2020

UNIQUE MAXIMAL IDEAL IN THE ALGEBRA L ((l_q)_c_0) WITH 1 < q

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UNIQUE MAXIMAL IDEAL IN THE ALGEBRA $\mathscr{L}((\sum \ell_q)_{c_0})$ WITH $1 < q < \infty$

by

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A Dissertation

Submitted in Partial Fulfillment of the

Requirements for the Degree of

Doctor of Philosophy

Major: Mathematical Sciences

The University of Memphis

May 2020

ACKNOWLEDGEMENTS

I am grateful to the University of Memphis, especially the department of mathematical sciences for financial and academic support so that, I could complete my studies.

Additionally, I would like to thank all of my mentors in the department who taught me the relevant topics and the necessary tools to succeed in Functional Analysis. In particular, I would like to thank my advisor, Dr. Bentuo Zheng, who taught me the fundamental courses in Banach space theory and guided me through the process of my dissertation.

Undoubtedly, I would like to thank my family for the spiritual support that I received during my years of study in the Master and Ph.D levels, they were my inspiration to continue this process and not give up.

Finally, I am blessed for having the opportunity of realizing the dream I had ever since I was young but was not able to accomplish until now.

ABSTRACT

Unique maximal ideal in the algebra $\mathscr{L}((\sum \ell_q)_{c_0})$ with $1 < q < \infty$. (May 2020)
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An important problem in Banach space theory since the 1950's has been the study of the structure of closed algebraic ideals in the algebra $\mathscr{L}(X)$ where X is a Banach space.

The Banach spaces X for which that structure is well-known are very few. It is known that every non-zero ideal in $\mathscr{L}(X)$ contains the ideal of all finite-rank operators on X and that if X has a Schauder basis every non-zero closed ideal in $\mathscr{L}(X)$ contains the ideal of all compact operators on X.

In this dissertation I study the structure of the space $(\sum \ell_q)_{c_0}$, for $1 < q < \infty$ and I find the unique proper maximal ideal in the algebra $\mathscr{L}((\sum \ell_q)_{c_0})$.

Let T be a bounded linear operator on $X = \left(\sum \ell_q\right)_{c_0}$ with $1 < q < \infty$. The operator T is said to be X-strictly singular if the restriction of T on any subspace of X that is isomorphic to X is not an isomorphism. In this dissertation I prove that the unique proper maximal ideal in $\mathscr{L}(X)$ is the set of all X-strictly singular operators on X.

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Chapter 1

Introduction

Banach space theory, as a branch of functional analysis, has been extensively studied since the first half of the 20th century. The work of Stefan Banach and published in his book "Theory of Linear Operations" was the foundation of functional analysis. The research work in this field grew very fast after the publication of his book. Many important problems in functional analysis were solved and a variety of subdisciplines were developed. In this dissertation, I study the structure of the space $X = (\sum \ell_q)_{c_0}$, with $1 < q < \infty$ and I find the unique maximal ideal in the algebra $\mathscr{L}(X)$.

1.1 Description of the Research Problem

First, recall that $(\sum \ell_q)_{c_0}$ is the space of all sequences $x = (x_n)_{n=1}^{\infty}$, with $x_n \in \ell_q$ for each $n \in \mathbb{N}$ and with $(\|x_n\|)_{n=1}^{\infty} \in c_0$, under the norm given by the formula: $\|x\| = \sup_n \left\{ \|x_n\|_{\ell_q} \right\}$.

The main result in this dissertation is that the unique maximal ideal in $\mathscr{L}((\sum \ell_q)_{c_0})$ is the set of all $(\sum \ell_q)_{c_0}$-strictly singular operators on $(\sum \ell_q)_{c_0}$.

The largest obstacle to prove the main result of this dissertation is to show structural results for $(\sum \ell_q)_{c_0}$. Unfortunately, some of the key lemmas used in [10] and in [53] for proving the structural results in $Z_{p,q}$ and $(\sum \ell_q)_{\ell_1}$ respectively, fail in the space $(\sum \ell_q)_{c_0}$.

I begin with a discussion of how isomorphic copies of ℓ_q in $(\sum \ell_q)_{c_0}$ with $1 < q < \infty$ are situated in $(\sum \ell_q)_{c_0}$.

We have that if X is a subspace of $(\sum \ell_q)_{c_0}$ isomorphic to ℓ_q, then for all $\varepsilon > 0$ there is a subspace Y of X that is $(1+\varepsilon)$-isomorphic to ℓ_q. This follows from a general result of Krivine and Maurey [30] about stable Banach spaces, but can be proved in an elementary way using James's proof of the non-distortability of the norm on ℓ_1.

We also need that copies of ℓ_q in $(\sum \ell_q)_{c_0}$ contain almost isometric copies of ℓ_q which are almost norm one complemented in $(\sum \ell_q)_{c_0}$. This result follows by Schechtman's and

Johnson's lemmas 2.5 and 2.6 [10].

The strategy for proving the main theorem will consist in showing that for each $\varepsilon > 0$, every subspace of $(\sum \ell_q)_{c_0}$ containing a copy of $(\sum \ell_q)_{c_0}$ must contain a copy of $(\sum \ell_q)_{c_0}$ that is $C+\varepsilon$-isomorphic to $(\sum \ell_q)_{c_0}$ and $C+\varepsilon$ complemented in $(\sum \ell_q)_{c_0}$. Since this is enough for our purposes, we will not attempt to find the best complementation constant. Thus, we will conclude that the space $(\sum \ell_q)_{c_0}$ is complementably homogeneous.

To prove the above mentioned result, we will use lemmas 2.5 and 2.6 [10] proved by Johnson and Schechtman.

Continuing with the strategy, we will show that if T is $(\sum \ell_q)_{c_0}$- strictly singular operator on $(\sum \ell_q)_{c_0}$, then for each $\varepsilon > 0$ we can find a copy X of $(\sum \ell_q)_{c_0}$ in $(\sum \ell_q)_{c_0}$ such that T restricted to X has a small norm. Consequently, the set of all $(\sum \ell_q)_{c_0}$- strictly singular operators on $(\sum \ell_q)_{c_0}$ is a linear subspace of $\mathscr{L}((\sum \ell_q)_{c_0})$. Thus, it is an ideal.

Since $(\sum \ell_q)_{c_0}$ is complementably homogeneous, then every element in $\mathscr{M}_X$ with $X = (\sum \ell_q)_{c_0}$ is $(\sum \ell_q)_{c_0}$- strictly singular. The other inclusion is easy to prove. Hence, we will have $\mathscr{M}_X$ is equal to the set of all X-strictly singular operators.

From [13] we know that if $\mathscr{M}_X$ is closed under addition, then $\mathscr{M}_X$ is the unique maximal ideal of $\mathscr{L}(X)$. In fact, since $\mathscr{M}_X$ is equal to the set of all X-strictly singular operators on X, then $\mathscr{M}_X$ is closed under addition and so the main theorem will follow.

1.2 Historical Results

An important problem that arises when studying derivations on a general Banach algebra $\mathscr{A}$ is to classify the commutators in the algebra $\mathscr{A}$, i.e., elements that can be expressed as $AB - BA$ with $A, B \in \mathscr{A}$.

In 1947, Wintner [52] proved that the identity is not a commutator and this implies that no non-zero scalar is a commutator.

In 1949, Wielandt [51] proved the same result that Wintner proved but in a different way and pointed out that this much at least is still valid in an arbitrary normed ring with unit.

In 1963, Halmos [22] showed that the theorem of Wintner-Wielandt when applied to the residue class ring of the ring of bounded operators modulo the ideal of compact operators, produce the result that no element of the form $\lambda + K$, where $\lambda \neq 0$ and K is compact, is a commutator.

There seem to be no general conditions for checking whether an element of a Banach algebra is a commutator. The situation changes if we consider $\mathscr{L}(X)$, the algebra of all bounded linear operators on a Banach space X. Different authors have classified the commutators in $\mathscr{L}(X)$ for several classical Banach spaces X.

The first complete classification of the commutators in the algebra $\mathscr{L}(\mathscr{H})$ was given in 1965 by Brown and Pearcy [8] when $\mathscr{H}$ is an arbitrary infinite dimensional Hilbert space. In their paper they first considered the case when $\mathscr{H}$ is a separable infinite dimensional Hilbert space and then generalized the result to an arbitrary infinite dimensional Hilbert space. They proved that the only operators in $\mathscr{L}(\mathscr{H})$ that are not commutators have the form $\lambda + K$, where K is compact and $\lambda \neq 0$. In their paper (J) denotes the ideal of the compact operators, (S) denotes the class of all operators that are congruent mod (J) to non-zero scalars and (F) denotes the complement in $\mathscr{L}(\mathscr{H})$ of the disjoint union of (J) and (S). Thus, the set (S) consists entirely of non-commutators so to prove the main result they proved that the union of (J) and (F) consists entirely of commutators. Since in 1965 Brown, Halmos and Pearcy proved [7] that every compact operator is a commutator, they only had to prove that the family (F) consists entirely of commutators. To prove that, they gave a characterization of operators of type (F) and then applied it to give a standard form of such operators in a matricial fashion, thus the proof of the main result relies, to a large extent, on matricial calculations. Hence, they proved that an operator on a separable Hilbert space is a commutator if and only if it is not of type (S).

In 1972, Apostol [3] generalized the results of Schneeberger [48] concerning compact operators to the case of a sequence space $\ell_p(X)$, where X is a Banach space over the complex field. Schneeberger had showed that some of the techniques used for describing the structure of commutators in a Hilbert space hold in an ℓ_p-space too. Apostol also proved that operators having a compact restriction to a suitable subspace are commutators. In the case X is finite dimensional, Apostol proved that if $T \in \mathscr{L}(\ell_p(X))$, $1 \leq p < \infty$, and λ belongs to the left essential spectrum of T, then $T - \lambda$ is a commutator, and if $T - \lambda$ is not compact for any λ in the complex field, then T is a commutator. Both proofs reduce to the case X is the complex field. In particular, Apostol proved that the only operators in $\mathscr{L}(\ell_p(\mathbb{C}))$ for $1 < p < \infty$ that are not commutators have the form $\lambda + K$ where K is compact and $\lambda \neq 0$.

In 1973, Apostol [4] proved that if X denotes a Banach space and $T \in \mathscr{L}(c_0(X))$ is compact, then T is a commutator. He obtained for operators acting on $c_0(\mathbb{C})$ the same results that he obtained in [3] making minor modifications to the proofs. Thus, he proved that if $T \in \mathscr{L}(c_0(\mathbb{C}))$ and if $T - \lambda$ is not compact for any $\lambda \in \mathbb{C}$, then T must be a commutator. Hence, using both results described above he concluded that any operator in $\mathscr{L}(c_0(\mathbb{C}))$, which is not of the form $T = \lambda + K$, with $\lambda \in \mathbb{C}$, $\lambda \neq 0$, K compact, is a commutator. He also proved that any compact operator $T \in \mathscr{L}(C([0,1]))$ is a commutator, where $C([0,1])$ is the set of the functions in $C((0,1])$ whose limit exists.

While Apostol [3] gave some information about the commutators in $\mathscr{L}(\ell_1)$, he was unable to give a complete characterization. He used in his proofs the fact that the unit vector basis of ℓ_p for $1 < p < \infty$ is shrinking, but as we know for ℓ_1 this is not hold.

In 2009, Dosev [15] proved that the commutators on ℓ_1 are the operators not of the form $\lambda I + K$ with $\lambda \neq 0$ and K is compact. He overcame the obstacle that the unit vector basis of ℓ_1 is not shrinking by using instead the structure of the infinite dimensional subspaces of ℓ_1. Dosev also studied spaces X such that X is isomorphic to $(\sum X)_p$ for $1 \leq p < \infty$ or $p = 0$ and defined the concept of a decomposition of X and used this concept to obtain partial results about commutators on these spaces. He also showed that the compact operators on X, where X is isomorphic to $(\sum X)_p$ for $1 \leq p < \infty$ or $p = 0$, are in fact commutators. In particular, for $p = 1$, he proved that every compact operator on L_1 or ℓ_1 is a commutator. He also showed that the ideal of strictly singular operators is the largest ideal in $\mathscr{L}(\ell_\infty)$ by proving that if T is not strictly singular operator, then I_{ℓ_∞} factor through T and hence any ideal that contains T must be $\mathscr{L}(\ell_\infty)$. He also proved that if $T \in \mathscr{L}(\ell_\infty)$ and T is strictly singular operator, then T is a commutator.

A common feature of all these spaces is that X is isomorphic to $(\sum X)_p$ for $1 \leq p < \infty$ or $p = 0$ and that the ideal of compact operators on X is the largest nontrivial ideal in $\mathscr{L}(X)$. Thus, the classification of the commutators on ℓ_p, $1 \leq p < \infty$ and c_0, suggests that if a Banach space X is isomorphic to $(\sum X)_p$ for $1 \leq p < \infty$ or c_0 and $\mathscr{L}(X)$ has a largest ideal $\mathscr{M}$, then every noncommutator on X has the form $\lambda I + K$, where $K \in \mathscr{M}$ and $\lambda \neq 0$.

It seems that a largest ideal in $\mathscr{L}(X)$ plays an important role in the classification of the commutators in the algebra.

In 1958, Tosio Kato [29] introduced the concept of a strictly singular operator and proved some properties of those operators. First, he showed that if B is a strictly singular operator, the same is true for AB and BC whenever the products are significant and A and C are closed, bounded linear operators. Then, he showed that the set of all strictly singular operators is a subspace of the algebra of all bounded linear operators. Last, he showed that if B_n is a sequence of strictly singular operators with common domain which converges in norm to some bounded linear operator B with the same domain of those of the B_ns, then B is strictly singular. Thus, he proved that the set of all strictly singular operators on a Banach space X is a closed two-sided ideal of the algebra $\mathscr{L}(X)$ of all bounded linear operators on X. From this and the fact that the identity operator of X is not strictly singular if dimension of X is infinity, he concluded that strict singularity implies complete continuity if X is a Hilbert space. In general, he showed that if X and Y are Hilbert spaces and $T : X \to Y$ is strictly singular, then T is completely continuous, in other words T is compact.

In 1960, I.A. Feldman, I.C. Gohberg, A.S. Markus [18] proved that the compact operators form the only closed proper ideal in $\mathscr{L}(X)$ for $X = \ell_p$, $1 \leq p < \infty$, or $X = c_0$.

In 1964, Whitley [50] defined an h-space as an infinite dimensional Banach space X where each closed infinite-dimensional subspace of X contains a complemented subspace isomorphic to X. He showed that each h-space is separable, that c_0 and ℓ_p, for $1 \leq p < \infty$, are h-spaces and that any Banach space isomorphic to an h-space is an h-space; in particular, c is an h-space. Whitley also proved that if X is an h-space, then the set of all strictly singular operators is the greatest ideal in $\mathscr{L}(X)$. He proved this theorem showing that any ideal which contains elements not in the set of all strictly singular operators must contain the identity map so it must coincide with $\mathscr{L}(X)$. As c_0 and ℓ_p, for $1 \leq p < \infty$, are h-spaces, then the set of all strictly singular operators is the greatest ideal in $\mathscr{L}(c_0)$ and in $\mathscr{L}(\ell_p)$. However, Whitley pointed out that in [18] it was shown that the compact operators are the only closed two-sided ideal in $\mathscr{L}(X)$ for $X = c_0$ and $X = \ell_p$, for $1 \leq p < \infty$. Thus, in these spaces the set of all strictly singular operators coincide with the set of all compact operators. He also proved that if X is an h-space and Y is a Banach space and all the maps from X to Y are strictly singular, then so are all the maps from Y to X.

In 2009, D. Dosev and W. B. Johnson [13] followed the ideas in [15] but in this paper they presented them from a different point of view, in a more general setting, and they included the case $p = \infty$. They proved that the non-commutators on ℓ_∞ have the form $\lambda I + S$ where S is a strictly singular operator on ℓ_∞.

They considered the set $\mathscr{M}_X = \{T \in \mathscr{L}(X) : I_X \text{ does not factor through } T\}$ which comes naturally from the work of the commutators on ℓ_p, $1 < p < \infty$, done by Dosev, Apostol [15], [3], [4]. In this paper they also showed that $\mathscr{M}_X$ equals the ideal of the strictly singular operators on ℓ_∞.

In [24], W. B. Johnson, B. Maurey, G. Schechtman and L. Tzafriri showed that $\mathscr{M}_X$ is the largest ideal in $\mathscr{L}(X)$ if $X = L_p(0,1)$ and $1 < p < \infty$. The same can be said for the case $p = 1$ from [16] by P. Enflo and T. W. Starbird.

A natural question can be posed. For which Banach spaces X is the set $\mathscr{M}_X$ the largest ideal in $\mathscr{L}(X)$?

It is clear that the set $\mathscr{M}_X$ is closed under left and right multiplication with operators from $\mathscr{L}(X)$, so $\mathscr{M}_X$ is an ideal if and only if $\mathscr{M}_X$ is closed under addition. It is also clear that if $\mathscr{M}_X$ is an ideal then is the largest ideal in $\mathscr{L}(X)$.

Dosev and Johnson in proposition 5.1 [13] gave a sufficient condition that a Banach space X must satisfy in order to have $\mathscr{M}_X$ is an ideal in $\mathscr{L}(X)$.

Theorem 1.1 (Proposition 5.1 in [13])**.** *Let X be a Banach space such that for every $T \in \mathscr{L}(X)$ we have $T \notin \mathscr{M}_X$ or $I - T \notin \mathscr{M}_X$. Then $\mathscr{M}_X$ is the largest ideal in $\mathscr{L}(X)$.*

The following is an example of theorem 1.1:

Example 1.2. $X = \mathscr{C}([0,1])$ *satisfies the conditions in the theorem, hence $\mathscr{M}_X$ is the largest ideal in $\mathscr{L}(\mathscr{C}([0,1]))$. This result was proved by J. Lindenstrauss, A. Pelczynski, in [38].*

In 2011 in [14], D. Dosev, W. B. Johnson and G. Schechtman considered operators $T : L_p \to L_p$, $1 \leq p < \infty$, that preserve a complemented copy of L_p; that is, there exists a complemented subspace $X \subseteq L_p$, $X \simeq L_p$ such that $T_{|X}$ is an isomorphism. They proved also from the definition of $\mathscr{M}_{L_p}$ that $T \notin \mathscr{M}_{L_p}$ if and only if T maps a copy of L_p isomorphically onto a complemented copy of L_p. Thus, they proved that the class of operators from L_p to L_p that do not preserve a complemented copy of L_p coincides with the class of L_p-strictly singular operators; hence, the class of L_p-strictly singular operators is the largest ideal in $\mathscr{L}(L_p)$.

In 2012, in [10], D. Chen, W. B. Johnson and B. Zheng proved structural results for $Z_{p,q}$. For instance in proposition 2.8, they proved that for every $\varepsilon > 0$, there exists a subspace X of $Z_{p,q}$, isometric to $Z_{p,q}$ so that the restriction to X of an operator that is $Z_{p,q}$-strictly singular on $Z_{p,q}$ has small norm. This lemma actually says that the set of all $Z_{p,q}$-strictly singular operators on $Z_{p,q}$ is an ideal. They proved also in lemma 2.7 that for each $\varepsilon > 0$, every subspace of $Z_{p,q}$ isomorphic to $Z_{p,q}$ contains a subspace that is $(1+\varepsilon)$-isomorphic to $Z_{p,q}$ and $(1+\varepsilon)$-complemented in $Z_{p,q}$. This lemma actually says that the ideal of all $Z_{p,q}$-strictly singular operators on $Z_{p,q}$ is maximal. As an immediate consequence of these two results they showed that the set of all $Z_{p,q}$-strictly singular on $Z_{p,q}$ coincide with the set $\mathscr{M}_{Z_{p,q}}$ and forms the unique maximal ideal in $\mathscr{L}(Z_{p,q})$.

In 2014 in [53], B. Zheng obtained structural results about the space $(\sum \ell_q)_{\ell_1}$. Since some of the key lemmas for proving the structural results for $Z_{p,q}$ fail in the space $(\sum \ell_q)_{\ell_1}$, he had to develop some new techniques to prove in lemma 2.8 that for every $\varepsilon > 0$, there exists a subspace X of $(\sum \ell_q)_{\ell_1}$, isometric to $(\sum \ell_q)_{\ell_1}$ so that the restriction to X of an operator that is $(\sum \ell_q)_{\ell_1}$-strictly singular on $(\sum \ell_q)_{\ell_1}$ has small norm. This lemma actually says that the set of all $(\sum \ell_q)_{\ell_1}$-strictly singular operators on $(\sum \ell_q)_{\ell_1}$ is an ideal. He proved also in lemma 2.5 that for each $\varepsilon > 0$, every subspace of $(\sum \ell_q)_{\ell_1}$ isomorphic to $(\sum \ell_q)_{\ell_1}$ contains a subspace that is $(1+\varepsilon)$-isomorphic to $(\sum \ell_q)_{\ell_1}$ and $(1+\varepsilon)$-complemented in $(\sum \ell_q)_{\ell_1}$. This lemma actually says that the ideal of all $(\sum \ell_q)_{\ell_1}$-strictly singular operators on $(\sum \ell_q)_{\ell_1}$ is maximal. As an immediate consequence of these two results he showed that the set of all $(\sum \ell_q)_{\ell_1}$-strictly singular on $(\sum \ell_q)_{\ell_1}$ coincide with the set $\mathscr{M}_{(\sum \ell_q)_{\ell_1}}$ and forms the unique maximal ideal in $\mathscr{L}((\sum \ell_q)_{\ell_1})$.

Even though there are a few Banach spaces X for which the lattice of closed ideals in $\mathscr{L}(X)$ is totally determined, there are quite a few partial results described. Some of them are the following.

Calkin [9] classified all the ideals in $\mathscr{L}(\ell_2)$. In particular, he proved that the only non-trivial closed ideal in the algebra $\mathscr{L}(\ell_2)$ is the ideal of compact operators.

For each non-separable Hilbert space H, Gramsch [21] and Luft [41] described independently all the closed ideals in $\mathscr{L}(H)$ and showed that they are well-ordered by inclusion.

Daws [11] extended the results obtained by Gramsch [21] and Luft [41] to the non-separable versions of ℓ_p and c_0. They defined for an infinite set I, the set $I^{<\infty} = \{A \subseteq I : |A| < \infty\}$. Thus, $c_0(I) = \{(x_i)_{i\in I} : \forall \varepsilon > 0, \{i \in I : |x_i| \geq \varepsilon\} \in I^{<\infty}\}$ is a Banach space with the supremum norm. Similarly, for $1 \leq p < \infty$ and

$$\ell_p(I) = \left\{ (x_i)_{i \in I} : \|(x_i)\|_p = \left(\sum_{i \in I} |x_i|^p \right)^{1/p} < \infty \right\}. \tag{1.1}$$

The main result in this paper is that for an infinite set I and for $E = \ell_p(I)$ or $E = c_0$, $\mathscr{L}(E)$ has perfect compact ideal structure, that is, the closed ideals in $\mathscr{L}(E)$ form an ordered chain:

$$\{0\} \subsetneq \mathscr{K}(E) \subsetneq \mathscr{K}_{\aleph_1}(E) \subsetneq .... \subsetneq \mathscr{K}_{|I|}(E) \subsetneq \mathscr{K}_{|I|^+}(E) = \mathscr{L}(E) \tag{1.2}$$

Volkmann [49] proved that there are exactly two maximal ideals in $\mathscr{L}(\ell_p \oplus \ell_q)$ for $1 \leq p, q < \infty$, $p \neq q$. These two ideals are generated by the operators that factor through ℓ_p and ℓ_q, respectively. Their intersection is the ideal of the strictly singular operators. He also proved that a similar result holds if either one of the spaces is replaced by c_0.

Diestel, Jarchow and Pietsch [12] proved that there are infinitely many closed ideals in $\mathscr{L}(L_p[0,1])$ for $1 < p < \infty$, $p \neq 2$. They also proved that there are uncountably many closed ideals in $\mathscr{L}(C[0,1])$.

Edelstein and Mityagin [43] showed that the ideal of weakly compact operators is a maximal ideal of codimension one in $\mathscr{L}(J)$, where J is the James's quasi-reflexive Banach space defined in [23]. In [31], Laustsen proved that this ideal is the only maximal ideal in $\mathscr{L}(J)$ and constructed Banach spaces X so that $\mathscr{L}(X)$ has any specified finite number of maximal ideals of any specified codimensions.

In [19], Gowers and Maurey constructed a hereditarily indecomposable Banach space and proved that the ideal of the strictly singular operators is a maximal ideal of codimension one in $\mathscr{L}(X)$ for each subspace X. This maximal ideal is unique (see [31]).

Androulakis and Schlumprecht showed in [2] that the non-compact, strictly singular operators exist in the space constructed by Gowers and Maurey, so the ideal of the strictly singular operators in this space is not the only non-trivial closed ideal in the algebra of the linear bounded operators on that space.

In 2018, Leung [36] identified a number of Banach spaces X for which $\mathscr{M}_X$ is the unique maximal ideal in the the algebra $\mathscr{L}(X)$. In particular, he proved that when X is a subsymmetric direct sum of ℓ_p, $1 \leq p < \infty$, c_0 or the Schlumprecht space S, then $\mathscr{M}_X$ is closed under addition and hence is the unique maximal ideal in $\mathscr{L}(X)$. In this paper, the set of all operators $T \in \mathscr{L}(X)$ that factor through Y is denoted by $\zeta_Y(X)$. If Y is isomorphic to $Y \oplus Y$, then $\zeta_Y(X)$ is an ideal in $\mathscr{L}(X)$ and hence, it is its closure. The space of weakly compact operators on J_p is denoted $\mathscr{W}(J_p)$, also $G_p = (\oplus \ell_\infty)_{\ell_p}$ and $J_p^{(\infty)} = (\oplus J_p(n))_{\ell_p}$. He proved that both $\mathscr{W}(J_p)$ and the closure of $\zeta_{G_p}(J_p)$ have a unique maximal ideal, namely, $I_1 = \{T \in \mathscr{W}(J_p) : I \in \mathscr{L}(J_p^{(\infty)})$ doesn't factor through $T\}$ and $I_2 = \{T \in \overline{\zeta_{G_p}(J_p)} : I \in \mathscr{L}(G_p)$ does not factor through $T\}$, respectively.

Gowers constructed a Banach space G so that the space ℓ_∞ / c_0 is a quotient of $\mathscr{L}(G)$, (see [20]). In [31], Laustsen classified the maximal ideals in $\mathscr{L}(G)$ and proved that each such ideal is the preimage of a maximal ideal in ℓ_∞ / c_0.

Anna Kaminska, Popov, Spinu, Tcaciuc and Troitsky studied in [26] the structure of closed algebraic ideals in the algebra of operators acting on a Lorentz sequence space.

Kania and Laustsen investigated in [27] the structure of the lattice of all closed ideals of $\mathscr{L}(C([0, \omega_1]))$ where ω_1 is the first uncountable ordinal.

For the space $X = \left(\oplus_{n=1}^\infty \ell_2^n\right)_{c_0}$, it has been shown in [32] that there are exactly four closed ideals in $\mathscr{L}(X)$, and they are totally ordered by inclusion. They are $\{0\}$, the closure of the finite-rank operators, the closure of the set of operators that factor through c_0 and $\mathscr{L}(X)$ itself.

In [34], it was shown that there are exactly four closed ideals in $\mathscr{L}(X)$ for the space $X = \left(\oplus_{n=1}^\infty \ell_2^n\right)_{\ell_1}$, namely, $\{0\}$, the compact operators, the closure of the set of operators factoring through ℓ_1, and $\mathscr{L}(X)$ itself.

In [33], Laustsen, Odell, Schlumprecht and Zsák proved results that provide information on the closed ideal structure of the Banach algebra of all operators on the space $X = \left(\oplus_{n=1}^{\infty} \ell_1^n\right)_{c_0}$. The closed ideals in $\mathscr{L}(X)$ are $\{0\}$, the compact operators, the norm closure of the ideal of operators factoring through c_0, the norm closure of the surjective hull of the ideal of operators factoring through c_0 and $\mathscr{L}(X)$ itself. They said that if there is another closed ideal, then it must lie between the third and fourth ideal described above. In particular the norm closure of the surjective hull of the ideal of operators factoring through c_0 is the unique maximal ideal of $\mathscr{L}(X)$.

In [37], Lin, Sari and Bentuo Zheng considered for $1 < p < \infty$, a class of p-regular Orlicz sequence spaces ℓ_M that are close to ℓ_p and studied the structure of the norm closed ideals in $\mathscr{L}(\ell_M)$. They showed that the unique maximal ideal in that algebra is the set of all ℓ_M-strictly singular operators and the immediate successor of the ideal of compact operators in that algebra is the closed ideal generated by the formal identity from ℓ_M into ℓ_p.

More results toward the description of maximal ideals on different Banach spaces can be found in [28], [5], [35], [47], [45].

Chapter 2

Preliminaries

In this chapter, we record some well known facts that will be used in the sequel. In this dissertation, any Banach space is built on a real vector space.

2.1 Notation and Basic Definitions

If X is a Banach space, then the space of all bounded linear operators on X will be denoted by $\mathscr{L}(X)$.

If $(x_n)_{n=1}^{\infty}$ is a sequence in a Banach space X, then the closed linear span of $(x_n)_{n=1}^{\infty}$ will be denoted by $[x_n]$.

Definition 2.1 (Block Subspace of ℓ_q)**.** *We say that Z is a block subspace of ℓ_q if Z is the closed linear span of a block basis of the unit vector basis for ℓ_q.*

Definition 2.2 (Strictly Singular Operator)**.** *A bounded linear operator T from a Banach space X into a Banach space Y is strictly singular if there is no infinite-dimensional subspace $E \subset X$ such that $T|_E$ is an isomorphism onto its range.*

Definition 2.3 (f(T, Z))**.** *If $T : X \to Y$ is an operator between Banach spaces and Z is a subspace of X, then:*

$$f(T,Z) = inf\{\|Tz\| : z \in Z, \|z\| = 1\} \tag{2.1}$$

Definition 2.4 ($(\sum X_n)_p$ space)**.** *Given a sequence $(X_n)_{n=1}^{\infty}$ of Banach spaces and $p \in [1,\infty] \cup \{0\}$, $(\sum X_n)_p$ is the space of all sequences $(x_n)_{n=1}^{\infty}$ with $x_n \in X_n$ for each n and $\|(x_n)_{n=1}^{\infty}\| = \|(\|x_n\|)_{n=1}^{\infty}\|_p < \infty$, and the formula $\|(x_n)_{n=1}^{\infty}\| = \|(\|x_n\|)_{n=1}^{\infty}\|_p < \infty$ for $p = 0$ is used in the sense $(\|x_n\|)_{n=1}^{\infty} \in c_0$.*

The projection from $(\sum \ell_q)_{c_0}$ onto the direct sum from the m-th ℓ_q to the n-th ℓ_q is denoted $P_{[m,n]}$.

When $X = (\sum \ell_q)_{c_0}$, we use $\ell_q^{(n)}$ to denote the n-th ℓ_q in the corresponding direct sum.

Definition 2.5 (Z-strictly Singular Operator)**.** *Let X, Y and Z be Banach spaces. An operator from X to Y is said to be Z-strictly singular provided that there is no subspace Z_0 of X which is isomorphic to Z for which $T_{|Z_0}$ is an isomorphism.*

Definition 2.6 (Factor Through)**.** *We say that $S \in \mathscr{L}(X)$ factors through $T \in \mathscr{L}(X)$ if there are $A, B \in \mathscr{L}(X)$ such that $S = ATB$.*

Definition 2.7 (Complementably Homogeneous)**.** *Let X be a Banach space. The space X is said to be complementably homogeneous if for any subspace Y of X that is isomorphic to X, there is a subspace Z of Y such that Z is isomorphic to X and Z is complemented in X.*

Definition 2.8 (C-isomorphism)**.** *A bounded linear operator $T : X \to Y$ between two Banach spaces X and Y is a C-isomorphism if T is an isomorphism on X and $\|T\|\|T^{-1}_{|TX}\| \leq C$.*

2.2 Banach Spaces and Examples

We say that a nonnegative real-valued function $\|\cdot\| : X \to \mathbb{R}$, where X is a vector space, is a norm on X, if $\|\cdot\|$ satisfies:

1. $\|x\| \geq 0$ for all x in X
2. $\|x\| = 0$ if and only if $x = 0$
3. $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$ and for all scalar $\alpha \in \mathbb{F}$
4. $\|x + y\| \leqslant \|x\| + \|y\|$ for all x and for all y in X.

A vector space X with a norm $\|\cdot\|$ defined on it, is called a normed space and is denoted as $(X, \|\cdot\|)$. The balls centered in the origin and with radius one play an important role in the normed spaces theory so we define the following balls:

1. The open unit ball of X, denoted B_X and defined as $B_X = \{x \in X : \|x\| < 1\}$.

2. The closed unit ball of X, denoted B_X and defined as $B_X = \{x \in X : \|x\| \leq 1\}$.

3. The unit sphere of X, denoted S_X and defined as $S_X = \{x \in X : \|x\| = 1\}$.

Let X be a linear space with a norm $\|\cdot\|$ defined on it. We define the metric in X induced by this norm as $d(x,y) = \|x-y\|$ for all $x, y \in X$. If the metric d is a complete metric, then X is called a Banach space and if Y is a subspace of a Banach space X, then Y is a Banach space if and only if Y is closed in X.

The following lemma gives a necessary and sufficient condition for X, a normed space, be a Banach space.

Lemma 2.9 (Lemma 1.15 in [17]). *A normed space X is a Banach space if and only if every absolutely convergent series in X is convergent.*

We have some very well known spaces that are Banach spaces. Here we have a few examples.

1. $C[0,1]$: The space of all scalar-valued continuous functions on $[0,1]$, endowed with the norm defined as $\|f\|_\infty = sup\{|f(x)| : x \in [0,1]\}$, which is called the supremum norm. In general the space $C(K)$ is a Banach space when K is a compact Hausdorff space.

2. ℓ_∞^n: The space of all n-tuples of scalars, endowed with a norm called the supremum norm and defined as $\|x\|_\infty = max\{|x_i| : i = 1, 2....n\}$, where $x = (x_1, ..., x_n) \in \ell_\infty^n$.

3. ℓ_p^n : The space of all n-tuples of scalars, endowed with the norm defined as $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$, where $x = (x_1, ..., x_n) \in \ell_p^n$ and $1 \leqslant p < \infty$.

4. ℓ_p : The space of all infinite scalar-valued sequences $x = (x_i)_{i=1}^\infty$ such that $\sum_{i=1}^{\infty} |x_i|^p < \infty$, endowed with the norm $\|x\|_{\ell_p} = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}$, where $1 \leqslant p < \infty$.

5. ℓ_∞ : The space of all infinite bounded scalar-valued sequences $x = (x_i)_{i=1}^\infty$, endowed with the norm $\|x\|_{\ell_\infty} = sup\{|x_i| : i \in \mathbb{N}\}$.

6. c : The subspace of ℓ_∞ of all sequences such that its limit exists and is finite.

7. c_0 : The subspace of ℓ_∞ of all sequences that converges to zero.

8. $L_p(X,\mu)$, $1 \leqslant p < \infty$: Let (X,S,μ) be a measure space. Let $\mathscr{F}$ be the collection of all measurable extended real-valued functions on X that are finite almost everywhere on X. We say that two functions f and g in $\mathscr{F}$ are equivalent if $f = g$ a.e on X. This relation is an equivalence relation so it induces a partition of $\mathscr{F}$ in disjoint equivalence classes. The collection of these equivalence classes has a linear structure and the equivalence classes are independent of the choice of representatives of each one. $L_p(X,\mu)$, $1 \leqslant p < \infty$ is the space of all equivalence classes of functions satisfying $\int_X |f(x)|^p < \infty$, endowed with the norm defined as $\|f\|_{L_p} = \left(\int_X |f(x)|^p\right)^{1/p}$.

9. $L_\infty(X,\mu)$: The space $L_\infty(X,\mu)$ is the space of all equivalence classes of functions for which f is essentially bounded, that is, there is some $M \geq 0$, called an essential upper bound for $[f]$ for which $|f| \leqslant M$ almost everywhere on X, endowed with the norm $\|f\|_{L_\infty}$ defined as the infimum of the essential upper bounds for f.

2.3 Linear Bounded Operators

We have that a subset B of a normed space X is called bounded if and only if there exists a non-negative number C such that $\|x\| \leq C$ for all $x \in B$, and a linear operator T between normed spaces X and Y is called bounded if the image of a bounded subset of X is a bounded subset of Y. We also say that T is a bounded operator if the image of the closed unit ball of X is a bounded subset of Y.

The following proposition gives several different ways of saying that a linear mapping between normed spaces is continuous.

Proposition 2.10 (Proposition 1.17 in [17]). *Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be normed spaces, and let T be a linear mapping from X into Y. The following are equivalent:*

1. *T is continuous on X.*

2. *T is continuous at the origin.*

3. *There is* $C > 0$ *such that:*
$$\|T(x)\|_Y \leqslant C\|x\|_X \tag{2.2}$$
for every $x \in X$.

4. *T is Lipschitz.*

5. $T(B_X)$ *is a bounded set in Y.*

The proposition 2.10 says that the collection of all continuous linear operators between normed spaces coincides with the collection of all bounded linear operators between normed spaces.

The set $\mathscr{L}(X,Y)$ is a vector space and if we endow this space with a norm called the operator norm and defined as

$$\|T\| = sup\{\|T(x)\|_Y : x \in B_X\}, \tag{2.3}$$

then the space becomes a normed space.

Notice that the norm of T, denoted $\|T\|$, is the smallest number that satisfies the inequality in (2.2) of 2.10. The norm of a bounded linear operator is also defined taking the supreme in (2.3) over all vectors $x \in S_X$.

The following proposition gives a sufficient condition for $\mathscr{L}(X,Y)$ be a Banach space.

Proposition 2.11 (Proposition 1.19 in [17]). *Let X, Y be normed linear spaces. If Y is a Banach space, then* $\mathscr{L}(X,Y)$ *is also a Banach space*

If we consider $\mathscr{L}(X,F)$, the operators in this space are called continuous linear functionals and in this case we define its norm as:

$$\|f\| = sup\{|f(x)| : x \in B_X\} \tag{2.4}$$

2.4 Isomorphisms and Isometries

Basically an isomorphism between normed spaces is a map that identifies the linear space structure and the topology of the domain with those of the range and a linear isometry does the same but also identifies the norm of the domain with the norm of the range.

Let X,Y be two normed spaces and let T be a linear operator from X into Y. We say that T is an isomorphism into Y if it is one to one, continuous and T^{-1} is continuous on the range of T. Moreover, X and Y are said isomorphic spaces if there is an isomorphism from X onto Y. The operator T is called an isometric isomorphism or linear isometry if:

$$\|Tx\|_Y = \|x\|_X, \forall x \in X \tag{2.5}$$

The following proposition gives a sufficient condition for a bounded linear operator between two Banach spaces have a closed range and be an into isomorphism.

Proposition 2.12 (See Chapter 1 in [17])**.** *Let X, Y be Banach spaces and $T \in \mathscr{L}(X,Y)$. If there is a $\delta > 0$ such that $\|T(x)\| \geq \delta\|x\|$ for all $x \in X$, then $T(X)$ is closed in Y, and moreover, T is an isomorphism from X into Y.*

We also say that an operator T between normed spaces X, Y is an isomorphism if and only if T is onto and there exist positive constants c, C so that

$$c\|x\|_X \leq \|Tx\|_Y \leq C\|x\|_X \tag{2.6}$$

for all $x \in X$.

2.5 Fundamental Theorems of Functional Analysis

There are several results in the Banach space theory that are called the Hahn-Banach theorem. All of them tell us that under particular conditions a linear space always has a large enough number of linear functionals. Here we give the linear space version of this theorem.

Before we state the linear version of the Hahn-Banach theorem we need to define a positively homogeneous sublinear functional. Let X be a vector space and f a real-valued function, f is called a positively homogeneous sublinear functional if:

1. $f(\delta x) = \delta f(x)$ for all $x \in X$ and $\delta \geq 0$

2. $f(x+y) \leq f(x) + f(y)$ for all $x, y \in X$

if, moreover, $f(\delta x) = |\delta| f(x)$ for all $x \in X$ and scalars δ, then f is called a seminorm on X.

The linear space version of the Hahn-Banach theorem says that a linear functional on a subspace, which is dominated by a sublinear functional, can be extended to a linear functional on all of the space, which is still dominated by the same sublinear functional.

Theorem 2.13 (Theorem 2.1 in [17])**.** *Let Y be a subspace of a real linear space X, and let f be a positively homogeneous sublinear functional on X. If g is a linear functional on Y such that $g(x) \leq f(x)$ for every $x \in Y$, then there is a linear functional G on X such that $G = g$ on Y and $G(x) \leq f(x)$ for every $x \in X$.*

Let X be a normed space. The space of all continuous linear functionals on X endowed with the norm defined in (2.4) is called the dual of X and is denoted X^*. We have that X^* is a Banach space when X is a normed space.

The following theorem is the normed space version of the Hahn-Banach and says that every functional on a subspace of a normed space can be extended on all of the space with the same norm. Notice that this theorem do not say anything about the uniqueness of the extension.

Theorem 2.14 (Theorem 2.4 in [17]). *Let Y be a subspace of a normed space X. If $f \in Y^*$, then there exists $F \in X^*$ such that $F|_Y = f$ and $\|F\|_{X^*} = \|f\|_{Y^*}$*

If we consider the span of a nonzero element in X and we apply theorem 2.14, we get the following corollary.

Corollary 2.15 (Corollary 2.5 in [17]). *Let X be a normed space. For every $x \neq 0$, there is $f \in S_{X^*}$ such that $f(x) = \|x\|$. In particular, $\|x\| = max\{|f(x)| : f \in B_{X^*}\}$ for every $x \in X$.*

A lot of the Banach space theory is based in three theorems which are mutually related, they are the Open Mapping Theorem, Uniform Boundedness Principle and the Closed Graph Theorem.

Now we turn our attention to the Open Mapping Theorem. Before we state the theorem we need to define an open map. Let X and Y be topological spaces and let f be a map from X into Y. If f maps open sets in X onto open sets in Y, then f is called an open map.

We have that when X and Y are normed spaces and T is a linear operator from X into Y, if T is open, then T is onto. The open mapping principle tell us that when T is a bounded linear operator the converse is true.

Theorem 2.16 (Theorem 2.24 in [17]). *Let X, Y be Banach spaces and $T \in \mathscr{L}(X,Y)$. If T is onto Y, then T is an open mapping.*

The following corollary says that every one to one bounded linear operator between Banach spaces which is also onto, is an isomorphism.

Corollary 2.17 (Corollary 2.25 in [17]). *Let X, Y be Banach spaces, and let $T \in \mathscr{L}(X,Y)$ be onto Y.*

1. *If T is one-to-one, then T^{-1} is a bounded linear operator.*

2. *There is a constant $M > 0$ such that for every $y \in Y$ there is $x \in T^{-1}(y)$ satisfying $\|x\|_X \leq M\|y\|_Y$.*

3. *Y is isomorphic to $X/Ker(T)$.*

Now we turn our attention to the Closed Graph Theorem. Before we state this theorem we need to define the direct sum of normed spaces and the graph of a linear operator. Let X and Y be normed spaces. The space of all ordered pairs (x,y) with $x \in X$ and $y \in Y$ endowed with the norm $\|(x,y)\| = \|x\|_X + \|y\|_Y$ is called a direct sum of X and Y and is denoted $X \bigoplus Y$.

For a linear operator T from X into Y, the set $G = \{(x,T(x)) : x \in X\}$ is called the graph of T and moreover, G is a subspace of $X \bigoplus Y$.

Theorem 2.18 (Theorem 2.26 in [17])**.** *Let X, Y be Banach spaces, and let T be a linear operator from X into Y. T is a bounded operator if and only if its graph is closed in $X \bigoplus Y$.*

Now we turn our attention to the Uniform Boundedness Principle. Before we state the principle we need to define a bounded and a pointwise bounded family of bounded linear operators. Let X, Y be normed spaces and $\mathscr{F}$ a family of operators such that $\mathscr{F} \subset \mathscr{L}(X,Y)$. We say that $\mathscr{F}$ is pointwise bounded if $sup\{\|T(x)\|_Y ; T \in \mathscr{F}\} < \infty$ for all $x \in X$. We say that $\mathscr{F}$ is bounded in $\mathscr{L}(X,Y)$ if there is $C > 0$ so that $\|T\| \leq C$ for all $T \in \mathscr{F}$.

We know that the norm of each operator in a family of bounded operators must be finite by the definition of the norm, but we do not know if those norms form an increasing sequence of numbers. The Uniform Boundedness Principle give us a sufficient condition to determine whether the norms of a given family of bounded linear operators have a finite least upper bound.

Theorem 2.19 (Theorem 3.12 in [17])**.** *Let X, Y be Banach spaces and $\mathscr{F} \subset \mathscr{L}(X,Y)$. If $\mathscr{F}$ is pointwise bounded, then $\mathscr{F}$ is bounded in $\mathscr{L}(X,Y)$.*

2.6 Weak Topology and Weak Star Topology

Let X be a normed space. The dual of X^* is denoted by X^{**} and the norm of this space is given by $\|F\| = \sup_{f \in B_{X^*}} |F(f)|$. The canonical embedding π of X into X^{**} is defined for $x \in X$ by $\pi(x)(f) = f(x)$.

From the Hahn-Banach theorem we get that the canonical embedding π is a linear into isometry and when the map π is onto, we say that the space X is reflexive. In particular a reflexive space X is isometric to X^{**}. The converse is not true since there are nonreflexive spaces which are isometric to its double dual but the isometry is not the canonical embedding.

As examples of spaces that are reflexive we have the spaces ℓ_p for $1 < p < \infty$. As examples of spaces that are not reflexive we have the spaces c_0 and ℓ_1.

Now we turn our attention to topologies for normed spaces that are weaker than the norm topology, that is, they have fewer open sets but they still have enough open sets in order to have useful properties. We have two important topologies called the weak topology and the weak star topology.

The weak topology is the weakest topology in a normed space X such that the linear funcionals in the dual of X are still continuos and the weak star topology is the weakest topology in the dual of X such that the elements in the range of the canonical embedding are still continuous.

Definition 2.20 (Definition 3.3 in [17])**.** *Let X be a normed space. The weak topology on X is the topology generated by a basis consisting of the sets:*

$$\mathscr{O} = \{x \in X : |f_i(x - x_0)| < \varepsilon, \textit{ for } i = 1, \ldots.., n\} \tag{2.7}$$

for all choices of $x_0 \in X$, $f_1, \ldots.. f_n \in X^*$ *and* $\varepsilon > 0$

Similarly, the weak star topology on the dual X^* *of X is generated by a basis consisting* *of the sets:*

$$\mathscr{O}^* = \{f \in X^* : |(f - f_0)(x_i)| < \varepsilon, \text{ for } i = 1, ..., n\} \tag{2.8}$$

for all choices of $f_0 \in X^*$, $x_1, ..., x_n \in X$ *and* $\varepsilon > 0$.

Now we characterize the weak convergence and the weak star convergence for sequences in a normed space and in its dual.

Proposition 2.21 (Proposition 3.7 in [17])**.** *Let X be a normed space.*

1. *Let* $f, f_1, f_2, ... \in X^*$. *Then* $f_n \overset{\omega^*}{\to} f$ *if and only if* $\lim_{n\to\infty}(f_n(x)) = f(x)$ *for every* $x \in X$.
2. *Let* $x, x_1, x_2, ... \in X$. *Then* $x_n \overset{\omega}{\to} x$ *if and only if* $\lim_{n\to\infty}(f(x_n)) = f(x)$ *for every* $f \in X^*$.

For finite-dimensional spaces the weak topology of X and the norm topology are the same and the same happens in the dual of X with the weak star topology and the norm topology, instead for infinite-dimensional spaces this is not the case.

We say that a subset M of a normed space X is weakly bounded provided $x^*(M)$ is a bounded set of numbers for each $x^* \in X^*$ and we say that a subset M of the dual of X is ω^*-bounded provided $sup\{|f(x)| : f \in M\} < \infty$ for every $x \in X$. The following theorem is due to Banach and Steinhaus.

Theorem 2.22 (Theorem 3.15 in [17])**.** *Let X be a Banach space. If* $M \subset X^*$ *is* ω^**-bounded, then M is bounded. If* $M \subset X$ *is* ω*-bounded, then M is bounded.*

We have that every weakly closed subset of a set X is strongly closed, but in general the converse is not true. The following theorem due to Mazur establishes the equivalence between these concepts for convex subsets of a normed space X.

Theorem 2.23 (Theorem 3.19 in [17])**.** *The closure and the weak closure of a convex subset of a normed space are the same. In particular, a convex subset of a normed space is closed if and only if it is weakly closed.*

The following theorem due to Alaoglu is one of the most important results in the normed spaces theory. This theorem reinforce us the concept that the weak star topology is different from both the norm and the weak topology. When we try to extend properties from finite-dimensional normed spaces to infinite-dimensional spaces we face the problem that the closed unit ball of a normed space is compact only when the space is finite-dimensional. When the space is infinite-dimensional, since the weak topology is included in the norm topology, then it is easier for the closed unit ball to be weakly compact than compact. Alaoglu tell us that the closed unit ball of the dual is always compact.

Theorem 2.24 (Theorem 3.21 in [17])**.** *Let X be a Banach space. Then B_{X^*} is compact in the ω^*-topology.*

When X is a reflexive Banach space we always have that B_X is compact in the weak topology of X.

The following proposition will be very useful in some key lemmas of our research problem and it says that if a operator T is bounded and $(x_n)_{n=1}^{\infty}$ converges weakly to x then $(Tx_n)_{n=1}^{\infty}$ converges weakly to Tx.

Proposition 2.25 (See Chapter 3 [17])**.** *Let X and Y be Banach spaces. If a linear operator T from X into Y is $\omega-\omega$-continuous, then $T \in \mathscr{L}(X,Y)$. On the other hand, every bounded linear operator is $\omega-\omega$-continuous.*

2.7 Projections

We know that if Y and Z are subspaces of a vector space X so that $X = Y + Z$ and $Y \cap Z = \{0\}$, then we say that X is the direct sum of Y and Z and we denote it by $X = Y \bigoplus Z$.

We also have that if X is a vector space and Y is a subspace of X, then a linear map $P : X \to X$ is called a projection onto Y if $P(X) = Y$ and $P(y) = y$ for every $y \in Y$. We say that Y is a complemented subspace of X if there is a bounded linear projection P of X onto Y. It is easy to show that if Y is a complemented subspace of a Banach space X, then Y is closed, however the converse is not true. But when Y is a closed finite-dimensional space or a closed finite-codimensional space we have that Y is complemented.

We also have that if Y_1 is a closed subspace of a Banach space X, then Y_2 is called a (topological) complement of Y_1 if X is the direct sum of Y_1 and Y_2 and Y_2 is a closed subspace of X.

Proposition 2.26 (Fact 5.4 in [17])**.** *Let X, Y be Banach spaces, let T be an isomorphism of X onto Y. if X_1 is a complemented subspace of X with topological complement X_2, then $T(X_1)$ is a complemented subspace of Y with topological complement $T(X_2)$.*

This proposition says that an isomorphism sends complemented subspaces of X onto complemented subspaces of Y.

Using proposition 2.26 one can show that if $P : X \to X_1$ is an onto projection with kernel X_2, then $TPT^{-1} : Y \to T(X_1)$ is an onto projection with kernel $T(X_2)$, (see [17]).

Proposition 2.27 (Proposition 5.5 in [17])**.** *Let Y be a closed subspace of a Banach space X.*

1. *If Y is complemented and Z is a complement of Y in X, then X/Y is isomorphic to Z.*
2. *The dual X^* is then isomorphic to $Y^* \bigoplus Z^*$.*
3. *If Y is a closed subspace of a Hilbert space H and Z is the orthogonal complement of Y in H, then H/Y is isometric to Z.*

This proposition says that the dual of the direct sum of Y and Z is the direct sum of the duals of Y and Z and that in the case of a Hilbert space H, the orthogonal complement of Y is a quotient of H.

The next theorem is due to Lindenstrauss and Tzafriri and characterizes the Banach spaces that are isomorphic to a Hilbert space.

Theorem 2.28 (Theorem 5.7 in [17])**.** *Let X be a Banach space. Every closed subspace of X is complemented in X if and only if X is isomorphic to a Hilbert space.*

The next proposition and the way it is proved are called the Pelczyński's decomposition method. This proposition gives necessary conditions for two Banach spaces be isomorphic.

Proposition 2.29 (Proposition 5.8 in [17]). *Let X and Y be Banach spaces. Assume that X is isomorphic to $X \bigoplus X$, Y is isomorphic to $Y \bigoplus Y$, X is isomorphic to a complemented subspace of Y, and Y is isomorphic to a complemented subspace of X. Then X is isomorphic to Y.*

2.8 Schauder Bases

Now we turn our attention to the concepts of bases and basic sequences which are essential in the study of the isomorphic theory of Banach spaces.

Definition 2.30 (Definition 1.a.1 in [39]). *A sequence $(x_n)_{n=1}^{\infty}$ in a Banach space X is called a Schauder basis of X if for each $x \in X$, there is a unique sequence of scalars $(a_n)_{n=1}^{\infty}$ so that $x = \sum\limits_{n=1}^{\infty} a_n x_n$*

Notice that in the definition it is required that the sequence of partial sums converges to x in norm. We also have that the vectors in the basis are nonzero and linearly independent. We have that if $(x_n)_{n=1}^{\infty}$ is a basis of X, then the closed linear span of $(x_n)_{n=1}^{\infty}$ equal X and X will be separable since the set of the rational finite linear combinations of $(x_n)_{n=1}^{\infty}$ is dense in X. We must point out that the order of the elements in the basis is important. If we permute the elements in a basis, it is possible that the new sequence won't be a basis. We also define the canonical projections P_n associated to a Schauder basis as $P_n(x) = \sum\limits_{i=1}^{n} a_i x_i$ for each $n \in \mathbb{N}$ and for $x = \sum\limits_{i=1}^{\infty} a_i x_i \in X$.

Lemma 2.31 (Lemma 6.2 in [17]). *Let $(e_i)_{i=1}^{\infty}$ be a Schauder basis of a normed space X, then the canonical projections P_n satisfy:*

1. $dim(P_n(X)) = n$;

2. $P_n P_m = P_m P_n = P_{min(m,n)}$;

3. $P_n(x) \to x$ *in X for every* $x \in X$.

conversely, if bounded linear projections $\{P_n\}_{n=1}^{\infty}$ *in a normed space X satisfy the items in 2.31, then* P_n *are canonical projections associated with some Schauder basis of X.*

The last lemma give us the conditions that the canonical projections associated to a Schauder basis satisfy and also a way of constructing a Schauder basis for a Banach space X when we have a family of projections that satisfy the conditions of the canonical projections.

The following proposition says that the canonical projections are uniformly bounded.

Proposition 2.32 (Proposition 1.1.4 in [1])**.** *Let* $(e_n)_{n=1}^{\infty}$ *be a Schauder basis for a Banach space X and* $(S_n)_{n=1}^{\infty}$ *the natural projections associated with it. Then* $\sup_n \|S_n\| < \infty$.

The number $K = \sup_n \|S_n\|$ is called the basis constant. If $K = 1$, the basis is called monotone.

Now we turn our attention to basic sequences in Banach spaces.

Definition 2.33 (Definition 1.1.8 in [1])**.** *Let X be a Banach space and let* $(e_k)_{k=1}^{\infty}$ *be a sequence in X. If* $(e_k)_{k=1}^{\infty}$ *is a basis for* $[e_k]$ *then* $(e_k)_{k=1}^{\infty}$ *is called a basic sequence.*

The following proposition will give us a test to find out when a sequence in X is a basic sequence. This test is called the Grunblum's criterion.

Proposition 2.34 (Proposition 1.1.9 in [1])**.** *A sequence* $(e_k)_{k=1}^{\infty}$ *of nonzero elements of a Banach space is basic if and only if there is a positive constant K such that*

$$\left\| \sum_{k=1}^{m} a_k e_k \right\| \le K \left\| \sum_{k=1}^{n} a_k e_k \right\| \tag{2.9}$$

for any sequences of scalars $(a_k)_{k=1}^{\infty}$ *and any integers* m, n *such that* $m \le n$

In the Banach space theory is often useful to know if a Banach space Y contains an isomorphic copy of another Banach space X with basis $(x_n)_{n=1}^{\infty}$. The natural way to find out this is looking for a basic sequence in Y such that its closed linear span is isomorphic to X. This idea take us to the concept of equivalent bases.

Definition 2.35 (Definition 1.3.1 in [1]). *Let X and Y be two Banach spaces. Let $(x_n)_{n=1}^{\infty}$ be a basis (basic sequence) of X and $(y_n)_{n=1}^{\infty}$ be a basis (basic sequence) of Y, then $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are called equivalent provided $\sum_{n=1}^{\infty} a_n x_n$ converges if and only if $\sum_{n=1}^{\infty} a_n y_n$ converges, whenever $(a_n)_{n=1}^{\infty}$ is a sequence of scalars. In this case we write $(x_n)_{n=1}^{\infty} \sim (y_n)_{n=1}^{\infty}$.*

As a consequence of the Closed Graph Theorem we have the following theorem.

Theorem 2.36 (Theorem 1.3.2 in [1]). *Two bases (or basic sequences) $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are equivalent if and only if there is an isomorphism $T : [x_n] \to [y_n]$ such that $Tx_n = y_n$ for each n.*

A corollary of this theorem will be very useful in this dissertation.

Corollary 2.37 (Corollary 1.3.3 in [1]). *Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be two bases for the Banach spaces X and Y respectively. Then $(x_n)_{n=1}^{\infty} \sim (y_n)_{n=1}^{\infty}$ if and only if there exists a constant $C > 0$ such that for all finitely nonzero sequences of scalars $(a_i)_{i=1}^{\infty}$ we have:*

$$C^{-1} \left\| \sum_{i=1}^{\infty} a_i y_i \right\| \leq \left\| \sum_{i=1}^{\infty} a_i x_i \right\| \leq C \left\| \sum_{i=1}^{\infty} a_i y_i \right\| \tag{2.10}$$

in the case $C = 1$, the bases are called isometrically equivalent.

A very useful way of getting new basic sequences from a basis or a basic sequence is constructing block bases. The following definition and the lemma that come after it express this idea.

Definition 2.38 (Definition 1.3.4 in [1]). *Let X be a Banach space, $(e_n)_{n=1}^{\infty}$ be a basis of X, $(p_n)_{n=1}^{\infty}$ be a strictly increasing sequence of integers with $p_0 = 0$ and $(a_n)_{n=1}^{\infty}$ a sequence of scalars. Let $\mu_n = \sum_{j=p_{n-1}+1}^{p_n} a_j e_j$ be a nonzero sequence of vectors in X. Then $(\mu_n)_{n=1}^{\infty}$ is called a block basic sequence of $(e_n)_{n=1}^{\infty}$.*

Lemma 2.39 (Lemma 1.3.5 in [1]). *Suppose $(e_n)_{n=1}^{\infty}$ is a basis for the Banach space X with basis constant K. Let $(\mu_k)_{k=1}^{\infty}$ be a block basic sequence of $(e_n)_{n=1}^{\infty}$. Then $(\mu_k)_{k=1}^{\infty}$ is a basic sequence with basis constant less than or equal to K.*

Definition 2.40 (Definition 1.3.8 in [1]). *Let X and Y be two Banach spaces and let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be two sequences in X and Y respectively. Suppose there is an invertible operator $T : X \to Y$ such that $T(x_n) = y_n$ for all $n \in \mathbb{N}$, then $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are called congruent.*

The following Theorem says that Schauder bases have stability properties, that is, if we perturb each element of a basis by a sufficiently small vector then we get again a basis and the new basis is equivalent to the original one. this theorem is called the principle of small perturbations.

Theorem 2.41 (Theorem 1.3.9 in [1]). *Let $(x_n)_{n=1}^{\infty}$ be a basic sequence in a Banach space X with basis constant K. If $(y_n)_{n=1}^{\infty}$ is a sequence in X such that*

$$2K \sum_{n=1}^{\infty} \frac{\|x_n - y_n\|}{\|x_n\|} = \theta < 1, \tag{2.11}$$

then $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are congruent. In particular:

1. *If $(x_n)_{n=1}^{\infty}$ is a basis, so is $(y_n)_{n=1}^{\infty}$ (in which case the basis constant of $(y_n)_{n=1}^{\infty}$ is at most $K(1+\theta)(1-\theta)^{-1}$)*

2. *$(y_n)_{n=1}^{\infty}$ is a basic sequence (with basis constant at most $K(1+\theta)(1-\theta)^{-1}$)*

3. *If $[x_n]$ is complemented, then $[y_n]$ is complemented.*

The following proposition is an application of the principle of small perturbation and is known as the Bessaga-Pelczyński selection principle and the technique used in the proof which consists in finding vectors with almost successive supports is known as the gliding hump argument.

Proposition 2.42 (Proposition 1.3.10 in [1]). *Let $(e_n)_{n=1}^{\infty}$ be a basis for a Banach space X with basis constant K and dual functionals $(e_n^*)_{n=1}^{\infty}$. Suppose $(x_n)_{n=1}^{\infty}$ is a sequence in X such that:*

1. $\inf_n \|x_n\| > 0$, *but*

2. $\lim_{n\to\infty} e_k^*(x_n) = 0$ *for all* $k \in \mathbb{N}$,

then $(x_n)_{n=1}^{\infty}$ contains a subsequence $(x_{n_k})_{k=1}^{\infty}$ which is congruent to some block basic sequence $(y_k)_{k=1}^{\infty}$ of $(e_n)_{n=1}^{\infty}$. Furthermore, for every $\varepsilon > 0$ it is possible to choose $(n_k)_{k=1}^{\infty}$ so that $(x_{n_k})_{k=1}^{\infty}$ has a basis constant at most $K+\varepsilon$. In particular, the same result holds if $(x_n)_{n=1}^{\infty}$ converges to 0 weakly but not in the norm topology.

2.9 The isomorphic structure of the spaces ℓ_p for $1 \leq p < \infty$ and c_0

The techniques developed using bases are very useful to understand the structure of the spaces ℓ_p for $1 \leq p < \infty$ and c_0. These spaces are equipped with a canonical monotone Schauder basis $(e_n)_{n=1}^{\infty}$ given by $e_n(k) = 1$ if $k = n$ and 0 otherwise.

The following lemma will say that the closed linear span of a normalized block basic sequence of the canonical basis of $X = \ell_p$ or $X = c_0$ is isometric to X and complemented in X.

Lemma 2.43 (Lemma 2.1.1 in [1]). *Let $(\mu_n)_{n=1}^{\infty}$ be a normalized block basic sequence in c_0 or in ℓ_p for some $1 \leq p < \infty$. Then $(\mu_n)_{n=1}^{\infty}$ is isometrically equivalent to the canonical basis of the space and $[\mu_n]$ is the range of a contractive projection.*

We say that $(\mu_n)_{n=1}^{\infty}$ is seminormalized if for some constants a and b we have:

$$0 < a \leq \|\mu_n\| \leq b < \infty,\ n \in \mathbb{N} \tag{2.12}$$

if this is the case, lemma 2.43 can be used for $\left(\frac{\mu_n}{\|\mu_n\|}\right)_{n=1}^{\infty}$, but then $(\mu_n)_{n=1}^{\infty}$ is only equivalent to $(e_n)_{n=1}^{\infty}$, not isometrically.

Using lemma 2.43 and the gliding hump technique we can prove the following important proposition.

Proposition 2.44 (Proposition 2.1.3 in [1])**.** *Let $(x_n)_{n=1}^{\infty}$ be a normalized sequence in ℓ_p for $1 \leq p < \infty$ such that for each $j \in \mathbb{N}$ we have $\lim_{n\to\infty} x_n(j) = 0$, (for example suppose $(x_n)_{n=1}^{\infty}$ is weakly null). Then there is a subsequence $(x_{n_k})_{k=1}^{\infty}$ which is a basic sequence equivalent to the canonical basis of ℓ_p and such that $[x_{n_k}]$ is complemented in ℓ_p. The proposition is also true in c_0.*

We know that for any spaces X and Y, every compact operator $T : X \to Y$ is strictly singular. Also, if $r > p$, the formal identity $I : \ell_p \to \ell_r$ is an operator that is strictly singular but not compact. The following theorem is due to Pitt and give a sufficient condition for a bounded operator from ℓ_r to ℓ_p be compact.

Theorem 2.45 (Proposition 2.c.3 in [39])**.** *Let $1 \leq p < r < \infty$. Then every bounded linear operator from ℓ_r into ℓ_p is compact. The same is true for every linear operator from c_0 into ℓ_p.*

The proof of the last theorem shows also that if $T \in \mathscr{L}(\ell_p, \ell_p)$ is strictly singular, then is compact.

2.10 Complemented subspaces of ℓ_p for $1 \leq p < \infty$ and c_0

The results in this section are due to Pelczyński and they show the power of the basic sequences techniques.

The next proposition follows from lemma 2.43 by using the principle of small perturbation and says that every infinite-dimensional closed subspace of ℓ_p for $1 \leq p < \infty$ contains a complemented copy of ℓ_p.

Proposition 2.46 (Proposition 2.2.1 in [1])**.** *Every infinite-dimensional closed subspace Y of ℓ_p for $p \in [1, \infty)$ contains a closed subspace Z such that Z is isomorphic to ℓ_p and complemented in ℓ_p. The proposition works for c_0 too.*

By using the last proposition, Pelczyński in the next theorem gave a complete characterization of the complemented subspaces of ℓ_p for $1 \le p < \infty$ and c_0.

Theorem 2.47 (Theorem 2.2.4 in [1])**.** *Suppose Y is a complemented infinite-dimensional subspace of ℓ_p where $p \in [1,\infty)$. Then Y is isomorphic to ℓ_p. This theorem also is satisfied when Y is a complemented infinite-dimensional subspace of c_0.*

The method used by Pelczyński to prove the last theorem is called the Pelczyński decomposition method. In the proof is used that $X = \ell_p$ or $X = c_0$ and Y are each isomorphic to a complemented subspace of the other space and that X is isomorphic to an infinite direct sum of itself with respect to a suitable norm. As a consequence of this we have that $X \oplus X$ is also isomorphic to X.

Definition 2.48 (Definition 2.2.5 in [1])**.** *A Banach space X is called prime if every complemented infinite-dimensional subspace of X is isomorphic to X.*

Thus the ℓ_p-spaces and c_0 are prime. In fact ℓ_∞ is also a prime space.

2.11 Unconditional Bases

Before we give the definition of an unconditional basis we need to recall that if X is a Banach space, $(x_n)_{n=1}^{\infty}$ a sequence in X, and $\sum_{n=1}^{\infty} x_n$ a formal series in X and if the series $\sum_{n=1}^{\infty} x_{\pi(n)}$ converges for every permutation π of $\mathbb{N}$, then we say that the series is unconditionally convergent.

The next lemma give us different ways of saying that a series is unconditionally convergent.

Lemma 2.49 (Lemma 2.4.2 in [1])**.** *Given a series $\sum_{n=1}^{\infty} x_n$ in a Banach space X, the following are equivalent:*

1. $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent;

2. *The series* $\sum_{k=1}^{\infty} x_{n_k}$ *converges for every increasing sequence of integers* $(n_k)_{k=1}^{\infty}$*;*

3. *The series* $\sum_{n=1}^{\infty} \varepsilon_n x_n$ *converges for every choice of signs* $(\varepsilon_n)_{n=1}^{\infty}$*;*

4. *For every* $\varepsilon > 0$*, there exists an n, so that, if F is any finite subset of* $\{n+1, n+2, ...\}$*, then* $\left\| \sum_{j\in F} x_j \right\| < \varepsilon$.

A basis $(e_n)_{n=1}^{\infty}$ of a Banach space X is called unconditional provided for each $x \in X$ the series $\sum_{n=1}^{\infty} e_n^*(x)e_n$ converges unconditionally.

Some examples of unconditional bases and conditional bases are the following:

1. The canonical basis of c_0 and of ℓ_p for $p \in [1,\infty)$ is unconditional.

2. The summing basis $(f_n)_{n=1}^{\infty}$ of c_0 defined as $f_n = e_1 + e_2 + ... + e_n$ for all $n \in \mathbb{N}$, is not unconditional.

The following proposition give us a criterion to determine when a basis of a Banach space is unconditional.

Proposition 2.50 (Proposition 3.1.3 in [1])**.** *A basis* $(e_n)_{n=1}^{\infty}$ *of a Banach space X is unconditional if and only if there is a constant* $K \geq 1$ *such that for all* $N \in \mathbb{N}$*, whenever* $a_1, ..., a_N, b_1, ..., b_N$ *are scalars satisfying* $|a_n| \leq |b_n|$ *for* $n = 1, ..., N$*, then the following inequality holds:*

$$\left\| \sum_{n=1}^{N} a_n e_n \right\| \leq K \left\| \sum_{n=1}^{N} b_n e_n \right\| \tag{2.13}$$

We have also that if $(e_n)_{n=1}^{\infty}$ is an unconditional basic sequence and σ is a subset of integers, then the bounded linear projections P_σ defined by $P_\sigma\left(\sum_{n=1}^{\infty} a_n e_n\right) = \sum_{n\in\sigma} a_n e_n$ are called the natural projections associated to the unconditional basic sequence. Similarly, for every choice of signs $\theta = (\theta_n)_{n=1}^{\infty}$, we have a bounded linear operator M_θ defined by $M_\theta\left(\sum_{n=1}^{\infty} a_n e_n\right) = \sum_{n=1}^{\infty} a_n \theta_n e_n$. The Uniform Boundedness Principle implies that $\sup_\sigma \|P_\sigma\|$ and $\sup_\theta \|M_\theta\|$ are finite. The number $\sup_\theta \|M_\theta\|$ is called the unconditional constant of

$(e_n)_{n=1}^{\infty}$ and we have the following inequalities:

$$\sup_{\sigma} \|P_\sigma\| \leq \sup_{\theta} \|M_\theta\| \leq 2 \sup_{\sigma} \|P_\sigma\| \tag{2.14}$$

The last expression shows that the unconditional constant of a basis is always greater or equal to the basis constant. We can always make the unconditional constant equal one by renorming the space with a new norm defined as $|||x||| = \sup_{\theta} \|M_\theta x\|$.

Every block basic sequence of an unconditional basis is again unconditional with unconditional constant less or equal than the unconditional constant of the basis. Also, the sequence of biorthogonal functionals of an unconditional basis is an unconditional basis of the dual with unconditional constant equal to the unconditional constant of the basis of the space.

2.12 Boundedly-complete and shrinking bases

In this section we will present under which conditions the sequence of biorthogonal functionals of a basis of a Banach space is a basis for the dual space. In general that is not true, however, the next proposition says that the sequence of biorthogonal functionals is a basic sequence.

Proposition 2.51 (Proposition 3.2.1 in [1]). *Suppose that $(e_n^*)_{n=1}^{\infty}$ is the sequence of biorthogonal functionals associated to a basis $(e_n)_{n=1}^{\infty}$ of a Banach space X. Then $(e_n^*)_{n=1}^{\infty}$ is a basic sequence in X^* with basis constant no bigger than that of $(e_n)_{n=1}^{\infty}$.*

When the closed linear span of the biorthogonal functionals associated to a basis in a Banach space X equals X^*, we say that the basis is shrinking.

The following two propositions give us necessary and sufficient conditions for a basis be shrinking.

Proposition 2.52 (Proposition 3.2.6 in [1]). *A basis $(e_n)_{n=1}^{\infty}$ of a Banach space X is* *shrinking if and only if whenever $x^* \in X^*$*

$$\lim_{N\to\infty} \left\| x^*|_{[e_n]_{n>N}} \right\| = 0 \tag{2.15}$$

where

$$\left\| x^*|_{[e_n]_{n>N}} \right\| = sup\left\{ |x^*(y)| : y \in [e_n]_{n>N} \right\}. \tag{2.16}$$

Proposition 2.53 (Proposition 3.2.7 in [1])**.** *A basis $(e_n)_{n=1}^{\infty}$ of a Banach space X is shrinking if and only if every bounded block basic sequence of $(e_n)_{n=1}^{\infty}$ is weakly null.*

Now we turn our attention to boundedly-complete bases. We say that a basis $(e_n)_{n=1}^{\infty}$ of a Banach space X is boundedly-complete if the series $\sum_{n=1}^{\infty} a_n e_n$ converges whenever $(a_n)_{n=1}^{\infty}$ is a sequence of scalars such that $\sup_m \left\| \sum_{n=1}^{m} a_n e_n \right\| < \infty$.

Now we give some examples of shrinking and boundedly-complete bases:

1. The canonical basis of ℓ_p is both shrinking and boundedly-complete for $1 < p < \infty$.

2. The canonical basis of ℓ_1 is boundedly-complete but not shrinking.

3. The canonical basis of c_0 is shrinking but not boundedly complete.

4. The summing basis of c_0 is neither shrinking or boundedly-complete.

The following theorem give us equivalent statements to the statement that a basis for a Banach space is boundedly complete.

Theorem 2.54 (Theorem 3.2.10 in [1])**.** *Let $(e_n)_{n=1}^{\infty}$ be a basis for a Banach space X with biorthogonal functionals $(e_n^*)_{n=1}^{\infty}$. Then the following are equivalent:*

1. *$(e_n)_{n=1}^{\infty}$ is a boundedly-complete basis for X,*

2. *$(e_n^*)_{n=1}^{\infty}$ is a shrinking basis for $H = \{x^* \in X^* : \|S_N^*(x^*) - x^*\| \to 0\}$, where $(S_N^*)_{N=1}^{\infty}$ is the sequence of adjoint operators of the partial sum projections associated to $(e_n)_{n=1}^{\infty}$.*

3. *The canonical map $j : X \to H^*$ defined by $j(x)(h) = h(x)$, for all $x \in X$ and $h \in H$, is an isomorphism.*

The following theorem give us equivalent statements to the statement that a basis for a Banach space is shrinking.

Theorem 2.55 (Theorem 3.2.12 in [1])**.** *Let $(e_n)_{n=1}^{\infty}$ be a basis for a Banach space X with biorthogonal functionals $(e_n^*)_{n=1}^{\infty}$. Then the following are equivalent:*

1. *$(e_n)_{n=1}^{\infty}$ is a shrinking basis for X,*

2. *$(e_n^*)_{n=1}^{\infty}$ is a boundedly-complete basis for H,*

3. *$H = X^*$*

2.13 Banach Algebras

Definition 2.56 (Definition 3.3.1 in [42])**.** *Suppose that X is a set, that $+$ and $*$ are binary operations from $X \times X$ into X, and that $\cdot$ is a binary operation from $\mathbb{F} \times X$ into X such that:*

1. *$(X, +, .)$ is a vector space.*

2. *$x * (y * z) = (x * y) * z$.*

3. *$x * (y + z) = x * y + x * z$ and $(x + y) * z = x * z + y * z$.*

4. *$\alpha.(x * y) = (\alpha.x) * y = x * (\alpha.y)$,*

then $(X, +, *, .)$ is an algebra. This algebra is an algebra with identity if $X \neq \{0\}$ and there exists $e \in X$ such that $e * x = x * e = x$ for all $x \in X$.

If $\|\cdot\|$ is a norm on $(X, +, .)$ such that $\|x * y\| \leq \|x\|\|y\|$ for all $x, y \in X$, then X is a normed algebra, and is a Banach algebra if the norm is a Banach norm.

As examples, we have: the space ℓ_∞ is a Banach algebra with identity the sequence $(1, 1, 1, 1, ...)$ and the multiplication of elements in this algebra is done termwise. The space c_0 is a closed subalgebra of ℓ_∞ and therefore is itself a Banach algebra but with no multiplicative identity.

In an algebra X with identity e we say that $x \in X$ is invertible if there is a $y \in X$ such that $xy = yx = e$, y is called the inverse of x and is denoted x^{-1}. Moreover, x^n is defined inductively by saying $x^1 = x$ and $x^n = x^{n-1}x$ for $n \geq 2$, also $x^0 = e$ and $x^{-n} = (x^{-1})^n$.

The next proposition gives some important facts about algebras.

Proposition 2.57 (Proposition 3.3.10 in [42])**.** *Suppose that X is an algebra.*

1. *If $x \in X$ and 0 is the zero element of X, then $0x = 0x = 0$.*

2. *The element 0 of X is not a multiplicative identity for X.*

3. *The algebra X has at most one multiplicative identity.*

now suppose that X is an algebra with identity e.

4. *The element 0 of X is not invertible.*

5. *Each element of X has at most one inverse.*

6. *If x, y, z are elements of X such that $yx = xz = e$, then $y = z$ and x is invertible with inverse y. That is, every element of X that is both left-invertible and right-invertible is invertible and each of its left inverses and right inverses equals its inverse.*

7. *If x and y are invertible elements of X and α is a nonzero scalar, then xy, αx, x^{-1} are invertible and have respective inverses $y^{-1}x^{-1}$, $\alpha^{-1}x^{-1}$, x.*

8. *If x is an invertible element of X and $n \in \mathbb{N}$, then x^n is invertible, and $(x^n)^{-1} = (x^{-1})^n = x^{-n}$.*

suppose next that X is a normed algebra, possibly without identity.

9. *If $x \in X$ and $n \in \mathbb{N}$, then $\|x^n\| \leq \|x\|^n$.*

finally, suppose that X is a normed algebra with identity e.

10. $\|e\| \geq 1$.

11. *If x is an invertible element of X, then* $\|x^{-1}\| \geq \|x\|^{-1}$.

We know that the addition of vectors and the multiplication of a scalar by a vector are continuous operations in a normed space, thus , they are also continuous operations in a normed algebra. The following theorem says that the same is true for multiplication of vectors in a normed algebra.

Proposition 2.58 (Proposition 3.3.11 in [42])**.** *The multiplication of members of a normed algebra X is a continuous operation from* $X \times X$ *into X.*

As a consequence of this proposition we have that the multiplication in a normed algebra distributes over infinite sums and an application of this gives origin to the next definition and theorem.

Definition 2.59 (Definition 3.3.12 in [42])**.** *Suppose that x is an element of an algebra X with identity e. Then the resolvent set* $\rho(x)$ *of x is the set of all scalars* α *such that* $\alpha e - x$ *is invertible. The spectrum* $\sigma(x)$ *of x is the complement in* $\mathbb{F}$ *of* $\rho(x)$*. The resolvent function or resolvent* R_x *of x is the function from* $\rho(x)$ *into X defined by the formula* $R_x(\alpha) = (\alpha e - x)^{-1}$.

Theorem 2.60 (Theorem 3.3.13 in [42])**.** *Suppose that x is an element of a Banach algebra with identity and that* α *is a scalar such that* $|\alpha| > \|x\|$*. Then* $\alpha \in \rho(x)$ *and* $R_x(\alpha) = \sum_{n=0}^{\infty} \alpha^{-(n+1)} x^n$.

This theorem yields to two important corollaries.

Corollary 2.61 (Corollary 3.3.14 in [42])**.** *If x is an element of a Banach algebra with identity and* $\alpha \in \sigma(x)$*, then* $|\alpha| \leq \|x\|$.

Corollary 2.62 (Corollary 3.3.15 in [42])**.** *If x is an element of a Banach algebra with identity e and* $\|x - e\| < 1$*, then x is invertible and* $x^{-1} = \sum_{n=0}^{\infty} (e - x)^n$.

The following proposition sets that the map $x \to x^{-1}$ is also continuous.

Proposition 2.63 (Proposition 3.3.16 in [42])**.** *Suppose that X is a Banach algebra with identity e. Let $G(X)$ be the set of invertible elements of X. Then $G(X)$ is an open subset of X that is a group with identity e under the restriction of the multiplication operation of X to $G \times G$. Furthermore, the map $x \to x^{-1}$ is a topological homeomorphism of $G(X)$ onto itself.*

As a consequence of the continuity of the map $\alpha \to \alpha e - x$ we get the following proposition.

Proposition 2.64 (Proposition 3.3.17 in [42])**.** *If x is an element of a Banach algebra with identity, then $\rho(x)$ is open.*

Now we have that $\sigma(x)$ is closed and bounded, so we have the following theorem.

Theorem 2.65 (Theorem 3.3.18 in [42])**.** *If x is an element of a Banach algebra with identity, then $\sigma(x)$ is compact.*

Now we present the fact whether or not the spectrum of an element in a Banach algebra with identity is empty. If the scalar field is $\mathbb{R}$ this can happen, however , this cannot happen when the scalar field is $\mathbb{C}$.

Theorem 2.66 (Theorem 3.3.24 in [42])**.** *If x is an element of a complex Banach algebra with identity, then $\sigma(x)$ is nonempty.*

So far we have that the spectrum $\sigma(x)$ of an element x of a Banach algebra with identity is compact and that $\sigma(x)$ is included in a closed disk centered at zero and with radius $\|x\|$. In general one is interested in finding the smallest such a disk. The radius of that smallest disk is called the spectral radius of x, is denoted by $r_\sigma(x)$ and is defined as:

$$r_\sigma(x) = max\{|\alpha| : \alpha \in \sigma(x)\} \tag{2.17}$$

We have the following important formula to find the spectral radius of x:

$$r_\sigma(x) = \lim_n \|x^n\|^{1/n} \tag{2.18}$$

In this dissertation we are mostly interested in the particular Banach algebra $\mathscr{L}(X)$ whose elements are the linear bounded operators on a nontrivial Banach space X. The multiplication of two operators in this algebra is given by composition and since X is nontrivial then the identity operator on X is the multiplicative identity and we say that an operator $T \in \mathscr{L}(X)$ is invertible if and only if is an isomorphism from X onto itself.

Now we turn our attention to the concept of an ideal of a Banach algebra X with identity e such that $\|e\| = 1$, which is one of the the main topics in this dissertation.

A nonempty subset I of X is called an ideal if:

1. I is a subspace of X;

2. for any $x \in X$ and $y \in I$, $xy \in I$ and $yx \in I$

Any algebra X has two trivial ideals, the ideal $\{0\}$ and the ideal X itself. Any other ideal different from X is called a proper ideal. A proper ideal M of X is called a maximal ideal if it is not properly contained in any proper ideal of X.

We have that the closure of any proper ideal is also a proper ideal, thus any maximal ideal must be closed.

2.14 Banach Lattices

Now we turn our attention to Banach spaces which are partially ordered and whose order and norm are related by some axioms.

Definition 2.67 (See Definition 1.a.1 in [40]). *If X is a Banach space over the real numbers with an order relation defined in it and if X satisfies the following axioms, then X is called a Banach lattice:*

1. *$x \leq y$ implies $x+z \leq y+z$ for all $x,y,z \in X$,*

2. *$\alpha x \geq 0$ for all $x \geq 0$ and for all real number $\alpha \geq 0$,*

3. *For all $x,y \in X$ there exists a l.u.b denoted $x \vee y$ and a g.l.b denoted $x \wedge y$,*

4. *Define $|x| = x \vee (-x)$ for all $x \in X$.*

$$If\ |x| \leq |y|,\ \ then\ \|x\| \leq \|y\| \tag{2.19}$$

By these four axioms we get that the lattice operations are norm continuous, and this fact implies that the set $C = \{x \in X : x \geq 0\}$ is norm closed. This set is called the positive cone of X.

For all $x,y \in X$ we define the positive part of x as $x_+ = x \vee 0$, the negative part of x as $x_- = -(x \wedge 0)$, the absolute value of x as $|x| = x_+ + x_-$ and we say that x,y are disjoint provided $|x| \wedge |y| = 0$.

Let X be a space with an unconditional basis $(e_n)_{n=1}^{\infty}$ whose unconditional constant is one. Define in X an order by:

$$\sum_{n=1}^{\infty} \alpha_n e_n \geq 0\ if\ and\ only\ if\ \alpha_n \geq 0\ for\ all\ n \tag{2.20}$$

we say that X with this order is a Banach lattice and the order is called the order induced by the unconditional basis.

In any space with an unconditional basis and with the order defined as in (2.20), the Axiom (2.19) in the definition 2.67 of a Banach lattice is not satisfied. The new axiom says that if $|x| \leq |y|$, then there exists a constant M such that $\|x\| \leq M\|y\|$.

If a linear space X only satisfies the first three axioms in the definition 2.67, then X is called a vector lattice. If X and Y are vector lattices and $T : X \to Y$ is a linear operator that satisfies $Tx \geq 0$ for all $x \geq 0$ in X, then T is called a positive operator. If moreover, T is one to one, onto and T^{-1} is also positive, then we have:

$$T(x_1 \vee x_2) = Tx_1 \vee Tx_2 \text{ and } T(x_1 \wedge x_2) = Tx_1 \wedge Tx_2 \text{ for all } x_1, x_2 \in X \tag{2.21}$$

In this case we say that T preserves the lattice structure. A linear operator as this one is called an order isomorphism and we say that two vector lattices are order isomorphic if there exists an order isomorphism between them. We also have that when X and Y are Banach lattices, any positive linear operator T from X to Y is continuous. Now, if there exists a linear isometry T from X onto Y, with X and Y two Banach lattices, which is also an order isomorphism, then X and Y are called order isometric Banach lattices.

If X is a Banach lattice and Y is a linear subspace of X, then Y is called a sublattice of X if $x, y \in Y$ implies $x \vee y \in Y$ and $x \wedge y \in Y$. Notice that it is enough that $x \vee y \in Y$ in order to say that Y is a sublattice because of $x \wedge y = x + y - x \vee y$. It is assumed that any sublattice is also norm closed.

An ideal Y in a Banach lattice X is a sublattice that satisfies the condition that if $|y| \leq |x|$ for some $x \in Y$, then $y \in Y$. It is assumed also that an ideal is norm closed.

Let X be a Banach lattice and Y be an ideal in X. If we take the positive cone of X/Y as the image of the positive cone of X under the quotient map, then X/Y is also a Banach lattice.

If X is a Banach lattice then X^*, the dual of X, is also a Banach lattice when the positive cone of X^* is defined by $x^* \geq 0$ if and only if $x^*(x) \geq 0$ for all $x \geq 0 \in X$.

Definition 2.68 (Definition 1.b.1 in [40])**.** *Let X be a Banach Lattice and let $1 \leq p < \infty$. If for all $x, y \in X$ with $x \wedge y = 0$ we have $\|x+y\|^p = \|x\|^p + \|y\|^p$, then X is called an abstract L_p space. If we have $\|x+y\| = max(\|x\|, \|y\|)$, then X is called an abstract M space.*

We record the following theorem since it was used in the proof of a lemma proved in [10] and we use that lemma in this dissertation.

Theorem 2.69 (Theorem 1.b.2 in [40])**.** *An abstract L_p space X, $1 \leq p < \infty$, is order isometric to an $L_p(\mu)$ space over some measure space (Ω, Σ, μ). If X has a weak unit, then μ can be chosen to be a finite measure.*

Now we give the definitions for p-convex and p-concave linear operators.

Definition 2.70 (Definition 1.d.3 in [40])**.** *Let X be a Banach lattice, V an arbitrary Banach space and let $1 \leq p \leq \infty$. A linear operator $T : V \to X$ is called p-convex if there exists a constant $M < \infty$ so that:*

$$\left\| \left(\sum_{i=1}^{n} |Tv_i|^p \right)^{1/p} \right\| \leq M \left(\sum_{i=1}^{n} \|v_i\|^p \right)^{1/p}, \ if\ 1 \leq p < \infty \tag{2.22}$$

or

$$\|\vee_{i=1}^{n} |Tv_i|\| \leq M \max_{1 \leq i \leq n} \|v_i\|, \ if\ p = \infty \tag{2.23}$$

for every choice of vectors $\{v_i\}_{i=1}^{n}$ in V.

We say that X is p-convex if the identity operator on X is p-convex and the smallest value of M is called the p-convexity.

Definition 2.71 (Definition 1.d.3 in [40])**.** *Let X be a Banach lattice, V an arbitrary Banach space and let $1 \leq p \leq \infty$. A linear operator $T : X \to V$ is called p-concave if there exists a constant $M < \infty$ so that:*

$$M \left\| \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \right\| \geq \left(\sum_{i=1}^{n} \|Tx_i\|^p \right)^{1/p}, \ if\ 1 \leq p < \infty \tag{2.24}$$

or

$$M \|\vee_{i=1}^{n} |x_i|\| \geq \max_{1 \leq i \leq n} \|Tx_i\|, \ if\ p = \infty \tag{2.25}$$

for every choice of vectors $\{x_i\}_{i=1}^{n}$ in X.

We say that X is p-concave if the identity operator on X is p-concave and the smallest value of M is called the p-concavity.

Now we give the definitions for the p-convexification and the p-concavification of a Banach lattice X in which the algebraic operations and the norm are $+$, $\cdot$, $\|\cdot\|$.

Let $p > 1$, for $x, y \in X$ and for a scalar α define:

$$x \oplus y = \left(x^{1/p} + y^{1/p}\right)^p, \ \alpha \odot x = \alpha^p \ . \ x, \tag{2.26}$$

define in the set X with these operations a new order equivalent to the original order and a new norm as $|||x||| = \|x\|^{1/p}$. We denote $X^{(p)}$ to the new Banach lattice obtained in this way from X and we call it the p-covexification of X and as its name indicates it is p-convex and its p-convexity constant is 1.

Let X be a Banach lattice which is known in advance that is p-convex and in which the algebraic operations and the norm are $+$, $\cdot$, $\|\cdot\|$.

For $x, y \in X$ and for a scalar α define:

$$x \oplus y = (x^p + y^p)^{1/p}, \ \alpha \odot x = \alpha^{1/p} \ . \ x, \tag{2.27}$$

define in the set X with these operations a new order equivalent to the original order and provided that the p-convexity of X is 1, we define a new norm as $|||x||| = \|x\|^p$. We denote $X_{(p)}$ to the new Banach lattice obtained in this way from X and we call it the p-concavification of X.

2.15 Compact Operators

Definition 2.72 (Definition 3.4.1 in[42])**.** *Suppose that X and Y are Banach spaces. A linear operator T from X into Y is compact if $T(B)$ is a relatively compact subset of Y whenever B is a bounded subset of X.*

It is known that every compact operator from a Banach space X into a Banach space Y is bounded and that a finite-rank linear operator from X into Y is compact if and only if it is bounded.

The following proposition give us statements equivalent to the statement that T is compact.

Proposition 2.73 (Proposition 3.4.4 in [42])**.** *Suppose that T is a linear operator from a Banach space X into a Banach space Y. Then the following are equivalent:*

1. *The operator T is compact.*

2. *The set $T(B_X)$ is a relatively compact subset of Y.*

3. *The set $T(B)$ is a totally bounded subset of Y whenever B is bounded subset of X.*

4. *Every bounded sequence $(x_n)_{n=1}^{\infty}$ in X has a subsequence $(x_{n_j})_{j=1}^{\infty}$ such that the sequence $(Tx_{n_j})_{j=1}^{\infty}$ converges.*

The following proposition says that compact operators not having finite rank cannot have closed range.

Proposition 2.74 (Proposition 3.4.6 in [42])**.** *The range of a compact linear operator from a Banach space into a Banach space is closed if and only if the operator has finite rank.*

It is known that the range of a bounded finite-rank operator from a Banach space X into a Banach space Y is separable. The following proposition tells us that the range of the compact operators has also this property.

Proposition 2.75 (Proposition 3.4.7 in[42])**.** *Every compact linear operator from a Banach space into a Banach space has a separable range.*

The set of all compact linear operators from a Banach space X into a Banach space Y is denoted by $\mathscr{K}(X,Y)$. The following proposition establishes that this set is closed in the algebraic sense and in the topological sense.

Proposition 2.76 (Proposition 3.4.8 in [42]). *Suppose that X and Y are Banach spaces, that S and R are compact linear operators from X into Y, that $\alpha \in \mathbb{F}$, and that $(T_n)_{n=1}^{\infty}$ is a sequence of compact linear operators from X into Y that converges to some $T \in \mathscr{L}(X,Y)$. Then $R+S$, αR, and T are all compact.*

Since the zero operator is compact, then the set of the compact operators is not empty. Thus, the last proposition says that $\mathscr{K}(X,Y)$ is a closed subspace of $\mathscr{L}(X,Y)$.

It is clear that the product of two operators, where one of them is compact and the other is bounded, is compact. Thus we have that $\mathscr{K}(X)$ is a closed ideal in $\mathscr{L}(X)$.

2.16 Weakly Compact Operators

Now we turn our attention to other type of linear operators between Banach spaces called weakly compact operators. In order to define these operators the definition of compact operators is weakened as follows.

Definition 2.77 (Definition 3.5.1 in [42]). *Suppose that X and Y are Banach spaces. A linear operator T from X into Y is weakly compact if $T(B)$ is a relatively weakly compact subset of Y whenever B is a bounded subset of X.*

The set of all weakly compact operators is denoted by $\mathscr{K}^{\omega}(X,Y)$. Clearly we have that every compact operator between Banach spaces is weakly compact. Since every relatively weakly compact subset of a Banach space is bounded, then we also have that every weakly compact operator between Banach spaces is bounded.

We know that a subset of a finite-dimensional subspace of a normed space is relatively weakly compact if and only if it is relatively compact. Thus, a finite rank operator between Banach spaces is weakly compact if and only if it is compact and then we have the following proposition.

Proposition 2.78 (see [42]). *A finite-rank linear operator between Banach spaces is weakly compact if and only if it is bounded.*

The following proposition tell us that the property of boundedness implies the property of weakly compactness when one of the Banach spaces is reflexive.

Proposition 2.79 (Proposition 3.5.4 in [42])**.** *If X and Y are Banach spaces and either X or Y is reflexive, then every bounded linear operator from X into Y is weakly compact.*

The following proposition give equivalent statements for the statement that T is a linear weakly compact operator.

Proposition 2.80 (Proposition 3.5.5 in [42])**.** *Suppose that T is a linear operator from a Banach space X into a Banach space Y. Then the following are equivalent:*

1. *The operator T is weakly compact.*

2. *The set $T(B_X)$ is a relatively weakly compact subset of Y.*

3. *Every bounded sequence $(x_n)_{n=1}^{\infty}$ in X has a subsequence $(x_{n_j})_{j=1}^{\infty}$ such that the sequence $(Tx_{n_j})_{j=1}^{\infty}$ converges weakly.*

The following proposition gives a necessary and sufficient condition for the range of a weakly compact operator between Banach spaces be closed.

Proposition 2.81 (proposition 3.5.6 in [42])**.** *The range of a weakly compact linear operator from a Banach space into a Banach space is closed if and only if the range of the operator is reflexive.*

The following proposition says that the closure of the range of a weakly compact operator between Banach spaces is the closed linear hull of the closure of the image of the unit ball of the domain, which is, a weakly compact set.

Proposition 2.82 (Proposition 3.5.7 in [42])**.** *Suppose that T is a weakly compact linear operator from a Banach space X into a Banach space Y. Then The closure of $T(X)$ is weakly compactly generated.*

The following proposition establishes that if X and Y are Banach spaces, then the set $\mathscr{K}^{\omega}(X,Y)$ is closed in the algebraic sense and in the topological sense.

Proposition 2.83 (Proposition 3.5.9 in [42])**.** *Suppose that X and Y are Banach spaces, that R and S are weakly compact linear operators from X into Y, that $\alpha \in \mathbb{F}$, and that $(T_n)_{n=1}^{\infty}$ is a sequence of weakly compact linear operators from X into Y that converges to some $T \in \mathscr{L}(X,Y)$. Then R+S, αR, and T are all weakly compact.*

Since the operator zero is weakly compact, then the set of all weakly compact operators is not empty and thus the set of all weakly compact operators is a closed subspace of the space of all linear bounded operators.

Since norm to norm continuous implies weak to weak continuous we have that if $S \in \mathscr{L}(X,Y)$ and $T \in \mathscr{L}(Y,Z)$, where X, Y, Z are Banach spaces, and either S or T is weakly compact, then TS is weakly compact.

By these two last facts exposed, we can conclude that if X is a Banach space, then $\mathscr{K}^{\omega}(X)$ is a closed ideal in $\mathscr{L}(X)$.

Chapter 3

Unique maximal ideal in the algebra $\mathscr{L}((\sum \ell_q)_{c_0})$ with $1 < q < \infty$

In this dissertation I prove that for $1 < q < \infty$, the set of all $(\sum \ell_q)_{c_0}$-strictly singular operators on $(\sum \ell_q)_{c_0}$ forms the unique maximal ideal in $\mathscr{L}((\sum \ell_q)_{c_0})$.

3.1 Prerequisites

In order to prove the main result we will need that if X is a subspace of $(\sum \ell_q)_{c_0}$ that is isomorphic to ℓ_q, then for all $\varepsilon > 0$ there is a subspace Y of X that is $(1+\varepsilon)$-isomorphic to ℓ_q, see [30].

We also need that copies of ℓ_q in $(\sum \ell_q)_{c_0}$ contain almost isometric copies of ℓ_q. This follows from the general results of lemmas 2.5 and 2.6 in [10] proved by W. B. Johnson and G. Schechtman. We present the proofs of these two lemmas but we have written them in more detail.

Lemma 3.1 (Lemma 2.5 in [10])**.** *Suppose that X has an unconditionally monotone basis with p-convexity constant one, and $(x_k)_{k=1}^n$, with $n \in \mathbb{N} \cup \{\infty\}$, is a disjoint sequence in X such that for some $0 < \theta < 1$ and all scalars $(\alpha_k)_{k=1}^\infty$ we have:*

$$\theta \left(\sum |\alpha_k|^p\right)^{1/p} \leq \left\|\sum \alpha_k x_k\right\| \leq \left(\sum |\alpha_k|^p\right)^{1/p}. \tag{3.1}$$

Then there is an unconditionally monotone norm $!\cdot!$ on X with p-convexity constant one such that for all scalars $(\alpha_k)_{k=1}^\infty$ we have:

$$\theta !x! \leq \|x\| \leq !x! \tag{3.2}$$

for all $x \in X$, and

$$(\sum |\alpha_k|^p)^{1/p} = !\Sigma \alpha_k x_k! \tag{3.3}$$

Proof. Without loss of generality we assume that $x_k \geq 0$ for all k, so that $[x_k]$ is a sublattice of X.

We first prove the lemma for the case $p = 1$. Assume that we have:

$$\theta\left(\sum|\alpha_k|\right) \leq \left\|\sum \alpha_k x_k\right\| \leq \left(\sum|\alpha_k|\right) \tag{3.4}$$

hence, by the lattice version of the Hahn-Banach theorem and the hypothesis on $(x_k)_{k=1}^{\infty}$, for all $x_k \geq 0$ there exists $x_k^* \geq 0$ such that $x_k^*(x_k) = 1$. Each x_k^* is supported where x_k is supported.

Define $x^* = \sum x_k^*$ with $\|x^*\| \leq \theta^{-1}$. Since $x_k^* \geq 0$ for all k, we have $x^* \geq 0$. We also have for a fixed j:

$$< x^*, x_j >= \left\langle \sum x_k^*, x_j \right\rangle = \sum < x_k^*, x_j >= 1 \tag{3.5}$$

Notice that $(x_k)_{k=1}^{\infty}$ is equivalent to the canonical basis of ℓ_1, thus $\sum \alpha_k x_k$ is an ℓ_1- vector so $x^* = \sum x_k^*$ is an ℓ_∞- sum.

Define $!\cdot!$ on X by:

$$!x! := \|x\| \vee \langle x^*, |x| \rangle \tag{3.6}$$

We claim $!\cdot!$ is a norm on X:

1. Suppose $x = 0$, then $!0! = \|0\| \vee < x^*, 0 >= 0 \vee 0 = 0$. On the other hand, if $!x! = 0$, then $\|x\| \vee \langle x^*, |x| \rangle = 0$ so $\|x\| = 0$ and, then $x = 0$.

2. $!\alpha x! = \|\alpha x\| \vee \langle x^*, |\alpha x| \rangle = |\alpha|\|x\| \vee |\alpha| \langle x^*, |x| \rangle = |\alpha| !x!$

3. Now we show the triangle inequality.

$$\begin{aligned} !x+y! &= \|x+y\| \vee \langle x^*, |x+y| \rangle \\ &\leqslant (\|x\| + \|y\|) \vee (\langle x^*, |x| + |y| \rangle) \\ &\leqslant (\|x\| \vee \langle x^*, |x| \rangle) + (\|y\| \vee \langle x^*, |y| \rangle) \\ &= !x! + !y! \end{aligned} \tag{3.7}$$

Notice that:

$$\|x\| \leqslant \|x\| \vee \langle x^*, |x|\rangle \tag{3.8}$$

and

$$\|y\| \leqslant \|y\| \vee \langle x^*, |y|\rangle, \tag{3.9}$$

then

$$\|x\| + \|y\| \leqslant (\|x\| \vee \langle x^*, |x|\rangle) + (\|y\| \vee \langle x^*, |y|\rangle) \tag{3.10}$$

and also we have:

$$\langle x^*, |x|\rangle + \langle x^*, |y|\rangle \leqslant (\|x\| \vee \langle x^*, |x|\rangle) + (\|y\| \vee \langle x^*, |y|\rangle) \tag{3.11}$$

Now we claim that $!\cdot!$ satisfies the expressions (3.2) and (3.3) in lemma 3.1. For (3.2), we have:

$$\begin{aligned} \theta !x! &= \theta[\|x\| \vee \langle x^*, |x|\rangle] \leqslant \theta[\|x\| \vee |\langle x^*, |x|\rangle|] \\ &\leqslant \theta[\|x\| \vee \theta^{-1}\|x\|] = \theta\|x\| \vee \|x\| \\ &\leqslant \|x\| \vee \|x\| = \|x\| \end{aligned} \tag{3.12}$$

On the other hand, by the definition of $!\cdot!$, we have $!x! \geq \|x\|$

For (3.3), we have:

$$\begin{aligned} !\Sigma\alpha_k x_k! &= \|\sum \alpha_k x_k\| \vee \langle x^*, |\Sigma\alpha_k x_k|\rangle \\ &= \|\sum \alpha_k x_k\| \vee \left\langle x^*, \sum_{\alpha_k \geq 0} \alpha_k x_k + \sum_{\alpha_k < 0} (-\alpha_k) x_k \right\rangle \\ &= \|\sum \alpha_k x_k\| \vee \sum |\alpha_k| \\ &\leqslant \sum |\alpha_k| \vee \sum |\alpha_k| \\ &= \sum |\alpha_k| \end{aligned} \tag{3.13}$$

On the other hand, we have $!\Sigma\alpha_k x_k! = \|\sum \alpha_k x_k\| \vee \sum |\alpha_k| \geq \sum |\alpha_k|$.

In the general case we apply the case $p = 1$ to the p-concavification of X and take the p-convexification of the resulting norm.

Notice that if x and y are disjoint, the vectors x^p and y^p are disjoint too, and $x^p + y^p$ is a vector whose components are powers of p. Taking $1/p$ power in each component we get $x+y$.

Let (e_j) be the unconditional monotone basis of X. For each k, let I_k be the support of x_k. We have $I_k \cap I_m = \emptyset$ for $k \neq m$.

Taking p-power in (3.1) and letting $\beta_k = \alpha_k^p$ and because we also have:

$$\|\sum \alpha_k x_k\|^p = |||\sum_{\oplus} \alpha_k^p \odot x_k||| \tag{3.14}$$

we get:

$$\theta^p \sum |\beta_k| \leqslant |||\sum_{\oplus} \beta_k \odot x_k||| \leqslant \sum |\beta_k| \tag{3.15}$$

Thus, using the p-concavification of X, we have reduced the problem to the case $p = 1$.

Then there exists $x^* \in X^*_{(p)}$, $x^* \geq 0$, with norm less than or equal to θ^{-p} such that:

$$< x^*, x_k >= 1 \tag{3.16}$$

for all k.

Define $!\cdot!$ on $X_{(p)}$ by:

$$!x! = |||x||| \vee \langle x^*, |x| \rangle \tag{3.17}$$

which satisfies:

$$\theta^p !x! \leqslant |||x||| \leqslant !x! \tag{3.18}$$

for all $x \in X_{(p)}$, and

$$\sum |\beta_k| = !\sum_{\oplus} \beta_k \odot x_k! \tag{3.19}$$

Now we take the p-convexification of the norm $!\cdot!$. Define $||||x|||| = !x!^{1/p}$ then we have:

$$\begin{aligned}||||x|||| &= !x!^{1/p} = (|||x||| \vee \langle x^*, |x|\rangle)^{1/p} \\ &= (\|x\|^p \vee \langle x^*, |x|\rangle)^{1/p} \leqslant (||x||^p \vee \theta^{-p}|||x|||)^{1/p} \\ &= (\|x\|^p \vee \theta^{-p}\|x\|^p)^{1/p} = (\theta^{-p}\|x\|^p)^{1/p} = \theta^{-1}\|x\|\end{aligned} \tag{3.20}$$

On the other hand, we have:

$$\begin{aligned}||||x|||| &= !x!^{1/p} = (|||x||| \vee \langle x^*, |x|\rangle)^{1/p} \\ &\geq (|||x|||)^{1/p} = (\|x\|^p)^{1/p} = \|x\|\end{aligned} \tag{3.21}$$

thus, we have proved (3.2) in lemma 3.1.

Now, to prove (3.3) in lemma 3.1, we have:

$$\begin{aligned}||||\sum_{\oplus} \alpha_k \odot x_k|||| &= !\sum_{\oplus} \alpha_k \odot x_k!^{1/p} \\ &= !\Sigma \alpha_k^p \odot x_k!^{1/p} \\ &= (\sum |\alpha_k|^p)^{1/p}\end{aligned} \tag{3.22}$$

□

Lemma 3.2 (Lemma 2.6 in [10])**.** *Suppose that X has an unconditionally monotone basis with p-convexity constant one* $(1 \leqslant p < \infty)$ *and* $(x_k)_{k=1}^n$*, with* $n \in \mathbb{N} \cup \{\infty\}$*, is a disjoint sequence of unit vectors in X which is isometrically equivalent to the unit vector basis for* ℓ_p*. Then* $\overline{span} x_k$ *is norm one complemented in X.*

Proof. Since the unit ball of ℓ_p is weak-star compact, the case $n = \infty$ follows from the case $n < \infty$, so we assume $n < \infty$. We can also assume that $x_k \geq 0$ for all k and that the union of the supports of the x_k is the entire unconditional basis for X.

We claim that in the p-concavification of X, the sequence $(x_k^p)_{k=1}^{\infty}$ is a disjoint sequence that is 1-equivalent to the unit vector basis of $\ell_1^{(n)}$. In fact, because in the concavification of

X, x_k^p is the same as x_k in X, we have:

$$
\begin{aligned}
|||\sum_{\oplus} a_k \odot x_k^p||| &= \left\|\sum a_k^{1/p} x_k\right\|^p = \left\|\sum a_k^{1/p} e_k\right\|^p \\
&= \left[\left(\sum |a_k^{1/p}|^p\right)^{1/p}\right]^p = \sum |a_k^{1/p}|^p \\
&= \sum |a_k| = \|\sum a_k e_k\|_{\ell_1}
\end{aligned} \tag{3.23}
$$

Then there is a norm one functional $x^* \geq 0$ in the dual of the concavification of X with $\left\langle x^*, x_k^p \right\rangle = 1$, for all k.

Since $x \wedge y = 0$, then the greatest lower bound of x and y is zero, so $x \geq 0$ and $y \geq 0$, thus x and y belong to X^+, so $x + y \in X^+$. Define $\|x\|_p = \langle x^*, |x|^p \rangle^{1/p}$ and we claim that this norm turns X into an abstract L_p space.

In fact, suppose that x and y are in X. As x and y are greater than or equal to zero then $|x| \wedge |y| = 0$ so x and y are disjoint. Thus, we have:

$$
\begin{aligned}
\|x+y\|_p^p &= [\langle x^*, |x+y|^p \rangle^{1/p}]^p = \langle x^*, (x+y)^p \rangle \\
&= \langle x^*, x^p + y^p \rangle = \langle x^*, x^p \rangle + \langle x^*, y^p \rangle \\
&= \langle x^*, |x|^p \rangle + \langle x^*, |y|^p \rangle = \|x\|_p^p + \|y\|_p^p
\end{aligned} \tag{3.24}
$$

Notice that $(x_k)_{k=1}^\infty$ are disjoint unit vectors in this abstract L_p space. In fact, we have:

$$
\|x_k\|_p^p = \langle x^*, |x_k|^p \rangle = \left\langle x^*, x_k^p \right\rangle = 1 \tag{3.25}
$$

We claim that in this abstract L_p space, $(x_k)_{k=1}^\infty$ is 1-equivalent to the unit vector basis for ℓ_p^n. In fact, we have:

$$
\begin{aligned}
\|\sum a_k x_k\|_p &= \langle x^*, |\sum a_k x_k|^p \rangle^{1/p} \\
&= \left\langle x^*, \sum_{a_k \geq 0} a_k^p x_k^p + \sum_{a_k < 0} (-a_k)^p x_k^p \right\rangle^{1/p} \\
&= (\sum |a_k|^p)^{1/p} = \|\sum a_k e_k\|_{\ell_p}
\end{aligned} \tag{3.26}
$$

Using the Theorem 1.b.2 [40], we have that $[x_k]$ is norm one complemented in that abstract L_p space.

We claim that $\|\cdot\|_p \leqslant \|\cdot\|_X$. In fact, we have:

$$\begin{aligned}
\|x\|_p^p &= \langle x^*, |x|^p \rangle \leqslant |\langle x^*, |x|^p \rangle| \\
&\leqslant |||x^*|||_{(X_{(p)})^*} ||||x|^p|||_{X(p)} \\
&= |||x^*|||_{(X_{(p)})^*} |||x|||_X^p \\
&= |||x|||_X^p = \|x\|_X^p
\end{aligned} \tag{3.27}$$

Since in X the sequence $(x_k)_{k=1}^{\infty}$ is 1-equivalent to the unit vector basis for ℓ_p^n, we conclude that the closed linear span of $(x_k)_{k=1}^{\infty}$ is also norm one complemented in X. □

Recall that if $T : X \to Y$ is an operator between Banach spaces and Z is a subspace of X, we define:

$$f(T,Z) = inf\{\|Tz\| : z \in Z, \|z\| = 1\} \tag{3.28}$$

We have that the number $f(T,Z)$ satisfies the following properties:

1. $f(T,Z) > 0$ if and only if $T|_Z$ is an isomorphism

2. $f(T,Z) = \|T\| > 0$ if and only if $T|_Z$ is a multiple of an isometry

3. $\|T\| \geq f(T,Z_1) \geq f(T,Z_2)$ if $Z_1 \subset Z_2 \subset X$

3.2 Essential Lemmas

Now I present the lemmas that will carry us to the main result of this dissertation. The first lemma will give us an important inequality which will be used later in the proofs of other lemmas. This lemma tell us that given a bounded linear operator on ℓ_q and a positive number ε, it is possible to find a block subspace Z of ℓ_q so that the operator restricted to Z has a norm which is at most $f(T,Z) + \varepsilon$. In the proof we will consider the case when T is a strictly singular operator and when T is not so.

Lemma 3.3. *Let* $1 < q < \infty$ *and let T:* $\ell q \to \left(\sum \ell_q\right)_{c_0}$ *be a bounded linear operator. Then for all* $\varepsilon > 0$*, there exists a block subspace Z of* ℓq *so that:*

$$\|T|_Z\| \leq f(T,Z) + \varepsilon \tag{3.29}$$

Proof. **Case 1:** T is a strictly singular operator.

Let $\varepsilon > 0$ be given. Since T is strictly singular, then for all infinite-dimensional subspace Z of ℓ_q we have $T|_Z$ is not an isomorphism onto its range and hence, $f(T,Z) = 0$.

Let $\varepsilon_i > 0$ be chosen so that $\sum\limits_{i=1}^{\infty} \varepsilon_i < \varepsilon$. Let $(e_i)_{i=1}^{\infty}$ be the unit vector basis of ℓq. Since $f(T,Z) = 0$ for all infinite-dimensional subspaces Z of ℓ_q, in particular we have $f\left(T,\ell_q\right) = 0$.

Select a vector $x_1 = \sum\limits_{i=1}^{\infty} a_i^{(1)} e_i \in \ell_q$, with $\|x_1\| = 1$ so that $\|Tx_1\| < \varepsilon_1$. Notice that it is possible to select a vector satisfying these conditions because $f\left(T,\ell_q\right) = 0$. We can find $N_1 \in \mathbb{N}$ such that $x_1 = \sum\limits_{i=1}^{N_1} a_i^{(1)} e_i$.

Now, let Z_1 be the closed linear span of $(e_i)_{i=N_1+1}^{\infty}$. Since Z_1 is an infinite-dimensional subspace of ℓ_q, then $f(T,Z_1) = 0$.

Select a vector $x_2 = \sum\limits_{i=N_1+1}^{\infty} a_i^{(2)} e_i \in Z_1$ with $\|x_2\| = 1$, so that $\|Tx_2\| < \varepsilon_2$. As before, we can find $N_2 \in \mathbb{N}$, $N_2 > N_1$, such that $x_2 = \sum\limits_{i=N_1+1}^{N_2} a_i^{(2)} e_i$. Let Z_2 be the closed linear span of $(e_i)_{i=N_2+1}^{\infty}$.

We continue in an obvious manner. The sequence $(x_i)_{i=1}^{\infty}$ obtained in this way is a block basic sequence of the canonical basis of ℓ_q; therefore, it is a basic sequence.

Consider the space $Z = [x_i]$. Thus, Z is a block subspace of ℓ_q. Now, let $z \in S_Z$, then:

$$\begin{aligned} \|Tz\| &\leq \sum_{i=1}^{\infty} \|T(b_i x_i)\| = \sum_{i=1}^{\infty} |b_i| \|T(x_i)\| \\ &\leq \sum_{i=1}^{\infty} \|T(x_i)\| < \sum_{i=1}^{\infty} \varepsilon_i < \varepsilon, \end{aligned} \tag{3.30}$$

then $\|T|_Z\| < \varepsilon$ and since $f(T,Z) = 0$, we have (3.29) is satisfied.

Case 2: T is not a strictly singular operator.

If T is not strictly singular, then there is an infinite-dimensional subspace Z of ℓ_q, such that $T|_Z$ is an isomorphism onto its range. Since Z is an infinite-dimensional subspace of ℓ_q, then Z contains a subspace Z_1 which is isomorphic to ℓ_q.

Let $(z_i)_{i=1}^{\infty}$ be a unit vector basis of Z_1. Then $(z_i)_{i=1}^{\infty}$ is equivalent to the canonical basis of ℓ_q; therefore, it converges weakly to zero. By passing to a subsequence of $(z_i)_{i=1}^{\infty}$ and doing a small perturbation we may assume that $[z_i]$ is isometric to ℓ_q, moreover, $T|_{Z_1}$ is an isomorphism.

Then by passing to a suitable block subspace spanned by a block basic sequence of the unit vector basis $(e_i)_{i=1}^{\infty}$ of ℓ_q, we may assume that T is an isomorphism. Thus, we have ℓ_q is isomorphic to $T(\ell_q)$.

Using the fact that subspaces of $\left(\Sigma\ell_q\right)_{c_0}$ which are isomorphic to ℓ_q contain smaller subspaces almost isometric to ℓ_q, and keeping in mind that $\varepsilon > 0$ gives wiggle room, the lemma reduces to the case where T maps ℓ_q into an isometric copy Y of ℓ_q.

Since T is bounded $(Te_n)_{n=1}^{\infty}$ converges weakly to zero. By passing to a subsequence of $(e_n)_{n=1}^{\infty}$ and doing a small perturbation, we may assume that $(Te_n)_{n=1}^{\infty}$ is disjoint supported in Y.

Let $\delta > 0$ be equal to $\liminf\limits_{n\to\infty} \|Te_n\|$. Then by passing to a further subsequence of $(e_n)_{n=1}^{\infty}$ we may assume that $\lim\limits_{n\to\infty} \|Te_n\| = \delta$. Again, by passing to a further subsequence of $(e_n)_{n=1}^{\infty}$ and relabelling we may assume that $\delta - \varepsilon/2 < \|Te_n\| < \delta + \varepsilon/2$ for all n.

Let $Z = [e_n]$, which is a block subspace of ℓ_q and let $x \in Z$ with $\sum\limits_{n=1}^{\infty} |a_n|^q = 1$, then we have:

$$\left\| T\left(\sum_{n=1}^{\infty} a_n e_n\right) \right\| = \left(\sum_{n=1}^{\infty} |a_n|^q \|Te_n\|^q\right)^{1/q} > \delta - \varepsilon/2 \qquad (3.31)$$

thus, $f(T,Z) \geq \delta - \varepsilon/2$ and this implies that:

$$\delta \leq f(T,Z) + \varepsilon/2 \qquad (3.32)$$

On the other hand we have:

$$\left\| T\left(\sum_{n=1}^{\infty} a_n e_n\right)\right\| = \left(\sum_{n=1}^{\infty} |a_n|^q \|Te_n\|^q\right)^{1/q} < \delta + \varepsilon/2 \tag{3.33}$$

Then by (3.32) and (3.33) we get:

$$\begin{aligned} \|T|_Z\| &\leq \delta + \varepsilon/2 \\ &\leq f(T,Z) + \varepsilon/2 + \varepsilon/2 \\ &= f(T,Z) + \varepsilon \end{aligned} \tag{3.34}$$

which proves (3.29). □

The following two lemmas provide structural results of the space $(\sum \ell_q)_{c_0}$.

The next lemma says that if T is a bounded linear operator from $(\sum \ell_q)_{c_0}$ into $(\sum \ell_q)_{c_0}$, then for every $\varepsilon > 0$ and each $m \in \mathbb{N}$ we can find an $n \in \mathbb{N}$ sufficiently large (in other words we go far away in the direct sum) so that the norm of the restriction of the operator $P_{[1,m]}T$ to the range of $P_{[n,\infty]}$ is less than ε.

Lemma 3.4. *Let* $1 < q < \infty$ *and* $T : (\sum \ell_q)_{c_0} \to (\sum \ell_q)_{c_0}$ *be a bounded linear operator. Then* $\forall m \in \mathbb{N}$*:*

$$\lim_{n\to\infty} \left\| P_{[1,m]} T P_{[n,\infty)} \right\| = 0 \tag{3.35}$$

Proof. Suppose this is not the case. Then there is an $\delta > 0$ and $m_0 \in \mathbb{N}$ such that for any $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that $n_k \geq k$ and $\left\| P_{[1,m_0]} T P_{[n_k,\infty)} \right\| \geq \delta$. Since this norm is decreasing we can assume that:

$$\left\| P_{[1,m_0]} T P_{[n,\infty)} \right\| \geq \delta \tag{3.36}$$

for all $n \in \mathbb{N}$.

Thus, we get a normalized vector sequence $(x_n)_{n=1}^{\infty} \subset (\Sigma\ell_q)_{c_0}$ such that:

$$\left\|P_{[1,m_0]}TP_{[n,\infty)}x_n\right\| \geq \delta \tag{3.37}$$

for all $n \in \mathbb{N}$.

Consider the sequence $\left(P_{[n,\infty)}x_n\right)_{n=1}^{\infty}$. By passing to a subsequence of $\left(P_{[n,\infty)}x_n\right)_{n=1}^{\infty}$ and doing a small perturbation we get a block basic sequence $(P_{[n_k,\infty)}x_{n_k})_{k=1}^{\infty}$ of the natural basis of $(\Sigma\ell_q)_{c_0}$.

We claim that this block basic sequence does not converge to zero in norm.

Indeed, since $\left\|P_{[1,m_0]}\right\| = 1$, then by (3.37) we have:

$$\begin{aligned} \left\|TP_{[n_k,\infty)}x_{n_k}\right\| &= \left\|P_{[1,m_0]}\right\| \left\|TP_{[n_k,\infty)}x_{n_k}\right\| \\ &\geq \left\|P_{[1,m_0]}TP_{[n_k,\infty)}x_{n_k}\right\| \geq \delta \end{aligned} \tag{3.38}$$

Thus, we get that $\delta \leqslant \left\|TP_{[n_k,\infty)}x_{n_k}\right\| \leqslant \|T\| \left\|P_{[n_k,\infty)}x_{n_k}\right\|$ and this implies that $\left\|P_{[n_k,\infty)}x_{n_k}\right\| \geq \dfrac{\delta}{\|T\|} > 0$ for all $k \in \mathbb{N}$.

Let $(y_k)_{k=1}^{\infty} = \left(P_{[n_k,\infty)}x_{n_k}\right)_{k=1}^{\infty}$. Since this sequence is a block basic sequence of the canonical basis of $(\sum \ell_q)_{c_0}$ and does not converge to zero in norm, then is equivalent to the canonical basis of c_0.

Since $(y_k)_{k=1}^{\infty}$ converges weakly to zero and T is bounded, then $(Ty_k)_{k=1}^{\infty}$ converges weakly to zero and then $(P_{[1,m_0]}Ty_k)_{k=1}^{\infty}$ converges weakly to zero in ℓ_q.

By passing to a further subsequence of $(y_k)_{k=1}^{\infty}$ and doing a small perturbation we may assume that $(P_{[1,m_0]}Ty_k)_{k=1}^{\infty}$ is a block basic sequence of the canonical basis of ℓ_q.

Moreover, by (3.37), $(P_{[1,m_0]}Ty_k)_{k=1}^{\infty}$ does not converges to zero in norm, then it is equivalent to the canonical basis of ℓ_q.

Since $(y_k)_{k=1}^{\infty}$ is equivalent to the canonical basis of c_0, then there exists a constant C_1 such that for all $n \in \mathbb{N}$:

$$\left\|\sum_{k=1}^{n} y_k\right\| \leq C_1 \left\|\sum_{k=1}^{n} e_k\right\|_{c_0} = C_1 \tag{3.39}$$

Since $\left(P_{[1,m_0]}Ty_k\right)_{k=1}^{\infty}$ is equivalent to the canonical basis of ℓ_q, then there exists a constant C_2 such that for all $n \in \mathbb{N}$:

$$\left\|\sum_{k=1}^{n} P_{[1,m_0]}Ty_k\right\| \geq C_2 \left\|\sum_{k=1}^{n} e_k\right\|_{\ell_q} = C_2 n^{1/q} \tag{3.40}$$

Then by(3.39) and (3.40) we have:

$$\begin{aligned} C_2 n^{1/q} &\leq \left\|\sum_{k=1}^{n} P_{[1,m_0]}Ty_k\right\| = \|P_{[1,m_0]}T(\sum_{k=1}^{n} y_k)\| \\ &\leq \left\|P_{[1,m_0]}T\right\| \left\|\sum_{k=1}^{n} y_k\right\| \leq \left\|P_{[1,m_0]}T\right\| C_1 \end{aligned} \tag{3.41}$$

Thus, $\left\|P_{[1,m_0]}T\right\| \geq \dfrac{C_2 n^{1/q}}{C_1}$ for all $n \in \mathbb{N}$, which contradicts the boundedness of $P_{[1,m_0]}T$. □

The next lemma says that if T is a bounded linear operator from $(\sum \ell_q)_{c_0}$ into ℓ_q, then for each positive number ε, we can find a subspace isometric to $(\sum \ell_q)_{c_0}$ so that T restricted to that subspace has small norm. We will prove the lemma in two cases. First, when the restriction of T to each ℓ_q^n is compact for all natural number n in a infinite subset of the natural numbers. Second, when T restricted to all of the ℓ_q^n is not compact except for natural numbers n in a finite set of the natural numbers.

Lemma 3.5. *Let* $1 < q < \infty$ *and let T:* $(\sum \ell_q)_{c_0} \to \ell_q$ *be a bounded linear operator. Then for all* $\varepsilon > 0$*, there is a subspace X of* $(\sum \ell_q)_{c_0}$ *so that X is isometric to* $(\sum \ell_q)_{c_0}$ *and* $||T_{|_X}|| < \varepsilon$.

Proof. **Case 1:** There is an infinite subset M of the Natural numbers so that $T_{|_{\ell_q^{(n)}}}$ is compact for all $n \in M$.

Let $\varepsilon > 0$, let $(\delta_n)_{n=1}^{\infty}$ be a sequence of positive real numbers decreasing to zero fast such that:

$$\sum_{n \in M} \delta_n < \varepsilon \tag{3.42}$$

Since $T_{|_{\ell_q^{(n)}}}$ is compact for all $n \in M$, then $T_{|_{\ell_q^{(n)}}}$ is strictly singular for all $n \in M$. Thus, for all infinite-dimensional subspaces Z of $\ell_q^{(n)}$ with $n \in M$ we have that $T_{|Z}$ is not an isomorphism. Then $f(T,Z)=0$. In particular, $f(T,\ell_q^{(n)})=0$ for $n \in M$.

Fix $n_0 \in M$. Choose $(\varepsilon_i)_{i=1}^{\infty}$ converging to zero fast so that $\sum_{i=1}^{\infty} \varepsilon_i < \delta_{n_0}$. Pick $x_1 \in \ell_q^{(n_0)}$ with norm one so that $\|Tx_1\| < \varepsilon_1$.

Let $(e_i)_{i=1}^{\infty}$ be the canonical basis of $\ell_q^{(n_0)}$, then $x_1 = \sum_{i=1}^{\infty} a_i^{(1)} e_i$. Hence, we can find $N_1 \in \mathbb{N}$ so that $x_1 = \sum_{i=1}^{N_1} a_i^{(1)} e_i$.

Let Z_1 be the closed linear span of $(e_i)_{i=N_1+1}^{\infty}$. Since Z_1 is an infinite-dimensional subspace of $\ell_q^{(n_0)}$, then $f(T,Z_1)=0$.

Now, pick $x_2 \in Z_1$ with norm one so that $\|Tx_2\| < \varepsilon_2$. Since $x_2 = \sum_{i=N_1+1}^{\infty} a_i^{(2)} e_i$, then we can find $N_2 \in \mathbb{N}$ with $N_2 > N_1$ so that $x_2 = \sum_{i=N_1+1}^{N_2} a_i^{(2)} e_i$. Let Z_2 be the closed linear span of $(e_i)_{i=N_2+1}^{\infty}$.

We continue in an obvious manner. The sequence $(x_i)_{i=1}^{\infty}$ obtained in this way is a block basic sequence of the canonical basis of $\ell_q^{(n_0)}$. Let $X_{n_0} = [x_i]$, then X_{n_0} is a block subspace of $\ell_q^{(n_0)}$ which is isometric to $\ell_q^{(n_0)}$.

We can do the same process as we did above for each $n \in M$ so that we can find a block basic sequence $(x_i)_{i=1}^{\infty}$ of the canonical basis of $\ell_q^{(n)}$ such that $X_n = [x_i]$ is a block subspace of $\ell_q^{(n)}$ isometric to $\ell_q^{(n)}$.

Let $z \in S_{X_n}$, then:

$$\begin{aligned} \|Tz\| &\leq \sum_{i=1}^{\infty} |b_i| \|Tx_i\| \leq \sum_{i=1}^{\infty} \|Tx_i\| \\ &< \sum_{i=1}^{\infty} \varepsilon_i < \delta_n \end{aligned} \tag{3.43}$$

Thus, we get:

$$\|T|_{X_n}\| < \delta_n \tag{3.44}$$

for all $n \in M$.

Let $X = \sum_{n \in M} X_n$, then X is isometric to $(\sum \ell_q)_{c_0}$.

By (3.42) and (3.44) we have:

$$\begin{aligned} \|T_{|X}\| &= \left\|\sum_{n\in M} T_{|X_n}\right\| \leq \sum_{n\in M}\left\|T_{|X_n}\right\| \\ &< \sum_{n\in M}\delta_n < \varepsilon \end{aligned} \tag{3.45}$$

Case 2: For all but finitely many $n \in \mathbb{N}$, $T|_{\ell_q^{(n)}}$ is not compact.

Discard finitely many $n \in \mathbb{N}$. Fix $n_0 \in \mathbb{N}$. Since $T_{|_{\ell_q^{(n_0)}}}$ is not compact, then $T_{|_{\ell_q^{(n_0)}}}$ is not strictly singular. Then there exists an infinite-dimensional subspace Z_1 of $\ell_q^{(n_0)}$ such that $T_{|Z_1}$ is an isomorphism onto its range.

Since Z_1 is an infinite-dimensional subspace of $\ell_q^{(n_0)}$, then Z_1 contains another subspace Z_2 which is isomorphic to $\ell_q^{(n_0)}$.

Let $(x_i)_{i=1}^{\infty}$ be a unit vector basis of Z_2. Then $(x_i)_{i=1}^{\infty}$ is equivalent to the canonical basis of $\ell_q^{(n_0)}$; therefore, it converges weakly to zero. By passing to a subsequence of $(x_i)_{i=1}^{\infty}$ and doing a small perturbation we get that $Z_{n_0} = [x_{i_k}]$ is isometric to $\ell_q^{(n_0)}$. Moreover, $T_{|Z_{n_0}}$ is an isomorphism.

We can do the same process as we did above for each $n \in \mathbb{N}$ so that we can find a block basic sequence $(x_i)_{i=1}^{\infty}$ of the canonical basis of $\ell_q^{(n)}$ such that $Z_n = [x_i]$ is a block subspace of $\ell_q^{(n)}$ isometric to $\ell_q^{(n)}$ and $T_{|Z_{n_0}}$ is an isomorphism.

Hence, by discarding finitely many $n \in \mathbb{N}$ and passing to block subspaces of each $\ell_q^{(n)}$ without loss of generality we may assume that $T_{|_{\ell_q^{(n)}}}$ is an isomorphism for all $n \in \mathbb{N}$.

By passing to block subspaces of each Z_n and using lemma 3.3 we may assume that:

$$\|T_{|Z_n}\| < f(T, Z_n) + 2^{-n}(\varepsilon/2) \tag{3.46}$$

Now, we claim that $\lim_{n\to\infty} f(T, Z_n) = 0$.

Suppose this is not the case. Then there exists a $\delta > 0$ and a sequence of numbers $(n_k)_{k=1}^{\infty}$ such that $f(T, Z_{n_k}) \geq \delta > 0$. Thus, for each $k \in \mathbb{N}$ there exists $x_{n_k} \in Z_{n_k}$ with norm one such that $\|Tx_{n_k}\| \geq \delta > 0$.

Consider $(x_{n_k})_{k=1}^{\infty} \in \sum Z_{n_k}$ in the sense of a c_0 direct sum.. Then for all $m \in \mathbb{N}$ we have:

$$\begin{aligned} \left\| \sum_{k=1}^{m} a_{n_k} x_{n_k} \right\| &= \sup_k \{ \|a_{n_k} x_{n_k}\| \} \\ &= \sup_k \{ |a_{n_k}| \} = \left\| \sum_{k=1}^{m} a_{n_k} e_k \right\|_{c_0} \end{aligned} \tag{3.47}$$

Then $(x_{n_k})_{k=1}^{\infty}$ is isometrically equivalent to the canonical basis of c_0. Hence, $(x_{n_k})_{k=1}^{\infty}$ is weakly null and since T is bounded, then $(Tx_{n_k})_{k=1}^{\infty}$ is weakly null. By passing to a further subsequence of $(x_{n_k})_{k=1}^{\infty}$ and doing a small perturbation we may assume that $(Tx_{n_k})_{k=1}^{\infty}$ is a block basic sequence of the canonical basis of ℓ_q. We know also this sequence does not converges to zero in norm. Then this sequence is equivalent to the canonical basis of ℓ_q.

Since $(x_{n_k})_{k=1}^{\infty}$ is isometrically equivalent to the canonical basis of c_0, then for all $m \in \mathbb{N}$ we have:

$$\left\| \sum_{k=1}^{m} x_{n_k} \right\| = \left\| \sum_{k=1}^{m} e_k \right\|_{c_0} = 1 \tag{3.48}$$

Since $(Tx_{n_k})_{k=1}^{\infty}$ is equivalent to the canonical basis of ℓ_q, then there exists a constant $C > 0$ such that for all $m \in \mathbb{N}$ we have:

$$\left\| \sum_{k=1}^{m} Tx_{n_k} \right\| \geq C \left\| \sum_{k=1}^{m} e_k \right\|_{\ell_q} = Cm^{1/q} \tag{3.49}$$

Then for all $m \in \mathbb{N}$, using (3.48) and (3.49), we have:

$$\begin{aligned} Cm^{1/q} &\leq \left\| \sum_{k=1}^{m} Tx_{n_k} \right\| = \left\| T\left(\sum_{k=1}^{m} x_{n_k} \right) \right\| \\ &\leq \|T\| \left\| \sum_{k=1}^{m} x_{n_k} \right\| = \|T\| \end{aligned} \tag{3.50}$$

which contradicts the boundedness of T.

Since $f(T, Z_n)$ converges to zero as n goes to infinity, by passing to a subsequence of $(Z_n)_{n=1}^{\infty}$ and relabelling we may assume that $f(T, Z_n) < 2^{-n}(\varepsilon/2)$ for all n. Thus, using

(3.46) we get:

$$\|T_{|Z_n}\| < 2^{-n}(\varepsilon/2) + 2^{-n}(\varepsilon/2) = 2^{-n}\varepsilon \tag{3.51}$$

Let $X = (\sum Z_n)$, then X is isometric to $(\sum \ell_q)_{c_0}$ and using (3.51) we get:

$$\begin{aligned} \|T_{|X}\| &= \left\|T_{|\sum Z_n}\right\| = \left\|\sum T_{|Z_n}\right\| \\ &\leq \sum \|T_{|Z_n}\| < \sum 2^{-n}\varepsilon = \varepsilon \end{aligned} \tag{3.52}$$

□

The next four lemmas show that the set of all $(\sum \ell_q)_{c_0}$-strictly singular operators on $(\sum \ell_q)_{c_0}$ forms an ideal in $\mathscr{L}((\sum \ell_q)_{c_0})$, with $1 < q < \infty$.

The next lemma show that every subspace of $(\sum \ell_q)_{c_0}$ which contains a copy of $(\sum \ell_q)_{c_0}$ must contain a complemented copy of $(\sum \ell_q)_{c_0}$, in other words, the lemma prove that the space $(\sum \ell_q)_{c_0}$ is complementably homogeneous for $1 < q < \infty$.

Lemma 3.6. *Let X be a subspace of $(\sum \ell_q)_{c_0}$ which is C-isomorphic to $(\sum \ell_q)_{c_0}$. Then for all $\varepsilon > 0$, there is a subspace Y of X which is C+ε-isomorphic to $(\sum \ell_q)_{c_0}$ and C+ε-complemented in $(\sum \ell_q)_{c_0}$.*

Proof. Let $(\varepsilon_i)_{i=1}^{\infty}$ be a sequence of positive real numbers decreasing to zero fast. Write $X = \sum_i X_i$ where each X_i is isomorphic to ℓ_q and the sum is C-isomorphic to $(\sum \ell_q)_{c_0}$.

By the stability of the ℓ_q we may assume by passing to subspaces of each X_i that X_i is $1+\varepsilon_i$ isomorphic to ℓ_q.

Let $(x_{i,k})_{k=1}^{\infty}$ be a unit vector sequence in X_i so that $(x_{i,k})_{k=1}^{\infty}$ is equivalent to the canonical basis of ℓ_q; therefore, it converges weakly to zero. Hence, by passing to a further subsequence of $(x_{i,k})_{k=1}^{\infty}$ and doing a small perturbation we can assume that X_i is a block subspace of $(\sum \ell_q)_{c_0}$.

By passing to a further subsequence of each $(x_{i,k})_{k=1}^{\infty}$ and doing a small perturbation and by a process of diagonalization we can assume that all of the spaces X_i are disjoint supported. Hence, their respective basis are disjoint supported with respect to the canonical basis of $(\sum \ell_q)_{c_0}$

Now let $J_i = \cup_{k=1}^{\infty} supp(x_{i,k})$. For any $x = \sum_{k,j} a_{k,j} e_{k,j} \in (\sum \ell_q)_{c_0}$ define $P_{J_i} : (\sum \ell_q)_{c_0} \to (\sum \ell_q)_{c_0}$ by $P_{J_i}(x) = \sum_{(k,j)\in J_i} a_{k,j} e_{k,j}$.

Since $P_{J_i}^2 = P_{J_i}$ for all $x \in (\sum \ell_q)_{c_0}$, then P_{J_i} is a projection. We also have that P_{J_i} has norm one.

Let $(e_{k,j})_{k,j}$ be the canonical basis of $(\sum \ell_q)_{c_0}$ and consider the space $A_i = [(e_{k,j})_{(k,j)\in J_i}]$ which is a subspace of $(\sum \ell_q)_{c_0}$; therefore, it has an unconditionally monotone basis with q-convexity constant one. Since the support of $X_i \in J_i$, then X_i is a subspace of A_i.

By lemma 2.5 in [10] we have a new norm $!\cdot!$ so that under this new norm the sequence $(x_{i,k})_{k=1}^{\infty}$ is isometrically equivalent to the canonical basis of ℓ_q. This implies that X_i is isometric to ℓ_q under the new norm.

By lemma 2.6 in [10] there is a projection $Q_i : A_i \to X_i$ such that $!Q_i! = 1$. We also have by the lemma 2.5 in [10] that $\|\cdot\|$ and $!\cdot!$ are equivalent norms. Moreover, the formal identity $I : (A_i, !\cdot!) \to (A_i, \|\cdot\|)$ is an onto isomorphism.

Thus, by the principle of small perturbations we have that $(X_i, \|\cdot\|)$ is also complemented and $\|Q_i\| \leqslant 1 + \varepsilon_i$.

Now consider the operator $\sum Q_i P_{J_i} : (\Sigma \ell_q)_{c_0} \to \Sigma X_i$. We claim that $\left\|\sum Q_i P_{j_i}\right\| < C + \varepsilon$.

Indeed, we have:

$$
\begin{aligned}
\left\|\sum Q_i P_{j_i}\right\| &= \sup\left\{\left\|\sum Q_i P_{j_i} x\right\| : \|x\| = 1\right\} \\
&= sup\left\{C \sup_i \|Q_i P_{j_i} x\| : \|x\| = 1\right\} \qquad (3.53) \\
&< C + \varepsilon
\end{aligned}
$$

□

The next lemma actually says that the set of all $(\sum \ell_q)_{c_0}$-strictly singular operators on $(\sum \ell_q)_{c_0}$ is an ideal.

Lemma 3.7. *Let $1 < q < \infty$ and let T be a bounded linear operator on $(\sum \ell_q)_{c_0}$ which is $(\sum \ell_q)_{c_0}$-strictly singular. Then for all $\varepsilon > 0$, there is a subspace X of $(\sum \ell_q)_{c_0}$ which is isometrically isomorphic to $(\sum \ell_q)_{c_0}$ and such that $\|T_{|X}\| < \varepsilon$.*

Proof. **case 1:** There is an infinite subset M of the Natural numbers so that $T_{|_{\ell_q^{(n)}}}$ is compact for all $n \in M$.

Let $\varepsilon > 0$ and let $(\delta_n)_{n=1}^{\infty}$ be a sequence of positive real numbers decreasing to zero fast such that:

$$\sum_{n \in M} \delta_n < \varepsilon \tag{3.54}$$

Since $T_{|_{\ell_q^{(n)}}}$ is compact for all $n \in M$, then $T_{|_{\ell_q^{(n)}}}$ is strictly singular for all $n \in M$. Then for all infinite-dimensional subspaces Z of $\ell_q^{(n)}$ with $n \in M$ we have that $T_{|Z}$ is not an isomorphism. Thus, $f(T,Z) = 0$. In particular $f(T, \ell_q^{(n)}) = 0$.

Fix $n_0 \in M$. Let $(\varepsilon_i)_{i=1}^{\infty}$ be a sequence of positive real numbers decreasing to zero fast so that $\sum_{i=1}^{\infty} \varepsilon_i < \delta_{n_0}$.

Pick $x_1 \in \ell_q^{(n_0)}$ with norm one so that $\|Tx_1\| < \varepsilon_1$. Let $(e_i)_{i=1}^{\infty}$ be the canonical basis of $\ell_q^{(n_0)}$, then $x_1 = \sum_{i=1}^{\infty} a_i^{(1)} e_i$. Thus, we can find $N_1 \in \mathbb{N}$ so that $x_1 = \sum_{i=1}^{N_1} a_i^{(1)} e_i$.

Let Z_1 be the closed linear span of $(e_i)_{i=N_1+1}^{\infty}$. Since Z_1 is an infinite-dimensional subspace of $\ell_q^{(n_0)}$, then $f(T, Z_1) = 0$.

Now pick $x_2 \in Z_1$ with norm one such that $\|Tx_2\| < \varepsilon_2$. Since $x_2 = \sum_{i=N_1+1}^{\infty} a_i^{(2)} e_i$, then we can find $N_2 \in \mathbb{N}$, $N_2 > N_1$, such that $x_2 = \sum_{i=N_1+1}^{N_2} a_i^{(2)} e_i$. Let Z_2 be the closed linear span of $(e_i)_{i=N_2+1}^{\infty}$.

We continue in an obvious manner. The sequence $(x_i)_{i=1}^{\infty}$ obtained in this way is a block basic sequence of the canonical basis of $\ell_q^{(n_0)}$. Thus, $(x_i)_{i=1}^{\infty}$ is isometrically equivalent to $(e_i)_{i=1}^{\infty}$ and $X_{n_0} = [x_i]$ is a block subspace which is isometric to $\ell_q^{(n_0)}$.

For each $n \in M$ we can do the same process as we did above and we will get a block basic sequence $(x_i)_{i=1}^{\infty}$ of the canonical basis of $\ell_q^{(n)}$ and a subspace $X_n = [x_i]$ which is a block subspace of $\ell_q^{(n)}$ isometric to $\ell_q^{(n)}$.

Let $z \in S_{X_n}$, then we have:

$$\begin{aligned} \|Tz\| &\leq \sum_{i=1}^{\infty} |b_i| \|T(x_i)\| \leq \sum_{i=1}^{\infty} \|T(x_i)\| \\ &< \sum_{i=1}^{\infty} \varepsilon_i < \delta_n \end{aligned} \tag{3.55}$$

Then:

$$\|T_{|X_n}\| < \delta_n \tag{3.56}$$

for all $n \in M$.

Let $X = \sum_{n \in M} X_n$, then X is an isometric subspace of $(\sum \ell_q)_{c_0}$ and using (3.54) and (3.56) we have:

$$\begin{aligned} \|T_{|X}\| &= \|T_{|\sum_{n \in M} X_n}\| = \left\| \sum_{n \in M} T_{|X_n} \right\| \\ &\leq \sum_{n \in M} \left\| T_{|X_n} \right\| < \sum_{n \in M} \delta_n < \varepsilon \end{aligned} \tag{3.57}$$

case 2: For all but finitely many $n \in \mathbb{N}$, $T_{|\ell_q^{(n)}}$ is not compact.

Discard finitely many $n \in \mathbb{N}$ and fix $n_0 \in \mathbb{N}$. Since $T_{|\ell_q^{(n_0)}}$ is not compact, then $T_{|\ell_q^{(n_0)}}$ is not strictly singular. Thus, there exists an infinite-dimensional subspace Z_1 of $\ell_q^{(n_0)}$ such that $T_{|Z_1}$ is an isomorphism.

Since Z_1 is an infinite-dimensional subspace of $\ell_q^{(n_0)}$, then Z_1 contains a subspace Z_2 which is isomorphic to $\ell_q^{(n_0)}$.

Let $(x_i)_{i=1}^{\infty}$ be a unit basis of Z_2, then $(x_i)_{i=1}^{\infty}$ is equivalent to the canonical basis of $\ell_q^{(n_0)}$; thus, it converges weakly to zero. By passing to a subsequence $(x_{i_k})_{k=1}^{\infty}$ of $(x_i)_{i=1}^{\infty}$ and doing a small perturbation we get a block basic sequence of the canonical basis of $\ell_q^{(n_0)}$.

Let $Z_{n_0} = [x_{i_k}]$ which is an isometric block subspace of $\ell_q^{(n_0)}$. Moreover, $T|_{Z_{n_0}}$ is an isomorphism.

We can do the same process as we did above for each $n \in \mathbb{N}$ and we will get a block basic sequence $(x_{i_k})_{k=1}^{\infty}$ of the canonical basis of $\ell_q^{(n)}$ and an isometric block subspace $Z_n = [x_{i_k}]$ of $\ell_q^{(n)}$ such that $T_{|Z_n}$ is an isomorphism.

Hence, by discarding finitely many $n \in \mathbb{N}$ and passing to block subspaces of each $\ell_q^{(n)}$

without loss of generality we may assume that $T_{|_{\ell_q^{(n)}}}$ is an isomorphism for all $n \in \mathbb{N}$.

By passing to block subspaces of each Z_n and using 3.3 we may assume that:

$$||T|_{Z_n}|| < f(T, Z_n) + 2^{-n}\varepsilon \tag{3.58}$$

for each $n \in \mathbb{N}$.

We claim that $\lim_{n\to\infty} f(T, Z_n) = 0$.

Suppose that is not the case. Then there exists a $\delta > 0$ and a sequence of numbers $(n_k)_{k=1}^{\infty}$ such that $f(T, Z_{n_k}) \geq \delta > 0$. Then $T_{|Z_{n_k}}$ is an isomorphism.

Consider the operator $T : (\sum Z_{n_k})_{c_0} \to \left(\sum \ell_q\right)_{c_0}$.

By passing to further subspaces of each Z_{n_k} and doing a small perturbation we may assume Tx_1 and Tx_2 are disjoint supported in $(\sum \ell_q)_{c_0}$ if $x_1 \in Z_{n_{k_1}}$, $x_2 \in Z_{n_{k_2}}$ and $k_1 \neq k_2$.

Let $x \in (\sum Z_{n_k})_{c_0}$, then $x = \sum_k x_k$ with $x_k \in Z_{n_k}$. Let $k_0 \in \mathbb{N}$ be such that:

$$\left\|x_{k_0}\right\| \geq \frac{1}{2}\|x\| \tag{3.59}$$

Then using (3.59) we have:

$$\begin{aligned} \|Tx\| &= \left\|\sum_k Tx_k\right\| \geq \left\|Tx_{k_0}\right\| \\ &\geq \alpha\left\|x_{k_0}\right\| \geq \frac{\alpha}{2}\|x\| \end{aligned} \tag{3.60}$$

Thus, we have that $T|_{\left(\sum Z_{n_k}\right)_{c_0}}$ is an isomorphism. This contradicts that T is $\left(\sum \ell_q\right)_{c_0}$-strictly singular on $\left(\sum \ell_q\right)_{c_0}$.

Since $f(T, Z_n)$ converges to zero, by passing to a subsequence of $(Z_n)_{n=1}^{\infty}$, and relabelling, we can assume that:

$$f(T, Z_n) < 2^{-n}\varepsilon \tag{3.61}$$

for all $n \in \mathbb{N}$.

Thus, using (3.58) and (3.61) we have:

$$\|T_{|Z_n}\| < 2^{-n}\varepsilon + 2^{-n}\varepsilon \tag{3.62}$$

Let $X = (\sum Z_n)$, then using (3.62) we have:

$$\begin{aligned} \|T_{|X}\| &= \left\|\sum T_{|Z_n}\right\| \leq \sum \left\|T_{|Z_n}\right\| \\ &< \sum(2^{-n}\varepsilon + 2^{-n}\varepsilon) = 2\varepsilon \end{aligned} \tag{3.63}$$

□

In the next lemma we use the last lemma to prove that the set of all $(\sum \ell_q)_{c_0}$-strictly singular operators on $(\sum \ell_q)_{c_0}$ is closed under addition and under scalar multiplication.

Lemma 3.8. *The set of all $(\sum \ell_q)_{c_0}$-strictly singular operators on $(\sum \ell_q)_{c_0}$ is a linear subspace of $\mathscr{L}((\sum \ell_q)_{c_0})$.*

Proof. Let T and Q be two $(\sum \ell_q)_{c_0}$- strictly singular operators on $(\sum \ell_q)_{c_0}$. We want to prove that T+Q is a $(\sum \ell_q)_{c_0}$-strictly singular operator on $(\sum \ell_q)_{c_0}$.

Suppose this is not the case. Then there exists a subspace X of $(\sum \ell_q)_{c_0}$, isomorphic to $(\sum \ell_q)_{c_0}$, such that $(T+Q)_{|X}$ is an isomorphism. Thus, there exists a $\delta > 0$ such that:

$$\|(T+Q)(x)\| \geq \delta\|x\| \tag{3.64}$$

for all $x \in X$.

Since T is $(\sum \ell_q)_{c_0}$- strictly singular on $(\sum \ell_q)_{c_0}$, then there exists a subspace Y of X which is isomorphic to $(\sum \ell_q)_{c_0}$ such that $\|T_{|Y}\| < \delta/2$.

Since Q is $(\sum \ell_q)_{c_0}$-strictly singular on $(\sum \ell_q)_{c_0}$, then there exists a subspace Z of Y which is isomorphic to $(\sum \ell_q)_{c_0}$ such that $\|Q_{|Z}\| < \delta/2$.

Then we have:

$$\begin{aligned} \|(T+Q)(z)\| &\leq \|T(z)\| + \|Q(z)\| \\ &< \delta/2\|z\| + \delta/2\|z\| = \delta\|z\| \end{aligned} \tag{3.65}$$

which contradicts the inequality (3.64).

Then $T+Q$ is a $(\sum \ell_q)_{c_0}$-strictly singular operator on $(\sum \ell_q)_{c_0}$.

Let α be a scalar. We want to prove αT is $(\sum \ell_q)_{c_0}$- strictly singular on $(\sum \ell_q)_{c_0}$. By the way of contradiction suppose this is not the case. Then there exists a subspace X of $(\sum \ell_q)_{c_0}$ which is isomorphic to $(\sum \ell_q)_{c_0}$ such that $(\alpha T)_{|X}$ is an isomorphism. Thus, there exists $\delta > 0$ such that:

$$\|(\alpha T)(x)\| \geq \delta\|x\| \tag{3.66}$$

for all $x \in X$.

Since T is $(\sum \ell_q)_{c_0}$-strictly singular operator on $(\sum \ell_q)_{c_0}$, then there exists a subspace Y of X which is isomorphic to $(\sum \ell_q)_{c_0}$ such that $\|T_{|_Y}\| < \delta/|\alpha|$.

Then we have:

$$\begin{aligned} \|(\alpha T)(y)\| &= |\alpha|\|T(y)\| \\ &< |\alpha|\frac{\delta}{|\alpha|}\|y\| \\ &= \delta\|y\| \end{aligned} \tag{3.67}$$

which contradicts the inequality (3.66). Then αT is a $(\sum \ell_q)_{c_0}$-strictly singular operator on $(\sum \ell_q)_{c_0}$.

Hence, the set of all $(\sum \ell_q)_{c_0}$-strictly singular operators on $(\sum \ell_q)_{c_0}$ is a linear subspace of $\mathscr{L}((\sum \ell_q)_{c_0})$. □

The next lemma shows that the set of all $(\sum \ell_q)_{c_0}$-strictly singular operators on $(\sum \ell_q)_{c_0}$ absorbs products (compositions) in $\mathscr{L}((\sum \ell_q)_{c_0})$.

Lemma 3.9. *Let $1 < q < \infty$. Then the set of all $(\sum \ell_q)_{c_0}$-strictly singular operators on $(\sum \ell_q)_{c_0}$ forms an ideal in $\mathscr{L}((\Sigma \ell_q)_{c_0})$.*

Proof. Let B be a $(\sum \ell_q)_{c_0}$-strictly singular operator on $(\sum \ell_q)_{c_0}$ and let A be an operator in $\mathscr{L}((\Sigma\ell_q)_{c_0})$.

We claim that AB is a $(\sum \ell_q)_{c_0}$-strictly singular operator on $(\sum \ell_q)_{c_0}$.

Suppose this is not the case. Then there exists a subspace Z of $(\sum \ell_q)_{c_0}$ which is isomorphic to $(\sum \ell_q)_{c_0}$ such that $AB_{|Z}$ is an isomorphism. Thus, there exists $\delta > 0$ such that $\|ABx\| \geq \delta\|x\|$ for all x in Z. Then $\|Bx\| \geq \|A\|^{-1}\delta\|x\|$ for all x in Z. Thus, $B_{|Z}$ is an isomorphism and this is a contradiction.

Now, we claim that BA is a $(\sum \ell_q)_{c_0}$-strictly singular operator on $(\sum \ell_q)_{c_0}$.

Suppose this is not the case. Then there exists a subspace Z of $(\sum \ell_q)_{c_0}$ which is isomorphic to $(\sum \ell_q)_{c_0}$ such that $BA_{|Z}$ is an isomorphism. Then there exists $\delta > 0$ such that $\|BAx\| \geq \delta\|x\|$ for all x in Z. Then $\|Ax\| \geq \|B\|^{-1}\delta\|x\|$ for all x in Z. Thus, $A_{|Z}$ is an isomorphism.

We have also that:

$$\begin{aligned} \|BAx\| &\geq \delta\left\|\frac{A}{\|A\|}\right\|\|x\| \\ &\geq \delta\|A\|^{-1}\|Ax\|, \end{aligned} \tag{3.68}$$

then $B_{|AZ}$ is an isomorphism.

Since $A_{|Z}$ is an isomorphism and Z is isomorphic to $(\Sigma\ell_q)_{c_0}$, then AZ is isomorphic to $(\Sigma\ell_q)_{c_0}$ and since $B_{|AZ}$ is an isomorphism, then we get a contradiction.

Then AB and BA are $(\sum \ell_q)_{c_0}$-strictly singular operator on $(\sum \ell_q)_{c_0}$. This result and the lemma 3.8 prove that the set of all $(\sum \ell_q)_{c_0}$- strictly singular operators on $(\sum \ell_q)_{c_0}$ form an ideal in $\mathscr{L}((\Sigma\ell_q)_{c_0})$. □

3.3 Main Result

Now we are ready to prove the main result. In order to prove the next theorem we will use the lemma 3.6 which actually says that the ideal of all $(\sum \ell_q)_{c_0}$-strictly singular operators on $(\sum \ell_q)_{c_0}$ is maximal in $\mathscr{L}((\sum \ell_q)_{c_0})$. We will use also the result of Dosev and Johnson in [13] about the set $\mathscr{M}_X$ to conclude the uniqueness of the ideal of all $(\sum \ell_q)_{c_0}$-strictly singular operators on $(\sum \ell_q)_{c_0}$.

Theorem 3.10. *Let $1 < q < \infty$. The set of all $(\sum \ell_q)_{c_0}$-strictly singular operators on $(\sum \ell_q)_{c_0}$ is the unique maximal ideal in $\mathscr{L}((\sum \ell_q)_{c_0})$.*

Proof. We will show that any ideal which contains elements not in the ideal of the $(\sum \ell_q)_{c_0}$-strictly singular operators on $(\sum \ell_q)_{c_0}$ must coincide with $\mathscr{L}((\sum \ell_q)_{c_0})$.

Let T be an operator in $\mathscr{L}((\sum \ell_q)_{c_0})$ but not in the ideal of the $(\sum \ell_q)_{c_0}$-strictly singular operators on $(\sum \ell_q)_{c_0}$. Then there exists a subspace X of $(\sum \ell_q)_{c_0}$ which is isomorphic to $(\sum \ell_q)_{c_0}$ and such that $T_{|X}$ is an isomorphism. Since X is isomorphic to $(\sum \ell_q)_{c_0}$, then TX is isomorphic to $(\sum \ell_q)_{c_0}$. Hence, by lemma 3.6 the subspace TX contains a subspace Z which is isomorphic to $(\sum \ell_q)_{c_0}$ and complemented in $(\sum \ell_q)_{c_0}$.

Let B: $Z \to (\sum \ell_q)_{c_0}$ be an onto isomorphism and let P: $(\sum \ell_q)_{c_0} \to Z$ be a continuous projection onto Z. Notice that $T^{-1}(Z) \subseteq X$ and since $T_{|X}$ is an isomorphism, then $T_{|_{T^{-1}(Z)}}$ is also an isomorphism. Since Z is isomorphic to $(\sum \ell_q)_{c_0}$, then $T^{-1}(Z)$ is isomorphic to $(\Sigma \ell_q)_{c_0}$. Let A: $(\Sigma \ell_q)_{c_0} \to T^{-1}(Z)$ be an onto isomorphism.

Notice that A and $B \circ P$ are in $\mathscr{L}((\sum \ell_q)_{c_0})$. Thus, any ideal containing T must contain $(B \circ P) \circ T \circ A$ which is a one to one map of $(\sum \ell_q)_{c_0}$ onto $(\sum \ell_q)_{c_0}$ which by the Open Mapping Theorem has a bounded inverse. Thus, any ideal containing T contains the identity map, therefore, it must coincide with $\mathscr{L}((\Sigma \ell_q)_{c_0})$. Thus, every proper ideal in $\mathscr{L}((\Sigma \ell_q)_{c_0})$ is contained in the ideal of $(\sum \ell_q)_{c_0}$-strictly singular operators on $(\sum \ell_q)_{c_0}$, in other words, the ideal of $(\sum \ell_q)_{c_0}$-strictly singular operators on $(\sum \ell_q)_{c_0}$ is maximal.

This proof shows that for $X = (\sum \ell_q)_{c_0}$, if T is not X-strictly singular on X, then I_X factor through T, that is, the set $\mathscr{M}_X = \{T \in \mathscr{L}(X) : I_X\ does\ not\ factor\ through\ T\}$ is included in the set $\{T \in \mathscr{L}(X) : T\ is\ X - strictly\ singular\ on\ X\}$.

Now we claim that the set $\{T \in \mathscr{L}(X) : T\ is\ X - strictly\ singular\ on\ X\}$ is included in the set $\mathscr{M}_X = \{T \in \mathscr{L}(X) : I_X\ does\ not\ factor\ through\ T\}$.

Indeed, suppose $T \notin \mathscr{M}_X$, then I_X factors through T. Thus, there exists $A, B \in \mathscr{L}(X)$ so that $I_X = ATB$.

Thus, for all $x \in X$ we have that $\|x\| = \|ATBx\| \leq \|A\|\|TBx\|$ and this implies that $\|TBx\| \geq \|A\|^{-1}\|x\|$ for all $x \in X$. Then TB is an isomorphism. Thus, B is an isomorphism.

Since B is an isomorphism, then X is isomorphic to BX and since TB is an isomorphism, then $T_{|_{BX}}$ is an isomorphism. This implies that T is not X-strictly singular on X. This prove the claim.

Hence, $\mathscr{M}_X = \{T \in \mathscr{L}(X) : T \ is \ X - strictly \ singular \ on \ X\}$.

Since the set of all X-strictly singular operators on X is closed under addition, then the set $\mathscr{M}_X$ is also closed under addition.

Thus, by the result of Dosev and Johnson in [13], the set $\mathscr{M}_X$ is the unique maximal ideal in $\mathscr{L}(X)$.

Then the set of all $(\sum \ell_q)_{c_0}$-strictly singular operators on $(\sum \ell_q)_{c_0}$ is the unique maximal ideal in $\mathscr{L}((\sum \ell_q)_{c_0})$. □

Chapter 4

Final remarks and Open Problems

In this dissertation we have found that the unique maximal ideal in $\mathscr{L}(X)$ for $X = (\sum \ell_q)_{c_0}$ and $1 < q < \infty$ is the set of all X-strictly singular operators on X but we have not described the commutators in this space. In my future research I would like to be able of describing the commutators on $\mathscr{L}(X)$. For now that description is still an open problem.

In [10] the authors defined a Wintner space X as a space where every non-commutator in $\mathscr{L}(X)$ is of the form $\lambda I + K$, where $\lambda \neq 0$ and K lies in a proper ideal. They established that the conjecture which says every infinite-dimensional Banach space is a Wintner space is not true since there may be an infinite-dimensional Banach space such that every finite rank commutator on X has zero trace. However, the conjecture that every Banach space that admits a Pelczyński decomposition is a Wintner space is much tamer. This is still an open problem. The authors also asked what happen if X is complementably homogeneous?.

The authors in [10] also asked questions about special cases:

1. Is every $C(K)$ space a Wintner space when K is a compact space?

2. What if also K is metrizable?

3. Is $C[0,1]$ a Wintner space?. The authors believes that a affirmative answer to the question when X is complementably homogeneous solves this open problem, however, they believe that is easier to solve the question about $C[0,1]$ than the one about X when X is complementably homogeneous.

4. Is every complemented subspace of $L_p(0,1)$, $1 \leq p < \infty$, a Wintner space?. It is even open whether every infinite-dimensional complemented subspace of L_1 is isomorphic to either L_1 or to ℓ_1. In [6] was shown that there are many different (up to isomorphisms) complemented subspaces of L_p for $1 < p \neq 2 < \infty$. In this spaces are included the Wintner spaces L_p, ℓ_p, $\ell_p \oplus \ell_2$, and $Z_{p,2}$ as was shown in [14], [3],

[15]. All of these spaces are complementably homogeneous and $\mathscr{M}_X$ is an ideal of $\mathscr{L}(X)$. We must remark that while ℓ_p and ℓ_2 admit a Pelczyński decomposition, its direct sum does not.

5. Is the Rosenthal's X_p space a Wintner space? (see [44]). In [10] the authors remarked that the space X_p for $2 < p < \infty$ was the first non-obvious complemented subspace of L_p and it has played an important role in the development of the structure theory of L_p. They say that this space is small in the sense that it embeds isomorphically into $\ell_p \oplus \ell_2$ but not as a complemented subspace.They remark that the space X_p does not admit a Pelczyński decomposition but it does admit something called a $p,2$ decomposition which serve as a substitute. They also remark that not every operator in $\mathscr{M}_{X_p}$ is X_p-strictly singular because you can map X_p isomorphically into a subspace of X_p that is isomorphic to $\ell_p \oplus \ell_2$ and no isomorphic copy of X_p in $\ell_p \oplus \ell_2$ can be complemented. They say that the ideas in [25] are useful to show that $\mathscr{M}_{X_p}$ is an ideal in $\mathscr{L}(X_p)$, but this is still open.

Another important open problem is due to Brown and Pearcy and establish wheather every compact operator on ℓ_2 is a commutator of compact operators. In fact for any Banach space X the same question is open.

The authors in [10] asked the questions: for what Banach spaces X is every compact operator a commutator of compact operators? . Is this true for every infinite dimensional space X? . Is it true for some infinite dimensional space X?.

Also in [10] the authors asked the following question. Assume that X is a complementably homogeneous Wintner space that it has a Pelczyński decomposition and that $\mathscr{M}_X$ is an ideal in $\mathscr{L}(X)$. If T is not in $\mathscr{M}_X$ and T is a commutator does there exist a complemented subspace X_1 of X that is isomorphic to X and such that $(I - P_{X_1})T|_{X_1}$ is an isomorphism?.

The answer to the last question is affirmative for $X = \ell_p$, $1 \leq p \leq \infty$, c_0, L_p, $1 \leq p \leq \infty$ and $Z_{p,q}$.

In [53] the author believes there are positive solutions to the problems of classifying all commutators on $(\sum \ell_q)_{\ell\infty}$ and $(\sum \ell_\infty)_{\ell_p}$.

In [32] the authors stated that there is a major open problem which asks whether or not there exists a Banach space X so that the ideal of the compact operators is a maximal ideal of codimension one in $\mathscr{L}(X)$. Schlumprecht in [46] have developed a new method of attacking this problem.

Bibliography

[1] F. Albiac and N. J. Kalton. *Topics in Banach Space Theory. Graduate Texts in Mathematics, 233*. Springer, New York, 2006.

[2] G. Androulakis and T. Schlumprecht. Strictly singular, non-compact operators exist on the space of Gowers and Maurey. *J. London Math. Soc.*, 64(3):655–674, 2001.

[3] C. Apostol. Commutators on ℓ_p spaces. *Rev. Roumaine Math. Pures Appl*, 17:1513–1534, 1972.

[4] C. Apostol. Commutators on c_0-spaces and on ℓ_∞-spaces. *Rev. Roumaine Math. Pures Appl*, 18:1025–1032, 1973.

[5] S. A. Argyros and R. G. Haydon. A hereditarily indecomposable L_∞-space that solves the scalar-plus-compact problem. *Acta Math.*, 206(1):1–54, 2011.

[6] J. Bourgain, H. P. Rosenthal, and G. Schechtman. An ordinal L_p-index for Banach spaces, with application to complemented subspaces of L_p. *Annals of Mathematic*, pages 193–228, 1981.

[7] A. Brown, P. R. Halmos, and C. Pearcy. Commutators of operators on Hilbert space. *Canadian J. Math.*, 17:695–708, 1965.

[8] A. Brown and C. Pearcy. Structure of commutators of operators. *Ann. of Math.*, 82:112–127, 1965.

[9] J. W. Calkin. Two-sided ideals and congruences in the ring of bounded operators in Hilbert space. *Ann. of Math.*, 42(2):839–873, 1941.

[10] D. Chen, W. B. Johnson, and B. Zheng. Commutators on $\left(\sum \ell_q\right)_p$. *Studia math*, 206(2):175–190, 2011.

[11] M. Daws. Closed ideals in the Banach algebra of operators on classical non-separable spaces. *Math. Proc. Cambridge Philos. Soc.*, 140(2):317–332, 2006.

[12] J. Diestel, H. Jarchow, and A. Pietsch. Operator ideals. *Handbook of the geometry of Banach spaces*, Vol.I:437–496, 2001.

[13] D. Dosev and W. B. Johnson. Commutators on ℓ_∞. *Bull. London Math. Soc.*, 42(1):155–169, 2010.

[14] D. Dosev, W. B. Johnson, and G. Schechtman. Commutators on L_p, $1 \leq p < \infty$. *J. Amer. Math. Soc.*, 26(1):101–127, 2013.

[15] D. T. Dosev. Commutators on ℓ_1. *J. Funct. Anal.*, 256(11):3490–3509, 2009.

[16] P. Enflo and T. W. Starbird. Subspaces of L_1 containing L_1. *Studia Math*, 65(2):203–225, 1979.

[17] M. Fabian, P. Habala, P. Hájek, V. Montesinos Santalucía, J. Pelant, and V. Zizler. *Functional Analysis and Infinite-Dimensional Geometry. CMS Books in Mathematics*. Springer, New York, 2001.

[18] I. A. Feldman, I. C. Gohberg, and A. S. Markus. Normally solvable operators and ideals associated with them. *Bul. Akad. Štiince RSS Moldoven.*, 10(76):51–70, 1960.

[19] W. T. Gowers and B. Maurey. The unconditional basic sequence problem. *J. Amer. Math. Soc.*, 6(4):851–874, 1993.

[20] W. T. Gowers and B. Maurey. Banach spaces with small spaces of operators. *Math. Ann.*, 307:543–568, 1997.

[21] B. Gramsch. Eine Idealstruktur Banachscher Operatoralgebren. *J. Reine Angew. Math.*, 225:97–115, 1967.

[22] P. R. Halmos. A Glimpse into Hilbert space. *Lectures on modern Mathematics, Wiley, New York*, Vol. I:pp. 1–22, 1963.

[23] R. C. James. Bases and reflexivity of Banach spaces. *Ann. Math.*, 52:518–527, 1950.

[24] W. B. Johnson, B. Maurey, G. Schechtman, and L. Tzafriri. Symmetric structures in Banach spaces. *Mem. Amer. Math. Soc.*, 19(217):v+298 pp, 1979.

[25] W. B. Johnson and E. Odell. Subspaces and quotients of $\ell_p \oplus \ell_2$ and X_p. *Acta Math.*, 147:117–147, 1981.

[26] A. Kaminska, A. I. Popov, E. Spinu, A. Tcaciuc, and V. G. Troitsky. Norm closed operator ideals in Lorentz sequence spaces. *J. Math. Anal. Appl.*, 389(1):247–260, 2012.

[27] T. Kania and N. J. Laustsen. Uniqueness of the maximal ideal of the Banach algebra of bounded operators on $C([0,\omega_1])$. *J. Funct. Anal.*, 262(11):4831–4850, 2012.

[28] T. Kania and N. J. Laustsen. Uniqueness of the maximal ideal of operators on the ℓ_p-sum of ℓ_∞^n, $n \in \mathbb{N}$ for $1 < p < \infty$. *Mathematical proceedings of the Cambridge Philosophical Society*, 160(3):413–421, 2016.

[29] T. Kato. Perturbation theory for nullity, deficiency and other quantities of linear operators. *J. Analyse Math.*, 6:261–322, 1958.

[30] J.-L. Krivine and B. Maurey. Espaces de Banach stables. *Israel J. Math.*, 39(4):273–295, 1981.

[31] N. J. Laustsen. Maximal ideals in the algebra of operators on certain Banach spaces. *Proc. Edinburgh Math. Soc.*, 45(3):523–546, 2002.

[32] N. J. Laustsen, R. J. Loy, and C. J. Read. The lattice of closed ideals in the Banach algebra of operators on certain Banach spaces. *J. Func. Anal.*, 214(1):106–131, 2004.

[33] N. J. Laustsen, E. Odell, T. Schlumprecht, and A. Zsák. Dichotomy theorems for random matrices and closed ideals of operators on $\left(\oplus_{n=1}^{\infty}\ell_1^n\right)_{c_0}$. *J. Lond. Math. Soc.*, 86(1):235–258, 2012.

[34] N. J. Laustsen, T. Schlumprecht, and A. Zsák. The lattice of closed ideals in the Banach algebra of operators on a certain dual Banach space. *J. Operator Theory*, 56:391–402, 2006.

[35] D. H. Leung. Ideals of operators on $\left(\oplus_{n=1}^{\infty}\ell_\infty(n)\right)_{\ell_1}$. *Proc. Amer. Math. Soc*, 143:3047–3053, 2015.

[36] D. H. Leung. Maximal ideals in some spaces of bounded linear operators. *Proceedings of the Edinburgh Mathematical society*, 61(1):251–264, 2018.

[37] P. K. Lin, B. Sarí, and B. Zheng. Norm closed ideals in the algebra of bounded linear operators on Orlicz sequence spaces. *Proceedings of the American Mathematical Society*, 142(5):1669–1680, 2014.

[38] J. Lindenstrauss and A. Pelczyński. Contributions to the theory of the classical Banach spaces. *J. of Func. Analysis*, 8(2):225–249, 1971.

[39] J. Lindenstrauss and L. Tzafriri. *Classical Banach Spaces. I. Sequence Spaces.* Springer, Berlin, 1977.

[40] J. Lindenstrauss and L. Tzafriri. *Classical Banach Spaces. II. Function Spaces.* Springer, Berlin, 1979.

[41] E. Luft. The two-sided closed ideals of the algebra of bounded linear operators of a Hilbert space. *Czechoslovak Math. J.*, 18(4):595–605, 1968.

[42] R. E. Megginson. *An Introduction to Banach Space Theory. Graduate Texts in Mathematics, 183.* Springer, New York, 1998.

[43] B. S. Mityagin and I. Edelstein. Homotopy type of linear groups of two classes of Banach spaces. *Funktsional 'nyi Analiz i ego Prilozheniya*, 4(3):61–72, 1970.

[44] H. P. Rosenthal. On the subspaces of L_p, $p > 2$, spanned by sequences of independent random variables. *Israel J. Math.*, 8(3):273–303, 1970.

[45] B. Sari, T. Schlumprecht, N. Tomczak-Jaegermann, and V. G. Troitsky. On norm closed ideals in $\mathscr{L}(\ell_p,\ell_q)$. *Studia Math*, 179(3):239–262, 2007.

[46] T. Schlumprecht. How many operators exist on a Banach space? Trends in Banach spaces and operator theory (Memphis, TN, 2001). *Contemp. Math., Amer. Math. Soc., Providence, RI., 2003*, 321:295–333, 2003.

[47] T. Schlumprecht. On the closed subideals of $\mathscr{L}(\ell_p \oplus \ell_q)$. *Oper. Matrices*, 6(2):311–326, 2012.

[48] C. H. Schneeberger. Commutators on a separable L^p-space. *Pro. A. M. S.*, 28:464–480, 1971.

[49] P. Volkmann. Operatoralgebren mit einer endlichen Anzahl von maximalen Idealen. *Studia Math.*, 55(2):151–156, 1976.

[50] R. J. Whitley. Strictly singular operators and their conjugates. *Trans. Amer. Math. Soc.*, 113:252–261, 1964.

[51] H. Wielandt. Uber die Unbeschranktheit der Operatoren der Quantenmechanik. *Math. Ann.*, 121:21, 1949.

[52] A. Wintner. The unboundedness of quantum-mechanical matrices. *Physical Review*, 71(2):738–739, 1947.

[53] B. Zheng. Commutators on $\left(\sum \ell_q\right)_{\ell_1}$. *Journal of Mathematical Analysis and applications*, 413(1):284–290, 2014.