arguments; for large arguments), denoted by α_φ^a (resp. α_φ^0; α_φ^∞), and the upper Matuszewska-Orlicz index on $\mathbb{R}_+$ (resp. for small arguments; for large arguments), denoted by β_φ^a (resp. β_φ^0; β_φ^∞), are defined by

$$\alpha_\varphi^a = \sup\left\{p : \sup_{\substack{0<a\le 1\\ t>0}} \frac{\varphi(at)}{\varphi(t)a^p} < \infty\right\}; \qquad \beta_\varphi^a = \inf\left\{p : \inf_{\substack{0<a\le 1\\ t>0}} \frac{\varphi(at)}{\varphi(t)a^p} > 0\right\}$$

$$\alpha_\varphi^0 = \sup\left\{p : \sup_{\substack{0<a\le 1\\ 0<t<1}} \frac{\varphi(at)}{\varphi(t)a^p} < \infty\right\}; \qquad \beta_\varphi^0 = \inf\left\{p : \inf_{\substack{0<a\le 1\\ 0<t<1}} \frac{\varphi(at)}{\varphi(t)a^p} > 0\right\}$$

$$\alpha_\varphi^\infty = \sup\left\{p : \sup_{\substack{0<a\le 1\\ t\ge 1}} \frac{\varphi(at)}{\varphi(t)a^p} < \infty\right\}; \qquad \beta_\varphi^\infty = \inf\left\{p : \inf_{\substack{0<a\le 1\\ t\ge 1}} \frac{\varphi(at)}{\varphi(t)a^p} > 0\right\}.$$

Since φ is convex, $\alpha_\varphi^i \ge 1$, where $i = a, 0, \infty$. Indeed, we only show this for $i = a$ since the other cases are similar. If $\alpha_\varphi^a < 1$, there exists $t > 0$ such that $\frac{\varphi(at)}{\varphi(t)a} \ge 1$ for all $0 < a \le 1$. Hence $\varphi(at) \ge a\varphi(t)$ for all $0 < a \le 1$, and this implies that φ has to be concave if $\alpha_\varphi^a < 1$.

Also, observe that $\alpha_\varphi^a = \min\{a_\varphi^i : i = 0, \infty\}$. To show this, notice that for every $p < \alpha_\varphi^a$,

$$\sup_{\substack{0<a\le 1\\ 0<t<1}} \frac{\varphi(at)}{\varphi(t)a^p} \le \sup_{\substack{0<a\le 1\\ t>0}} \frac{\varphi(at)}{\varphi(t)a^p} < \infty \quad \text{and} \quad \sup_{\substack{0<a\le 1\\ t\ge 1}} \frac{\varphi(at)}{\varphi(t)a^p} \le \sup_{\substack{0<a\le 1\\ t>0}} \frac{\varphi(at)}{\varphi(t)a^p} < \infty,$$

and so $p < \alpha_\varphi^i$, where $i = 0, \infty$. Hence $\alpha_\varphi^a \leq \min\{a_\varphi^i : i = 0, \infty\}$. Now, suppose that $\alpha_\varphi^a < \min\{a_\varphi^i : i = 0, \infty\}$. Let $M = \min\{a_\varphi^i : i = 0, \infty\}$ for convenience. Then there exists $\alpha_\varphi^a < p < M$ such that

$$\sup_{\substack{0<a\leq 1\\ 0<t<1}} \frac{\varphi(at)}{\varphi(t)a^p} < \infty \quad \text{and} \quad \sup_{\substack{0<a\leq 1\\ t\geq 1}} \frac{\varphi(at)}{\varphi(t)a^p} < \infty.$$

We see that

$$\sup_{\substack{0<a\leq 1\\ t>0}} \frac{\varphi(at)}{\varphi(t)a^p} \leq \sup_{\substack{0<a\leq 1\\ 0<t<1}} \frac{\varphi(at)}{\varphi(t)a^p} + \sup_{\substack{0<a\leq 1\\ t\geq 1}} \frac{\varphi(at)}{\varphi(t)a^p} < \infty,$$

Hence, $p < \alpha_\varphi^a$, but this is a contradiction. Thus, we have $\alpha_\varphi^a = \min\{a_\varphi^i : i = 0, \infty\}$. The following fact is well-known in the theory of Orlicz spaces.

Theorem 4.9. [42, 45] $\alpha_\varphi^i > 1$, where $i = a, 0, \infty$ if and only if φ_* satisfies the appropriate Δ_2 condition.

In Corollary 4.29, Corollary 4.30, and Corollary 4.31, we will see that the lower Matuszewska-Orlicz indices of φ is related to the Δ_2 condition on its complementary function φ_*. To show this, we need certain growth conditions, namely, Δ^p and Δ^{*p}.

Definition 4.10. [28] An Orlicz function φ is said to satisfy the Δ^p (resp. Δ^p_∞; Δ^p_0) condition, where $p \geq 1$, if there exists $K > 0$ (resp. there exist $K > 0$ and $u_0 \geq 0$; there exist $K > 0$ and $u_0 > 0$ with $\varphi(u_0) > 0$) such that $\varphi(au) \leq Ka^p\varphi(u)$ for all $a \geq 1$ and $u \geq 0$ (resp. $u \geq u_0$; $u \leq au \leq u_0$)

An Orlicz function φ is said to satisfy the Δ^{*p} (resp. Δ^{*p}_∞; Δ^{*p}_0) condition, where $p \geq 1$, if there exists $K > 0$ (resp. $K > 0$ and $u_0 \geq 0$; $K > 0$ and $u_0 > 0$ with $\varphi(u_0) > 0$) such that $\varphi(au) \geq Ka^p\varphi(u)$ for all $a \geq 1$ and $u \geq 0$ (resp. $u \geq u_0$; $0 \leq au \leq u_0$).

We provide some facts which will be useful:

Theorem 4.11. [28] Given $p \geq 1$ all conditions below are equivalent:

(1) φ satisfies the condition Δ^{*p};

(2) There exists an Orlicz function ψ equivalent to φ such that for all $a \geq 1, u \geq 0$, $\psi(au) \geq a^p \psi(u)$;

(3) There exists an Orlicz function ψ equivalent to φ such that $\psi(u^{1/p})$ is convex.

Theorem 4.12. [28] For $1 < p < \infty$, an Orlicz function φ satisfies the appropriate Δ^{*p} condition if and only if φ^* satisfies the appropriate Δ^q condition, where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 4.13. [28] An Orlicz function φ satisfies the appropriate Δ_2 condition if and only if it satisfies the appropriate Δ^p condition for some $p \geq 1$.

In view of Theorem 4.9, Theorem 4.12, and Theorem 4.13, we can describe a relationship between the lower Matuszewska-Orlicz indices and the Δ_2 condition on the complementary function φ_*.

Corollary 4.14. Let φ be an Orlicz function. Then $\alpha_\varphi^i > 1$, where $i = a, 0, \infty$, if and only if φ satisfies the appropriate Δ^{*p} condition for some $p > 1$.

4.1 u-ideal and strict u-ideals in Orlicz-Lorentz spaces

In this section, we assume φ to be an Orlicz N-function. From Theorem 3.6 and Remark 4.4, we obtain the following consequence.

Corollary 4.15. The order-continuous subspace $(\Lambda_{\varphi,w})_a$ is a u-ideal in $\Lambda_{\varphi,w}$.

In the case of $\Lambda^0_{\varphi,w}$, we have $(\Lambda^0_{\varphi,w})^* = (\Lambda^0_{\varphi,w})^*_r \oplus (\Lambda^0_{\varphi,w})^*_s$ from Theorem 3.10. Hence

$$\|H - S\| = \|H + (-S)\| = \inf\left\{\lambda > 0 : P_{\varphi,w}\left(\frac{h}{\lambda}\right) + \frac{1}{\lambda}\|S\| \leq 1\right\} = \|H + S\|.$$

Thus, the following result holds.

Corollary 4.16. The order-continuous subspace $(\Lambda^0_{\varphi,w})_a$ is a u-ideal in $\Lambda^0_{\varphi,w}$.

Notice that $(\Lambda^0_{\varphi,w})_a$ is a u-ideal in $\Lambda^0_{\varphi,w}$ even though $(\Lambda^0_{\varphi,w})_a$ is not an M-ideal in $\Lambda^0_{\varphi,w}$. Since the Köthe dual spaces $((\Lambda_{\varphi,w})_a)' = \mathcal{M}^0_{\varphi_*,w}$ and $((\Lambda^0_{\varphi,w})_a)' = \mathcal{M}_{\varphi_*,w}$ are norming subspaces of $(\Lambda_{\varphi,w})^*$ and $(\Lambda^0_{\varphi,w})^*$ by Proposition 4.7, we have the following consequence.

Corollary 4.17. The order-continuous subspace $(\Lambda_{\varphi,w})_a$ is a strict u-ideal in $\Lambda_{\varphi,w}$. Also, the order-continuous subspace $(\Lambda^0_{\varphi,w})_a$ is a strict u-ideal in $\Lambda^0_{\varphi,w}$. The sequence analogues also hold.

4.2 u-ideals and strict u-ideals in biduals on Orlicz-Lorentz spaces

We start with the following fact.

Proposition 4.18. [19, Example 1] An order-continuous Banach lattice X is a u-ideal in its bidual X^{**}.

From Proposition 4.18, we have

Corollary 4.19. The order-continuous subspaces $(\Lambda_{\varphi,w})_a$, $(\Lambda^0_{\varphi,w})_a$, $(\lambda_{\varphi,w})_a$, and $(\lambda^0_{\varphi,w})_a$ are u-ideals in their biduals.

For strict u-ideals in biduals, we consider the following fact.

Proposition 4.20. [19, Example 2] An M-embedded Banach lattice X is a strict u-ideal in X^{**}.

Hence in view of Theorem 3.17 and Proposition 4.20, we have the following consequence.

Corollary 4.21. Let φ be an Orlicz N-function.

(1) If both φ and φ_* satisfy the appropriate Δ_2 condition, then $(\Lambda_{\varphi,w})_a$ is a strict u-ideal in its bidual $((\Lambda_{\varphi,w})_a)^{**} \simeq \Lambda_{\varphi,w}$.

(2) If φ does not satisfy the appropriate Δ_2 condition while φ_* does, then $(\Lambda_{\varphi,w})_a$ is a strict u-ideal in its bidual $((\Lambda_{\varphi,w})_a)^{**} \simeq \Lambda_{\varphi,w}$.

We also have the sequence analogue of Corollary 4.21 from Theorem 3.18 and Proposition 4.20.

Corollary 4.22. Let φ be an Orlicz N-function.

(1) If both φ and φ_* satisfy the Δ_2^0 condition, then $(\lambda_{\varphi,w})_a$ is a strict u-ideal in its bidual $((\lambda_{\varphi,w})_a)^{**} \simeq \lambda_{\varphi,w}$.

(2) If φ does not satisfy the Δ_2^0 condition while φ_* does, then $(\lambda_{\varphi,w})_a$ is a strict u-ideal in its bidual $((\lambda_{\varphi,w})_a)^{**} \simeq \lambda_{\varphi,w}$.

Now we use the general results to obtain the characterization of strict u-ideals in biduals for Orlicz-Lorentz spaces in both norms.

Proposition 4.23. [19, Example 1] An order-continuous Banach lattice X is a strict u-ideal in X^{**} if and only if X does not contain an isomorphic copy of l_1.

We have the characterizations of the order-continuous subspaces of Orlicz-Lorentz spaces containing an isomorphic copy of l_1. The following result is for $\lambda_{\varphi,w}$.

Theorem 4.24. [29, Theorem 6.13] Let φ be an Orlicz function. The following are equivalent:

(1) l^1 is isomorphic to a subspace of $(\lambda_{\varphi,w})_a$;

(2) $1 \in [\alpha_\varphi^0, \beta_\varphi^0]$.

Theorem 4.25. [29, Theorem 7.8] Let φ be an Orlicz function and $I = [0, \gamma)$, where $\gamma < \infty$. The following are equivalent:

(1) l^1 is isomorphic to a subspace of $(\Lambda_{\varphi,w})_a$;

(2) $1 \in [\alpha_\varphi^\infty, \beta_\varphi^\infty]$.

When $\gamma = \infty$, we have a slight different description.

Theorem 4.26. [29, Theorem 7.18] Let φ be an Orlicz function and $I = [0, \gamma)$, where $\gamma = \infty$. The following are equivalent:

(1) l^1 is isomorphic to a subspace of $(\Lambda_{\varphi,w})_a$;

(2) $1 \in [\alpha_\varphi^0, \beta_\varphi^0] \cup [\alpha_\varphi^\infty, \beta_\varphi^\infty]$ or there exists $c > 0$ such that $\int_0^\infty \varphi\left(\frac{c}{t}\right) w(t)dt < \infty$.

From Theorem 4.24 and Theorem 4.25, the order-continuous subspaces $(\lambda_{\varphi,w})_a$ and $(\Lambda_{\varphi,w})_a$ over $I = (0, \gamma)$, where $\gamma < \infty$, do not contain an isomorphic copy of l^1 if and only if $\alpha_\varphi^i > 1$, $i = 0, \infty$.

Now, we consider when $\gamma = \infty$. Notice that $1 \in [\alpha_\varphi^0, \beta_\varphi^0] \cup [\alpha_\varphi^\infty, \beta_\varphi^\infty]$ is equivalent to say $\alpha_\varphi^a = 1$ because $\alpha_\varphi^a = \min\{a_\varphi^i : i = 0, \infty\}$. So in view of Theorem 4.26, the order-continuous subspace $(\Lambda_{\varphi,w})_a$ does not contain an isomorphic copy of l^1 if and only if $\alpha_\varphi^a > 1$ and for every $c > 0$, $\int_0^\infty \varphi\left(\frac{c}{t}\right) w(t)dt = \infty$. In fact, the former implies the latter.

To show this, first we observe that for an Orlicz function $\psi \sim_i \varphi$, where $i = a, 0, \infty$, the lower Matuszewska-Orlicz indices of ψ and φ are the same. The proof of the following lemma is essentially the same as the proof of Lemma in [28] (See pg 117), so we will just state the result without proof.

Lemma 4.27. Let φ and ψ be Orlicz functions such that $\varphi \sim_a \psi$. Then, $\alpha_\varphi^a = \alpha_\psi^a$.

So, we are ready to prove our claim.

Lemma 4.28. Let φ be an Orlicz function. If $\alpha_\varphi^a > 1$, then $\int_0^\infty \varphi\left(\frac{c}{t}\right) w(t)dt = \infty$ for every $c > 0$.

Proof. Let $\alpha_\varphi^a > 1$. Then by Lemma 4.14 and Theorem 4.11, there exists $\psi \sim_a \varphi$ such that $\psi(u^{1/p})$ for some $p > 1$ is convex. In view of Lemma 4.27, we have

$\alpha_\varphi^a = \alpha_\psi^a$. Hence, without loss of generality, we assume that $\varphi(u^{1/p})$ is convex. Then we have

$$\int_0^\infty \varphi\left(\frac{c}{t}\right) w(t)dt = \int_0^\infty \varphi\left(\left(\left(\frac{c}{t}\right)^p\right)^{1/p}\right) w(t)dt \geq \int_0^c \varphi\left(\left(\left(\frac{c}{t}\right)^p\right)^{1/p}\right) w(t)dt.$$

When $t < c$, $\left(\frac{c}{t}\right)^p \geq 1$. From the fact that $\varphi(u^{1/p})$ is convex $p > 1$, and w is decreasing,

$$\int_0^c \varphi(((c/t)^p)^{1/p})w(t)dt \geq \int_0^c \frac{c^p}{t^p}\varphi(1)w(t)dt \geq c^p\varphi(1)\int_0^c \frac{w(t)}{t^p}dt \geq c^p\varphi(1)w(c)\int_0^c \frac{1}{t^p}dt = \infty.$$

Therefore, $\int_0^\infty \varphi\left(\frac{c}{t}\right) w(t) = \infty$. □

Based on what we discussed in this chapter, we provide a characterization of strict u-ideals on the bidual of the order-continuous subspaces of Orlicz-Lorentz spaces $\Lambda_{\varphi,w}$.

Corollary 4.29. Let φ be an Orlicz function. The following are equivalent:

(1) φ_* satisfies the Δ_2^0 condition;

(2) $\alpha_\varphi^0 > 1$;

(3) $(\lambda_{\varphi,w})_a$ does not contain an isomorphic copy of l^1;

(4) $(\lambda_{\varphi,w})_a$ is a strict u-ideal in its bidual $((\lambda_{\varphi,w})_a)^{**}$.

Corollary 4.30. Let φ be an Orlicz function and $I = [0, \gamma)$, where $\gamma < \infty$. The following are equivalent:

(1) φ_* satisfies the Δ_2^∞ condition;

(2) $\alpha_\varphi^\infty > 1$;

(3) $(\Lambda_{\varphi,w})_a$ does not contain an isomorphic copy of l^1;

(4) $(\Lambda_{\varphi,w})_a$ is a strict u-ideal in its bidual $((\Lambda_{\varphi,w})_a)^{**}$.

Corollary 4.31. Let φ be an Orlicz function and $I = [0, \gamma)$, where $\gamma = \infty$. Assume $W(\infty) = \infty$. The following are equivalent:

(1) φ_* satisfies the Δ_2 condition;

(2) $\alpha_\varphi > 1$;

(3) $(\Lambda_{\varphi,w})_a$ does not contain an isomorphic copy of l^1;

(4) $(\Lambda_{\varphi,w})_a$ is a strict u-ideal in its bidual $((\Lambda_{\varphi,w})_a)^{**}$.

CHAPTER 5

DIAMETER TWO PROPERTIES AND THE RADON-NIKODÝM PROPERTY IN ORLICZ-LORENTZ SPACES

Let X be a Banach space. For $\epsilon > 0$ and $x^* \in S_{X^*}$, we define a slice $S_{x^*,\epsilon} = \{x \in B_X : x^*(x) > 1 - \epsilon\}$ of the unit ball B_X. A slice $S_{x^*,\epsilon}$ is a relatively weakly open subset in B_X.

Definition 5.1. [1]

(1) A Banach space X is said to have the strong diameter two property when every convex combination of slices has diameter two.

(2) A Banach space X is said to have the diameter two property when every nonempty relatively weakly open subset of the unit ball B_X has diameter two.

(3) A Banach space X is said to have the local diameter two property when every slice of the unit ball B_X has diameter two.

Any nonempty relatively weakly open subset of B_X contains a convex combination of slices, which is shown by Bourgain [17]. Hence the strong diameter two property implies the diameter two property. Since a slice is a relatively weakly open subset of B_X, the diameter two property implies the local diameter two property. We also note that these properties are opposite from the Radon-Nikodým property due to the fact that a space with the Radon-Nikodým property has a slice with arbitrarily small diameter on the unit ball [9, 14].

In this chapter, φ is always an Orlicz N-function. The materials in Section 5.1 has been published in [35].

5.1 The Radon-Nikodým property and diameter two properties in Orlicz-Lorentz spaces

We provide a sufficient condition for Orlicz-Lorentz spaces to have Radon-Nikodým property.

Theorem 5.2. If φ satisfies the Δ_2 condition for $\gamma = \infty$, or φ satisfies the Δ_2^∞ condition for $\gamma < \infty$, then $\Lambda_{\varphi,w}$ is a separable dual space. Consequently $\Lambda_{\varphi,w}$ has the Radon-Nikodým property.

Proof. If a Banach function lattice X has the Fatou property and $X_a = X_b$ then $(X_a)^* = X'$ [8, Corollary 4.2, p. 23]. Now, in view of Theorem 1.6, consider the space $\mathcal{M}^0_{\varphi,w}$, which is the Köthe dual space of $\Lambda_{\varphi_*,w}$ since $\varphi_{**} = \varphi$. By the general theory [54], any Köthe dual space must satisfy the Fatou property. Hence $\mathcal{M}^0_{\varphi,w}$ satisfies this property.

By Theorem 5.5 in [8], if X is a r.i. Banach space on a non-atomic measure space and $\lim_{t\to 0^+} \phi_X(t) = 0$ then $X_a = X_b$. Thus, in view of Proposition 2.8, we have that $\lim_{t\to 0+} \phi_{\mathcal{M}}(t) = 0$ for the space $\mathcal{M}_{\varphi,w}$. Furthermore, this is also true for $\mathcal{M}^0_{\varphi,w}$ since the Luxemburg and Orlicz norms are equivalent [27]. Therefore, we have $(\mathcal{M}^0_{\varphi,w})_a = (\mathcal{M}^0_{\varphi,w})_b$.

Note that φ is an Orlicz N-function if and only if φ_* is an Orlicz N-function [11]. Therefore, all above facts remain true if we substitute φ by φ_*. By Theorem 1.6 and from the fact that $\Lambda_{\varphi,w}$ has the Fatou property, we see that $[(\mathcal{M}^0_{\varphi_*,w})_a]^* = (\mathcal{M}^0_{\varphi_*,w})' = (\Lambda_{\varphi,w})'' = \Lambda_{\varphi,w}$, so $\Lambda_{\varphi,w}$ is a dual space. By the Δ_2 condition, $W(\infty) = \infty$, and separability of the Lebesgue measure, $\Lambda_{\varphi,w}$ is separable [24, Theorem 2.4]. Hence from the well known result [14, 40], it must satisfy the Radon-Nikodým property. $\square$

From the fact that a Banach space X with the Radon-Nikodým property has a slice of arbitrarily small diameter on the unit ball [9, 14], we have the following consequence.

Corollary 5.3. If φ satisfies the Δ_2 condition for $\gamma = \infty$ or the Δ_2^∞ condition for $\gamma < \infty$, then there exists a slice of the unit ball $B_{\Lambda_{\varphi,w}}$ with arbitrarily small diameters.

For the sequence spaces, we obtain the similar result to the function spaces.

Theorem 5.4. If φ satisfies the Δ_2^0 condition then $\lambda_{\varphi,w}$ is a separable dual space. Consequently $\lambda_{\varphi,w}$ has the Radon-Nikodým property, and there exist relatively weakly open subsets of the unit ball $B_{\lambda_{\varphi,w}}$ with arbitrarily small diameters.

Proof. By Proposition 1 in [30], under the assumption of the Δ_2^0 condition of φ and $W(\infty) = \infty$, $(\lambda_{\varphi,w})_a = \lambda_{\varphi,w}$ and the unit vectors e_n form a boundedly complete basis in $\lambda_{\varphi,w}$. It follows that $(\lambda_{\varphi,w})_b = \lambda_{\varphi,w}$ and the space has the Fatou property. Then clearly the space is separable.

We also have by Theorem 5.4 in [8] that $(\lambda_{\varphi,w})^* = (\lambda_{\varphi,w})'$. Hence in view of Theorem 1.7, $\mathfrak{m}^0_{\varphi,w} = (\lambda_{\varphi_*,w})'$, so the space $\mathfrak{m}^0_{\varphi,w}$ has the Fatou property.

Let $x = (x(i)) \in (\mathfrak{m}^0_{\varphi,w})_b$. Then $\|\sum_{i=1}^m x(i)e_i - x\|^0_{\mathfrak{m}} = \|x\chi_{\{m+1,m+2,\dots\}}\|^0_{\mathfrak{m}} \to 0$ as $m \to \infty$. Hence $x \in (\mathfrak{m}^0_{\varphi,w})_a$ [8, Proposition 3.2, p. 14]. Thus $(\mathfrak{m}^0_{\varphi,w})_a = (\mathfrak{m}^0_{\varphi,w})_b$. Now similarly as in the function case in view of [8, Corollary 4.2, p. 23], $[(\mathfrak{m}^0_{\varphi,w})_a]^* = (\mathfrak{m}^0_{\varphi,w})'$. Finally by Theorem 1.7, $[(\mathfrak{m}^0_{\varphi_*,w})_a]^* = (\mathfrak{m}^0_{\varphi_*,w})' = \lambda_{\varphi,w}$, which shows that $\lambda_{\varphi,w}$ is a dual space. □

We show when Orlicz-Lorentz function spaces satisfy the diameter two property.

Theorem 5.5. If $\gamma = \infty$ and φ does not satisfy the Δ_2 condition, or $\gamma < \infty$ and φ does not satisfy the Δ_2^∞ condition, then the diameter of any nonempty relatively weakly open subset of the unit ball in Orlicz-Lorentz space $\Lambda_{\varphi,w}$ equipped with the Luxemburg norm is equal to two.

Proof. Let Z be a nonempty relatively weakly open subset of the unit ball in Orlicz-Lorentz space $\Lambda_{\varphi,w}$. Since m is non-atomic, $\Lambda_{\varphi,w}$ is infinite-dimensional. Then we can find an element $x \in Z$ such that $\|x\| = 1$. We have $d_x(\lambda) < \infty$ for any $\lambda > 0$. In fact

$$1 \geq \rho(x) = \int_0^\gamma \varphi(x^*)w \geq \int_{\{s\in[0,\gamma):\ x^*(s)>\lambda\}} \varphi(\lambda)w = \varphi(\lambda)\int_0^\beta w,$$

where $\beta \leq \infty$ is such that the intervals $(0,\beta)$ and $\{s \in [0,\gamma) : \ x^*(s) > \lambda\}$ have equal measure. By the assumption $W(\infty) = \infty$ if $\gamma = \infty$, we must have $\beta < \infty$ and so $d_x(\lambda) = d_{x^*}(\lambda) = \beta < \infty$. Choose $c > 0$ and a Lebesgue measurable set $E \subset [0,\gamma)$ with $mE > 0$ and $|x(t)| \leq c$ on E.

Suppose φ does not satisfy the Δ_2 condition when $\gamma = \infty$, or φ does not satisfy the Δ_2^∞ condition when $\gamma < \infty$. Then by Lemma 1.5 there exists $(t_n) \subset (0,\infty)$ such that for all $n \in \mathbb{N}$,

$$\varphi\left(\left(1 + \frac{1}{n}\right)t_n\right) > 2^n\varphi(t_n). \tag{5.1}$$

Assume without loss of generality that $t_n \uparrow \infty$ when $\gamma < \infty$ and $t_n \uparrow \infty$ or $t_n \downarrow 0$ when $\gamma = \infty$.

We consider first the case when $t_n \uparrow \infty$. Since $2^n\varphi(t_n) \to \infty$, so we can assume that $mE < \infty$. Then we choose a disjoint sequence of measurable sets $E_n \subset E$ such that for $n \in \mathbb{N}$,

$$\int_0^{m(E_n)} w = \frac{1}{2^n\varphi(t_n)}. \tag{5.2}$$

Indeed, since $\frac{1}{2^n\varphi(t_n)} \leq \frac{1}{2^n\varphi(t_1)}$, $\sum_{n=1}^\infty \frac{1}{2^n\varphi(t_n)} < \infty$, and so $\sum_{n=n_0}^\infty \frac{1}{2^n\varphi(t_n)} < \int_0^{m(E)} w$ for some $n_0 \in \mathbb{N}$. Then by $1/(2^n\varphi(t_n)) \downarrow 0$ and by nonatomicity of m, we can find a disjoint sequence of measurable sets $E_k \subset E$ such that $\int_0^{m(E_k)} w = \frac{1}{2^{n_0+k}\varphi(t_{n_0+k})}$ where $k \in \mathbb{N}$. Without loss of generality we can assume further that $n_0 = 0$. Clearly $mE_n \to 0$.

Now let $t_n \downarrow 0$ and $\gamma = \infty$. Then we can still choose a disjoint sequence of measurable sets (E_n) satisfyting equation (5.2) and $E_n \subset E$, where $mE = \infty$. Indeed there exists (I_n), which is an increasing sequence of measurable subsets of $[0, \infty)$ such that $\cup_{n=1}^{\infty} I_n = [0, \infty)$ and $m(I_n) < \infty$. Thus $E = \cup_{n=1}^{\infty} E \cap I_n$, where $m(E \setminus I_n) = \infty$. By continuity of W, $W(0) = 0$, and $W(m(E \setminus I_n)) = W(\infty) = \infty$, there exist $a_n > 0$ such that $\int_0^{a_n} w = \frac{1}{2^n \varphi(t_n)}$ for $n \in \mathbb{N}$. From the fact that $m(E \setminus I_n) = \infty$, there exists a disjoint sequence (E_n) of measurable sets satisfying (5.2) and such that $E_n \subset E \setminus I_n$ with $mE_n = a_n$. In this case, by (5.1), we have $mE_n \to \infty$.

Define

$$x_n' = x\chi_{I \setminus E_n} + t_n \chi_{E_n} \quad \text{and} \quad x_n'' = x\chi_{I \setminus E_n} - t_n \chi_{E_n}.$$

Note first that $x_n' \to x$ and $x_n'' \to x$ m-a.e. on I from the fact that E_n are disjoint. By the Fatou property of $\Lambda_{\varphi,w}$, $1 = \|x\| \le \liminf \|x_n'\|$ and $1 = \|x\| \le \liminf \|x_n''\|$. We will show that $\lim_{n\to\infty} \|x_n'\| = 1$ and $\lim_{n\to\infty} \|x_n''\| = 1$. Indeed, by orthogonal subadditivity of $\rho_{\varphi,w}$,

$$\rho_{\varphi,w}(x_n') = \int_0^\gamma \varphi(x\chi_{I\setminus E_n} + t_n\chi_{E_n})^* w \le \int_0^\gamma \varphi(x\chi_{I\setminus E_n})^* w + \int_0^\gamma \varphi(t_n\chi_{E_n})^* w.$$

Then Lemma 1.3 gives us

$$\int_0^\gamma \varphi(x\chi_{I\setminus E_n})^* w + \int_0^\gamma \varphi(t_n\chi_{E_n})^* w \le \int_0^\gamma \varphi(|x|)^* \chi_{[0, m(I\setminus E_n))} w + \int_0^{mE_n} \varphi(t_n) w.$$

From (5.2),

$$\int_0^\gamma \varphi(|x|)^* \chi_{[0, m(I\setminus E_n))} w + \int_0^{mE_n} \varphi(t_n) w \le \int_0^\gamma \varphi(x^*) w + \frac{1}{2^n} = \rho_{\varphi,w}(x) + \frac{1}{2^n}.$$

Hence $\limsup_{n\to\infty} \rho_{\varphi,w}(x_n') \le \rho(x) \le 1$. Then for any $\epsilon > 0$, there exists n_0 such that for all $n \ge n_0$, $\sup_{k\ge n} \rho_{\varphi,w}(x_k') \le 1 + \epsilon$. It follows from convexity of $\rho_{\varphi,w}$ that $\sup_{k\ge n} \rho_{\varphi,w}\left(\frac{x_k'}{1+\epsilon}\right) \le 1$. Therefore, for all $n \ge n_0$, $\|x_n'\| \le 1 + \epsilon$. This implies that

$\limsup_{n\to\infty} \|x_n'\| \leq 1$ and proves that $\lim_{n\to\infty} \|x_n'\| = 1$. Analogously, we get that $\lim_{n\to\infty} \|x_n''\| = 1$.

Let F be a bounded linear functional on $\Lambda_{\varphi,w}$. Then $F = H + S$, where H is a regular functional associated to $h \in \mathcal{M}^0_{\varphi_*,w}$ and S a singular functional which corresponding value is zero on $(\Lambda_{\varphi,w})_a$. We show that $x - x_n' \in (\Lambda_{\varphi,w})_a$. Since the function $|x|$ is bounded by c on $E_n \subset E$, we have

$$|(x - x_n')(t)| = |x(t) - t_n|\chi_{E_n}(t) \leq (|x(t)| + t_n)\chi_{E_n}(t) \leq (c + t_n)\chi_{E_n}(t)$$

on I. Then for any $\lambda > 0$, by Lemma 1.3,

$$\rho(\lambda(x - x_n')) \leq \int_0^\gamma \varphi(\lambda(c + t_n)\chi_{E_n})^* w = \varphi(\lambda(c + t_n)) \int_0^{mE_n} w < \infty,$$

which shows that $x - x_n' \in (\Lambda_{\varphi,w})_a$. Hence $S(x - x_n') = 0$ and $F(x - x_n') = H(x - x_n') = \int_{E_n} xh - \int_{E_n} t_n h$.

Assume first when $t_n \uparrow \infty$. So $E_n \subset E$ and $mE_n \to 0$. Since $h \in \mathcal{M}_{\varphi_*,w}$, $P_{\varphi_*,w}(\lambda h) < \infty$ for some $\lambda > 0$, and if we let $0 \leq v \in L^0$ and $v \prec w$, we obtain

$$\begin{aligned} |F(x - x_n')| &\leq \lambda^{-1} \int_I |x\chi_{E_n}|\,|\lambda h| dm + \lambda^{-1} \int_I t_n \chi_{E_n} |\lambda h| dm \\ &\leq \lambda^{-1} \int_0^\gamma (x\chi_{E_n})^* \,(\lambda h)^* + \lambda^{-1} \int_0^\gamma (t_n\chi_{E_n})^* (\lambda h)^* \quad \text{(by Lemma 1.3)} \\ &\leq \lambda^{-1} \int_0^{mE_n} x^* \,(\lambda h^*) + \lambda^{-1} \int_0^{mE_n} t_n (\lambda h^*) \\ &= \lambda^{-1} \int_0^{mE_n} x^* \left(\frac{\lambda h^*}{v}\right) v + \lambda^{-1} \int_0^{mE_n} t_n \left(\frac{\lambda h^*}{v}\right) v. \end{aligned}$$

From Young's inequality,

$$\lambda^{-1} \int_0^{mE_n} x^* \left(\frac{\lambda h^*}{v}\right) v \leq \lambda^{-1} \left(\int_0^{mE_n} \varphi(x^*) v + \int_0^{mE_n} \varphi_* \left(\frac{\lambda h^*}{v}\right) v \right),$$

and

$$\lambda^{-1}\int_0^{mE_n} t_n\left(\frac{\lambda h^*}{v}\right)v \le \lambda^{-1}\left(\int_0^{mE_n}\varphi(t_n)v+\int_0^{mE_n}\varphi_*\left(\frac{\lambda h^*}{v}\right)v\right).$$

Due to the fact that $v \prec w$, where $v\downarrow$, we have $\int_0^{m(E_n)}\varphi(x^*)v \le \int_0^{m(E_n)}\varphi(x^*)w$ by Lemma 1.1 and $\varphi(t_n)\int_0^{m(E_n)} v \le \varphi(t_n)\int_0^{m(E_n)} w$. Taking the infimum of $\int_0^{m(E_n)}\varphi_*\left(\frac{\lambda h^*}{v}\right)v$ over $v \prec w$, we get

$$|F(x-x_n')| \le \frac{1}{\lambda}\left(\int_0^{mE_n}\varphi(x^*)w + 2\inf\left\{\int_0^{mE_n}\varphi_*\left(\frac{\lambda h^*}{v}\right)v : v\prec w\right\}+\varphi(t_n)\int_0^{mE_n} w\right).$$

By (5.2), we have $\varphi(t_n)\int_0^{mE_n} w = 1/2^n \to 0$. Moreover, $\rho_{\varphi,w}(x)$ is finite and $mE_n \to 0$, so $\int_0^{mE_n}\varphi(x^*)w \to 0$. Since $P_{\varphi_*,w}(\lambda h)$ is finite, there exists $v_1 \prec w$ such that $\int_0^{\gamma}\varphi_*\left(\frac{\lambda h^*}{v_1}\right)v_1 \le P_{\varphi_*,w}(\lambda h^*)+1<\infty$. Hence $\int_0^{mE_n}\varphi_*\left(\frac{\lambda h^*}{v_1}\right)v_1 \to 0$ as $n\to\infty$, so $\inf\left\{\int_0^{mE_n}\varphi_*\left(\frac{\lambda h^*}{v}\right)v : v\prec w\right\} \le \int_0^{mE_n}\varphi_*\left(\frac{\lambda h^*}{v_1}\right)v_1 \to 0$. Thus $|F(x-x_n')|\to 0$ as $n\to\infty$. We can show the similar claim for x_n'', so both (x_n') and (x_n'') converge weakly to x.

Now consider when $\gamma=\infty$ and $t_n\downarrow 0$. For some $\lambda>0$, $P_{\varphi_*,w}(\lambda h)<\infty$. Then there exists $0\le v\in L^0$, $v\prec w$ with $\int_I \varphi_*\left(\lambda|h|/v\right)v \le P_{\varphi_*,w}(\lambda h)+1<\infty$. By the form of F and Young's inequality,

$$\begin{aligned}
|F(x-x_n')| &\le \lambda^{-1}\int_I |x\chi_{E_n}||\lambda h| + \lambda^{-1}\int_I t_n\chi_{E_n}|\lambda h| \\
&= \lambda^{-1}\int_{E_n}|x|\left(\frac{|\lambda h|}{v}\right)v+\lambda^{-1}\int_{E_n}t_n\left(\frac{|\lambda h|}{v}\right)v \\
&\le \lambda^{-1}\left(\int_{E_n}\varphi(|x|)v + 2\int_{E_n}\varphi_*\left(\frac{\lambda|v|}{v}\right)v+\varphi(t_n)\int_{E_n}v\right).
\end{aligned}$$

In view of Lemma 1.1, $\int_I\varphi(|x|)v \le \int_I\varphi(x^*)v^* \le \int_I\varphi(x^*)w=\rho(x)<\infty$. Due to the construction of E_n we get $E_n\subset I\setminus I_n$, where the sequence $(I\setminus I_n)$ is decreasing with $m\bigcap(I\setminus I_n)=0$. Therefore as $n\to\infty$,

$$\int_{E_n}\varphi(|x|)v \le \int_{I\setminus I_n}\varphi(|x|)v \to 0.$$

By the choice of v, $\int_I \varphi_*(\lambda|h|/v)v < \infty$, so we have

$$\int_{E_n} \varphi_* \left(\frac{\lambda|h|}{v}\right) v \le \int_{I \setminus I_n} \varphi_* \left(\frac{\lambda|h|}{v}\right) v \to 0.$$

Finally by the fact that $\int_{E_n} v \le \int_0^{mE_n} v^* \le \int_0^{mE_n} w$ and by (5.2),

$$\varphi(t_n) \int_{E_n} v \le \varphi(t_n) \int_0^{mE_n} w = 1/2^n \to 0.$$

Consequently in both cases we have that $x'_n \to x$ and $x''_n \to x$ weakly.

Now, we compute the diameter of Z. For $n \in \mathbb{N}$,

$$\begin{aligned} \|x'_n - x''_n\| &= 2\|t_n \chi_{E_n}\| = 2 \inf \left\{ \lambda > 0 : \int_0^{mE_n} \varphi\left(\frac{t_n}{\lambda}\right) w \le 1 \right\} \\ &= 2 \inf \left\{ \lambda > 0 : \frac{t_n}{\lambda} \le \varphi^{-1}\left(\frac{1}{W(mE_n)}\right) \right\} \\ &= \frac{2t_n}{\varphi^{-1}\left(1/W(m(E_n))\right)} \overset{\text{by (5.2)}}{=} \frac{2t_n}{\varphi^{-1}(2^n \varphi(t_n))} \overset{\text{by (5.1)}}{\ge} \frac{2n}{n+1}. \end{aligned}$$

Hence $\|x'_n - x''_n\| \to 2$ as $n \to \infty$. Taking $f'_n = \frac{x'_n}{\|x'_n\|}$ and $f''_n = \frac{x''_n}{\|x''_n\|}$, for any bounded linear functional F on $\Lambda_{\varphi,w}$ we get

$$\begin{aligned} |F(x - f'_n)| &\le |F(x - x'_n)| + \left| F\left(x'_n - \frac{x'_n}{\|x'_n\|}\right) \right| \\ &\le |F(x - x'_n)| + \left|1 - \frac{1}{\|x'_n\|}\right| \|F\| \|x'_n\| \to 0, \end{aligned}$$

as $n \to \infty$, due to $\|x'_n\| \to 1$. Thus $f'_n \to x$ weakly. Similarly $f''_n \to x$ weakly. We also show that $\|f'_n - f''_n\| \to 2$. Indeed,

$$\begin{aligned} \|f'_n - f''_n\| &= \left\| \frac{x'_n}{\|x'_n\|} - x'_n + x'_n - \frac{x''_n}{\|x''_n\|} \right\| \ge \left\| \frac{x'_n}{\|x'_n\|} - x'_n \right\| - \left\| \frac{x''_n}{\|x''_n\|} - x'_n \right\| \\ &= \left| \frac{1}{\|x'_n\|} - 1 \right| \|x'_n\| - \left\| \frac{x''_n}{\|x''_n\|} + x''_n - x''_n - x'_n \right\| \\ &\ge \left| \frac{1}{\|x'_n\|} - 1 \right| \|x'_n\| - \left| \frac{1}{\|x''_n\|} - 1 \right| \|x''_n\| + \|x'_n - x''_n\|. \end{aligned}$$

The last expression approaches 2 as $\|x_n'\|, \|x_n''\| \to 1$ and $\|x_n' - x_n''\| \to 2$ as $n \to \infty$. We have constructed two sequences $(f_n'), (f_n'') \subset Z$ whose distance goes to 2, and this shows that the diameter of Z is two. This completes the proof. □

The analogous result for the sequence spaces holds.

Theorem 5.6. Suppose that φ does not satisfy the Δ_2^0 condition. Then any nonempty relatively weakly open subset of the unit ball in the Orlicz-Lorentz sequence space $\lambda_{\varphi,w}$ equipped with the Luxemburg norm has the diameter two.

Proof. Let Z be a weakly open subset of the unit ball in $\lambda_{\varphi,w}$. Since the space is infinite dimensional, there exists $x \in Z$ such that $\|x\| = 1$. By the Fatou property of $\lambda_{\varphi,w}$, $\|x\chi_{\{1,\dots,n\}}\| \to \|x\|$ as $n \to \infty$. Suppose φ does not satisfy Δ_2^0. Then there exists $(t_n) \to 0$ such that

$$\varphi\left\{\left(1 + \frac{1}{n}\right) t_n\right\} > 2^n \varphi(t_n). \tag{5.3}$$

We claim that there exists a sequence of subsets $(E_j) \subset \mathbb{N} \setminus \{1, \dots, j\}$ and a subsequence $(n_j) \subset \mathbb{N}$ such that for all $j \in \mathbb{N}$,

$$\frac{1}{2^j} \le \varphi(t_{n_j}) \sum_{k=1}^{m(E_j)} w(k) \le \frac{1}{2^{j-2}}. \tag{5.4}$$

Indeed, without loss of generality, assume $w(1) \le 1$. Then $w(k) \le 1$ for any $k \ge 1$. Since $\varphi(t_n) \to 0$, there exists the largest natural number n_1 such that $\varphi(t_{n_1}) \le 1$. Also there exists $k_1 \ge 1$ such that $\frac{1}{2^{k_1}} \le \varphi(t_{n_1}) \le \frac{1}{2^{k_1 - 1}}$. Hence for all $j \in \mathbb{N}$,

$$\frac{W(j)}{2^{k_1}} \le \varphi(t_{n_1}) W(j) \le \frac{W(j)}{2^{k_1-1}}. \tag{5.5}$$

Since $W(\infty) = \infty$, we can find $j \in \mathbb{N}$ such that

$$W(j) \ge 2^{k_1 - 1},$$

and let

$$m_1 = \min\{j \in \mathbb{N} : W(j) \geq 2^{k_1-1}\}.$$

By $w(1) \leq 1$ and $k_1 \geq 1$, we have that $m_1 \geq 2$. By definition of m_1 we get that $W(m_1 - 1) < 2^{k_1-1}$. But $W(m_1) = W(m_1 - 1) + w(m_1) < 2^{k_1-1} + 1 \leq 2^{k_1}$. Now by (5.5),

$$\frac{1}{2} = \frac{2^{k_1-1}}{2^{k_1}} \leq \frac{W(m_1)}{2^{k_1}} \leq \varphi(t_{n_1})W(m_1) \leq \frac{W(m_1)}{2^{k_1-1}} \leq \frac{2^{k_1}}{2^{k_1-1}} = 2.$$

Finally let $E_1 \subset \mathbb{N} \setminus \{1\}$ be such that $|E_1| = m_1$, and so we get (5.4) for $j = 1$.

As a second step let $n_2 > n_1$. There exists $k_2 > k_1$ such that $\frac{1}{2^{k_2}} \leq \varphi(t_{n_2}) \leq \frac{1}{2^{k_2-1}}$. Let

$$m_2 = \min\{j \in \mathbb{N} : W(j) \geq 2^{k_2-2}\}.$$

Then $2^{k_2-2} \leq W(m_2) = W(m_2 - 1) + w(m_2) < 2^{k_2-2} + 1 \leq 2^{k_2-1}$. Hence

$$\frac{1}{2^2} = \frac{2^{k_2-2}}{2^{k_2}} \leq \frac{W(m_2)}{2^{k_2}} \leq \varphi(t_{n_2})W(m_2) \leq \frac{W(m_2)}{2^{k_2-1}} \leq \frac{2^{k_2-1}}{2^{k_2-1}} = 1.$$

Thus, there exists $E_2 \subset \mathbb{N} \setminus \{1, 2\}$ of size $m_2 = |E_2|$ satisfying (5.4) for $j = 2$. Now proceeding analogously by induction we can find $E_j \subset \mathbb{N} \setminus \{1, \ldots, j\}$ and a subsequence (n_j) satisfying (5.4).

Define now the sequences $(x'_j)_{j=1}^{\infty}, (x''_j)_{j=1}^{\infty}$ by $x'_j = x\chi_{\mathbb{N}\setminus E_j} + t_{n_j}\chi_{E_j}$ and $x''_j = x\chi_{\mathbb{N}\setminus E_j} - t_{n_j}\chi_{E_j}$. By the orthogonal subadditivity of $\alpha_{\varphi,w}$,

$$\begin{aligned}
\alpha_{\varphi,w}(x'_j) &= \sum_{k=1}^{\infty} \varphi((x\chi_{\mathbb{N}\setminus E_j} + t_{n_j}\chi_{E_j})^*(k))w(k) \\
&\leq \sum_{k=1}^{\infty} \varphi((x\chi_{\mathbb{N}\setminus E_j})^*(k))w(k) + \sum_{k=1}^{\infty} \varphi((t_{n_j}\chi_{E_j})^*(k)w(k)).
\end{aligned}$$

Notice that

$$\sum_{k=1}^{\infty} \varphi((x\chi_{\mathbb{N}\setminus E_j})^*(k))w(k) \leq \alpha_{\varphi,w}(x),$$

and

$$\sum_{k=1}^{\infty} \varphi((t_{n_j}\chi_{E_j})^*(k)w(k)) \leq \sum_{k=1}^{\infty} \varphi(t_{n_j})\chi_{\{1,\ldots,|E_j|\}}(k)w(k).$$

Hence we have

$$\alpha_{\varphi,w}(x'_j) \leq \alpha_{\varphi,w}(x) + \varphi(t_{n_j})\sum_{k=1}^{|E_j|} w(k).$$

From (5.4),

$$\alpha_{\varphi,w}(x) + \varphi(t_{n_j})\sum_{k=1}^{|E_j|} w(k) \leq \alpha_{\varphi,w}(x) + \frac{1}{2^{j-2}} \leq 1 + \frac{1}{2^{j-2}}.$$

By dividing each side of the above inequality by $1 + \frac{1}{2^{j-2}}$ and by the convexity of the modular $\alpha_{\varphi,w}$,

$$\alpha_{\varphi,w}\left((1+1/2^{j-2})^{-1}x'_j\right) \leq \left(1+1/2^{j-2}\right)^{-1}\alpha(x'_j) \leq 1.$$

So $\|x'_j\| \leq 1 + 1/2^{j-2}$. In addition, $\|x'_j\| \geq \|x\chi_{\mathbb{N}\setminus E_j}\| \geq \|x\chi_{\{1,\ldots,j\}}\|$. By the Fatou property of $\lambda_{\varphi,w}$ we have $\|x\chi_{\{1,\ldots,j\}}\| \to \|x\| = 1$, so $\|x'_j\| \to 1$ as $n \to \infty$. Similarly $\|x''_j\| \to 1$.

We claim that $x'_j \to x$ and $x''_j \to x$ weakly. Because $mE_j < \infty$, $x - x'_j = x\chi_{E_j} - t_{n_j}\chi_{E_j} \in (\lambda_{\varphi,w})_a$. Then by Theorem 1.7, $F(x - x'_j) = H(x - x'_j)$ for any $F \in (\lambda_{\varphi,w})^*$. Let H be a regular functional which corresponds to an element $(\eta(k))_{k=1}^{\infty} = \eta \in \mathfrak{m}^0_{\varphi_*,w}$. We can write $H(x) = \sum_{k=1}^{\infty} x(k)\eta(k)$ for $x \in \lambda_{\varphi,w}$. Since $\eta \in \mathfrak{m}^0_{\varphi_*,w}$, $p_{\varphi_*,w}(\delta\eta) < \infty$ for some $\delta > 0$. Let v be a positive sequence such that $v \prec w$ and

$$\sum_{k=1}^{\infty} \varphi_*\left(\frac{\delta|\eta(k)|}{v(k)}\right)v(k) \leq p_{\varphi_*,w}(\delta\eta) + 1 < \infty. \tag{5.6}$$

Observe that

$$\begin{aligned}
|H(x-x_j')| &= \left|\sum_{k=1}^{\infty}(x(k)\chi_{E_j}(k)-t_{n_j}\chi_{E_j}(k))\eta(k)\right| \\
&\leq \sum_{k=1}^{\infty}|x(k)\chi_{E_j}(k)\eta(k)|+\sum_{k=1}^{\infty}|t_{n_j}\chi_{E_j}\eta(k)| \\
&= \delta^{-1}\left(\sum_{k=1}^{\infty}\frac{|x(k)\delta\eta(k)|v(k)}{v(k)}\chi_{E_j}(k)+\sum_{k=1}^{\infty}\frac{t_{n_j}\delta|\eta(k)|v(k)}{v(k)}\chi_{E_j}(k)\right).
\end{aligned}$$

By Young's inequality

$$\sum_{k=1}^{\infty}\frac{|x(k)\delta\eta(k)|v(k)}{v(k)}\chi_{E_j}(k)\leq\sum_{k=1}^{\infty}\varphi(|x(k)|)v(k)\chi_{E_j}(k)+\sum_{k=1}^{\infty}\varphi_*\left(\frac{\delta|\eta(k)|}{v(k)}\right)v(k)\chi_{E_j}(k),$$

and

$$\sum_{k=1}^{\infty}\frac{t_{n_j}\delta|\eta(k)|v(k)}{v(k)}\chi_{E_j}(k)\leq\varphi(t_{n_j})\sum_{k=1}^{\infty}v(k)\chi_{E_j}+\sum_{k=1}^{\infty}\varphi_*\left(\frac{\delta|\eta(k)|}{v(k)}\right)v(k)\chi_{E_j}(k).$$

Hence,

$$|H(x-x_j')|\leq\frac{1}{\delta}\sum_{k=1}^{\infty}\varphi(|x(k)|)v(k)\chi_{E_j}(k)+\frac{2}{\delta}\sum_{k=1}^{\infty}\varphi_*\left(\frac{\delta|\eta(k)|}{v(k)}\right)v(k)\chi_{E_j}(k)+\frac{1}{\delta}\varphi(t_{n_j})\sum_{k=1}^{\infty}v(k)\chi_{E_j}$$

Since $v\prec w$, by Lemma 1.1,

$$\sum_{k=1}^{\infty}\varphi(|x(k)|)v(k)\leq\sum_{k=1}^{\infty}\varphi(x^*(k))v^*(k)\leq\sum_{k=1}^{\infty}\varphi(x^*(k))w(k)=\alpha_{\varphi,w}(x)<\infty.$$

From Lemma 1.3, we have

$$\sum_{k=1}^{\infty}\varphi(|x(k)|)v(k)\chi_{E_j}(k)\leq\sum_{k=j+1}^{\infty}\varphi(|x(k)|)v(k)\to 0\quad\text{as}\quad j\to\infty.$$

Also, from the fact that $\sum_{k=1}^{\infty} v(k)\chi_{E_j}(k) \le \sum_{k=1}^{m(E_j)} v^*(k) \le \sum_{k=1}^{m(E_j)} w(k)$ and from (5.4), we get

$$\varphi(t_{n_j})\sum_{k=1}^{\infty} v(k)\chi_{E_j}(k) \le \varphi(t_{n_j})\sum_{k=1}^{m(E_j)} w(k) \le 1/2^{j-2} \to 0.$$

In addition, by (5.6),

$$\sum_{k=1}^{\infty} \varphi_*\left(\frac{\delta|\eta(k)|}{v(k)}\right) v(k)\chi_{E_j}(k) \le \sum_{k=j+1}^{\infty} \varphi_*\left(\frac{\delta|\eta(k)|}{v(k)}\right) v(k) \to 0.$$

Thus, $|H(x - x_n')| \to 0$, which implies that $x_n' \to x$ weakly. Similarly, $x_n'' \to x$ weakly.

Now, we show that $\|x_j' - x_j''\| \to 2$. First, notice that $\|x_j' - x_j''\| = \|2t_{n_j}\chi_{E_j}\|$. Then,

$$\|2t_{n_j}\chi_{E_j}\| = 2\inf\left\{\delta : \sum_{k=1}^{|E_j|} \varphi\left(\frac{t_{n_j}}{\delta}\right) w \le 1\right\} = 2\inf\left\{\delta : \frac{t_{n_j}}{\delta} \le \varphi^{-1}\left(\frac{1}{W(m(E_j))}\right)\right\}.$$

Moreover, we have

$$2\inf\left\{\delta : \frac{t_{n_j}}{\delta} \le \varphi^{-1}\left(\frac{1}{W(m(E_j))}\right)\right\} = 2\inf\left\{\delta : \frac{t_{n_j}}{\varphi^{-1}\left(1/W(m(E_j))\right)} \le \delta\right\}.$$

Hence,

$$\|2t_{n_j}\chi_{E_j}\| = \frac{2t_{n_j}}{\varphi^{-1}\left(1/W(m(E_j))\right)}. \tag{5.7}$$

From (5.4),

$$\varphi^{-1}(2^j\varphi(t_{n_j})) \ge \varphi^{-1}\left(\frac{1}{W(m(E_j))}\right) \ge \varphi^{-1}(2^{j-2}\varphi(t_{n_j})).$$

By (5.7), we finally obtain

$$\|2t_{n_j}\chi_{E_j}\| = \frac{2t_{n_j}}{\varphi^{-1}(1/W(m(E_j)))} \ge \frac{2t_{n_j}}{\varphi^{-1}(2^j\varphi(t_{n_j}))} \ge \frac{2t_{n_j}}{\varphi^{-1}(2^{n_j}\varphi(t_{n_j}))} \overset{\text{by (5.3)}}{\ge} \frac{2n_j}{n_j+1},$$

and as $j \to \infty$, $\frac{2n_j}{n_j+1} \to 2$. Now, taking $f_j' = \frac{x_j'}{\|x_j'\|}$ and $f_j'' = \frac{x_j''}{\|x_j''\|}$, $f_j', f_j'' \in B_{\lambda_{\varphi,w}}$. We can show analogously, as for function case, that $f_j' \to x$, $f_j'' \to x$ weakly and $\|f_j' - f_j''\| \to 2$ as $j \to \infty$, and this completes the proof. □

We have the following consequence.

Theorem 5.7. The following are equivalent:

(1) φ does not satisfy the appropriate Δ_2 condition;

(2) $\Lambda_{\varphi,w}$ has the diameter two property;

(3) $\Lambda_{\varphi,w}$ has the local diameter two property.

Proof. (2) implies (3) from the fact that a slice of the unit ball is a relatively weakly open subset of the unit ball. If φ satisfies the appropriate Δ_2 condition, then $\Lambda_{\varphi,w}$ satisfies the Radon-Nikodým property. So by Corollary 5.3, there exists a slice with arbitrarily small diameter. Thus, (3) implies (1). By Theorem 5.5, (1) implies (2). □

Also, we can characterize Orlicz-Lorentz function spaces with the Radon-Nikodým property.

Corollary 5.8. An Orlicz-Lorentz space $\Lambda_{\varphi,w}$ on (I, m) has the Radon-Nikodým property if and only if φ satisfies the Δ_2 condition if $\gamma = \infty$, and φ satisfies the Δ_2^∞ condition if $\gamma < \infty$.

The analogous results hold for the sequence spaces from Theorem 5.4 and Theorem 5.6.

Theorem 5.9. Let w be a weight sequence φ be an Orlicz N-function. Then the following are equivalent:

(1) φ does not satisfy the Δ_2^0 condition;

(2) $\lambda_{\varphi,w}$ has the diameter two property;

(3) $\lambda_{\varphi,w}$ has the local diameter two property.

Corollary 5.10. An Orlicz-Lorentz sequence space $\lambda_{\varphi,w}$ has the Radon-Nikodým property if and only if φ satisfies the Δ_2^0 condition.

5.2 M-embedded Orlicz-Lorentz spaces and their relation to the strong diameter two property and octahedral norms

In this section, we show results on the strong diameter two property of $\Lambda_{\varphi,w}$. Recall that a Banach space X is said to have the strong diameter two property if every convex combination of slices has diameter two. Also, we recall definitions of the dual versions of diameter two properties.

Definition 5.11. [21]

(1) The dual space X^* has the weak* strong diameter two property if every convex combination of weak* slices of B_{X^*} has diameter two.

(2) The dual space X^* has the weak* diameter two property if every nonempty relatively weak* open subset of B_{X^*} has diameter two.

(3) The dual space X^* has the weak* local diameter two property if every weak* slices of B_{X^*} has diameter two.

The following result shows the relationship between proper M-embedded spaces and the strong diameter two property.

Theorem 5.12. [1, Theorem 4.10] If X is properly M-embedded, then both X and X^{**} have the strong diameter two property.

We have the following consequence for Orlicz-Lorentz spaces.

Corollary 5.13. Suppose that φ does not satisfy the appropriate Δ_2 conditions while φ_* does. Then, $\Lambda_{\varphi,w} \simeq ((\Lambda_{\varphi,w})_a)^{**}$ and $(\Lambda_{\varphi,w})_a$ satisfy the strong diameter two property. The similar result holds for the sequence spaces $\lambda_{\varphi,w} \simeq ((\lambda_{\varphi,w})_a)^{**}$ and $(\lambda_{\varphi,w})_a$.

Proof. If φ does not satisfy the appropriate Δ_2 condition but φ does, then $(\Lambda_{\varphi,w})_a$ is properly M-embedded by Theorem 3.17. Also notice that $\Lambda_{\varphi,w} \simeq ((\Lambda_{\varphi,w})_a)^{**}$. Hence in view of Theorem 5.12, $\Lambda_{\varphi,w}$ and $(\Lambda_{\varphi,w})_a$ have the strong diameter two property. The sequence case can be proven similarly by using Theorem 3.18 and Theorem 5.12. □

For the rest of the section, we look at the relationship between octahedral norms and diameter two properties. Octahedral norms were first introduced in [18].

Definition 5.14. For a Banach space X, the norm on X is said to be octahedral if for every finite-dimensional subspace Y of X and every $\epsilon > 0$, there exists $x \in S_X$ such that for all $y \in Y$, $\|x + y\| \geq (1 - \epsilon)(\|x\| + \|y\|)$.

The relationship between diameter two properties and octahedral norms has been recently investigated in [21].

Theorem 5.15. [21, Theorem 3.6] Let X be a Banach space. Then the following are equivalent:

(1) X has the strong diameter two property;

(2) X^* is octahedral.

Now we have the following consequence.

Corollary 5.16. Suppose that φ does not satisfy the appropriate Δ_2 condition while φ_* does. Then $\mathcal{M}^0_{\varphi_*,w} \simeq ((\Lambda_{\varphi,w})_a)^*$ and $((\Lambda_{\varphi,w})_a)^{***} \simeq (\Lambda_{\varphi,w})^*$ are octahedral. The similar result holds for the sequence spaces.

Proof. From Corollary 5.13, the spaces $(\Lambda_{\varphi,w})_a$ and $\Lambda_{\varphi,w}$ have the strong diameter two property. Hence by Theorem 5.15, $((\Lambda_{\varphi,w})_a)^* \simeq \mathcal{M}^0_{\varphi_*,w}$ and $(\Lambda_{\varphi,w})^*$ are octahedral. □

Theorem 5.17. [21, Theorem 3.5] Let X be a Banach space. Then the following are equivalent:

(1) X^* has the weak* strong diameter two property;

(2) X is octahedral.

We conclude this section with a sufficient condition for $\Lambda_{\varphi,w}$ to have the weak* strong diameter two property.

Corollary 5.18. Suppose that φ does not satisfy the appropriate Δ_2 condition while φ_* does. Then, $((\Lambda_{\varphi,w})_a)^{**} \simeq \Lambda_{\varphi,w}$ has the weak* strong diameter two property. The similar result holds for the sequence spaces.

Proof. In view of Corollary 5.16, $\mathcal{M}^0_{\varphi_*,w} \simeq ((\Lambda_{\varphi,w})_a)^*$ and $(\Lambda_{\varphi,w})^*$ are octahedral. From the fact that φ_* satisfies the appropriate Δ_2 condition, $(\mathcal{M}^0_{\varphi_*,w})^* = (\mathcal{M}^0_{\varphi_*,w})' = \Lambda_{\varphi,w}$ by Theorem 3.16. Hence $\Lambda_{\varphi,w}$ and $(\Lambda_{\varphi,w})^{**}$ have the weak* strong diameter two property by Theorem 5.17. □

5.3 On weak* strong diameter two properties of Orlicz-Lorentz spaces

The next theorem is stronger than Corollary 5.18 since the condition on φ_* is dropped. Recall that for $x \in S_X$ and $\epsilon > 0$, a weak*-slice of the unit ball B_{X^*} is defined by $S_{x,\epsilon} = \{x^* \in B_{X^*} : x^*(x) > 1 - \epsilon\}$.

Theorem 5.19. If φ does not satisfy the appropriate Δ_2 condition, then any convex combination of weak*-slices of $B_{\Lambda_{\varphi,w}}$ has the diameter two.

Proof. Suppose that φ does not satisfy the Δ_2 condition when $\gamma = \infty$, or φ does not satisfy the Δ_2^∞ condition when $\gamma < \infty$. In view of Lemma 1.5, there exists a sequence $(t_n) \subset (0, \infty)$ such that

$$\varphi\left(\left(1 + \frac{1}{n}\right) t_n\right) > 2^n \varphi(t_n). \tag{5.8}$$

for all $n \in \mathbb{N}$. Without loss of generality, assume $t_n \uparrow \infty$ or $t_n \downarrow 0$ when $\gamma = \infty$ and $t_n \uparrow \infty$ when $\gamma < \infty$.

If $t_n \uparrow \infty$, we have $2^n \varphi(t_n) \uparrow \infty$. So there exists a sequence of pairwise disjoint measurable sets E_n such that for every $n \in \mathbb{N}$,

$$\int_0^{mE_n} w = \frac{1}{2^n \varphi(t_n)}. \tag{5.9}$$

Indeed, notice that $\frac{1}{\varphi(t_1)} < \infty$. Hence $\sum_{n=1}^\infty \frac{1}{2^n \varphi(t_n)} < \frac{1}{\varphi(t_1)} < \infty$. Choose $n_0 \in \mathbb{N}$ such that $\sum_{n=n_0}^\infty \frac{1}{2^n \varphi(t_n)} < W(\gamma) \le \infty$. Since $W(t)$ is continuous over $(0, \gamma)$ and the Lebesgue measure m is nonatomic, there exists a sequence of disjoint measurable sets $E_k \subset I$ such that $\int_0^{mE_k} w = \frac{1}{2^{n_0+k} \varphi(t_{n_0+k})}$ for every $k \in \mathbb{N}$. By assuming $n_0 = 0$, we showed (5.9). Moreover, $\frac{1}{2^n \varphi(t_n)} \downarrow 0$ so $mE_n \to 0$.

For the case of $t_n \downarrow 0$ and $\gamma = \infty$, we can also find a sequence of pairwise disjoint measurable sets E_n satisfying (5.9) for every $n \in \mathbb{N}$. From (5.8), we see that $\frac{1}{2^n \varphi(t_n)} \uparrow \infty$ as $t_n \downarrow 0$. Let $(I_n)_{n=1}^\infty$ be a sequence of measurable subsets of $I = [0, \infty)$ such that $I = \cup I_n$, $I_1 \subset I_2 \subset I_3 \subset ...$ and $mI_n < \infty$. Then, $m(I \setminus I_n) = \infty$ for all $n \in \mathbb{N}$. Since m is nonatomic and $W(t)$ is continuous on $(0, \infty)$, there exists a sequence of pairwise disjoint measurable subsets E_n, where $E_n \subset I \setminus I_n$, such that (5.9) is satisfied. From the fact that $\frac{1}{2^n \varphi(t_n)} \uparrow \infty$, we have $mE_n \uparrow \infty$.

Let $S_1, S_2, ...S_k$, $k \in \mathbb{N}$, be weak*-slices of $B_{\Lambda_{\varphi,w}}$. Since a weak*-slice is a weak*-open subset of $B_{\Lambda_{\varphi,w}}$, there exists $f_i \in S_i$ such that $\|f_i\| = 1$, where $i = 1, 2, ..., k$. Let $f = \sum_{i=1}^k \lambda_i f_i$ where $\sum_{i=1}^k \lambda_i = 1$ with each $\lambda_i > 0$ for

$i = 1, 2, ...k$. Define

$$f'_n = \sum_{i=1}^{k} \lambda_i f_i \chi_{I\setminus E_n} + t_n \chi_{E_n} = \sum_{i=1}^{k} \lambda_i \left(f_i \chi_{I\setminus E_n} + t_n \chi_{E_n} \right) = \sum_{i=1}^{k} \lambda_i f'_{i,n},$$

where $f'_{i,n} = f_i \chi_{I\setminus E_n} + t_n \chi_{E_n}$, and

$$f''_n = \sum_{i=1}^{k} \lambda_i f_i \chi_{I\setminus E_n} - t_n \chi_{E_n} = \sum_{i=1}^{k} \lambda_i \left(f_i \chi_{I\setminus E_n} - t_n \chi_{E_n} \right) = \sum_{i=1}^{k} \lambda_i f''_{i,n},$$

where $f''_{i,n} = f_i \chi_{I\setminus E_n} - t_n \chi_{E_n}$. Notice that $f'_{i,n} \to f_i$ a.e. and $f''_{i,n} \to f_i$ a.e. Hence $1 = \|f_i\| \le \liminf \|f'_{i,n}\|$ and $1 = \|f_i\| \le \liminf_n \|f''_{i,n}\|$ by the Fatou property for every $i = 1, 2, 3, ..., k$. By the orthogonal subadditivity of $\rho_{\varphi,w}$,

$$\begin{aligned} \rho_{\varphi,w}(f'_{i,n}) = \int_0^\gamma \varphi((f_i \chi_{I\setminus E_n} + t_n \chi_{E_n})^*) w &\le \int_0^\gamma \varphi(|f_i|)\chi_{I\setminus E_n})^* w + \int_0^{mE_n} \varphi(t_n) w \\ &\le \int_0^\gamma \varphi(f_i^*)) w + \frac{1}{2^n} \text{ by (5.8)} \\ &\le 1 + \frac{1}{2^n} \to 1, \end{aligned}$$

as $n \to \infty$. Hence $\limsup_n \rho_{\varphi,w}(f'_{i,n}) \le 1$. So for any $\epsilon > 0$, there exists N such that for all $n \ge N$, $\rho_{\varphi,w}(f'_{i,n}) \le 1 + \epsilon$. This shows that $\|f'_{i,n}\| \le 1 + \epsilon$ for such $n \in \mathbb{N}$, in other words, $\limsup_n \|f'_{i,n}\| \le 1$. Thus, $\lim_{n\to\infty} \|f'_{i,n}\| = 1$ for $i = 1, 2, ..., k$. Similarly, $\lim_{n\to\infty} \|f''_{i,n}\| = 1$ for $i = 1, 2, ..., k$.

Notice that $\Lambda_{\varphi,w} = (\mathcal{M}^0_{\varphi_*,w})' \simeq ((\mathcal{M}^0_{\varphi_*,w})_a)^*$ by Theorem 3.16. Hence in order to show f'_n and f''_n converges to f weakly*, we only have to check

$$\left| \int_0^\gamma h(f - f'_n) \right| \to 0 \ \text{ and } \ \left| \int_0^\gamma h(f - f''_n) \right| \to 0.$$

When $t_n \uparrow \infty$,

$$\begin{aligned}\left|\int_0^\gamma h(f_i - f'_{i,n})\right| &= \left|\int_0^\gamma h(f_i - f_i\chi_{I\setminus E_n} - t_n\chi_{E_n})\right| = \left|\int_0^\gamma hf_i\chi_{E_n} - ht_n\chi_{E_n}\right| \\ &\leq \int_0^\gamma |hf_i|\chi_{E_n} + \int_0^\gamma |h|t_n\chi_{E_n}.\end{aligned}$$

By the property of decreasing rearrangement and Lemma 1.3,

$$\begin{aligned}\int_0^\gamma |hf_i|\chi_{E_n} + \int_0^\gamma |h|t_n\chi_{E_n} &\leq \int_0^\gamma h^*(f_i\chi_{E_n})^* + \int_0^\gamma h^*(t_n\chi_{E_n})^* \\ &\leq \int_0^{mE_n} h^* f_i^* + \int_0^{mE_n} t_n h^*.\end{aligned}$$

Let v be a decreasing, non-negative measurable function such that $v \prec w$. Then, we have

$$\int_0^{mE_n} h^* f_i^* + \int_0^{mE_n} t_n h^* \leq \int_0^{mE_n} f_i^* \frac{h^*}{v} v + \int_0^{mE_n} t_n \frac{h^*}{v} v.$$

From Young's inequality,

$$\int_0^{mE_n} f_i^* \frac{h^*}{v} v \leq \int_0^{mE_n} \left\{\varphi(f_i^*)v + \varphi_*\left(\frac{h^*}{v}\right) v\right\},$$

and

$$\int_0^{mE_n} t_n \frac{h^*}{v} v \leq \int_0^{mE_n} \left\{\varphi(t_n)v + \varphi_*\left(\frac{h^*}{v}\right) v\right\}.$$

Hence

$$\left|\int_0^\gamma h(f_i - f'_{i,n})\right| \leq \int_0^{mE_n} \varphi(f_i^*)v + \int_0^{mE_n} \varphi(t_n)v + 2\int_0^{mE_n} \varphi_*\left(\frac{h^*}{v}\right) v.$$

Since $v \prec w$, from Hardy's lemma,

$$\left|\int_0^\gamma h(f_i - f'_{i,n})\right| \leq \int_0^{mE_n} \varphi(f_i^*)w + \int_0^{mE_n} \varphi(t_n)w + 2\inf\left\{\int_0^{mE_n} \varphi_*\left(\frac{h^*}{v}\right) v : v \prec w\right\}.$$

By the choice of E_n and (5.9), $mE_n \to 0$ and $\varphi(t_n)\int_0^{mE_n} w = \frac{1}{2^n} \to 0$ as $n \to \infty$. Since $f_i \in S_{\Lambda_{\varphi,w}}$ for all i, $\rho_{\varphi,w}(f_i) < \infty$ so $\int_0^{mE_n} \varphi(f_i^*)w \to 0$. Notice that $P_{\varphi_*,w}(h) < \infty$ because $h \in (\mathcal{M}_{\varphi_*,w})_a$. Then there exists $v_1 \prec w$ such that $\int_0^\gamma \varphi_*\left(\frac{h^*}{v_1}\right) v_1 \le P_{\varphi_*,w}(h) + 1 < \infty$. Hence

$$\inf\left\{\int_0^{mE_n} \varphi_*\left(\frac{h^*}{v}\right) v : v \prec w, v\downarrow\right\} \le \int_0^{mE_n} \varphi_*\left(\frac{h^*}{v_1}\right) v_1 \to 0$$

as $n \to \infty$. So $|\int_0^\gamma h(f_i - f'_{i,n})| \to 0$. Furthermore, by the fact that $\sum_{i=1}^k \lambda_i = 1$,

$$\int_0^\gamma h\left(\sum_{i=1}^k \lambda_i f_i - \sum_{i=1}^k \lambda_i f'_{i,n}\right) \to 0 \quad \text{as} \quad n \to \infty. \tag{5.10}$$

The similar argument shows that $|\int_0^\gamma h(f_i - f''_{i,n})| \to 0$ and

$$\int_0^\gamma h\left(\sum_{i=1}^k \lambda_i f_i - \sum_{i=1}^k \lambda_i f''_{i,n}\right) \to 0 \quad \text{as} \quad n \to \infty. \tag{5.11}$$

Now, consider the case $t_n \downarrow 0$ and $\gamma = \infty$. Let $h \in (\mathcal{M}^0_{\varphi,w})_a$ and $v \prec w$, $v \ge 0$ such that $\int_I \varphi_*\left(\frac{|h|}{v}\right) v < 1 + P_{\varphi_*,w}(h) < \infty$. From Young's inequality,

$$\begin{aligned}\left|\int_{E_n} h(f_i - t_n)\right| &\le \int_{E_n} |hf_i| + \int_{E_n} |h| t_n = \int_{E_n} \frac{|hf_i|}{v} v + \int_{E_n} \frac{|h|t_n}{v} v \\ &\le \int_{E_n} \varphi(|f_i|) v + 2\int_{E_n} \varphi_*\left(\frac{|h|}{v}\right) v + \varphi(t_n)\int_{E_n} v.\end{aligned}$$

Notice that $E_n \subset I \setminus I_n$ by the construction. Since $I \setminus I_{n+1} \subset I \setminus I_n$ for all $n \in \mathbb{N}$, $m(I \setminus I_n) \to 0$. Hence

$$\int_{E_n} \varphi(|f_i|) v \le \int_{I\setminus I_n} \varphi(|f_i|) v \to 0 \ \text{ and } \ \int_{E_n} \varphi_*\left(\frac{|h|}{v}\right) v \le \int_{I\setminus I_n} \varphi_*\left(\frac{|h|}{v}\right) v \to 0$$

as $n \to \infty$. From Lemma 1.1 and Lemma 1.3, $\int_{E_n} v \leq \int_0^{mE_n} w$ so by (5.9),

$$\varphi(t_n) \int_{E_n} v \leq \varphi(t_n) \int_0^{mE_n} w = \frac{1}{2^n} \to 0$$

as $n \to \infty$. This implies (5.10). By the similar argument, we also obtain (5.11). Therefore, f_n' and f_n'' weakly* converges to f.

Furthermore, for every $i = 1, 2, ..., k$,

$$\begin{aligned} \left| \int_I h \left(f_i - \frac{f_{i,n}'}{\|f_{i,n}'\|} \right) \right| &\leq \left| \int_I h(f_i - f_{i,n}') \right| + \left| \int_I h \left(f_{i,n}' - \frac{f_{i,n}'}{\|f_{i,n}\|} \right) \right| \\ &\leq \left| \int_I h(f_i - f_{i,n}') \right| + \|h\|_{\mathcal{M}_{\varphi,w}} \left| 1 - \frac{1}{\|f_{i,n}\|} \right| \|f_{i,n}'\| \to 0 \end{aligned}$$

as $n \to \infty$. Hence by the fact that $\sum_{i=1}^k \lambda_i = 1$,

$$\left| \int_I h \left(\sum_{i=1}^k \lambda_i f_i - \sum_{i=1}^k \lambda_i \frac{f_{i,n}'}{\|f_{i,n}'\|} \right) \right| \leq \sum_{i=1}^k \lambda_i \left| \int_I h \left(f_i - \frac{f_{i,n}'}{\|f_{i,n}'\|} \right) \right| \to 0$$

as $n \to \infty$. Then $\frac{f_{i,n}'}{\|f_{i,n}'\|} \in S_i$ for sufficiently large n. So we have $\sum_{i=1}^k \lambda_i \frac{f_{i,n}'}{\|f_{i,n}'\|} \in \sum_{i=1}^k \lambda_i S_i$. By the similar argument, since $\left| \int_I h \left(f_i - \frac{f_{i,n}''}{\|f_{i,n}''\|} \right) \right| \to 0$ as $n \to \infty$, for sufficiently large n, $\frac{f_{i,n}''}{\|f_{i,n}''\|} \in S_i$ so $\sum_{i=1}^k \lambda_i \frac{f_{i,n}''}{\|f_{i,n}''\|} \in \sum_{i=1}^k \lambda_i S_i$.

From the construction of f_n' and f_n'',

$$\begin{aligned} \|f_n' - f_n''\| = \left\| \sum_{i=1}^k \lambda_i f_{i,n}' - \sum_{i=1}^k \lambda_i f_{i,n}'' \right\| &= \left\| \sum_{i=1}^k \lambda_i (f_i \chi_{I \setminus E_n} + t_n \chi_{E_n} - f_i \chi_{I \setminus E_n} + t_n \chi_{E_n}) \right\| \\ &= \left\| \sum_{i=1}^k \lambda_i 2 t_n \chi_{E_n} \right\| = \|2 t_n \chi_{E_n}\|. \end{aligned}$$

By the definition of the Luxemburg norm on $\Lambda_{\varphi,w}$ and

$$\begin{aligned}
\|2t_n\chi_{E_n}\| = \inf\left\{\eta > 0 : \rho_{\varphi,w}\left(\frac{2t_n\chi_{E_n}}{\eta}\right) \le 1\right\} &= \inf\left\{\eta > 0 : \int_I \varphi\left(\frac{2t_n\chi_{(0,mE_n)}}{\eta}\right) w \le 1\right\} \\
&= \inf\left\{\eta > 0 : \varphi\left(\frac{2t_n}{\eta}\right) \le \frac{1}{W(mE_n)}\right\} \\
&= \inf\left\{\eta > 0 : \eta \ge \frac{2t_n}{\varphi^{-1}(1/W(mE_n))}\right\} \\
&= \frac{2t_n}{\varphi^{-1}(1/W(mE_n))}.
\end{aligned}$$

From (5.8) and (5.9),

$$\frac{2t_n}{\varphi^{-1}(1/W(mE_n))} = \frac{2t_n}{\varphi^{-1}(2^n\varphi(t_n))} \ge \frac{2n}{n+1} \to 2 \quad \text{as} \quad n \to \infty.$$

Hence $\|f_n' - f_n''\| \to 2$ as $n \to \infty$.

Let $f' = \sum_{i=1}^k \lambda_i \frac{x_{i,n}'}{\|x_{i,n}'\|}$ and $f'' = \sum_{i=1}^k \lambda_i \frac{x_{i,n}''}{\|x_{i,n}''\|}$. From the reverse triangle inequality,

$$\begin{aligned}
\|f' - f''\| &= \left\| \sum_{i=1}^k \lambda_i \frac{f_{i,n}'}{\|f_{i,n}'\|} - \sum_{i=1}^k \lambda_i f_{i,n}' + \sum_{i=1}^k \lambda_i f_{i,n}' - \sum_{i=1}^k \lambda_i \frac{f_{i,n}''}{\|f_{i,n}''\|} \right\| \\
&\ge \left\| \sum_{i=1}^k \lambda_i \frac{f_{i,n}'}{\|f_{i,n}'\|} - \sum_{i=1}^k \lambda_i f_{i,n}' \right\| - \left\| \sum_{i=1}^k \lambda_i \frac{f_{i,n}''}{\|f_{i,n}''\|} - \sum_{i=1}^k \lambda_i f_{i,n}' \right\|.
\end{aligned}$$

Since $\left\|\sum_{i=1}^k \lambda_i f_{i,n}'' - \sum_{i=1}^k \lambda_i f_{i,n}'\right\| = \|f_n' - f_n''\| \to 2$ and $\|f_{i,n}''\| \to 1$, we get

$$\begin{aligned}
\left\| \sum_{i=1}^k \lambda_i \frac{f_{i,n}''}{\|f_{i,n}''\|} - \sum_{i=1}^k \lambda_i f_{i,n}' \right\| &\ge \left\| \sum_{i=1}^k \lambda_i f_{i,n}'' - \sum_{i=1}^k \lambda_i f_{i,n}' \right\| - \left\| \sum_{i=1}^k \lambda_i f_{i,n}'' - \sum_{i=1}^k \lambda_i \frac{f_{i,n}''}{\|f_{i,n}''\|} \right\| \\
&= \left\| \sum_{i=1}^k \lambda_i f_{i,n}'' - \sum_{i=1}^k \lambda_i f_{i,n}' \right\| - \sum_{i=1}^k \lambda_i \left| 1 - \frac{1}{\|f_{i,n}''\|} \right| \|f_{i,n}''\| \\
&= 2t_n\|\chi_{E_n}\| - \sum_{i=1}^k \lambda_i \left| 1 - \frac{1}{\|f_{i,n}''\|} \right| \|f_{i,n}''\| \to 2
\end{aligned}$$

as $n \to \infty$. Therefore, with $\|f_{i,n}'\| \to 1$,

$$\begin{aligned}\|f' - f''\| &\geq \left\|\sum_{i=1}^{k} \lambda_i \frac{f'_{i,n}}{\|f'_{i,n}\|} - \sum_{i=1}^{k} \lambda_i f'_{i,n}\right\| - \left\|\sum_{i=1}^{k} \lambda_i \frac{f''_{i,n}}{\|f''_{i,n}\|} - \sum_{i=1}^{k} \lambda_i f'_{i,n}\right\| \\ &= \sum_{i=1}^{k} \lambda_i \left|\frac{1}{\|f'_{i,n}\|} - 1\right| \|f'_{i,n}\| - \left\|\sum_{i=1}^{k} \lambda_i \frac{f''_{i,n}}{\|f''_{i,n}\|} - \sum_{i=1}^{k} \lambda_i f'_{i,n}\right\| \to 2\end{aligned}$$

as $n \to \infty$, and the proof is finished. □

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