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VANISHING RELAXATION TIME DYNAMICS OF THE JORDAN MOORE-GIBSON-THOMPSON (JMGT) EQUATION ARISING IN HIGH FREQUENCY ULTRASOUND (HFU)

by

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ABSTRACT

The (third-order in time) JMGT equation is a nonlinear (quasilinear) Partial Differential Equation (PDE) model introduced to describe the acoustic velocity potential in ultrasound wave propagation. One begins with the parabolic Westervelt equation governing the dynamics of the pressure in nonlinear acoustic waves. In its derivation from constitutive laws, one then replaces the Fourier law with the Maxwell-Cattaneo law, to avoid *the paradox of the infinite speed of propagation*. This process then gives rise to a new third time derivative term, with a small constant coefficient τ, referred to as relaxation time. As a consequence, the mathematical structure of the underlying model changes drastically from the parabolic character of the Westervelt model (whose linear part generates a s.c, analytic semigroup) to the hyperbolic-like character of the JMGT model (whose linear part generates a s.c, group on a suitable function space). This is a particularly delicate issue since the $\tau-$ dynamics is governed by a generator which is singular as $\tau \to 0$. It is therefore of both mathematical and physical interest to analyze the asymptotic behavior of hyperbolic solutions of the JMGT model as the relaxation parameter $\tau \geq 0$ tends to zero. In particular, it will be shown that for suitably calibrated initial data one obtains at the limit exponentially time-decaying solutions. The rate of convergence allows one then to estimate the relaxation time needed for the signal to reach the target. The interest in studying this type of problems is motivated by a large array of applications arising in engineering and medical sciences. These include applications to welding, lithotripsy, ultrasound technology, noninvasive treatment of kidney stones.

Keywords: Jordan-Moore-Gibson-Thompson equation; third-order evolutions; strong convergence of semigroup; rate of convergence; uniform exponential decays; acoustic waves.

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CHAPTER 1

PART I - LINEAR

Abstract

Moore-Gibson-Thompson (MGT) equations, which describe acoustic waves in a heterogeneous medium, are considered. These are the third order in time evolutions of a predominantly hyperbolic type. MGT models account for a finite speed propagation due to the appearance of thermal relaxation coefficient $\tau > 0$ in front of the third order time derivative. Since the values of τ are relatively small and often negligible, it is important to understand the asymptotic behavior and characteristics of the model when $\tau \to 0$. This is a particularly delicate issue since the $\tau-$ dynamics is governed by a generator which is singular as $\tau \to 0$. It turns out that the limit dynamics corresponds to the linearized Westervelt equation which is of a parabolic type. In this paper, we provide a rigorous analysis of the asymptotics which includes strong convergence of the corresponding evolutions over infinite horizon. This is obtained by studying convergence rates along with the uniform exponential stability of the third order evolutions. Spectral analysis for the MGT-equation along with a discussion of spectral uppersemicontinuity for both equations (MGT and linearized Westervelt) will also be provided.

Keywords: Moore-Gibson-Thompson equation; third order evolutions, singular limit; strong convergence of semigroup; uniform exponential decays; acoustic waves,spectral analysis.

1.1 Introduction

In this paper, we consider PDE system describing the propagation of acoustic waves in a heterogeneous medium. The corresponding models, referred as Westervelt, Kuznetsov or Moore-Gibson-Thompson equation (MGT), have attracted considerable attention triggered by important applications in medicine, engineering and life sciences (see [13, 14, 19, 21, 22, 39]). Processes such as welding, lithotripsy or high frequency focused ultrasound depend on accurate modeling involving acoustic equations. From a mathematical point of view, these are either second-order-in -time equations with strong diffusion or third-order-in-time dynamics. While in the first case the equation is of strongly parabolic type (diffusive effects are dominant), in the second case the system displays partial hyperbolic effect which can be easily attested by spectral analysis. From a physical point of view, the difference between two types manifests itself by accounting for finite speed of propagation for the MGT equation vs infinite speed of propagation for diffusive phenomena. By accounting for thermal relaxation in the process, MGT equation resolves infinite speed of propagation paradox associated with Westervelt-Kuznetsov equation. The goal of this work is a careful asymptotic analysis of MGT equation with respect to vanishing relaxation parameter. We will show that the Westervelt-Kuznetsov equation is a limit (in terms of projected semigroups) of MGT equation, when the relaxation parameter vanishes. A quantitative rate of convergence of the corresponding solutions will be derived as well.

1.1.1 Physical motivation, modeling and thermal relaxation parameter

Physical models for nonlinear acoustics depend on what constitutive law we choose to describe the dynamics of the heat conduction. According to Fourier (classical continuum mechanics), the dynamics of the thermal flux in a homogeneous and isotropic thermally conducting medium obeys the relation

$$\overrightarrow{q} = -K\nabla\theta, \tag{1.1.1}$$

where $\overrightarrow{q}$ is the flux vector, $\theta = \theta(t,x)$ is the absolute temperature and the constant $K > 0$ is the thermal conductivity, see [10] for more details.

Along with the conservation of mass, momentum and energy, the use of Fourier's law for the heat flux lead us to obtain a number of known equations among which we find the classic second order (in time) nonlinear Westervelt's equation for the acoustic pressure $u = u(t,x)$ which can be written as

$$(1-2ku)u_{tt} - c^2\Delta u - \delta\Delta u_t = 2k(u_t)^2, \tag{1.1.2}$$

where k is a parameter that depends on the mass density and the constant of the nonlinearity of the medium and c and δ denote the speed and diffusivity of the sound, respectively. There are many references addressing various modeling aspects. Within the context of this paper, we refer to [10, 21, 22, 23] and references therein.

Unfortunately, Fourier's law does not fully describe the heat diffusion process. Physically, Fourier's law predicts the propagation of the thermal signals at infinite speed, which is unrealistic (see [17]). Mathematically, the so-called *paradox of infinite speed of propagation* (or *paradox of heat conduction*, in physics) intuitively means that initial data has an instantaneous effect on the entire space. Quantitatively we translate this notion in terms of support.

In order to make the notion clear, consider the linearized homogeneous Westervelt's equation

$$\alpha u_{tt} - c^2 \Delta u - \delta \Delta u_t = 0, \tag{1.1.3}$$

(α being a real constant) with initial conditions $u(0,x) = u_0(x)$ and $u_t(0,x) = u_1(x)$. We can simply assume, for the time being, that $x \in \mathbb{R}^n$.

Suppose that $\mathrm{supp}(u_0) \cup \mathrm{supp}(u_1) \subset B(z,R)$, that is, u_0 and u_1 have supports inside the ball of radius R and center $z \in \mathbb{R}^n$. We say that the Partial Differential Equation (PDE) above has *finite speed of propagation* if the solution u is such that $u(t,\cdot)$ has compact support for every $t > 0$. More precisely we call *speed of propagation* the number C defined as the infimum of the values $c > 0$ such that $\mathrm{supp}(u(t,\cdot)) \subset B(z, R+ct)$. If there is no finite c with the above property, we say that the PDE has infinite speed of propagation.

One can fairly easily see why Fourier's law leads to infinite speed of propagation. In general lines, neglecting internal dissipation and all sort of thermal sources, the authors of [31, 10] used the equation of continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \overrightarrow{u}) = 0,$$

(here $\overrightarrow{u}$ is the velocity vector of the material point, t is time and ρ is the mass density) to write the balance law for internal heat energy as

$$\rho C_p \frac{D\theta}{Dt} + \nabla \cdot \overrightarrow{q} = 0, \tag{1.1.4}$$

where C_p is the specific heat at constant pressure and the operator

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \overrightarrow{u} \cdot \nabla$$

represents the material derivative.

Therefore, by simply replacing the flux vector in (1.1.4) by the Fourier's law and using the definition of the material derivative we end up with what is called *heat transport equation*

$$\theta_t + \overrightarrow{u} \cdot \nabla \theta - \eta \nabla^2 \theta = 0, \tag{1.1.5}$$

where $\eta = \frac{K}{\rho C_p}$ is the constant of thermal diffusivity. Assuming for one moment that the material point does not move (i.e., $\overrightarrow{u} = 0$) the heat transport equation (1.1.5) reduces to the classical diffusion equation which is a PDE with parabolic behavior. The solution for the heat equation, as we know, is given by a convolution $u = \phi \star u_0$ where ϕ is the fundamental solution of the Laplace equation and u_0 is the initial data. From this structure, we can see that small disturbance on the initial data has the potential to affect the whole solution in the entire space.

In order to address this *defect* and account for a finite speed to the heat conduction, several improvements and modifications to the Fourier's law were studied (see [42]). Although different, all the modifications agree with the fact that it is unrealistic to consider that any change of temperature is immediatly felt regardless of position. It is interesting to note that the first work to notice this phenomena with a derivation of a new third order in time model is [41] by Professor G. G. Stokes. After 97 years, in [6], C. Cattaneo derived what today is known as the Maxwell-Cattaneo law (see also [17, 42, 7]).

The Maxwell-Cattaneo Law is given by

$$\overrightarrow{q} + \tau \frac{\partial \overrightarrow{q}}{\partial t} = -K\nabla\theta, \tag{1.1.6}$$

and managed to remove the infinite speed paradox by adding the so called *thermal inertia* term which is proportional to the time derivative of the flux vector.

It is important to observe that Maxwell-Cattaneo law as we presented in (1.1.6) resolves the paradox of infinite speed of propagation, but the diffusion process is only free of paradoxes in the case where the body (or the object) of the dynamic is resting. In the moving frame, this same constitutive law gives rise to another paradox related to the Galilean relativity regarding the invariance of physical laws in all frames. This latter and last paradox can be resolved by replacing the time derivative in (1.1.6) by the material derivative. More details about this issue can be found in [10].

The material-dependent constant τ is known as the *thermal relaxation parameter* or *time relaxation parameter* and is the center of this paper. Physically τ represents the time necessary to achieve steady heat conduction once a temperature gradient is imposed to a volume element. This time lag can be (and in fact it is) translated to different phenomena and contexts, as is the case where the models are used to study problems of High-Frequency Ultrasound (HFU) in lithotripsy, thermotherapy, ultrasound cleaning and sonochemistry. See [27, 22, 10].

The goal of this paper is to *quantify* the sensitivity the thermal relaxation parameter τ on a variety of materials by studying a singular perturbation problem, which makes sense since a number of experiments found this parameter to be small in several mediums, although not all. Among the ones where τ is not small we find biological tissue (1-100 seconds), sand (21 seconds), H acid (25 seconds) and $NaHCO_3$ (29 seconds) (see [10]). Among the ones with τ small we find cells and melanosome (order of milliseconds), blood vessels (order of microseconds, depending on the diameter) (see [46]) and most metals (order of picoseconds) (see [10]).

The same procedure as to obtain the Westervelt's equation leads us now to the third-order (in time) nonlinear Moore-Gibson-Thompson (MGT) equation

$$\tau u_{ttt} + (1-2ku)u_{tt} - c^2\Delta u - b\Delta u_t = 2k(u_t)^2, \tag{1.1.7}$$

where k and c has the same meaning as the ones in the Westervelt's equation but the diffusivity of the sound δ also suffers a change due to the presence of the thermal relaxation parameter τ and gives place to a new parameter $b = \delta + \tau c^2$. The operator Δ is understood as the Laplacian subject to suitable boundary conditions-Dirichlet, Neumann or Robin. It should be noted that the original version of this model dates back to Stokes paper [41]. A typical JMGT equation is equipped with the additional more precise physical parameters [27], however, for the sake of transparency only the canonical abstract form is retained.

Let we introduce some definitions and notations that will be used for this work.

Definition: Let Ω is an open set of $\mathbb{R}^n, 1 \leq p < \infty$. The Sobolev space $W^{k,p}(\Omega)$ [29] consists of functions $u \in L^p(\Omega)$ such that the weak derivative $D^\alpha u$ exists and $D^\alpha u \in L^p(\Omega), \forall |\alpha| \leq k$.

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), \forall |\alpha| \leq k\},$$

$$\|u\|_{W^{k,p}} = \left(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p dx \right)^{1/p}.$$

For $p = 2$, we have

$$H^k(\Omega) = W^{k,2}(\Omega).$$

Example: $H^0(\Omega) = L^2(\Omega) \to \|u\|_2 = \left(\int_\Omega |u|^2 dx \right)^{1/2}$ and $H^1(\Omega) = W^{1,2}(\Omega)$.

Let Ω be a bounded domain in $\mathbb{R}^n$ $(n = 2,3)$ with a C^2-boundary $\Gamma = \partial\Omega$. We work with $L^2(\Omega)$ but the treatment could be similarly carried out to any (separable) Hilbert space H. We consider a positive selfadjoint operator, $A = -\Delta$ with homogeneous Dirichlet boundary conditions, $u|_{\partial\Omega=0}$ where

$$\mathcal{D}(A) = \{u \in L_2, \Delta u \in L_2; u|_{\partial\Omega=0}\},$$

which was shown that $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$ in [33] and $A^\theta, \theta \in [0,1]$ denotes fractional power of A, see [36],[40]. It was also shown that

$$\mathcal{D}(A^{1/2}) = H_0^1(\Omega) \text{ which is the completion of } C^\infty \text{ with compact support in } H^1(\Omega).$$

In other words if $u \in H_0^1(\Omega)$, then $u \in H^1(\Omega)$ where u vanishes outside a compact set contained in Ω.

Notation: We denote

$$(u,v) \equiv (u,v)_{\mathcal{H}}, \quad \|u\| \equiv |u| \equiv |u|_{\mathcal{H}}.$$

$$L_p(Z) \equiv L_p(0,T;Z).$$

The following three spaces are important for the development for our result. We define $\mathbb{H}_0$ as

$$\mathbb{H}_0 \equiv \mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2}) \times L^2(\Omega);$$

which is the cross product of three Hilbert spaces namely $\mathcal{D}(A)$, $\mathcal{D}(A^{1/2})$ as defined above and $L^2(\Omega)$ and we impose the product topology as if $y \in (y_1, y_2, y_3) \in \mathbb{H}_0 \equiv \mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2}) \times L^2(\Omega)$, then

$|y|_{\mathbb{H}_0} = |(y_1,y_2,y_3)|_{\mathbb{H}_0} = |y_1|_{\mathcal{D}(A^{1/2})} + |y_2|_{\mathcal{D}(A^{1/2})} + |y_3|_{L^2(\Omega)}$. The same arguments apply to the following spaces $\mathbb{H}_1$ and $\mathbb{H}_2$.

$$\mathbb{H}_1 \equiv \mathcal{D}(A) \times \mathcal{D}(A^{1/2}) \times L^2(\Omega);$$

$$\mathbb{H}_2 \equiv \mathcal{D}(A) \times \mathcal{D}(A) \times \mathcal{D}(A^{1/2}).$$

The following Sobolev's embeddings hold

$$\mathcal{D}(A^{1/2}) \subset H^1(\Omega) \subset L_6(\Omega) \text{ with } |w|_{L_6} \leqslant C_1 |A^{1/2} w|, \quad w \in \mathcal{D}(A^{1/2}).$$

$$\mathcal{D}(A) \subset H^2(\Omega) \subset L_\infty \text{ with } |w|_{L_\infty} \leqslant C_2 |Aw|, \quad w \in \mathcal{D}(A).$$

1.1.2 Past literature and introduction of the problem

This section collects the relevant past results pertinent to the model under consideration.

Let Ω be a bounded domain in $\mathbb{R}^n$ $(n = 2,3)$ with a C^2-boundary $\Gamma = \partial\Omega$ immersed in a resting medium. We work with $L^2(\Omega)$ but the treatment could be similarly carried out to any (separable) Hilbert space H. Consider $A : \mathcal{D}(A) \subset L^2(\Omega) \to L^2(\Omega)$ defined as the Dirichlet Laplacian, i.e., $A = -\Delta$ with $\mathcal{D}(A) = H_0^1(\Omega) \cap H^2(\Omega)$. All the results remain true if we assume A to be any unbounded positive self-adjoint operator with compact resolvent defined on H.

Consider the nonlinear third order evolution

$$\begin{cases} \tau u_{ttt} + (1+2ku)u_{tt} + c^2Au + bAu_t = 0, \\ u(0,\cdot) = u_0, u_t(0,\cdot) = u_1, u_{tt}(0,\cdot) = u_2, \end{cases} \tag{1.1.8}$$

and its linearization

$$\begin{cases} \tau u_{ttt} + \alpha u_{tt} + c^2Au + bAu_t = 0, \\ u(0,\cdot) = u_0, u_t(0,\cdot) = u_1, u_{tt}(0,\cdot) = u_2. \end{cases} \tag{1.1.9}$$

The natural phase spaces associated with these evolutions are the following:

$$\mathbb{H}_0 \equiv \mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2}) \times L^2(\Omega) \ \ and \ \ \mathbb{H}_1 \equiv \mathcal{D}(A) \times \mathcal{D}(A^{1/2}) \times L^2(\Omega). \tag{1.1.10}$$

Generation of linear semigroups associated with (1.1.9) has been studied in [26, 38] where it was shown that for any $\tau > 0, b > 0$ (1.1.9) generates a strongly continuous group on either $\mathbb{H}_0$ or $\mathbb{H}_1$. This result depends on $b > 0$. When $b = 0$ the generation of semigroups fails [18].

The nonlinear (quasilinear) model (1.1.7) has been treated in [27] where it was shown that for the initial data sufficiently small in $\mathbb{H}_1$, i.e., in a ball $B_{\mathbb{H}_1}(r)$ there exists nonlinear semigroup operator defined on $\mathbb{H}_1$ for all $t > 0$. The value of r depends only on the values of the physical parameters in the equation and not on $t > 0$. The aforementioned result depends on uniform stability of the dynamics of (1.1.9) and this holds for $\gamma = \alpha - \tau c^2 b^{-1} > 0$.

Subsequently, the authors in [38] showed that the linear equation generates a C_0-group in four different spaces with exponential stability provided $\gamma = \alpha - \tau c^2 b^{-1} > 0$. In case $\gamma = 0$ the system is conservative and in case $\gamma < 0$, by assuming very regular energy spaces the authors in [12] showed that (1.1.9) generates a chaotic semigroup.

Spectral analysis of the linear problem was also studied [38, 27, 26]. The spectrum consists of continuous spectrum and point spectrum. The location of the eigenvalues confirms partially hyperbolic character of the dynamics.

The same model with added memory, where the latter accounts for molecular relaxation, was considered in [35, 34, 16, 2, 4] for linear case and in [32] for the nonlinear case.

All the results obtained and mentioned above pertain to the situation when $\tau > \tau_0 > 0$. Since the parameter τ in many applications is typically very small, it is essential to understand the effects of diminishing values of relaxation parameter on quantitative properties of the underlined dynamics. This will provide important information on sensitivity of the model with respect to time relaxation. The goal of this paper is precisely to consider the vanishing parameter $\tau \to 0$ and its consequences on the resulting evolution. Specific questions we ask are the following:

- Convergence of semigroups with respect to vanishing relaxation parameter $\tau \geq 0$.
- Uniform (with respect to $\tau > 0$) asymptotic stability properties of the "relaxed" groups.
- Asymptotic (in τ) behavior of the spectrum for the family of the generators.

To our best knowledge, this is the first work that takes into consideration asymptotic properties of the MGT dynamics with respect to the vanishing relaxation parameter. The limiting evolution changes the character from a hyperbolic group to a parabolic semigroup. This change is expected to be reflected by the asymptotic properties of the spectrum and quantitative estimates for the corresponding evolutions. It should also be noted that the problem under consideration does not fit the usual Trotter-Kato type of the framework. This is due to the fact that the limit problem corresponds formally to degenerated structure. Thus, convergence of the resolvents (condition required by Trotter Kato framework) does not have a natural interpretation.

1.2 Main Results

1.2.1 Convergence of the projected semigroup solutions

As before, let Ω be a bounded domain in $\mathbb{R}^n$ $(n = 2,3)$ with a C^2-boundary $\Gamma = \partial\Omega$ immersed in a resting medium and $A : \mathcal{D}(A) \subset L^2(\Omega) \to L^2(\Omega)$ defined as the Dirichlet Laplacian, i.e., $A = -\Delta$ with $\mathcal{D}(A) = H_0^1(\Omega) \cap H^2(\Omega)$.

Let $T > 0$. We consider a family of "hyperbolic" abstract third order problems

$$\begin{cases} \tau u^\tau_{ttt} + \alpha u^\tau_{tt} + c^2 A u^\tau + b^\tau A u^\tau_t = 0, \ t > 0, \\ u^\tau(0,\cdot) = u_0, u^\tau_t(0,\cdot) = u_1, u^\tau_{tt}(0,\cdot) = u_2, \end{cases} \tag{1.2.1}$$

where $b^\tau = \delta + \tau c^2$ and $\alpha, c, \delta, \tau > 0$.

We rewrite (1.2.1) abstractly by using a mass operator M_τ as below:

$$\begin{cases} M_\tau U^\tau_t(t) = \mathcal{A}^\tau_0 U^\tau(t), \ t > 0, \\ U^\tau(0) = U_0 = (u_0, u_1, u_2)^T, \end{cases} \tag{1.2.2}$$

or equivalently with $\mathcal{A}^\tau = M_\tau^{-1} \mathcal{A}^\tau_0$

$$\begin{cases} U^\tau_t(t) = \mathcal{A}^\tau U^\tau(t), \ t > 0, \\ U^\tau(0) = U_0 = (u_0, u_1, u_2)^T, \end{cases} \tag{1.2.3}$$

where

$$U^\tau \equiv \begin{pmatrix} u^\tau \\ u^\tau_t \\ u^\tau_{tt} \end{pmatrix}; \ \mathcal{A}^\tau \equiv \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\tau^{-1} c^2 A & -\tau^{-1} b^\tau A & -\tau^{-1}\alpha \end{pmatrix}; \ M_\tau \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \tau \end{pmatrix}. \tag{1.2.4}$$

The evolution described in (1.2.3) can be considered on several product spaces with the results depending on the space and the domain where $\mathcal{A}^\tau$ is defined.

Remark 1.2.1. The generator $\mathcal{A}^\tau$ "blows up" when $\tau \to 0$.

We define $\mathbb{H}_0, \mathbb{H}_1, \mathbb{H}_2$ as

$$\mathbb{H}_0 \equiv \mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2}) \times L^2(\Omega);$$

$$\mathbb{H}_1 \equiv \mathcal{D}(A) \times \mathcal{D}(A^{1/2}) \times L^2(\Omega);$$

$$\mathbb{H}_2 \equiv \mathcal{D}(A) \times \mathcal{D}(A) \times \mathcal{D}(A^{1/2}).$$

The operators $\mathcal{A}^\tau$ are considered on each of these spaces with natural domains induced by the given topology.

For instance, $\mathcal{A}^\tau : \mathcal{D}(\mathcal{A}^\tau) \subset \mathbb{H}_0 \to \mathbb{H}_0$ has the domain defined by

$$\mathcal{D}(\mathcal{A}^\tau) = \{(u,v,w) \in \mathbb{H}_0; c^2 u + b^\tau v \in \mathcal{D}(A)\}.$$

Clearly, the domains are not compact in $\mathbb{H}_0$. Analogous setups are made for $\mathbb{H}_1$ and $\mathbb{H}_2$.

For each $\tau > 0$, we will consider weighted norms defined by the means of the mass operator M_τ.

$$||M_\tau^{1/2} U||^2_{\mathbb{H}_0}, ||M_\tau^{1/2} U||^2_{\mathbb{H}_1}, ||M_\tau^{1/2} U||^2_{\mathbb{H}_2},$$

that is,

$$\|(u,v,w)\|^2_{\tau,0} = \|u\|^2_{\mathcal{D}(A^{1/2})} + \|v\|^2_{\mathcal{D}(A^{1/2})} + \tau\|w\|^2_2 = ||M_\tau^{1/2} U||^2_{\mathbb{H}_0};$$

$$\|(u,v,w)\|^2_{\tau,1} = \|u\|^2_{\mathcal{D}(A)} + \|v\|^2_{\mathcal{D}(A^{1/2})} + \tau\|w\|^2_2 = ||M_\tau^{1/2} U||^2_{\mathbb{H}_1};$$

$$\|(u,v,w)\|^2_{\tau,2} = \|u\|^2_{\mathcal{D}(A)} + \|v\|^2_{\mathcal{D}(A)} + \tau\|w\|^2_{\mathcal{D}(A^{1/2})} = ||M_\tau^{1/2} U||^2_{\mathbb{H}_2}$$

with $\|\cdot\|_2$ representing the standard L^2-norm. We shall also use the rescaled notation: $\mathbb{H}_0^\tau = M_\tau^{1/2}\mathbb{H}_0$, $\mathbb{H}_1^\tau = M_\tau^{1/2}\mathbb{H}_1$, $\mathbb{H}_2^\tau = M_\tau^{1/2}\mathbb{H}_2$ with an obvious interpretation for the composition where the elements of $\mathbb{H}_0^\tau$ coincide with the elements of $\mathbb{H}_0$ and induced topology given by $\|(u,v,w)\|_{\tau,0}$.

Theorem 1.2.1. **(Generation of a group on $\mathbb{H}_0$ and $\mathbb{H}_2$).** *Let $\alpha, c, \delta > 0$. Then, for each $\tau > 0$ the operator $\mathcal{A}^\tau$ generates a C_0-group $\{T^\tau(t)\}_{t\geqslant 0}$ on $\mathbb{H}_0$ and also on $\mathbb{H}_2$.*

Theorem 1.2.1 follows from [38] applied to $\mathbb{H}_0$ space. The invariance of the generator under the multiplication by fractional powers of A leads to the result stated for $\mathbb{H}_2$.

Theorem 1.2.2. **(Equi-boundedness and uniform (in τ) exponential stability in $\mathbb{H}_0^\tau$).** *Consider the family $\mathcal{F} = \{T^\tau(t)\}_{\tau>0}$ of groups generated by $\mathcal{A}^\tau$ on $\mathbb{H}_0$. Assume that $\gamma^\tau \equiv \alpha - c^2\tau(b^\tau)^{-1} \geq \gamma_0 > 0$. Then, there exists $\tau_0 > 0$ and constants $M = M(\tau_0), \omega = \omega(\tau_0) > 0$ (both independent on τ) such that*

$$\|T^\tau(t)\|_{\mathcal{L}(\mathbb{H}_0^\tau)} \leqslant Me^{-\omega t} \text{ for all } \tau \in (0, \tau_0] \text{ and } t \geqslant 0.$$

Theorem 1.2.3. *Let $\alpha, c, \delta > 0$. Then*

(a) **(generation on $\mathbb{H}_1$)** *For each $\tau > 0$ the operator $\mathcal{A}^\tau$ generates a C_0-group $\{T^\tau(t)\}_{t\geqslant 0}$ on $\mathbb{H}_1$.*

(b) (**equi-boundedness and uniform (in τ) exponential stability**) *Consider the family* $\mathcal{F}_1 = \{T^\tau(t)\}_{\tau>0}$ *of groups generated by* $\mathcal{A}^\tau$ *on* $\mathbb{H}_1$. *Assume* $\gamma^\tau > \gamma_0 > 0$. *Then, there exists* $\tau_0 > 0$ *and constants* $\overline{M}_1 = \overline{M}_1(\tau_0)$, $\overline{\omega}_1 = \overline{\omega}_1(\tau_0) > 0$, *both independent on* τ *such that*

$$\|T^\tau(t)\|_{L(\mathbb{H}_i^\tau)} \leqslant \overline{M}_1 e^{-\overline{\omega}_1 t} \text{ for all } \tau \in (0, \tau_0] \text{ and } t \geqslant 0.$$

Remark 1.2.2. Notice that the space $\mathbb{H}_2$ is obtained by multiplication of elements in $\mathbb{H}_0$ by $A^{1/2}$ (componentwise), therefore, if we assume initial data in $\mathbb{H}_2$, it follows that uniform (in τ) boundedness and stability of the dynamics remain true.

In order to characterize asymptotic behavior of the family $\mathcal{F}$ of the groups $T^\tau(t)$ we introduce the space $\mathbb{H}_0^0 \equiv \mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2})$ and the projection operator $P : \mathbb{H}_0 \to \mathbb{H}_0^0$ defined as

$$\mathbb{H}_0 \ni (u, v, w) \mapsto (u, v) \in \mathbb{H}_0^0.$$

With this notation Theorem 1.2.2 implies uniform boundedness of the sequence

$$||PT^\tau(t)E||_{L(\mathbb{H}_0^0)} \leq Me^{-\omega t}, t > 0, \tag{1.2.5}$$

where E denotes the extension operator from $\mathbb{H}_0^0 \to \mathbb{H}_0^\tau$ defined by $E(u,v) \equiv (u,v,0)$. From (1.2.5) we deduce that for every $U^0 = (u_0, u_1) \in \mathbb{H}_0^0$ the corresponding projected solutions $PT^\tau(t)EU^0$ have a weakly convergent subsequence in $\mathbb{H}_0^0$ and weakly star in $L^\infty(0, \infty; \mathbb{H}_0^0)$. By standard distributional calculus one shows that such subsequence converges *weakly* to $U^0(t) = (u^0(t), u_t^0(t))$ which satisfies (distributionally) the following **limit** equation

$$\begin{cases} \alpha u_{tt}^0 + c^2 A u^0 + \delta A u_t^0 = 0, \\ u^0(0, \cdot) = u_0, u_t^0(0, \cdot) = u_1, \end{cases} \tag{1.2.6}$$

which rewritten as first order system becomes

$$\begin{cases} U_t^0(t) = \mathcal{A} U^0(t),\ t > 0, \\ U^0(0) = U_0^0 = (u_0, u_1)^T, \end{cases} \tag{1.2.7}$$

where

$$U^0 \equiv \begin{pmatrix} u^0 \\ u_t^0 \end{pmatrix}; \ \mathcal{A} = \begin{pmatrix} 0 & I \\ -c^2\alpha^{-1}A & -\delta\alpha^{-1}A \end{pmatrix} \tag{1.2.8}$$

and

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathbb{H}_0^0 \to \mathbb{H}_0^0$$

with

$$\mathcal{D}(\mathcal{A}) = \{(u,v) \in \mathcal{D}(\mathcal{A}^{1/2}) \times \mathcal{D}(\mathcal{A}^{1/2}); c^2u + \delta v \in \mathcal{D}(A^{3/2})\}.$$

Equation (1.2.6) is a known and well studied in the literature strongly damped wave equations. In fact, generation of an analytic and exponentially decaying semigroup on the space $\mathcal{D}(A^{1/2}) \times L^2(\Omega)$ is a standard by now result [37, 9, 8]. Less standard is the analysis on $\mathbb{H}_0^0 \equiv \mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2})$, where contractivity and dissipativity are no longer valid. This latter is the framework relevant to our analysis.

Proposition 1.2.1. (a) (**generation of a semigroup on** $\mathbb{H}_0^0$) *Let* $\mathbb{H}_0^0 \equiv \mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2})$ *and* $\alpha, \delta, c > 0$. *Then the operator* $\mathcal{A}$ *generates an (noncontractive) analytic semigroup* $\{T(t)\}_{t \geqslant 0}$ *in* $\mathbb{H}_0^0$.

(b) (**exponential stability**) *There exist constants* $M_0, \omega_0 > 0$ *such that*

$$\|T(t)\|_{\mathcal{L}(\mathbb{H}_0^0)} \leqslant M_0 e^{-\omega_0 t}, \ t \geqslant 0.$$

Proof. The well-posedness and analyticity of the associated generator on the space $L^2(\Omega) \times L^2(\Omega)$ is a direct consequence of [33] (Theorem 3B.6, p. 293) and [45] (Proposition 2.2, p. 387). Invariance of the semigroup under the action of $A^{1/2}$ implies the same result in $\mathbb{H}_0^0$, hence justifying the part (a) of Proposition 1.2.1. As to the exponential stability, while this is a well known fact proved by energy methods on the space $\mathcal{D}(A^{1/2}) \times L^2(\Omega)$, the decay rates on $\mathbb{H}_0^0$ (nondissipative case) need a justification. In our case, this follows from the estimate in (1.2.5) along with weak lower semicontinuity of $||PT^\tau(t)EU^0||^2_{\mathbb{H}_0^0}$. The conclusion on exponential stability can also be derived independently of the family $\mathcal{F}$, by evoking analyticity of the generator [9] along with the spectrum growth determined condition and the analysis of the location of the spectrum (see section 2.2 below). □

Remark 1.2.3. Proposition 1.2.1 also holds with $\mathbb{H}_0^0$ replaced by $\mathbb{H}_1^0 \equiv \mathcal{D}(A) \times \mathcal{D}(A)$.

Our main interest and goal of this work is to provide a quantitative description of *strong* convergence, when $\tau \to 0$, of hyperbolic groups $T^\tau(t)$ to the parabolic like semigroup $T(t)$. Our result is formulated below.

Theorem 1.2.4. (a) **(Rate of convergence)** *Let* $U_0 \in \mathbb{H}_2$. *Then there exists* $C = C(T, \tau_0)$ *such that*

$$\|PT^\tau(t)U_0 - T(t)PU_0\|^2_{\mathbb{H}_0^0} \leqslant C\tau\|U_0\|^2_{\mathbb{H}_2}$$

uniformly for $t \in [0, T]$.

(b) **(Strong convergence)** *Let* $U_0 \in \mathbb{H}_0$. *Then the following strong convergence holds*

$$\|PT^\tau(t)U_0 - T(t)PU_0\|_{\mathbb{H}_0^0} \to 0 \text{ as } \tau \to 0 \tag{1.2.9}$$

uniformly for all $t \geqslant 0$.

Remark 1.2.4. Note that Theorem 1.2.4 pertains to uniform (in time) strong convergence on infinite time horizon. This fact is essential in infinite horizon optimal control theory [33].

Remark 1.2.5. A standard tool for proving strong convergence of semigroups is Trotter-Kato Theorem [28]. However, this approach does not apply to the problem under consideration due to the singularity of the family of generators. A consistency requirement (convergence of the resolvents) is problematic due to specific framework where the family of $\mathcal{A}^\tau$ becomes singular when $\tau = 0$. More refined approach applicable to this particular framework will be developed.

The Theorem 1.2.4 provides the information about strong convergence of the solution u^τ and its first derivative in time. Regarding the second time derivative, we have the following.

Proposition 1.2.2. *Let* $U_0 \in \mathbb{H}_0$, *then we have*

$$\tau^{1/2}u^\tau_{tt} \to 0 \text{ weakly}^* \text{ in } L^\infty(0,\infty;L^2(\Omega)).$$

1.2.2 Spectral Analysis and Comparison Between $\sigma(\mathcal{A}^\tau)$ and $\sigma(\mathcal{A})$

Recall that A is assumed to be a positive self-adjoint operator with compact resolvent defined on a infinite-dimensional Hilbert space H ($L^2(\Omega)$ for instance). This allow us to infer that the spectrum of A

consists purely of the point spectrum. Moreover, it is countable and positive. In other words:

$$\sigma(A) = \sigma_p(A) = \{\mu_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+^*$$

and $\mu_n \to \infty$ as $n \to \infty$.

We begin with the characterization of the spectrum of $\mathcal{A}$, the operator corresponding to the limit problem. See Figure 1.1.

Proposition 1.2.3. (a) *The residual spectrum is empty:* $\sigma_r(\mathcal{A}) = \emptyset$.

(b) *The continuous spectrum consists of one single real value:* $\sigma_c(\mathcal{A}) = \left\{-\dfrac{c^2}{\delta}\right\}$.

(c) *The point spectrum is given by*

$$\sigma_p(\mathcal{A}) = \left\{\lambda \in \mathbb{C}; \alpha\lambda^2 + \delta\mu_n\lambda + c^2\mu_n = 0,\ n \in \mathbb{N}\right\} = \left\{\lambda_n^0, \lambda_n^1, n \in \mathbb{N}\right\},$$

where $\mathrm{Re}(\lambda_n^i) \in \mathbb{R}_-^*$ *for all* $n \in \mathbb{N}$ *and* $i = 1,2$. *Moreover, both branches are eventually real and the following limits hold:*

$$\lim_{n\to\infty} \lambda_n^0 = -\frac{c^2}{\delta} \text{ and } \lim_{n\to\infty} \lambda_n^1 = -\infty.$$

Regarding the spectrum of $\mathcal{A}^\tau$ we have, see Figure 1.2 for each $\tau > 0$.

Proposition 1.2.4. (a) *The residual spectrum is empty* $\sigma_r(\mathcal{A}^\tau) = \emptyset$ *for all* $\tau > 0$.

(b) *The continuous spectrum is either empty or consists of a single real value:*

$$\sigma_c(\mathcal{A}^\tau) = \begin{cases} \left\{-\dfrac{c^2}{b^\tau}\right\} & \text{if } \gamma^\tau > 0, \\ \emptyset & \text{if } \gamma^\tau = 0, \end{cases}$$

where $\gamma^\tau \equiv \alpha - c^2\tau(b^\tau)^{-1}$.

(c) *The point spectrum is given by*

$$\sigma_p(\mathcal{A}^\tau) = \{\lambda \in \mathbb{C};\ \tau\lambda^3 + \alpha\lambda^2 + b^\tau\mu_n\lambda + c^2\mu_n = 0,\ n \in \mathbb{N}\} = \left\{\lambda_n^{0,\tau}, \lambda_n^{1,\tau}, \lambda_n^{2,\tau}, n \in \mathbb{N}\right\}.$$

One of the branches, say $\lambda_n^{0,\tau}$, *is eventually real while the other two branches are eventually complex,* *conjugate of each other and the following limits hold:*

$$\lim_{n\to\infty} \lambda_n^{0,\tau} = -\frac{c^2}{b^\tau}, \lim_{n\to\infty} \mathrm{Re}(\lambda_n^{1,\tau}) = -\frac{\gamma^\tau}{2\tau} \textit{ and } \lim_{n\to\infty} |\mathrm{Img}(\lambda_n^{1,\tau})| = \infty.$$

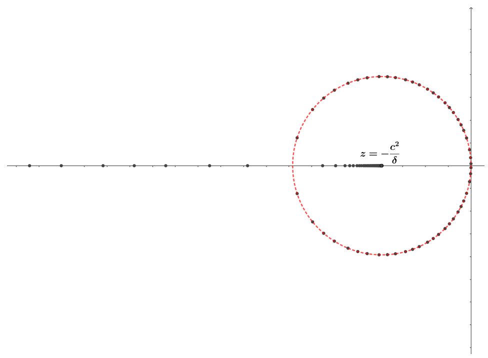

Figure 1.1: **Graphical representation of $\sigma_p(\mathcal{A})$ and $\sigma_c(\mathcal{A})$**: the circle in red is centered at $\left(-\frac{c^2}{\delta}, 0\right)$ and has radius $r = \frac{c^2}{\delta}$. The shape of the eigenvalues is represented by the curve of black dots. Clearly, the bigger is the radius, the bigger is number of complex roots. Nevertheless, the asymptotic behavior of the eigenvalues are the same: when n becomes sufficiently large, all the eigenvalues are real and we have one branch converging to $-\infty$ while the other converge to the point in the continuous spectrum $z = -\frac{c^2}{\delta}$.

The next lemma allows us to establish quantitative relation between $\sigma(\mathcal{A})$ and $\sigma(\mathcal{A}^\tau)$ for small τ.

Lemma 1.2.5. *Let $n \in \mathbb{N}$ fixed. Then, among the three roots $\{\lambda_n^{0,\tau}, \lambda_n^{1,\tau}, \lambda_n^{2,\tau}\}_{\tau>0}$ of the equation*

$$\tau\lambda^3 + \alpha\lambda^2 + b^\tau \mu_n \lambda + c^2 \mu_n = 0,$$

there are two converging, as $\tau \to 0$, to the two roots $\{\lambda_n^1, \lambda_n^2\}$ of the equation

$$\alpha\lambda^2 + \delta\mu_n\lambda + c^2\mu_n = 0.$$

Proof. Write

$$\tau\lambda^3 + \alpha\lambda^2 + b^\tau \mu_n \lambda + c^2 \mu_n = p(\lambda)(\alpha\lambda^2 + \delta\mu_n\lambda + c^2\mu_n) + q_\tau(\lambda)$$

with

$$p(\lambda) = \frac{1}{\alpha^2}\left[\tau\alpha\lambda + \alpha^2 - \tau\delta\mu_n\right]$$

$$q_\tau(\lambda) = \frac{\tau\mu_n}{\alpha^2}\left[\alpha(\alpha-1)c^2 - \delta^2\mu_n\right]\lambda + \frac{\tau\mu_n^2 c^2 \delta}{\alpha^2}$$

and notice that

$$\lim_{\tau\to 0} q_\tau(\lambda) = 0 \text{ and } \lim_{\tau\to 0} p_\tau(\lambda) = 1$$

for every λ. □

The above statement implies the following corollary.

Corollary 1.2.6. (Uppersemicontinuity of the spectrum) *Let* $\varepsilon > 0$ *given. Then, for each* $z^0 \in \sigma_p(\mathcal{A})$ *there exists* $\delta = \delta_\varepsilon > 0$ *and* $\tau < \delta$ *such that the set*

$$\{z^\tau \in \sigma(\mathcal{A}^\tau); |z^\tau - z^0| < \varepsilon\}$$

is nonempty.

Remark 1.2.6. The goal of Propositition 1.2.4 is to localize the vertical asymptote in the spectrum explicitly and support the later claim that, as τ vanishes, it becomes arbitrarily far from the imaginary axis. Notice that the proofs of Lemma 1.2.5 and part (c) of Proposition 1.2.4 (see page 38 [Section 1.3.6]) have some similarities of algebraic manipulation but have different meaning. The proof of part (c) of Proposition 1.2.4 takes advantage of the known single-point continuous spectrum to conclude that for n very large the third degree polynomial must have no more then one real root and then quantify the imaginary and real parts of the complex roots. However, the proof of Lemma 1.2.5 makes use of the quadratic structure of the point spectrum of $\mathcal{A}$ in order to conclude the expected approximation.

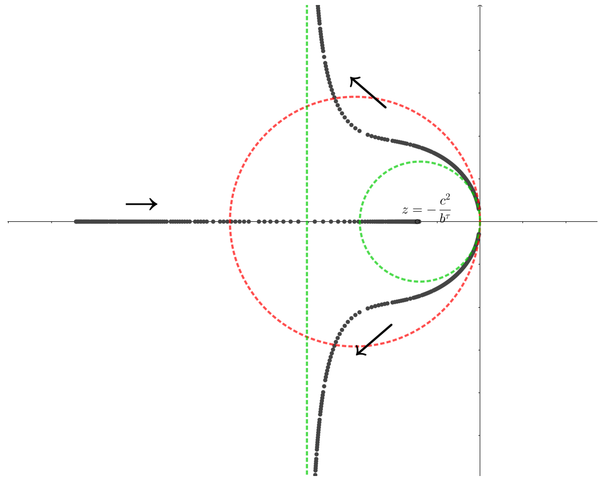

Figure 1.2: **Graphical representation of $\sigma_p(\mathcal{A}^\tau)$ and $\sigma_c(\mathcal{A}^\tau)$ for a $\tau=\tau_0\in(0,1]$ fixed**: the circle in red is the same as in Figure 1.1 while the circle in green is centered at $\left(-\frac{c^2}{b^\tau},0\right)$ and has radius $r=\frac{c^2}{b^\tau}$. The shape of the eigenvalues is represented by the curve of black dots. The green vertical line is given by $x=-\frac{\gamma^\tau}{2\tau}$ (see Theorem 1.2.4). The asymptotic behavior of the eigenvalues is exactly as we described in Theorem 1.2.4: as n becomes large, two branches of the eigenvalues have their respective real parts converging to $-\frac{\gamma^\tau}{2\tau}$ while their imaginary parts split into $\pm\infty$, and the other branch converges to the continuous spectrum, which in this case is given by the single point $z=-\frac{c^2}{b^\tau}$.

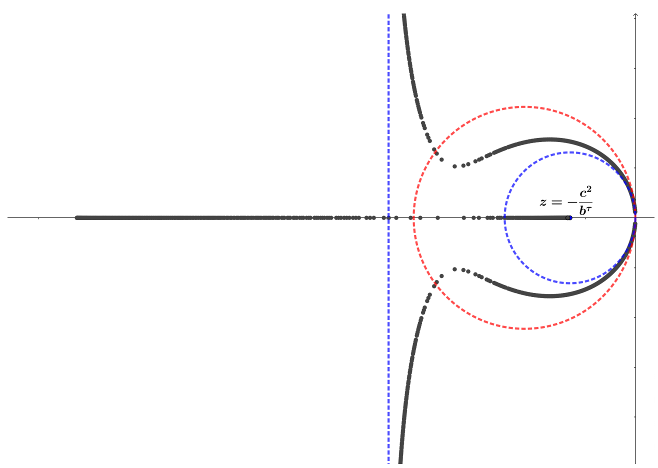

Figure 1.3: **Graphical representation of $\sigma_p(\mathcal{A}^\tau)$ and $\sigma_c(\mathcal{A}^\tau)$ for a $\tau=\tau_0/10$.**

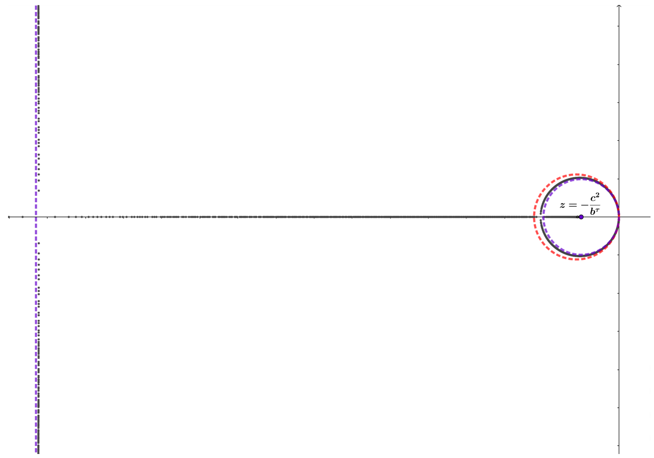

Figure 1.4: **Graphical representation of $\sigma_p(\mathcal{A}^\tau)$ and $\sigma_c(\mathcal{A}^\tau)$ for a $\tau = \tau_0/100$** : It is important to observe how the "vertical" spectrum escapes to $-\infty$ as τ becomes small.

The remaining part of the paper is devoted to the proofs of the main results.

1.3 Proofs

1.3.1 Proof of Theorem 1.2.2

Part I: Equi-boundedness of the groups

Let u^τ be the solution for (1.2.1) and consider the energy functional $E^\tau(t)$ defined as

$$E^\tau(t) = E_0^\tau(t) + E_1^\tau(t),$$

where

$$E_0^\tau(t) = \frac{\alpha}{2}\|u_t^\tau(t)\|_2^2 + \frac{c^2}{2}\|A^{1/2}u^\tau(t)\|_2^2 \tag{1.3.1}$$

and

$$E_1^\tau(t) = \frac{b^\tau}{2}\left\|A^{1/2}\left(u_t^\tau(t) + \frac{c^2}{b^\tau}u^\tau(t)\right)\right\|_2^2 + \frac{\tau}{2}\left\|u_{tt}^\tau(t) + \frac{c^2}{b^\tau}u_t^\tau(t)\right\|_2^2 + \frac{c^2\gamma^\tau}{2b^\tau}\|u_t^\tau(t)\|_2^2. \tag{1.3.2}$$

The following differential identity can be derived for the functional $E_1^\tau(t)$:

$$\frac{d}{dt}E_1^\tau(t) + \gamma^\tau\|u_{tt}^\tau(t)\|_2^2 = 0, \tag{1.3.3}$$

where $\gamma^\tau = \alpha - c^2\tau(b^\tau)^{-1}$. The proof of (1.3.3) follows along the same lines as in Lemma 3.1 [27] . The equality is derived first for smooth solution and then extended by density to the "energy" level solutions. For readers convenience the details of the derivation are given in the Appendix. Moreover, for each fixed value of $\tau > 0$, the authors in [27] establish exponential decay (with decay rates depending on τ) provided $\gamma^\tau > 0$.

We aim to prove that the family of semigroups $\mathcal{F}$ is equi-bounded in τ. In other words, there exists τ_0 small enough such that if we consider $\tau \in (0,\tau_0]$, we can provide an uniform (in τ) bound for the norm of the solutions in $(\mathbb{H}_0, \|\cdot\|_{\tau,0}) = \mathbb{H}_0^\tau$. To achieve this, we establish first topological equivalence of energy function with respect to the topology defined on $\mathbb{H}_0^\tau$. This is given in the lemma to follow.

Lemma 1.3.1. *Let $U_0 \in \mathbb{H}_0$ and define $\tau_0 \equiv \inf\{c > 0; \gamma^\tau > 0$ for all $\tau \in (0,c]\} > 0$. Then there exist $k = k(\tau_0), K = K(\tau_0) > 0$ such that*

$$k\|T^\tau(t)U_0\|^2_{\tau,0} \leqslant E^\tau(t) \leqslant K\|T^\tau(t)U_0\|^2_{\tau,0},\ \tau \in (0,\tau_0] \tag{1.3.4}$$

for all $t \geqslant 0$.

Proof. We begin with the second inequality-as an easier one. In order to get it, we observe that from (1.3.2) we have[1]:

$$\begin{aligned}
E^\tau(t) &= \frac{b^\tau}{2}\left\|A^{1/2}\left(u_t^\tau + \frac{c^2}{b^\tau}u^\tau\right)\right\|_2^2 + \frac{\tau}{2}\left\|u_{tt}^\tau + \frac{c^2}{b^\tau}u_t^\tau\right\|_2^2 + \left(\frac{c^2\gamma^\tau}{2b^\tau} + \frac{\alpha}{2}\right)\|u_t^\tau\|_2^2 + \frac{c^2}{2}\|A^{1/2}u^\tau\|_2^2 \\
&\leqslant \left(\frac{b^\tau + c^2}{2}\right)\|A^{1/2}u_t^\tau\|_2^2 + \frac{c^2}{2}\left(2 + \frac{c^2}{b^\tau}\right)\|A^{1/2}u^\tau\|_2^2 + \frac{c^2}{2b^\tau}\left(\tau + \frac{c^2\tau}{b^\tau} + \frac{\alpha b^\tau}{c^2} + \gamma^\tau\right)\|u_t^\tau\|_2^2 + \frac{\tau}{2}\left(1 + \frac{c^2}{b^\tau}\right)\|u_{tt}^\tau\|_2^2 \\
&\leqslant \frac{c^2}{2}\left(2 + \frac{c^2}{b^\tau}\right)\|A^{1/2}u^\tau\|_2^2 + \frac{c^2}{2b^\tau}\left[C^*\left(\tau + \frac{c^2\tau}{b^\tau} + \gamma^\tau\right) + \frac{(\alpha C^* + b^\tau)b^\tau}{c^2} + b^\tau\right]\|A^{1/2}u_t^\tau\|_2^2 + \frac{\tau}{2}\left(1 + \frac{c^2}{b^\tau}\right)\|u_{tt}^\tau\|_2^2 \\
&\leqslant \frac{c^2}{2}\left(2 + \frac{c^2}{\delta}\right)\|A^{1/2}u^\tau\|_2^2 + \frac{c^2}{2\delta}\left[C^*\left(\tau_0 + \frac{c^2\tau_0}{\delta} + \gamma^{\tau_0}\right) + \frac{(\alpha C^* + \delta)\delta}{c^2} + \delta\right]\|A^{1/2}u_t^\tau\|_2^2 + \frac{\tau}{2}\left(1 + \frac{c^2}{\delta}\right)\|u_{tt}^\tau\|_2^2 \\
&\leqslant \max\left\{\frac{c^2}{2}\left(2 + \frac{c^2}{\delta}\right); \frac{c^2}{2\delta}\left[C^*\left(\tau_0 + \frac{c^2\tau_0}{\delta} + \gamma^{\tau_0}\right) + \frac{(\alpha C^* + \delta)\delta}{c^2} + \delta\right]; \frac{1}{2} + \frac{c^2}{2\delta}\right\}\|U^\tau(t)\|^2_{\tau,0},
\end{aligned}$$

[2] where $C^* = C^*(n,\Omega)$ is the constant that appears in Poincaré's Inequality.

Now the second inequality in (1.3.4) follows after we define

$$K(\tau_0) \equiv \max\left\{\frac{c^2}{2}\left(2 + \frac{c^2}{\delta}\right); \frac{c^2}{2\delta}\left[C^*\left(\tau_0 + \frac{c^2\tau_0}{\delta} + \gamma^{\tau_0}\right) + \frac{(\alpha C^* + \delta)\delta}{c^2} + \delta\right]; \frac{1}{2} + \frac{c^2}{2\delta}\right\}. \tag{1.3.5}$$

[1] we have omitted the obvious dependence on t.
[2] we have omitted the obvious dependence on t.

For the first inequality, fix $\varepsilon > 0$ to be determined later. Peter-Paul inequality then implies that

$$-\frac{c^2\varepsilon\|A^{1/2}u_t^\tau\|_2^2}{2} - \frac{c^2\|A^{1/2}u^\tau\|_2^2}{2\varepsilon} \leqslant c^2(A^{1/2}u^\tau, A^{1/2}u_t^\tau) \text{ and } -\frac{\varepsilon\tau c^2}{2b^\tau}\|u_{tt}^\tau\|_2^2 - \frac{\tau c^2}{2\varepsilon b^\tau}\|u_t^\tau\|_2^2 \leqslant \frac{\tau c^2}{b^\tau}(u_{tt}^\tau, u_t^\tau). \tag{1.3.6}$$

Then we have

$$\begin{aligned}
E^\tau(t) &= \frac{b^\tau}{2}\left\|A^{1/2}\left(u_t^\tau + \frac{c^2}{b^\tau}u^\tau\right)\right\|_2^2 + \frac{\tau}{2}\left\|u_{tt}^\tau + \frac{c^2}{b^\tau}u_t^\tau\right\|_2^2 + \left(\frac{c^2\gamma^\tau}{2b^\tau} + \frac{\alpha}{2}\right)\|u_t^\tau\|_2^2 + \frac{c^2}{2}\|A^{1/2}u^\tau\|_2^2 \\
&= \frac{c^2}{2}\left(1+\frac{c^2}{b^\tau}\right)\|A^{1/2}u^\tau\|_2^2 + \frac{b^\tau}{2}\|A^{1/2}u_t^\tau\|_2^2 + c^2(A^{1/2}u^\tau, A^{1/2}u_t^\tau) + \frac{\tau}{2}\|u_{tt}^\tau\|_2^2 + \frac{\alpha}{2}\left(1+\frac{c^2}{b^\tau}\right)\|u_t^\tau\|_2^2 + \frac{\tau c^2}{b^\tau}(u_{tt}^\tau, u_t^\tau) \\
&\geqslant \frac{c^2}{2}\left(1+\frac{c^2}{b^\tau}-\frac{1}{\varepsilon}\right)\|A^{1/2}u^\tau\|_2^2 + \left(\frac{b^\tau}{2} - \frac{c^2\varepsilon}{2}\right)\|A^{1/2}u_t^\tau\|_2^2 + \frac{\tau}{2}\left(1 - \frac{c^2\varepsilon}{b^\tau}\right)\|u_{tt}^\tau\|_2^2 + \left(\frac{\alpha}{2}\left(1+\frac{c^2}{b^\tau}\right) - \frac{\tau c^2}{2\varepsilon b^\tau}\right)\|u_t^\tau\|_2^2,
\end{aligned}$$

Pick an $\varepsilon > 0$ such that $\frac{b^\tau}{b^\tau + c^2} < \varepsilon < \frac{b^\tau}{c^2}$ and observe that

$$\frac{b^\tau}{2} - \frac{c^2\varepsilon}{2} > 0 \text{ and } \frac{c^2}{2}\left(\frac{c^2}{2}{}_1\right) - \frac{c^2}{2\varepsilon} = \frac{c^2}{2}\left(\frac{c^2}{b^\tau} + 1 - \frac{1}{\varepsilon}\right) > 0$$

and similarly

$$\frac{\tau}{2} - \frac{\varepsilon\tau c^2}{2b^\tau} > 0 \text{ and } \frac{\alpha}{2}\left(\frac{c^2}{b^\tau}+1\right) - \frac{\tau c^2}{2\varepsilon b^\tau} = \frac{\alpha}{2}\left(\frac{c^2}{b^\tau} + 1 - \frac{1}{\varepsilon}\right) = \frac{\gamma^\tau}{2}\left(\frac{c^2}{b^\tau}+1\right) + \frac{\tau c^2}{2b^\tau}\left(\frac{c^2}{b^\tau} + 1 - \frac{1}{\varepsilon}\right) > 0.$$

Hence, picking

$$\varepsilon = \frac{2b^\tau}{b^\tau + 2c^2}$$

and continuing the lower bound estimate of $E^\tau(t)$ we have

$$\begin{aligned}
E^\tau(t) &\geqslant \frac{c^2}{2}\left(1+\frac{c^2}{b^\tau}-\frac{1}{\varepsilon}\right)\|A^{1/2}u^\tau\|_2^2 + \left(\frac{b^\tau}{2} - \frac{c^2\varepsilon}{2}\right)\|A^{1/2}u_t^\tau\|_2^2 + \frac{\tau}{2}\left(1 - \frac{c^2\varepsilon}{b^\tau}\right)\|u_{tt}^\tau\|_2^2 \\
&\geqslant \frac{c^2}{4}\|A^{1/2}u^\tau\|_2^2 + \frac{\delta^2}{(4+\tau_0)c^2 + 2\delta}\|A^{1/2}u_t^\tau\|_2^2 + \frac{\tau\delta}{2\delta + (4+\tau_0)c^2}\|u_{tt}^\tau\|_2^2 \\
&\geqslant \min\left\{\frac{c^2}{4}; \frac{\delta^2}{(4+\tau_0)c^2 + 2\delta}; \frac{\delta}{2\delta + (4+\tau_0)c^2}\right\}\|U^\tau(t)\|_{\tau,0}^2.
\end{aligned}$$

where we have used

$$\frac{c^2}{2}\left(1 + \frac{c^2}{b^\tau} - \frac{1}{\varepsilon}\right) = \frac{c^2}{4},$$

$$\frac{b^\tau}{2}-\frac{c^2\varepsilon}{2}=\frac{(b^\tau)^2}{4c^2+2b^\tau}\geqslant\frac{\delta^2}{(4+\tau_0)c^2+2\delta}$$

and similarly

$$\frac{\tau}{2}\left(1-\frac{c^2\varepsilon}{b^\tau}\right)=\frac{\tau b^\tau}{2b^\tau+4c^2}\geqslant\frac{\tau\delta}{2\delta+(4+\tau_0)c^2}.$$

Setting

$$k(\tau_0)\equiv\min\left\{\frac{c^2}{4};\frac{\delta^2}{(4+\tau_0)c^2+2\delta};\frac{\delta}{2\delta+(4+\tau_0)c^2}\right\} \tag{1.3.7}$$

gives the first part of the inequality in Lemma 1.3.1. This completes the proof of the Lemma. □

Lemma 1.3.1 along with the identity (1.3.3) imply the equi-boundedness of the family $\mathcal{F}$. Indeed,

Lemma 1.3.2. *Let $U_0\in\mathbb{H}_0$. For given k in* (1.3.7) *and K in* (1.3.5)*, there exists a constant $L_1>0$ (independent on $\tau\in(0,\tau_0)$) such that*

$$\|T^\tau(t)U_0\|_{\tau,0}^2\leqslant\frac{1}{k}E^\tau(t)\leqslant\frac{L_1K}{k}\|U_0\|_{\tau,0}^2.$$

Proof. We work with sufficiently smooth solutions guaranteed by the well-posedness-regularity theory. The final estimates are obtained via density.

Taking the L^2-inner product of (1.2.1) with u_t^τ we obtain

$$b^\tau\|A^{1/2}u_t^\tau(t)\|_2^2=\tau\|u_{tt}^\tau(t)\|_2^2-\frac{d}{dt}\left[\frac{\alpha}{2}\|u_t^\tau\|_2^2+\frac{c^2}{2}\|A^{1/2}u^\tau\|_2^2+\tau(u_{tt}^\tau,u_t^\tau)\right]. \tag{1.3.8}$$

Multiplying (1.3.8) by γ^τ with using (1.3.1) gives

$$\gamma^\tau b^\tau\|A^{1/2}u_t^\tau(t)\|_2^2=\gamma^\tau\tau\|u_{tt}^\tau(t)\|_2^2-\gamma^\tau\frac{d}{dt}E_0^\tau(t)-\gamma^\tau\tau\frac{d}{dt}(u_{tt}^\tau,u_t^\tau) \tag{1.3.9}$$

Combine (2.3.18) with the identity (1.3.3) we obtain

$$\frac{d}{dt}E_1^\tau(t)+\gamma^\tau\frac{d}{dt}E_0^\tau(t)+b^\tau\gamma^\tau\|A^{1/2}u_t^\tau(t)\|_2^2=\gamma^\tau(\tau-1)\|u_{tt}^\tau(t)\|_2^2-\gamma^\tau\tau\frac{d}{dt}(u_{tt}^\tau,u_t^\tau). \tag{1.3.10}$$

Since τ is very small and we assume $0<\tau<1$, we have $\gamma^\tau(\tau-1)\|u_{tt}^\tau(t)\|_2^2<0$ for all $t\in[0,T]$.

Then integrating w.r.t. time from 0 to t we have

$$E_1^\tau(t)+\gamma^\tau E_0^\tau(t)+\gamma^\tau b^\tau\int_0^t\|A^{1/2}u_t^\tau(s)\|_2^2ds\leqslant E_1^\tau(0)+\gamma^\tau E_0^\tau(0)+\gamma^\tau\tau(u_{tt}^\tau,u_t^\tau)\big|_0^t. \tag{1.3.11}$$

From (1.3.3), we have $E_1^\tau(t) \leqslant E_1^\tau(0)$ and notice that

$$\begin{aligned}\tau(u_{tt}^\tau, u_t^\tau)\big|_0^t &= \tau(u_{tt}^\tau(t), u_t^\tau(t)) - \tau(u_2, u_1)\\ &\leqslant \frac{\tau}{2}\|u_{tt}^\tau(t)\|_2^2 + \frac{\tau}{2}\|u_t^\tau(t)\|_2^2 + \frac{\tau}{2}\|u_2\|_2^2 + \frac{\tau}{2}\|u_1\|_2^2\\ &\leqslant \left(1 + \frac{\tau b}{\alpha c^2}\right)\left[E_1^\tau(t) + E_1^\tau(0)\right] \leqslant 2\left(\frac{\alpha c^2 + \tau b}{\alpha c^2}\right)E_1^\tau(0).\end{aligned}$$

Then

$$E_1^\tau(t) + \gamma^\tau E_0^\tau(t) + \gamma^\tau b^\tau \int_0^t \|A^{1/2}u_t^\tau(s)\|_2^2 ds \leqslant E_1^\tau(0) + \gamma^\tau E_0^\tau(0) + 2\gamma^\tau\left(\frac{\alpha c^2 + \tau b^\tau}{\alpha c^2}\right)E_1^\tau(0).$$

$$E_1^\tau(t) + \gamma^\tau E_0^\tau(t) \leqslant \max\left\{\frac{\alpha c^2 + 2\gamma^\tau(\alpha c^2 + \tau b^\tau)}{\alpha c^2}, \gamma^\tau\right\}E^\tau(0)$$

Therefore

$$E^\tau(t) \leqslant \frac{\max\left\{\dfrac{\alpha c^2 + 2\gamma^{\tau_0}\alpha c^2 + \tau b^\tau}{\alpha c^2}, \gamma^{\tau_0}\right\}}{\min\{1, \gamma^{\tau_0}\}}E^\tau(0) = L_1 E^\tau(0) \tag{1.3.12}$$

This means that the *total* energy $E^\tau(t)$ is bounded in time by the initial *total* energy. Thus the proof is obtained. □

From Lemma 1.3.2 we conclude

Corollary 1.3.3. There exists a constant $M > 0$ (independent on $\tau \in (0, \tau_0)$) such that

$$\|T^\tau(t)\|_{\mathcal{L}(\mathbb{H}_0^\tau)} \leqslant M, \forall t > 0. \tag{1.3.13}$$

Part II: Uniform (in τ) decay rates

In order to prove the uniformity of the decay rates we use the Pazy-Datko Theorem. The first step consists of showing the the map $t \mapsto \|T^\tau(t)U_0\|_{\tau,0}^2$ belongs to $L^1(0,\infty;\mathbb{H}_0)$. This is the statement of the next Lemma.

Lemma 1.3.4. *There exists $\overline{K} > 0$ independent on τ such that*

$$\int_0^\infty \|T^\tau(s)U_0\|_{\tau,0}^2 ds \leqslant \overline{K}\|U_0\|_{\tau,0}^2 < \infty. \tag{1.3.14}$$

Proof. Multiplying the identity (1.3.3) by 2τ gives

$$2\tau\frac{d}{dt}E_1^\tau(t)+2\tau\gamma^\tau\|u_{tt}^\tau(t)\|_2^2=0 \tag{1.3.15}$$

Multiply (1.3.8) by γ^τ we have

$$\gamma^\tau b^\tau\|A^{1/2}u_t^\tau(t)\|_2^2=\gamma^\tau\tau\|u_{tt}^\tau(t)\|_2^2-\gamma^\tau\frac{d}{dt}E_0^\tau(t)+\gamma^\tau\tau\frac{d}{dt}(u_{tt}^\tau,u_t^\tau) \tag{1.3.16}$$

With (1.3.15) and (1.3.16) we get

$$2\tau\frac{d}{dt}E_1^\tau(t)+\gamma^\tau\frac{d}{dt}E_0^\tau(t)+\tau\gamma^\tau\|u_{tt}^\tau(t)\|_2^2+b^\tau\gamma^\tau\|A^{1/2}u_t^\tau(t)\|_2^2=-\gamma^\tau\tau\frac{d}{dt}(u_{tt}^\tau,u_t^\tau). \tag{1.3.17}$$

Then integrating w.r.t. time from 0 to t we have

$$2\tau E_1^\tau(t)+\gamma^\tau E_0^\tau(t)+\gamma^\tau\int_0^t\left[\tau\|u_{tt}^\tau(s)\|_2^2+b^\tau\|A^{1/2}u_t^\tau(s)\|_2^2\right]ds=2\tau E_1^\tau(0)+\gamma^\tau E_0^\tau(0)-\gamma^\tau\tau(u_{tt}^\tau,u_t^\tau)\big|_0^t. \tag{1.3.18}$$

Notice that from Lemma 1.3.2 we have

$$\tau(u_{tt}^\tau,u_t^\tau)\big|_0^t\leqslant(1+\tau_0)\left(\|T^\tau(t)U_0\|_\tau^2+\|U_0\|_\tau^2\right)\leqslant\frac{(1+\tau_0)(L_1K+k)}{k}\|U_0\|_\tau^2.$$

Therefore,

$$\begin{aligned}\int_0^t\left[\tau\|u_{tt}^\tau(s)\|_2^2+b^\tau\|A^{1/2}u_t^\tau(s)\|_2^2\right]ds&\leqslant\frac{\gamma^\tau(1+\tau_0)(L_1K+k)+(2\tau_0+\gamma^\tau)L_1Kk}{k\gamma^\tau}\|U_0\|_{\tau,0}^2\\&\leqslant\frac{\gamma^{\tau_0}(1+\tau_0)(L_1K+k)+(2\tau_0+\gamma^{\tau_0})L_1Kk}{k\gamma^{\tau_0}}\|U_0\|_{\tau,0}^2=M_1\|U_0\|_{\tau,0}^2.\end{aligned} \tag{1.3.19}$$

Similarly, taking the L^2-inner product of (1.2.1) with u^τ we have

$$\frac{b^\tau}{2}\frac{d}{dt}\|A^{1/2}u^\tau\|_2^2+c^2\|A^{1/2}u^\tau(t)\|_2^2=\alpha\|u_t^\tau(t)\|_2^2+\frac{d}{dt}\left[\frac{\tau}{2}\|u_t^\tau|\|_2^2-\tau(u_{tt}^\tau,u^\tau)-\alpha(u_t^\tau,u^\tau)\right] \tag{1.3.20}$$

and integrating (1.3.20) w.r.t time from 0 to t we have

$$
\begin{aligned}
&\frac{b^\tau}{2}\|A^{1/2}u^\tau(t)\|_2^2+c^2\int_0^t \|A^{1/2}u^\tau(s)\|_2^2 ds\\
&\qquad = \frac{b^\tau}{2}\|A^{1/2}u_0\|_2^2+\alpha\int_0^t \|u_t^\tau(s)\|_2^2 ds+\Big[\frac{\tau}{2}\|u_t^\tau\|\|_2^2-\tau(u_{tt}^\tau,u^\tau)-\alpha(u_t^\tau,u^\tau)\Big]\Big|_0^t\\
&\qquad \leqslant b^\tau\|U_0\|_{\tau,0}^2+\frac{\alpha M_1}{b^\tau}\|U_0\|_{\tau,0}^2+\frac{\big[(\alpha+\tau_0)C^*+1\big](L_1K+k)}{k}\|U_0\|_{\tau,0}^2\\
&\qquad = \frac{kb^\tau+\alpha kM_1+b^\tau\big[(\alpha+\tau_0)C^*+1\big](L_1K+k)}{b^\tau k}\|U_0\|_{\tau,0}^2\\
&\qquad \leqslant \frac{k\delta+\alpha kM_1+\delta\big[(\alpha+\tau_0)C^*+1\big](L_1K+k)}{\delta k}\|U_0\|_{\tau,0}^2=M_2\|U_0\|_{\tau,0}^2. \qquad (1.3.21)
\end{aligned}
$$

Hence, by (1.3.19) and (1.3.21) we get

$$\int_0^\infty \|T^\tau(s)U_0\|_\tau^2 ds\leqslant \frac{M_1+M_2}{1+\delta+c^2}\|U_0\|_\tau^2=M_3\|U_0\|_\tau^2<\infty. \qquad (1.3.22)$$

Therefore, according to Theorem 4.1 ([40], p. 116) the rate ω can be determined as follows: We first chose a number ρ such that $0<\rho<M_3^{-1}$, then we define a number $\eta_0=M_3\rho^{-1}$ and choose another number η such that $\eta>\eta_0$. The rate is then given by

$$\omega=-\frac{1}{\eta}\log(M_3\rho)>0,$$

and is clearly independent on τ. The proof is thus completed. □

1.3.2 Proof of Theorem 1.2.3

(a) Well-posedness

The well-posedness follows directly from Theorem 1.4 in [26] and the fact that on the space $\mathbb{H}_1$ the standard sum norm $\|\cdot\|$ and $\|\cdot\|_{\tau,1}$ are equivalent for each $\tau\in(0,1]$.

(b) Equi-boundedness on $\mathbb{H}_1^\tau$ and uniform (in τ) exponential stability

Let u^τ be the solution for (1.2.1) and consider the energy functional $\mathcal{E}^\tau(t)$ defined as

$$\mathcal{E}^\tau(t)=E^\tau(t)+\|Au^\tau(t)\|_2^2. \qquad (1.3.23)$$

By Theorem 1.3 ([26]) the energy functional above (which is equivalent to the $\|(u^\tau,u_t^\tau,u_{tt}^\tau)\|$ for all $t\geqslant 0$)

decays exponentially with time [for each fixed $\tau > 0$, provided $\gamma^\tau \equiv \alpha - c^2\tau(b^\tau)^{-1} > 0$.

As in the previous theorem, the equi-boundedness of the family $\mathcal{F}_1$ follows from the lemma:

Lemma 1.3.5. *Let $U_0 \in \mathbb{H}_1$ and define $\tau_0 \equiv \inf\{c > 0; \gamma^\tau > 0$ for all $\tau \in (0,c]\} > 0$. Then there exist $k_1 = k_1(\tau_0)$ and $K_1 = K_1(\tau_0)$ such that*

$$k_1\|T^\tau(t)U_0\|^2_{\tau,1} \leqslant \mathcal{E}^\tau(t) \leqslant K_1\|T^\tau(t)U_0\|^2_{\tau,1},\ \tau \in (0,\tau_0]$$

for all $t \geqslant 0$.

The proof of Lemma 1.3.5 as well as the conclusion of the equi-boundedness in $\mathbb{H}_1$ capitalizes on the estimates already derived for $\mathbb{H}_0$. We shall focus on additional terms which need to be estimated additionally.

Lemma 1.3.5 is obtained from Lemma 1.3.1 and the estimates already derived for the space $\mathbb{H}_0^\tau$ by adding the term $||Au^\tau(t)||_2^2$. This gives the inequality stated in Lemma 1.3.5. Recall (1.3.12), we will obtain the apriori bound for $\mathcal{E}^\tau(t)$ from the relation

$$\mathcal{E}^\tau(t) = E^\tau(t) + ||Au^\tau(t)||_2^2 \leq L_1E^\tau(0) + \sup_{t\geq 0}||Au^\tau(t)||_2^2.$$

To achieve the goal, the second term needs to be accounted for. To estimate the second term we employ the equality by taking the L^2-inner product of (1.2.1) with the multiplier Au^τ and integrating w.r.t time from 0 to t. Then

$$\begin{aligned}\frac{b^\tau}{2}\|Au^\tau(t)\|_2^2 + c^2\int_0^t \|Au^\tau(s)\|_2^2 ds = \frac{b^\tau}{2}\|Au_0\|_2^2 \\ + \alpha\int_0^t \|A^{1/2}u_t^\tau(s)\|_2^2 ds + \left[\frac{\tau}{2}\|A^{1/2}u_t^\tau\|\|_2^2 - \tau(u_{tt}^\tau, Au^\tau) - \alpha(u_t^\tau, Au^\tau)\right]\Bigg|_0^t\end{aligned} \tag{1.3.24}$$

and by using the already obtained estimates in $\mathbb{H}_0^\tau$

$$\begin{aligned}\frac{(b^\tau-\varepsilon)}{2}\|Au^\tau(t)\|_2^2 + c^2\int_0^t \|Au^\tau(s)\|_2^2 ds &\leqslant b^\tau\|U_0\|^2_{\tau,1} + \alpha M3\|U_0\|^2_{\tau,1} + \frac{\left(\frac{\tau}{2}+\tau C_\varepsilon+\alpha\right)(K_1+k_1)}{k_1}\|U_0\|^2_{\tau,1} \\ &= \frac{2k_1b^\tau + 2k_1\alpha M_3 + \left[\tau(1+2C_\varepsilon)+2\alpha\right](K_1+k_1)}{2k_1}\|U_0\|^2_{\tau,1} \\ &\leqslant \frac{2k_1\delta + 2k_1\alpha M_3 + \left[\tau_0(1+2C_\varepsilon)+2\alpha\right](K_1+k_1)}{2k_1}\|U_0\|^2_{\tau,1} \\ &= \hat{C}\|U_0\|^2_{\tau,1}.\end{aligned} \tag{1.3.25}$$

Rescaling ε allows to estimate $sup_t||Au^\tau(t)||$, hence $\mathcal{E}^\tau(t)$. Then we have

$$\|T^\tau(t)U_0\|^2_{\tau,1} \leqslant \frac{1}{k_1}\mathcal{E}^\tau(t) \leqslant \left(\frac{\hat{C}K_1}{k_1}\right)\|U_0\|^2_{\tau,1},$$

from where it follows that the groups are equibounded also on $\mathbb{H}^\tau_1$, i.e.,

Corollary 1.3.6. There exists a constant $N > 0$ [independent on $\tau \in (0,\tau_0]$] such that

$$\|T^\tau(t)\|_{\mathcal{L}(\mathbb{H}^\tau_1)} \leqslant \left(\frac{\hat{C}K_1}{k_1}\right)^{1/2} = N. \tag{1.3.26}$$

As for exponential uniform decays, we shall evoke again the Pazy-Datko Theorem. This shows the existence of uniform (in τ) decay rate with existence of $\overline{K_1} > 0$ independent on τ such that

Lemma 1.3.7.

$$\int_0^\infty \|T^\tau(s)U_0\|^2_{\tau,1}ds \leqslant \overline{K_1}\|U_0\|^2_{\tau,1} < \infty. \tag{1.3.27}$$

Proof. Direct from (1.3.25), we have

$$c^2\int_0^t \|Au^\tau(s)\|^2_2 ds \leqslant N_1\|U_0\|^2_{\tau,1}. \tag{1.3.28}$$

Hence, by (1.3.22) and (1.3.24) we have

$$\int_0^\infty \|T^\tau(s)U_0\|^2_{\tau,1}ds \leqslant \frac{M_3+N_1}{c^2}\|U_0\|^2_{\tau,1} = N_2\|U_0\|^2_{\tau,1} < \infty. \tag{1.3.29}$$

Thus by Theorem 4.1 ([40], p. 116) the rate ω can be taken as we first chose a number ρ such that $0 < \rho < N_2^{-1}$, then we define a number $\eta_0 = N_2\rho^{-1}$ and choose another number η such that $\eta > \eta_0$. The rate is then given by

$$\omega = -\frac{1}{\eta}\log(N_2\rho) > 0,$$

and is clearly independent on τ. Then the proof is completed. □

1.3.3 Proof of Theorem 1.2.4

Proof of part (a)-convergence rates .

Let $x^\tau = u^\tau - u^0$ where u^τ and u^0 are the solutions for the problems (1.2.1) and (1.2.6) respectively with the same initial values for $u(t=0)$ and $u_t(t=0)$. By taking the difference of the two problems we can write a $x^\tau-$problem given by

$$\begin{cases} \alpha x_{tt}^\tau + c^2 A x^\tau + \delta A x_t^\tau = -\tau u_{ttt}^\tau - \tau c^2 A u_t^\tau \quad \text{ in } (0,T)\times\Omega, \\ x^\tau(0) = 0, x_t^\tau(0) = 0. \end{cases} \tag{1.3.30}$$

Observe that since $\mathbb{H}_2$ is $\mathbb{H}_0$ subject to the multiplication by $A^{1/2}$ where the latter leaves the dynamics invariant and $\mathcal{A}^\tau$ generates a C_0-group in $\mathbb{H}_0$, we also have $\mathcal{A}^\tau$ generating a C_0-group in $\mathbb{H}_2$.

We aim to prove that

$$\|PT^\tau(t)U_0 - T(t)PU_0\|_{\mathbb{H}_0^0}^2 \leqslant \tau C\|U_0\|_{\mathbb{H}_2}^2$$

which is the same as showing that

$$\|A^{1/2}x^\tau(t)\|_2^2 + \|A^{1/2}x_t^\tau(t)\|_2^2 \leqslant \tau C\|U_0\|_{\mathbb{H}_2}^2$$

for all $t \in [0,T]$.

Step 1: Reconstruction of $\|A^{1/2}x^\tau(t)\|_2^2$

We start by taking the L^2-inner product of $x^\tau-$equation (1.3.30) with x_t^τ. This gives

$$\alpha(x_{tt}^\tau(t), x_t^\tau(t)) + c^2(Ax^\tau(t), x_t^\tau(t)) + \delta(Ax^\tau(t)_t, x_t^\tau(t)) = -\tau(u_{ttt}^\tau(t), x_t^\tau(t)) - \tau c^2(Au_t^\tau(t), x_t^\tau(t))$$

which can be rewritten as

$$\frac{\alpha}{2}\frac{d}{dt}\|x_t^\tau\|_2^2 + \frac{c^2}{2}\frac{d}{dt}\|A^{1/2}x^\tau\|_2^2 + \delta\|A^{1/2}x_t^\tau(t)\|_2^2 = -\tau(u_{ttt}^\tau(t), x_t^\tau(t)) - \tau c^2(A^{1/2}u_t^\tau(t), A^{1/2}x_t^\tau(t)). \tag{1.3.31}$$

We now integrate (1.3.31) with respect to time from 0 to $t \in (0,T]$. This gives

$$\begin{aligned} &\frac{\alpha}{2}\|x_t^\tau(t)\|_2^2 + \frac{c^2}{2}\|A^{1/2}x^\tau(t)\|_2^2 + \delta\int_0^t \|A^{1/2}x_t^\tau(s)\|_2^2 ds \\ &\qquad \leqslant \frac{\delta}{2}\int_0^t \|A^{1/2}x_t^\tau(s)\|_2^2 ds + \frac{\tau C^*}{\delta}\int_0^t \|\sqrt{\tau}u_{ttt}^\tau(s)\|_2^2 ds + \frac{\tau^2 c^4 T}{\delta}\sup_{t\in[0,T]}\|A^{1/2}u_t^\tau(t)\|_2^2 \end{aligned}$$

where we have used the zero initial conditions of the $x^\tau-$equation and C^* is the Poincaré's constant. Then

$$\frac{\alpha}{2}\|x_t^\tau(t)\|_2^2+\frac{c^2}{2}\|A^{1/2}x^\tau(t)\|_2^2+\frac{\delta}{2}\int_0^t\|A^{1/2}x_t^\tau(s)\|_2^2ds \\ \leqslant \frac{\tau C^*}{\delta}\int_0^t\|\sqrt{\tau}u_{ttt}^\tau(s)\|_2^2ds+\frac{\tau^2c^4T}{\delta}\sup_{t\in[0,T]}\|A^{1/2}u_t^\tau(t)\|_2^2 \tag{1.3.32}$$

This was the reconstruction we needed.

Step 2: Reconstruction of $\|A^{1/2}x_t^\tau(t)\|_2^2$

We start by taking the L^2-inner product of $x^\tau-$equation (1.3.30) with Ax_t^τ. This gives

$$\alpha(x_{tt}^\tau(t),Ax_t^\tau(t))+c^2(Ax^\tau(t),Ax_t^\tau(t))+\delta(Ax_t^\tau(t),Ax_t^\tau(t))=-\tau(u_{ttt}^\tau(t),Ax_t^\tau(t))-\tau c^2(Au_t^\tau(t),Ax_t^\tau(t))$$

which can be rewritten as

$$\frac{\alpha}{2}\frac{d}{dt}\|A^{1/2}x_t^\tau\|_2^2+\frac{c^2}{2}\frac{d}{dt}\|Ax^\tau\|_2^2+\delta\|Ax_t^\tau(t)\|_2^2=-\tau(u_{ttt}(t),Ax_t^\tau(t))-\tau c^2(Au_t^\tau(t),Ax_t^\tau(t)). \tag{1.3.33}$$

We now integrate (1.3.33) with respect to time from 0 to $t\in(0,T]$. This gives

$$\frac{\alpha}{2}\|A^{1/2}x_t^\tau(t)\|_2^2+\frac{c^2}{2}\|Ax^\tau(t)\|_2^2+\delta\int_0^t\|Ax_t^\tau(s)\|_2^2ds \\ \leqslant \frac{\delta}{2}\int_0^t\|Ax_t^\tau(s)\|_2^2ds+\frac{\tau}{\delta}\int_0^t\|\sqrt{\tau}u_{ttt}^\tau(s)\|_2^2ds+\frac{\tau^2c^4T}{\delta}\sup_{t\in[0,T]}\|Au_t^\tau(t)\|_2^2 \tag{1.3.34}$$

Then

$$\frac{\alpha}{2}\|A^{1/2}x_t^\tau(t)\|_2^2+\frac{c^2}{2}\|Ax^\tau(t)\|_2^2+\frac{\delta}{2}\int_0^t\|Ax_t^\tau(s)\|_2^2ds\leqslant\frac{\tau}{\delta}\int_0^t\|\sqrt{\tau}u_{ttt}^\tau(s)\|_2^2ds+\frac{\tau^2c^4T}{\delta}\sup_{t\in[0,T]}\|Au_t^\tau(t)\|_2^2, \tag{1.3.35}$$

where we have used the zero initial conditions of the $x^\tau-$equation.

This was the reconstruction we needed.

Step 3: Uniform (in τ) bound for $\int_0^t\|\sqrt{\tau}u_{ttt}^\tau(s)\|_2^2ds$

Recall that the problem (1.2.1) is written abstractly as (see (1.2.4))

$$\begin{cases} M_\tau U_t^\tau(t) = M_\tau \mathcal{A}^\tau U^\tau(t),\ t>0, \\ U^\tau(0) = U_0 = (u_0, u_1, u_2)^T. \end{cases} \tag{1.3.36}$$

In order to estimate $\|\sqrt{\tau}u_{ttt}^\tau(t)\|_2^2$, we differentiate (1.3.36) in time, which leads us to

$$\begin{cases} M_\tau U_{tt}^\tau(t) = M_\tau \mathcal{A}^\tau U_t^\tau(t),\ t>0, \\ U_t^\tau(0) = \mathcal{A}^\tau U_0. \end{cases} \tag{1.3.37}$$

and by relabeling $V^\tau = U_t^\tau$ we can further rewrite

$$\begin{cases} M_\tau V_t^\tau(t) = M_\tau \mathcal{A}^\tau V^\tau(t),\ t>0, \\ V^\tau(0) = V_0 = \mathcal{A}^\tau U_0. \end{cases} \tag{1.3.38}$$

Now, since we are considering $U_0 \in \mathbb{H}_2$, which means $u_0, u_1 \in \mathcal{D}(A)$ and $u_2 \in \mathcal{D}(A^{1/2})$, we have $\mathcal{A}^\tau U_0 \in \mathbb{H}_1$. Therefore, by Theorem 1.2.2 and Remark 1.2.2 we get

$$\|\sqrt{\tau}u_{ttt}^\tau(t)\|_2^2 \leqslant \|T^\tau(t)V_0\|_{\tau,1}^2 \leqslant \overline{M_1}^2 \|V_0\|_{\tau,1}^2 \leqslant \frac{\overline{K}}{\tau}\|U_0\|_{\mathbb{H}_2}^2, \tag{1.3.39}$$

for all $t \in [0,T]$, where $\overline{K}$ does not depend on τ. We also obtain

$$\gamma^\tau \tau \int_0^t \|u_{ttt}^\tau(s)\|_2^2 ds \leqslant C\|U_0\|_{\mathbb{H}_2}^2, \tag{1.3.40}$$

where C does not depend on τ.

Proof. Apply (1.3.3) to time derivatives. This gives

$$\gamma \int_0^T |u_{ttt}^\tau|^2 \leq C|A^{1/2}u_{tt}^\tau(0)|^2 + |A^{1/2}u_t^\tau(0)|^2 + \tau|u_{ttt}^\tau(0)|^2$$

From the equation read of $\tau u^{\tau}_{ttt}(0)$

$$\tau |u^{\tau}_{ttt}(0)|^2 \leq \frac{C}{\tau}[|u_2|^2 + |Au_0|^2 + |Au_1|^2]$$

This gives the conclusion in (1.3.40). □

Step 4: Collecting the estimates

By adding (1.3.32) and (1.3.35) and using (1.3.40) and remark 1.2.2 we conclude

$$\|A^{1/2}x^{\tau}(t)\|_2^2 + \|A^{1/2}x^{\tau}_t(t)\|_2^2 \leqslant \tau C \|U_0\|^2_{\mathbb{H}_2}. \tag{1.3.41}$$

This finishes the proof of part (a) of Theorem 1.2.4.

Proof of part (b)-strong convergence. This amounts to showing that given $U_0 \in \mathbb{H}^{\tau}_0$ and given $\varepsilon > 0$ there exist $\delta > 0$ such that if $\tau < \delta$ then

$$\|PT^{\tau}(t)U_0 - T(t)PU_0\|^2_{\mathbb{H}^0_0} < \varepsilon$$

for all times $t > 0$. The strategy is prove that for any given fixed time T the above inequality is true and then to choose a suitable T such that for $t > T$ the energy is still bounded above by ε.

Step 1: Finite time. Let $U_0 \in \mathbb{H}_0$ and $T > 0$. Let $\varepsilon > 0$ be arbitrary.

Since $\mathbb{H}_2$ is dense in $\mathbb{H}_0$, if we define

$$\varepsilon' = \frac{\varepsilon}{2(M + M_0)},$$

we can find $U_{\varepsilon'} \in \mathbb{H}_2$ such that

$$\|U_0 - U_{\varepsilon'}\|^2_{\tau,0} < \varepsilon'.$$

Define

$$\delta = \delta_{\varepsilon} = \frac{\varepsilon}{2C\|U_{\varepsilon'}\|^2_{\mathbb{H}_2}}$$

(where C comes from Step 4 in the proof of part (a) considering $U_{\varepsilon'}$ as the initial condition) and notice that $\delta \to 0$ as $\varepsilon \to 0$.

Then, for $\tau < \delta$ we estimate by using (1.3.13),

$$\begin{aligned}
\|PT^{\tau}(t)U_0 - T(t)PU_0\|^2_{P(\mathbb{H}_0)} &\leqslant \|PT^{\tau}(t)(U_0 - U_{\varepsilon'}) - T(t)P(U_0 - U_{\varepsilon'})\|^2_{\mathbb{H}_0^0} + \|PT^{\tau}(t)U_{\varepsilon'} - T(t)PU_{\varepsilon'}\|^2_{\mathbb{H}_0^0} \\
&\leqslant \|T^{\tau}(t)\|^2_{\mathcal{L}(\mathbb{H}_0^{\tau})}\|U_0 - U_{\varepsilon'}\|^2_{\tau,0} + \|T(t)\|^2_{\mathcal{L}(\mathbb{H}_0^0)}\|U_0 - U_{\varepsilon'}\|^2_{\tau,0} + C\tau\|U_{\varepsilon'}\|^2_{\mathbb{H}_2} \\
&< \varepsilon'(M + M_0) + C\delta\|U_{\varepsilon'}\|^2_{\mathbb{H}_2} \\
&< \varepsilon.
\end{aligned}$$

Step 2: Infinite time. Integrating Equation (1.3.31) in time from s to t we have

$$\begin{aligned}
I(x) &= \frac{\alpha}{2}\|x_t^{\tau}(t)\|_2^2 + \frac{c^2}{2}\|A^{1/2}x^{\tau}(t)\|_2^2 = \left[\frac{\alpha}{2}\|x_t^{\tau}(s)\|_2^2 + \frac{c^2}{2}\|A^{1/2}x^{\tau}(s)\|_2^2\right] \\
&- \delta\int_s^t \|A^{1/2}x_t^{\tau}(\sigma)\|_2^2 d\sigma - \tau\int_s^t (u_{ttt}^{\tau}(\sigma), x_t^{\tau}(\sigma))d\sigma - \tau c^2\int_s^t (A^{1/2}u_t^{\tau}(\sigma), A^{1/2}x_t(\sigma))d\sigma \\
&\leqslant \left[\frac{\alpha}{2}\|x_t^{\tau}(s)\|_2^2 + \frac{c^2}{2}\|A^{1/2}x^{\tau}(s)\|_2^2\right] + \tau\int_s^t (u_{ttt}^{\tau}(\sigma), x_t^{\tau}(\sigma))d\sigma + \tau c^2\int_s^t (A^{1/2}u_t^{\tau}(\sigma), A^{1/2}x_t(\sigma))d\sigma \\
&\leqslant \left[\frac{\alpha}{2}\|x_t^{\tau}(s)\|_2^2 + \frac{c^2}{2}\|A^{1/2}x^{\tau}(s)\|_2^2\right] + \tau\int_s^t [\|\sqrt{\tau}u_{ttt}^{\tau}(\sigma)\|_2^2 + \|x_t^{\tau}(\sigma)\|_2^2]d\sigma \\
&+ \tau c^2\int_s^t [\|A^{1/2}u_t^{\tau}(\sigma)\|_2^2 + \|A^{1/2}x_t(\sigma)\|_2^2]d\sigma.
\end{aligned}$$

Now observe that all the terms on the right hand side above (both inside and outside the integral) are uniformly exponentially stable. Therefore, there exist positive constants L_1, L_2, a, b such that

$$\begin{aligned}
I(x) \leqslant \left[\frac{\alpha}{2}\|x_t^{\tau}(s)\|_2^2 + \frac{c^2}{2}\|A^{1/2}x^{\tau}(s)\|_2^2\right] + \tau\int_s^{\infty} [\|\sqrt{\tau}u_{ttt}^{\tau}(\sigma)\|_2^2 + \|x_t^{\tau}(\sigma)\|_2^2]d\sigma \\
+ \tau c^2\int_s^{\infty} [\|A^{1/2}u_t^{\tau}(\sigma)\|_2^2 + \|A^{1/2}x_t(\sigma)\|_2^2]d\sigma \leqslant L_1 e^{-as} + \frac{L_2}{b}e^{-bs} < \varepsilon,
\end{aligned}$$

as long as

$$s > T_{1,\varepsilon} \equiv \max\left\{-\frac{1}{a}\ln\left(\frac{\varepsilon}{2L_1}\right), -\frac{1}{b}\ln\left(\frac{\varepsilon b}{2L_2}\right)\right\}.$$

Similar estimates are valid when one integrates (1.3.33) in time from s to t. This will then gives rise to an $T_{2,\varepsilon}$.

Thus taking $T = T_{\varepsilon} \equiv \max\{T_{1,\varepsilon}, T_{2,\varepsilon}\}$ in Step 1 and combining it with control of the tail of the integral leads to the convergence in (1.2.9) uniformly for all $t \geqslant 0$. This completes the proof of part (b) of Theorem 1.2.4 .

1.3.4 Proof of Proposition 1.2.2

The uniform bounds imply, among other things, that there exist z_1 and z_2 such that

$$\begin{cases} u_t^\tau \to z_1 & \text{weakly}^* \text{ in } L^\infty(0,T;\mathcal{D}(A^{1/2})), \\ \tau^{1/2}u_{tt}^\tau \to z_2 & \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)). \end{cases} \tag{1.3.42}$$

An argument of Distributional Calculus shows that $u_{tt}^\tau \to z_{1,t}$ in $H^{-1}(\mathcal{D}(A^{1/2}))$ and therefore $\tau^{1/2}u_{tt}^\tau \to \tau^{1/2}z_{1,t} \to 0$ in $H^{-1}(\mathcal{D}(A^{1/2}))$. Uniqueness of the limit, $z_2 = 0$ then leads to the conclusion.

1.3.5 Proof of Proposition 1.2.3

Recall that $\{\mu_n\}_{n\in\mathbb{N}}$ is the set of eigenvalues of A and since A is unbounded we can assume $\mu_n \to \infty$ as $n \to \infty$. Proving the Proposition 1.2.3 amounts to the study of spectrum of $\mathcal{A}$ on the space $\mathbb{H}_0^0$.

Lemma 1.3.8.

$$\sigma_p(\mathcal{A}) = \sigma_p(\mathcal{A}^*) = \{\lambda \in \mathbb{C};\ \alpha\lambda^2 + \delta\mu_n\lambda + c^2\mu_n = 0,\ n \in \mathbb{N}\} = \left\{ \frac{-\delta\mu_n \pm \sqrt{\delta^2\mu_n^2 - 4\alpha c^2\mu_n}}{2\alpha},\ n \in \mathbb{N} \right\}.$$

Proof. Since A is a positive self-adjoint operator with compact resolvent,

$$\sigma(A) = \sigma_p(A) \subset \mathbb{R}_+^*.$$

The spectrum of A is countable and positive. So we assume the point spectrum is then a sequence (μ_n) such that $\mu_n \to +\infty$ as $n \to +\infty$. We shall consider the operator $\mathcal{A}$ acting on $\mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2})$ with the domain

$$\mathcal{D}(\mathcal{A}) = \{(u,v) \in \mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2}); c^2u + \delta v \in \mathcal{D}(A^{3/2})\}.$$

Let $\varphi = (\varphi_1, \varphi_2)^T \in \mathcal{D}(\mathcal{A})$. We seek to the describe the values of $\lambda \in \mathbb{C}$ such that

$$\mathcal{A}\varphi = \lambda\varphi. \tag{1.3.43}$$

We compute:

$$\mathcal{A}\varphi = \begin{pmatrix} 0 & I \\ -c^2\alpha^{-1}A & -\delta\alpha^{-1}A \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \varphi_2 \\ -c^2\alpha^{-1}A\varphi_1 - \delta\alpha^{-1}A\varphi_2 \end{pmatrix}.$$

Therefore the equation (1.3.43) will be satisfied if and only if

$$\varphi_2 = \lambda\varphi_1$$

and

$$c^2A\varphi_1 + \delta A\varphi_2 = -\alpha\lambda\varphi_2$$

which is the same as

$$c^2A\varphi_1 + \delta\lambda A\varphi_1 = -\alpha\lambda^2\varphi_1$$

or further

$$A\varphi_1 = \frac{-\alpha\lambda^2}{c^2+\delta\lambda}\varphi_1.$$

The last equation means that φ_1 is an eigenvector of A and because of that must be associated with some eigenvalue μ_n. Therefore, the relation between λ and μ_n can be easily derived to be the quadratic equation

$$\alpha\lambda^2 + \delta\mu_n\lambda + c^2\mu_n = 0$$

from where follows that

$$\lambda_n = -\frac{\delta\mu_n}{2\alpha} \pm \frac{\sqrt{\delta^2\mu_n^2 - 4\alpha c^2\mu_n}}{2\alpha}. \tag{1.3.44}$$

We now characterize the point spectrum of $\mathcal{A}^*$. Keeping in mind the following facts:

(i) A is self-adjoint in $H = L^2(\Omega)$.

(ii) Fractional powers preserve self-adjointness.

We begin by computing $\mathcal{A}^*$.

Let $\varphi, \psi \in \mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2})$, $\varphi = (\varphi_1, \varphi_2)^T$ and $\psi = (\psi_1, \psi_2)^T$. We have

$$\begin{aligned}
(\mathcal{A}\varphi, \psi)_{\mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2})} &= \left(\begin{pmatrix} 0 & I \\ -c^2\alpha^{-1}A & -\delta\alpha^{-1}A \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right)_{\mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2})} \\
&= \left(\begin{pmatrix} \varphi_2 \\ -c^2\alpha^{-1}A\varphi_1 - \delta\alpha^{-1}A\varphi_2 \end{pmatrix}, \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right)_{\mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2})} \\
&= (\varphi_2, \psi_1)_{\mathcal{D}(A^{1/2})} + \left(-c^2\alpha^{-1}A\varphi_1 - \delta\alpha^{-1}A\varphi_2, \psi_2\right)_{\mathcal{D}(A^{1/2})} \\
&= (\varphi_2, \psi_1)_{\mathcal{D}(A^{1/2})} + \left(\varphi_1, -c^2\alpha^{-1}A\psi_2\right)_{\mathcal{D}(A^{1/2})} + \left(\varphi_2, -\delta\alpha^{-1}A\psi_2\right)_{\mathcal{D}(A^{1/2})} \\
&= \left(\varphi_1, -c^2\alpha^{-1}A\psi_2\right)_{\mathcal{D}(A^{1/2})} + \left(\varphi_2, \psi_1 - \delta\alpha^{-1}A\psi_2\right)_{\mathcal{D}(A^{1/2})} \\
&= \left(\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \begin{pmatrix} -c^2\alpha^{-1}A\psi_2 \\ \psi_1 - \delta\alpha^{-1}A\psi_2 \end{pmatrix} \right)_{\mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2})} \\
&= \left(\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \begin{pmatrix} 0 & -c^2\alpha^{-1}A \\ I & -\delta\alpha^{-1}A \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right)_{\mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2})}.
\end{aligned}$$

Therefore

$$\mathcal{A}^* = \begin{pmatrix} 0 & -c^2\alpha^{-1}A \\ I & -\delta\alpha^{-1}A \end{pmatrix},$$

with

$$\mathcal{D}(\mathcal{A}^*) = \{(u,v) \in \mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2}), c^2u + \delta v \in \mathcal{D}(A^{3/2})\}.$$

We then find out the point spectrum of $\mathcal{A}^*$.

Let $\varphi = (\varphi_1, \varphi_2)^T \in \mathcal{D}(\mathcal{A}^*)$. We seek to the describe the values of $\lambda \in \mathbb{C}$ such that

$$\mathcal{A}^*\varphi = \lambda\varphi. \tag{1.3.45}$$

We compute:

$$\mathcal{A}^*\varphi = \begin{pmatrix} 0 & -c^2\alpha^{-1}A \\ I & -\delta\alpha^{-1}A \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} -c^2\alpha^{-1}A\varphi_2 \\ \varphi_1 - \delta\alpha^{-1}A\varphi_2 \end{pmatrix}.$$

Therefore the equation (1.3.45) will be satisfied if and only if

$$-c^2\alpha^{-1}A\varphi_2 = \lambda\varphi_1$$

and

$$\varphi_1 - \delta\alpha^{-1}A\varphi_2 = \lambda\varphi_2.$$

Decoupling gives:

$$A\varphi_2 = \frac{-\alpha\lambda^2}{c^2+\delta\lambda}\varphi_2$$

and the last equation means that φ_2 is an eigenvector of A and because of that must be associated with some eigenvalue μ_n through the quadratic equation

$$\alpha\lambda^2 + \delta\mu_n\lambda + c^2\mu_n = 0$$

which implies that

$$\sigma_p(\mathcal{A}^*) = \sigma_p(\mathcal{A}),$$

completing the proof. □

Part (a) then follows directly from Lemma 1.3.8 because we know that $\lambda \in \sigma_r(\mathcal{A})$ if and only if $\overline{\lambda} \in \sigma_p(\mathcal{A}^*)(= \sigma_p(\mathcal{A})$, in our case). However, we know that if $\overline{\lambda} \in \sigma_p(\mathcal{A})$, so is λ. Therefore, since $\sigma_p(\mathcal{A}) \cap \sigma_r(\mathcal{A}) = \emptyset$, it follows $\sigma_r(\mathcal{A}) = \emptyset$.

Now since the parameters $\delta, \alpha, c^2 > 0$ are fixed, we can see that $\sigma_p(\mathcal{A})$ is eventually real, which means that no matter how we pick those parameters, since $\mu_n \to +\infty$ as $n \to \infty$ we will always be able to find an index N such that from that index on all the eigenvalues will be real.

It is also clear to see that the point spectrum of $\mathcal{A}$ is on the left side of the complex plane. In fact, it follows from the formula (1.3.44) that in case λ_n is complex we have

$$\text{Re}(\lambda_n) = -\frac{\delta\mu_n}{2\alpha} < 0.$$

For the real ones, the "$-$" case of the formula (1.3.44) we have nothing to check because clearly $\lambda_n < 0$.

For the "$+$" case we just notice that

$$\sqrt{\delta^2\mu_n^2 - 4\alpha c^2\mu_n} < \delta\mu_n$$

and the strict inequality guarantees that $\lambda_n < 0$. Therefore, in order to describe the continuous spectrum of $\mathcal{A}$ we just analyze the limit

$$\lim_{n\to\infty} \frac{-\delta\mu_n \pm \sqrt{\delta^2\mu_n^2 - 4\alpha c^2\mu_n}}{2\alpha}.$$

Two basic limit arguments show that

$$\lim_{n\to\infty} \frac{-\delta\mu_n + \sqrt{\delta^2\mu_n^2 - 4\alpha c^2\mu_n}}{2\alpha} = -\frac{c^2}{\delta}$$

and

$$\lim_{n\to\infty} \frac{-\delta\mu_n - \sqrt{\delta^2\mu_n^2 - 4\alpha c^2\mu_n}}{2\alpha} = -\infty,$$

which implies $\sigma_c(\mathcal{A}) \supset \left\{-\frac{c^2}{\delta}\right\}$, since $\sigma(\mathcal{A})$ is closed and $\sigma_r(\mathcal{A}) = \emptyset$ with $-\delta^{-1}c^2$ not an eigenvalue of $\mathcal{A}$. In order to complete the proof of part (b) we need to show that $-\frac{c^2}{\delta}$ is *the only element* in the continuous spectrum of $\mathcal{A}$. To establish this we shall show that any $\lambda \notin \sigma_p(\mathcal{A}) \cup \left\{-\frac{c^2}{\delta}\right\}$ is in the resolvent set of $\mathcal{A}$. Let $(f,g)^T \in \mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2})$. We need to prove that there exists $(u,v)^T \in \mathcal{D}(\mathcal{A})$ such that

$$\mathcal{A}(u,v)^T - \lambda(u,v)^T = (f,g)^T.$$

After writing down explicitly the equation we obtain

$$\begin{aligned} \lambda u - v &= f \\ \alpha\lambda v + A(c^2u + \delta v) &= \alpha g \end{aligned} \tag{1.3.46}$$

which leads to solvability of

$$(c^2 + \delta\lambda)Au + \alpha\lambda^2 u = \alpha\lambda f + \alpha g + \delta A f, \ in\ [\mathcal{D}(A^{1/2})]'$$

or equivalently

$$(c^2 + \delta\lambda)u + \alpha\lambda^2 A^{-1}u = \alpha\lambda A^{-1}f + \alpha A^{-1}g + \delta f \equiv F \tag{1.3.47}$$

and further because $c^2+\delta\lambda \neq 0$,

$$u+\frac{\alpha\lambda^2}{c^2+\delta\lambda}A^{-1}u=(c^2+\delta\lambda)^{-1}F. \tag{1.3.48}$$

Note that $F\in\mathcal{D}(A^{1/2})$ and we are looking for a solution $u\in\mathcal{D}(A^{1/2})$. Since A^{-1} is compact on $L(\mathcal{D}(A^{1/2}))$, unique solvability of (1.3.48) if and only if the operator

$$I+\frac{\alpha\lambda^2}{c^2+\delta\lambda}A^{-1}$$

is injective. On the other hand, the latter takes place if and only if $-\frac{\alpha\lambda^2}{c^2+\delta\lambda}\notin\sigma_p(A)$. This is also true due to the fact that $\lambda\notin\sigma_p(\mathcal{A})$. In view of the above, we obtain $u\in\mathcal{D}(A^{1/2})$ and therefore $v=\lambda u-f\in\mathcal{D}(A^{1/2})$.

To conclude we need to assert that $(u,v)^T\in\mathcal{D}(\mathcal{A})$. For the latter we just notice that $A(c^2u+\delta v)=\alpha g-\alpha\lambda v\in\mathcal{D}(A^{1/2})$-as desired by the characterization of the domain of $\mathcal{A}$. The proof of the Proposition is thus complete.

1.3.6 Proof of Proposition 1.2.4

As in the proof of Proposition 1.2.3, proving proposition 1.2.4 relies on the analysis of point spectrum of $\mathcal{A}^\tau$ along with the asymptotics. To begin with, the point spectrum of $\mathcal{A}^\tau$ and $(\mathcal{A}^\tau)^*$ coincide and it is given by the set

$$\sigma_p(\mathcal{A}^\tau)=\sigma_p((\mathcal{A}^\tau)^*)=\{\lambda\in\mathbb{C};\ \tau\lambda^3+\alpha\lambda^2+b^\tau\mu_n\lambda+c^2\mu_n=0,\ n\in\mathbb{N}\},$$

which is a consequence of basic algebraic manipulation.

The exact same argument as for Part (a) in Proposition 1.2.3 shows Part (a) here.

Now, with empty residual spectrum we know that the points in the continuous spectrum, if any, needs to be in the approximate point spectrum. By using the exact same process as in [38] (Theorem 5.2, Part (b), (b_1) and (b_2), p.1913 and 1914) one can show that $-\frac{c^2}{b^\tau}$ is an eigenvalue of $\mathcal{A}^\tau$ in case $\gamma^\tau=0$ and a limit of eigenvalues in case $\gamma^\tau>0$. Thus $-\frac{c^2}{b^\tau}\in\sigma_c(\mathcal{A}^\tau)$ in case $\gamma^\tau>0$. We shall show now that $-\frac{c^2}{b^\tau}$ coincides with the point in continuous spectrum $\sigma_c(\mathcal{A}^\tau)$ in case $\gamma^\tau>0$. This is to say $\sigma_c(\mathcal{A}^\tau)=\left\{-\frac{c^2}{b^\tau}\right\}$. For this, it is sufficient to show that every $\lambda\in\mathbb{C}$ different from $-\frac{c^2}{b^\tau}$ and outside the point spectrum of $\mathcal{A}^\tau$ belongs to the resolvent set. We need to prove that there exist solution $(u,v,w)^T\in\mathcal{D}(\mathcal{A}^\tau)$ As before, we consider the system:

$$
\begin{aligned}
v - \lambda u &= f \in \mathcal{D}(A^{1/2}) \\
w - \lambda v &= g \in \mathcal{D}(A^{1/2}) \\
-\tau^{-1}[c^2 Au + b^\tau Av + \alpha w] - \lambda w &= h \in L^2(\Omega).
\end{aligned} \tag{1.3.49}
$$

Collecting the terms yields:

$$
(\tau^{-1}c^2 + \lambda\tau^{-1}b^\tau)Au + (\lambda^3 + \tau^{-1}\alpha\lambda^2)u = \tau^{-1}(\alpha\lambda + b^\tau)Af + \tau^{-1}\alpha Ag + \lambda^2 f + \lambda g + h,
$$

where the equation is defined on $[\mathcal{D}(A^{1/2})]'$. Since $\lambda \neq -\frac{c^2}{b^\tau}$, the above can be written as

$$
u + d(\lambda,\tau)A^{-1}u = F(f,g,h) \in \mathcal{D}(A^{1/2}), \tag{1.3.50}
$$

where

$$
d(\lambda,\tau) = \frac{\lambda^3 + \tau^{-1}\alpha\lambda^2}{\tau^{-1}c^2 + \lambda\tau^{-1}b^\tau} = \frac{\lambda^3\tau + \alpha\lambda^2}{c^2 + \lambda b^\tau}
$$

and

$$
F(f,g,h) = \tau^{-1}(\alpha\lambda + b^\tau)Af + \tau^{-1}\alpha Ag + \lambda^2 f + \lambda g + h.
$$

Since A^{-1} is compact in $L(\mathcal{D}(A^{1/2}))$, unique solvability [for $u \in \mathcal{D}(A^{1/2})$ of (1.3.50) is equivalent to the injectivity of $I + d(\lambda,\tau)A^{-1}$. The latter is equivalent to the fact that $-d(\lambda,\tau) \notin \sigma_p(A^\tau)$, which in turn is equivalent to $\mu_n + d(\lambda,\tau) \neq 0$. This last condition is guaranteed by the fact that $\lambda \notin \sigma_p(\mathcal{A}^\tau)$. Thus there exists a unique $u \in \mathcal{D}(A^{1/2})$ solving (1.3.50). Going back to (1.3.49) we obtain the improved regularity $v \in \mathcal{D}(A^{1/2}), w \in \mathcal{D}(A^{1/2})$ and also $c^2 u + b^\tau v \in \mathcal{D}(A)$. Hence $(u,v,w)^T \in \mathcal{D}(\mathcal{A}^\tau)$ as desired. The proof of equivalence $\sigma_c(\mathcal{A}^\tau) = \{-\frac{c^2}{b^\tau}\}$ is completed.

In order to complete the proof of Proposition 1.2.4 it suffices to prove the part (c). Here, the aim is to show that when $\tau \to 0$ the hyperbolic branch of the spectrum of $\mathcal{A}^\tau$ escapes to $-\infty$. For this we show that for n large, the equation

$$
\tau\lambda^3 + \alpha\lambda^2 + (b^\tau\mu_n)\lambda + c^2\mu_n = 0. \tag{1.3.51}
$$

has two complex roots whose imaginary parts approach $\pm\infty$.

The argument is as follows: define a number θ_n such that

$$\theta_n \approx -\frac{\gamma^\tau c^4}{\mu_n b^3} \text{ for } n \text{ large.}$$

We claim that

$$-\frac{c^2}{b^\tau} + \theta_n \approx \lambda_n^{0,\tau} \text{ for } n \text{ large,}$$

that is, for n large $-\frac{c^2}{b^\tau} + \theta_n$ is *almost* a root of (1.3.51).

Indeed, notice that (1.3.51) can be rewritten as

$$\tau\lambda^3 + \alpha\lambda^2 + (b^\tau\mu_n)\lambda + c^2\mu_n = \left(\lambda + \frac{c^2}{b^\tau} - \theta_n\right) q_n(\lambda) + r_n(\lambda),$$

where

$$q_n(\lambda) = \tau\lambda^2 + (\gamma^\tau + \tau\theta_n)\lambda + b^\tau\mu_n - (\gamma^\tau + \tau\theta_n)\left(\frac{c^2}{b^\tau} - \theta_n\right)$$

and

$$r_n(\lambda) = (\gamma^\tau + \tau\theta_n)\left(\frac{c^2}{b^\tau} - \theta_n\right)^2 + b^\tau\mu_n\theta_n.$$

Since $\mu_n \to \infty$ as $n \to \infty$, we have $\theta_n \approx 0$ which implies

$$r_n(\lambda) \approx \gamma^\tau\frac{c^4}{b^{2\tau}} - \gamma^\tau\frac{c^4}{b^{2\tau}} = 0 \text{ for } n \text{ large,}$$

where we have used $b^\tau\mu_n\theta_n \approx -\frac{\gamma^\tau c^4}{b^{2\tau}}$. This proves the claim made above.

As a consequence, for n large we have

$$\tau\lambda^3 + \alpha\lambda^2 + (b^\tau\mu_n)\lambda + c^2\mu_n \approx \left(\lambda + \frac{c^2}{b^\tau} - \theta_n\right)\left(\tau\lambda^2 + (\gamma^\tau + \tau\theta_n)\lambda + b^\tau\mu_n - (\gamma^\tau + \tau\theta_n)\left(\frac{c^2}{b^\tau} - \theta_n\right)\right).$$

Therefore, for n large the two other roots of the equation (which are complex) are approximately the two roots of

$$\tau\lambda^2 + (\gamma^\tau + \tau\theta_n)\lambda + b^\tau\mu_n - (\gamma^\tau + \tau\theta_n)\left(\frac{c^2}{b^\tau} - \theta_n\right).$$

Then, a basic result for quadratic equation yields

$$2\mathrm{Re}(\lambda_n^{\tau,1}) = 2\mathrm{Re}(\lambda_n^{\tau,2}) = -\frac{\gamma^\tau + \tau\theta_n}{\tau} \to -\frac{\gamma^\tau}{\tau} \text{ as } n \to \infty$$

and

$$|\mathrm{Img}(\lambda_n^{\tau,1})| = |\mathrm{Img}(\lambda_n^{\tau,1})| \to +\infty \text{ as } n \to \infty.$$

The proof of part (c) is then complete.

1.4 Appendix

Lemma 1.4.1. *(The energy identity) For all $U_0^\tau \in \mathbb{H}_0$ we have*

$$\frac{d}{dt}E_1^\tau(t) + \gamma^\tau \|u_{tt}^\tau(t)\|_2^2 = 0. \tag{1.4.1}$$

Proof. We first consider strong solutions with initial data in $\mathcal{D}(\mathcal{A}^\tau)$. This implies $U^\tau(t) \in \mathbb{H}_0$ and $u_t(t), u_{tt}(t) \in \mathcal{D}(A^{1/2})$, $u_{ttt}(t) \in H$. For these elements the following calculus is justifiable.

Notice that the expansion of $E_1^\tau(t)$ is

$$\begin{aligned} E_1^\tau(t) = {} & \frac{\tau}{2}\|u_{tt}^\tau(t)\|_2^2 + \frac{b^\tau}{2}\|A^{1/2}u_t^\tau(t)\|_2^2 + \frac{c^4}{2b^\tau}\|A^{1/2}u^\tau(t)\|_2^2 \\ & + c^2(Au_t^\tau(t), u^\tau(t)) + \frac{\tau c^2}{b^\tau}(u_{tt}^\tau(t), u_t^\tau(t)) + \frac{\alpha c^2}{2b^\tau}\|u_t^\tau(t)\|_2^2. \end{aligned} \tag{1.4.2}$$

Firs, taking the L^2-inner product of (1.2.1) with u_{tt}^τ gives

$$\frac{d}{dt}\left(\frac{\tau}{2}\|u_{tt}^\tau\|_2^2 + c^2(Au_t^\tau, u^\tau) + \frac{b^\tau}{2}\|A^{1/2}u_t^\tau\|_2^2\right) + \alpha\|u_{tt}^\tau(t)\|^2 - c^2(Au_t^\tau(t), u_t^\tau(t)) = 0. \tag{1.4.3}$$

Next similarly, taking the L^2-inner product of (1.2.1) with u_t^τ gives

$$\frac{d}{dt}\left(\tau(u_{tt}^\tau, u_t^\tau) + \frac{\alpha}{2}\|u_t^\tau\|_2^2 + \frac{c^2}{2}\|A^{1/2}u^\tau\|_2^2\right) - \tau\|u_{tt}^\tau(t)\|_2^2 + b^\tau\|A^{1/2}u_t^\tau(t)\|_2^2 = 0. \tag{1.4.4}$$

Combining (1.4.3) and $\frac{c^2}{b^\tau} \times (1.4.4)$, we get

$$\frac{d}{dt}\left[\frac{\tau}{2}\|u_{tt}^\tau\|_2^2 + \frac{b^\tau}{2}\|A^{1/2}u_t^\tau\|_2^2 + \frac{c^4}{2b^\tau}\|A^{1/2}u^\tau\|_2^2 + c^2(Au_t^\tau, u_t^\tau) + \frac{\tau c^2}{b^\tau}(u_{tt}^\tau, u_t^\tau) + \frac{\alpha c^2}{2b^\tau}\|u_t^\tau\|_2^2\right]$$

$$+(\alpha-\frac{c^2\tau}{b^\tau})\|u_{tt}^\tau(t)\|_2^2=0. \tag{1.4.5}$$

By (1.4.2) and the definition of $\gamma^\tau=\alpha-\frac{c^2\tau}{b^\tau}$, we obtain the identity

$$\frac{d}{dt}E_1^\tau(t)+\gamma^\tau\|u_{tt}^\tau(t)\|_2^2=0.$$

□

It is equivalent to say that

$$E_1^\tau(t)+\gamma^\tau\int_0^t\|u_{tt}^\tau(s)\|_2^2ds=E_1^\tau(0), \tag{1.4.6}$$

and the final conclusion is obtained by evoking density of $\mathcal{D}(\mathcal{A}^\tau)$ in $\mathbb{H}_0$.

Chapter 2

Part II - Nonlinear

2.1 Introduction

The physical problem of interest is a propagation of nonlinear waves in an acoustic environment with applications to ultrasound technology where the waves are supposed to ”hit” a given target [such as a tumor or a dense mass]. More generally, we are looking at high intensity focused waves which are used in the study of many applications such as High-Intensity-Frequency Ultrasound (HIFU) or HFU in lithotripsy, thermotherapy, ultrasound cleaning, and sonochemistry [22, 23, 25, 27]. These models are of great interest as a highly active field of research especially in medical and industrial applications. The goal of this work is a careful asymptotic analysis of the Jordan-Moore-Gibson-Thompson (JMGT) equation ,

$$\tau u_{ttt} + (1-2ku)u_{tt} - c^2\Delta u - b\Delta u_t = 2k(u_t)^2,$$

with respect to the vanishing relaxation parameter, $\tau > 0$. The interest of this problem is the present of τ. The constant τ which accounts for finite speed of propagation of the waves is known as the *time relaxation parameter*. Since the parameter $\tau > 0$ in applications is relatively small, it is essential to understand the effects of diminishing values of relaxation. This is a particularly delicate issue since the $\tau-$ dynamics is governed by a generator which is singular as $\tau \to 0$. This will provide important information on sensitivity and asymptotic analysis of the model with respect to time relaxation. The goal of this paper is to consider the vanishing parameter $\tau \to 0$ and its consequences on the resulting evolution. Consequently, we will show that the Westervelt-Kuznetsov equation,

$$(1-2ku)u_{tt} - c^2\Delta u - b\Delta u_t = 2k(u_t)^2,$$

is a limit of the JMGT equation, when the relaxation parameter vanishes and a quantitative rate of convergence of the corresponding solutions will be derived. In particular, it will provide information on how long one needs to wait for the signal to reach the target. This will help in constructing numerical schemes in order to capture a realistic behavior of nonlinear solutions. In order to investigate the nonlinear model, the estimate of variable viscosity coefficients $1-2ku$, and higher energy levels as well as a smallness of the initial data are required to avoid potential degeneracy where the model fails. At the same time, this raises numbers of mathematical difficulties at the level of estimates.

2.1.1 Physical motivation, modeling and thermal relaxation parameter

Based on recent developments in modeling of nonlinear acoustic waves [10, 19, 22, 30, 41] the physical model can be described with the main physical quantities being $\vec{v} =$ the acoustic particle velocity, $p =$ the acoustics pressure and $\rho =$ the mass density which can be decomposed as

$$\vec{v} = \vec{v}_0 + \vec{v}_\sim, \quad p = p_0 + p_\sim \quad \rho = \rho_0 + \rho_\sim.$$

The equations governing the propagation of sound in a fluid medium are;

- the Navier Stokes equation where ςv is the bulk viscosity and μv is the shear viscosity

$$\rho\big(\vec{v}_t + (\vec{v}\cdot\bigtriangledown)\vec{v}\big) + \bigtriangledown p = \mu_\text{V}\Delta\vec{v} + \big(\frac{\mu_\text{V}}{3} + \varsigma_\text{V}\big)\bigtriangledown(\bigtriangledown\cdot\vec{v}); \tag{2.1.1}$$

- the equation of continuity

$$\bigtriangledown\big(\rho\vec{v}\big) = -\rho_t; \tag{2.1.2}$$

- the state equation of relation between $p_\sim$ and $\rho_\sim$

$$\rho_\sim = \frac{p_\sim}{c^2} - \frac{1}{\rho_0 c^4}\frac{B}{2A}p_\sim^2 - \frac{k}{\rho_0 c^4}\big(\frac{1}{c_\text{V}} - \frac{1}{c_p}\big)p_{\sim t}, \tag{2.1.3}$$

where $\frac{B}{A}$ is the nonlinear parameter. We subtract the divergence of (2.1.1) from the time derivative of (2.1.2), add (2.1.3) and neglect the third and higher order terms. We arrive at the Kuznetsov equation. Then the Westervelt equation is obtained by omitting the quadratic velocity term.

$$\frac{1}{c^2}u_{tt} - \Delta u - \frac{b}{c^2}\Delta(u_t) = \frac{\beta_a}{\rho_0 c^4}(u^2)_{tt}, \tag{2.1.4}$$

where $\beta_a = 1 + \frac{B}{2A}$, u denotes the acoustic pressure fluctuations, c the speed of sound, d the diffusivity of sound, ρ_0 the mass density, and $\rho_0 v_t = -\bigtriangledown u$. Next, let u denotes the pressure. Then the potential degeneracy of (2.1.4) is expressed as

$$(1 - 2ku)u_{tt} - c^2\Delta u - \delta\Delta u_t = 2k(u_t)^2, \tag{2.1.5}$$

where $k = -\frac{\beta_a}{c^2}$ and $b = \delta + \tau c^2$. Then P.M. Jordan [21] extended these models based on the original derivation

$$\left(\frac{d}{dt} + q\right)\frac{d^2 s}{dt^2} = k\left\{(1+\alpha\beta)\frac{d}{dt} + q\right\}\frac{d^2 s}{dx^2}, \tag{2.1.6}$$

obtained by G.G. Stokes [41] in 1851 to a higher order PDE which is a *third-order* in time PDE model, where ψ denotes the acoustic velocity potential and τ is the positive constant accounting for relaxation time. We obtain

$$\tau\psi_{ttt} + \psi_{tt} - c^2\Delta\psi - b\Delta\psi = -\left(\frac{\beta_a}{c^2}(\psi_t)^2\right)_t. \tag{2.1.7}$$

The fact that the model is of third-order in time results from application of the Maxwell-Cattaneo law [17, 42, 7, 6], rather than the more traditional Fourier law in describing heat conductivity. This is to avoid the so-called infinite speed of propagation paradox -associated with the Fourier's law. Maxwell -Cattaneo law introduces a "small" parameter τ.

Synthesizing the model from the mathematical-semigroup standpoint we are dealing with the following nonlinear system: this is *third-order* in time equation which is a nonlinear (quasilinear) Partial Differential Equation (PDE) model used to describe the acoustic velocity potential in ultrasound wave propagation. It is referred as Jordan Moore Gibson Thompson (JMGT) equation

$$\tau u_{ttt} + (1-2ku)u_{tt} - c^2\Delta u - b\Delta u_t = 2k(u_t)^2, \tag{2.1.8}$$

where $u = u(t,x)$ is the acoustic pressure, k is a parameter that depends on the mass density and the constant of the nonlinearity, the parameters c and b denote the speed and diffusivity of the sound, respectively. They are required to be positive. There are boundary conditions associated with the model-say homogeneous Dirichlet boundary conditions. There are two immediate features of interest in the model as follows.

(a) The presence of the parameter $\tau > 0$ which corresponds to time relaxation and makes the problem third order in time.

(b) A possibly degenerate character due to the uncontrolled sign of $(1-2ku)$ where u is the unknown solution.

If one looks at the general theory of *linear* semigroups [18], we realize that the problem may be ill-posed even in the fully linear case. Third-order in time equations may be ill-posed in a sense that the associated semigroup may be not bounded. On the other hand, on physical grounds the parameter τ is very important - it accounts for finite speed of propagation of the waves. Thus, in order to respect physics, we must account for the presence of τ. The feature described in (b)-possible degeneracy tells us that we are dealing with a quasilinear PDE where solvability may be restricted to waves of small amplitude. However, this is consistent with the applications we have in mind.

2.1.2 Past literature and introduction to the problem

Consider the JMGT equation [27]

$$\begin{cases} \tau u^\tau_{ttt} + (1-2ku^\tau)u^\tau_{tt} + c^2Au^\tau + bAu^\tau_t = 2k(u^\tau_t)^2, \\ u^\tau(0,\cdot) = u_0, u^\tau_t(0,\cdot) = u_1, u^\tau_{tt}(0,\cdot) = u_2, \end{cases} \tag{2.1.9}$$

where τ, k, c, b, and $\tau > 0$.

The abstract equation (2.1.9) can be rewritten by using a mass operator M_τ as a first-order system of the following form

$$\begin{cases} M_\tau U^\tau_t(t) = \mathcal{A}^\tau_0(t)U^\tau(t) + F(u),\ t > 0, \\ U^\tau(0) = U_0 = (u_0, u_1, u_2)^T \end{cases} \tag{2.1.10}$$

or equivalently with $\mathcal{A}^\tau(t) = M_\tau^{-1}\mathcal{A}^\tau_0(t)$

$$\begin{cases} U^\tau_t(t) = \mathcal{A}^\tau(t)U^\tau(t) + F(u),\ t > 0, \\ U^\tau(0) = U_0 = (u_0, u_1, u_2)^T, \end{cases} \tag{2.1.11}$$

where

$$U^\tau \equiv \begin{pmatrix} u^\tau \\ u^\tau_t \\ u^\tau_{tt} \end{pmatrix};\ \mathcal{A}^\tau(t) \equiv \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\tau^{-1}c^2A & -\tau^{-1}b^\tau A & -\tau^{-1}(1-2ku^\tau) \end{pmatrix}; \tag{2.1.12}$$

and

$$M_\tau \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \tau \end{pmatrix};\ F(u) \equiv \begin{pmatrix} 0 \\ 0 \\ 2k(u^\tau_t)^2 \end{pmatrix}, \tag{2.1.13}$$

where $F(u)$ depends on the solution. The following three spaces are important for the development for our result. We define $\mathbb{H}_0, \mathbb{H}_1, \mathbb{H}_2$ as

$$\mathbb{H}_0 \equiv \mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2}) \times L^2(\Omega);$$

$$\mathbb{H}_1 \equiv \mathcal{D}(A) \times \mathcal{D}(A^{1/2}) \times L^2(\Omega);$$

$$\mathbb{H}_2 \equiv \mathcal{D}(A) \times \mathcal{D}(A) \times \mathcal{D}(A^{1/2}).$$

We employ the same notation as [5]. For each $\tau > 0$, we will consider weighted norms defined by the means of the mass operator M_τ.

$$|M_\tau^{1/2}U|^2_{\mathbb{H}_0}, |M_\tau^{1/2}U|^2_{\mathbb{H}_1}, |M_\tau^{1/2}U|^2_{\mathbb{H}_2},$$

that is,

$$|(u,v,w)|^2_{\tau,0} = |u|^2_{\mathcal{D}(A^{1/2})} + |v|^2_{\mathcal{D}(A^{1/2})} + \tau|w|^2_2 = |M_\tau^{1/2}U|^2_{\mathbb{H}_0};$$

$$|(u,v,w)|^2_{\tau,1} = |u|^2_{\mathcal{D}(A)} + |v|^2_{\mathcal{D}(A^{1/2})} + \tau|w|^2_2 = |M_\tau^{1/2}U|^2_{\mathbb{H}_1};$$

$$|(u,v,w)|^2_{\tau,2} = |u|^2_{\mathcal{D}(A)} + |v|^2_{\mathcal{D}(A)} + \tau|w|^2_{\mathcal{D}(A^{1/2})} = |M_\tau^{1/2}U|^2_{\mathbb{H}_2}.$$

We denote the standard L_2-norm by $\|.\|_{L_2} = |.|$ and other norms will be written clearly.
We shall also use the rescaled notation: $\mathbb{H}_0^\tau = M_\tau^{1/2}\mathbb{H}_0$, $\mathbb{H}_1^\tau = M_\tau^{1/2}\mathbb{H}_1$, $\mathbb{H}_2^\tau = M_\tau^{1/2}\mathbb{H}_2$ with an obvious interpretation for the composition where the elements of $\mathbb{H}_0^\tau$ coincide with the elements of $\mathbb{H}_0$ and induced topology given by $|(u,v,w)|_{\tau,0}$.

The analysis of the linear model will provide a critical step for the analysis of the nonlinear problem as the global solvability of the nonlinear problem depends on a good control of decay rates obtained for the linearized equation [27, 20, 37].

The linear model - MGT

From[5], In a general framework where $-\Delta$ with zero Dirichlet boundary conditions is replaced by a selfadjoint, positive operator A densely defined on a Hilbert space H [26, 38, 32].

We first consider the linearization of the JMGT where $F = 0$. This is the MGT model model with constant positive coefficients τ, α, b, and c,

$$
\begin{cases}
\tau u_{ttt}^\tau + \alpha u_{tt}^\tau + c^2 A u^\tau + b A u_t^\tau = 0, \\
u^\tau(0,\cdot) = u_0, u_t^\tau(0,\cdot) = u_1, u_{tt}^\tau(0,\cdot) = u_2,
\end{cases}
\tag{2.1.14}
$$

where we emphasize the dependence on τ. For the MGT, generation of linear semigroups has been studied in [26, 38]. It was shown that for any $\tau > 0, b > 0$ (2.1.14) generates a strongly continuous group on either $\mathbb{H}_0$ or $\mathbb{H}_1$. However, for $b = 0$, the generation of semigroups fails [18]. The same model with added memory was considered in [35, 34, 16, 2, 4]. Moreover, the authors in [38] showed that the linear equation generates a C_0-group in four different spaces with exponential stability provided $\gamma = \alpha - \tau c^2 b^{-1} > 0$. In case $\gamma = 0$ the system is conservative and in case $\gamma < 0$, the authors in [12] showed that (2.1.14) generates a chaotic semigroup.

The well-posedness and asymptotic behavior of the linearization is required and critical for the analysis of the nonlinear problem. We will recall pertinent results from chapter 1.

The system (2.1.14) can be rewritten abstractly as

$$
\begin{cases}
U_t^\tau(t) = \mathcal{A}^\tau U^\tau(t),\ t > 0, \\
U^\tau(0) = U_0 = (u_0, u_1, u_2)^T,
\end{cases}
\tag{2.1.15}
$$

where

$$
U^\tau \equiv \begin{pmatrix} u^\tau \\ u_t^\tau \\ u_{tt}^\tau \end{pmatrix};\ \mathcal{A}^\tau \equiv \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\tau^{-1}c^2 A & -\tau^{-1} b^\tau A & -\tau^{-1}\alpha \end{pmatrix}. \tag{2.1.16}
$$

The evolution described in (2.1.15) can be considered on several product spaces with the results depending on the space and the domain where $\mathcal{A}^\tau$ is defined. We note that $\mathcal{A}^\tau$ is singular when $\tau = 0$ which is the challenge of this work. To handle this challenge, we need to get an estimate to prevent this singularity. The natural phase spaces associated with these evolutions are

$$\mathbb{H}_0 \equiv \mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2}) \times L^2(\Omega);$$

$$\mathbb{H}_1 \equiv \mathcal{D}(A) \times \mathcal{D}(A^{1/2}) \times L^2(\Omega);$$

$$\mathbb{H}_2 = A^{1/2}\mathbb{H}_0 \equiv \mathcal{D}(A) \times \mathcal{D}(A) \times \mathcal{D}(A^{1/2}).$$

Let u^τ be the solution for (2.1.14) and consider the energy functional $E^\tau(t)$ defined as

$$E^\tau(t) = E_0^\tau(t) + E_1^\tau(t),$$

where

$$E_0^\tau(t) = \frac{\alpha}{2}|u_t^\tau|^2 + \frac{c^2}{2}|\mathcal{A}^{1/2}u^\tau|^2 \tag{2.1.17}$$

and

$$E_1^\tau(t) = \frac{b^\tau}{2}|\mathcal{A}^{1/2}(u_t^\tau + \frac{c^2}{b^\tau}u^\tau)|^2 + \frac{\tau}{2}|u_{tt}^\tau + \frac{c^2}{b^\tau}u_t^\tau|^2 + \frac{c^2\gamma^\tau}{2b^\tau}|u_t^\tau|^2. \tag{2.1.18}$$

where $\gamma^\tau = \alpha - c^2\tau(b^\tau)^{-1}$. Note that

$$E^\tau(t) \approx \tau|u_{tt}^\tau|^2 + |A^{1/2}u_t^\tau|^2 + |A^{1/2}u^\tau|^2.$$

The following results have been obtained in the first chapter, [5].

Theorem 2.1.1 (Generation of a group on $\mathbb{H}_0$ and $\mathbb{H}_2$)**.** *Let $\alpha, c, \delta > 0$. Then, for each $\tau > 0$ the operator $\mathcal{A}^\tau$ generates a C_0-group $\{T^\tau(t)\}_{t\geqslant 0}$ on $\mathbb{H}_0$ and also on $\mathbb{H}_2$.*

Theorem 2.1.2 (Generation of a group on $\mathbb{H}_1$)**.** *Let $\alpha, c, \delta > 0$. then for each $\tau > 0$ the operator $\mathcal{A}^\tau$ generates a C_0-group $\{T^\tau(t)\}_{t\geqslant 0}$ on $\mathbb{H}_1$.*

- Uniform (in τ) exponential stability

  Theorem 2.1.3 (uniform (in τ) exponential stability in $\mathbb{H}_0$ and $\mathbb{H}_2$)**.** *Consider the family $\mathcal{F} = \{T^\tau(t)\}_{\tau>0}$ of groups generated by $\mathcal{A}^\tau$ on $\mathbb{H}_0$. Assume that $\gamma^\tau \equiv \alpha - c^2\tau(b^\tau)^{-1} \geq \gamma_0 > 0$. Then, there exists $\tau_0 > 0$ and constants $M = M(\tau_0), \omega = \omega(\tau_0) > 0$ (both independent on τ) such that*

  $$|T^\tau(t)|_{\mathcal{L}(\mathbb{H}_0^\tau)} \leqslant Me^{-\omega t} \text{ for all } \tau \in (0, \tau_0] \text{ and } t \geqslant 0.$$

  Observe that since $\mathbb{H}_2$ is $\mathbb{H}_0$ subject to the multiplication by $A^{1/2}$ where the latter leaves the dynamics invariant and $\mathcal{A}^\tau$ generates a C_0-group in $\mathbb{H}_0$, we also have $\mathcal{A}^\tau$ generating a C_0-group in $\mathbb{H}_2$.

  Theorem 2.1.4 (uniform (in τ) exponential stability in $\mathbb{H}_1$)**.** *Consider the family $\mathcal{F}_1 = \{T^\tau(t)\}_{\tau>0}$ of groups generated by $\mathcal{A}^\tau$ on $\mathbb{H}_1$. Assume $\gamma^\tau > \gamma_0 > 0$. Then, there exist $\tau_0 > 0$ and constants $\overline{M}_1 =$*

$\overline{M}_1(\tau_0), \overline{\omega}_1 = \overline{\omega}_1(\tau_0) > 0$, *both independent on* τ *such that*

$$|T^{\tau}(t)|_{\mathcal{L}(\mathbb{H}_1^{\tau})} \leqslant \overline{M}_1 e^{-\overline{\omega}_1 t} \text{ for all } \tau \in (0, \tau_0] \text{ and } t \geqslant 0.$$

The above estimates allow to construct a limit semigroup $T(t)$ corresponding to $\tau = 0$. This corresponds to a second order equation, whose dynamics $T(t)$ generates an analytic semigroup on a relevant space because of the strong damping. This is in strong contrast to the $T^{\tau}(t)$ semigroups -which are of hyperbolic type and time reversible.

- Convergence of semigroups with respect to vanishing relaxation parameter $\tau \geq 0$.

Theorem 2.1.5 (rate of convergence)**.** *Let* $U_0 \in \mathbb{H}_2$. *Then there exists* $C = C(T, \tau_0)$ *such that*

$$|PT^{\tau}(t)U_0 - T(t)PU_0|^2_{\mathbb{H}_0^0} \leqslant C\tau |U_0|^2_{\tau,2},$$

uniformly for $t \in [0, T]$.

Theorem 2.1.6 (strong convergence)**.** *Let* $U_0 \in \mathbb{H}_0$. *Then the following strong convergence holds.*

$$|PT^{\tau}(t)U_0 - T(t)PU_0|_{\mathbb{H}_0^0} \to 0 \text{ as } \tau \to 0, \tag{2.1.19}$$

uniformly for all $t \geqslant 0$.

The nonlinear model - JMGT

We consider next the *nonlinear problem* called Jordan-Moore-Gibson-Thompson (JMGT). Since this is a quasilinear and a potentially degenerate PDE, one expects that the existence theory would involve some restrictions on the initial data. The nonlinear [quasilinear] model (2.1.8) has been studied in [27] where it was shown that for the initial data sufficiently small in $\mathbb{H}_1$, i.e., in a ball $B_{\mathbb{H}_1}(r)$ there exists nonlinear semigroup operator defined on $\mathbb{H}_1$ for all $t > 0$. The value of r depends on the physical parameters.

Let Ω be a bounded domain in $\mathbb{R}^n$ ($n = 2, 3$) with a C^2-boundary $\Gamma = \partial\Omega$ immersed in a resting medium and the operator A is an unbounded, positive self-adjoint and densely defined $A : \mathcal{D}(A) \subset L^2(\Omega) \to L^2(\Omega)$ defined as the Dirichlet Laplacian, i.e., $A = -\Delta$ with $\mathcal{H} = (L^2\Omega)$, $\mathcal{D}(A^{1/2}) = H_0^1(\Omega)$, and $\mathcal{D}(A) = H_0^1(\Omega) \cap$

$H^2(\Omega)$. Let $T > 0$. We consider a family of "hyperbolic" abstract third order problem as

$$\begin{cases} \tau u_{ttt}^\tau + (1 - 2ku^\tau)u_{tt}^\tau + c^2 Au^\tau + bAu_t^\tau = 2k(u_t^\tau)^2, \\ u^\tau(0,\cdot) = u_0, u_t^\tau(0,\cdot) = u_1, u_{tt}^\tau(0,\cdot) = u_2, \end{cases} \tag{2.1.20}$$

where $\tau, k, c, b > 0$ are physical constants from the derivation of the problem and the relaxation parameter τ will be of particular interest for us.

Recall the energy

$$E^\tau(t) \approx \tau |u_{tt}^\tau|^2 + |A^{1/2}u_t^\tau|^2 + |A^{1/2}u^\tau|^2.$$

For the space $\mathbb{H}_1^\tau \equiv \mathcal{D}(A) \times \mathcal{D}(A^{1/2}) \times L^2(\Omega)$, we define the corresponding lower energy $\mathcal{E}^\tau(t)$ as

$$\mathcal{E}^\tau(t) \approx E^\tau(t) + |Au^\tau(t)|^2.$$

For the space $\mathbb{H}_2^\tau \equiv \mathcal{D}(A) \times \mathcal{D}(A) \times \mathcal{D}(A^{1/2})$, we define the corresponding higher energy $\mathfrak{E}^\tau(t)$ as

$$\mathfrak{E}^\tau(t) \approx \mathcal{E}^\tau(t) + |Au_t^\tau(t)|^2 + |A^{1/2}u_{tt}^\tau|^2.$$

The analysis of the dynamics when $\tau > 0$ has been presented in [27]. The following results have been obtained.

- Well-posedness

  Theorem 2.1.7 (local (in time) well-posedness)**.** *Let $T > 0$ be arbitrary and $\gamma^* = 1 - \frac{c^2\tau}{b}$. There exists $\rho_T(\gamma^*) > 0$ such that if the initial data U_0 satisfies $\mathcal{E}(0) \leqslant \rho_T(\gamma^*)$, then there exists a unique solution and*

  $$\mathcal{E}(t) < \infty,$$

  for all $t \in [0, T]$.

  Theorem 2.1.8 (global (in time) well-posedness)**.** *Let $\gamma^* > 0$ and $C > 0$. Then there exists $\rho_C(\gamma^*) > 0$ such that if the initial data U_0 satisfies $\mathcal{E}(0) \leqslant \rho_C(\gamma^*)$, then there exists a unique solution and*

  $$\mathcal{E}(t) \leqslant C,$$

for all $t > 0$.

These results show in particular that there exists a nonlinear semigroup on a small ball (small radius) in $\mathbb{H}_1^\tau$. The proof are given in Theorem 1.4 in [27]. Problems which are open and will be obtained in this work are the following

- Uniform (in τ) exponential decay of nonlinear semigroup.
- Convergence of semigroups with respect to vanishing relaxation parameter $\tau \geq 0$

The difficulty of this work is the operator $\mathcal{A}^\tau$ (2.1.12) is singular when $\tau \to 0$. To our best knowledge, as in the linear case, this is the first work that takes into consideration asymptotic properties of the JMGT dynamics with respect to the vanishing relaxation parameter. The limiting evolution changes the character from a hyperbolic group to a parabolic semigroup. This change is expected to be reflected by the asymptotic properties of the spectrum and quantitative estimates for the corresponding evolutions.

In conclusion, the model of interest is (2.1.8) studied within the context of quasilinear theory with particular emphasis on the analysis of the solutions as varying with respect to small τ. This formulates our research as follows:

1. Develop existence, uniqueness and stability theory - within a proper functional analytic PDE setting, for solutions corresponding to (2.1.8) for each τ with quantitative analysis of dependence on τ.
2. Develop an asymptotic analysis of said solutions when the limiting parameter τ tends to zero. In particular, determine the limiting behavior and show how fast solutions are converging to this limit state. This particular aspect is particularly important since the values of time relaxation parameter are small and it is critically important to detect potential degeneracies of solutions of the model in this vanishing-limiting region.

In order to accomplish goal #1, we shall use PDE - energy/multipliers based techniques in order to construct an adequate quasilinear theory and trace dependence on τ. The final conclusion obtained by barrier's method, see appendix.

In order to accomplish goal #2, we shall use the estimates generated by step 1 in order to develop an appropriate nonlinear semigroup theory adequate for the characterization of our solutions and their limits. We note that nonlinear version of Trotter Kato framework does not apply due to singularity of the generator and the resolvent.

Notation: $(u,v) \equiv (u,v)_{L_2}$ and $|u|^2 \equiv \|u\|_{L_2}^2$ and otherwise it will be written specifically.

2.2 Main Results

Our first result is existence and uniqueness of solutions defined on space $\mathbb{H}_2$ for sufficiently small initial data U_0 in $\mathbb{H}_2$. The next question is asymptotic behavior with respect to the relaxation parameter τ.

- Uniform (in τ) exponential decay in low topology $\mathbb{H}_1^\tau$

  Theorem 2.2.1. *Let $U_0 \in \mathbb{H}_1$ be sufficiently small i.e., $\|U_0\|_{\mathbb{H}_1} \leqslant r$ for all $r > 0$. Then there exists $N_1 = N_1(r)$and $\omega_1 > 0$ (independent on $\tau \in (0,\tau_0)$) such that*

$$\mathcal{E}^\tau(t) \leqslant N_1(r)e^{-\omega_1 t},$$

  for all $t > 0$ and $U = (u,u_t,u_{tt})$ satisfies (2.1.20).

- Uniform (in τ) exponential decay in high topology $\mathbb{H}_2^\tau$.

  Theorem 2.2.2. *Let $U_0 \in \mathbb{H}_2$ be sufficiently small i.e., $\|U_0\|_{\mathbb{H}_2} \leqslant r$ for all $r > 0$. Then there exists $N_2 = N_2(r)$and $\omega_2 > 0$ (independent on $\tau \in (0,\tau_0)$) such that*

$$\mathfrak{E}^\tau(t) \leqslant N_2(r)e^{-\omega_2 t},$$

  for all $t > 0$ and $U = (u,u_t,u_{tt})$ satisfies (2.1.20).

  It leads us to the exponential stability as well as to the existence of an uniform (in τ) decay rate. These estimates allow to construct a limit nonlinear semigroup $T(t)$ corresponding to $\tau = 0$.

- Rate of Convergence/Strong Convergence

  The final stage is to show that the Westervelt-Kuznetsov equation (2.1.5), see [24] is a limit of the JMGT equation, when the relaxation parameter vanishes $(\tau \to 0)$,

$$\begin{cases} (1-2ku^0)u^0_{tt} + c^2Au^0 + bAu^0_t = 2k(u^0_t)^2, \\ u^0(0,\cdot) = u_0, u^0_t(0,\cdot) = u_1. \end{cases} \tag{2.2.1}$$

Let $x^\tau = u^\tau - u^0$ where u^τ and u^0 are the solutions for the problems (2.1.9) and (2.2.1) respectively with the same initial values for $u(t=0)$ and $u_t(t=0)$. Taking the difference of these two problems allows us to obtain a quantitative rate of convergence of the corresponding solutions with smooth data as well as strong convergence when $\tau \to 0$. If we take sufficiently small initial data, then the following holds.

Theorem 2.2.3. *(a)* ***Rate of convergence:*** *Let $U_0 \in \mathbb{H}_2$ be sufficiently small then*

$$|P(U^\tau(t,U_0)) - U^\tau(t,PU_0)|^2_{\mathbb{H}_1 \times L_2} \leqslant \tau \|U_0\|^2_{\mathbb{H}_2}$$

uniformly for $t \in [0,T]$.

(b) ***Strong convergence:*** *Let $U_0 \in \mathbb{H}_2$ be sufficiently small. Then the following strong convergence holds on $[0,\infty)$.*

$$|P(U^\tau(t,U_0)) - U^\tau(t,PU_0)|^2_{\mathbb{H}_1 \times L_2} \to 0 \text{ as } \tau \to 0$$

uniformly for all $t \geqslant 0$.

2.3 Proof

2.3.1 Proof of Theorem (2.2.1) - Uniform (in τ) exponential stability in $\mathbb{H}_1^\tau$

Let u^τ be the solution for (2.1.9) and consider the energy functional $\mathcal{E}^\tau(t)$ defined as

$$\mathcal{E}^\tau(t) \approx E^\tau(t) + |Au^\tau(t)|^2. \tag{2.3.1}$$

From (2.1.9), we rewrite as

$$\tau u^\tau_{ttt} + u^\tau_{tt} + c^2 A u^\tau + bAu^\tau_t = 2ku^\tau u^\tau_{tt} + 2k(u^\tau_t)^2. \tag{2.3.2}$$

Let u^τ be the solution for (2.1.9) and consider the energy functional $E^\tau(t)$ defined as

$$E^\tau(t) = E^\tau_0(t) + E^\tau_1(t),$$

where

$$E^\tau_1(t) = \frac{b}{2}|A^{1/2}(u^\tau_t + \frac{c^2}{b}u^\tau)|^2 + \frac{\tau}{2}|u^\tau_{tt} + \frac{c^2}{b}u^\tau_t|^2 + \frac{c^2}{2b}(\gamma^\tau(t)u^\tau_t, u^\tau_t). \tag{2.3.3}$$

$$E_0^\tau(t) = \frac{1}{2}|\alpha(t)u_t^\tau|^2 + \frac{c^2}{2}|A^{1/2}u^\tau|^2. \tag{2.3.4}$$

Notice that

$$E^\tau(t) \approx \tau|u_{tt}^\tau|^2 + |A^{1/2}u_t^\tau|^2 + |A^{1/2}u^\tau|^2.$$

For $\gamma^\tau = 1 - \dfrac{c^2\tau}{b}$, the expansion of (2.3.3) is

$$E_1^\tau(t) = \frac{\tau}{2}|u_{tt}^\tau|^2 + \frac{b}{2}|A^{1/2}u_t^\tau|^2 + \frac{c^4}{2b}|A^{1/2}u^\tau|^2 + c^2(\mathcal{A}u_t^\tau, u^\tau) + \frac{\tau c^2}{b}(u_{tt}^\tau, u_t^\tau) + \frac{c^2}{2b}|u_t^\tau|^2. \tag{2.3.5}$$

The first step is establishing the following energy identity.

Lemma 2.3.1. *The following identity holds*

$$\frac{d}{dt}E_1^\tau(t) + \gamma^\tau|u_{tt}^\tau|^2 = (2ku^\tau u_{tt}^\tau, u_{tt}^\tau + \frac{c^2}{b}u_t^\tau) + (2k(u_t^\tau)^2, u_{tt}^\tau + \frac{c^2}{b}u_t^\tau). \tag{2.3.6}$$

Proof. **Step 1:** Taking the L^2- inner product of (2.3.2) with u_{tt}^τ gives

$$\tau(u_{ttt}^\tau, u_{tt}^\tau) + (u_{tt}^\tau, u_{tt}^\tau) + c^2(Au^\tau, u_{tt}^\tau) + b(Au_t^\tau, u_{tt}^\tau) = (2ku^\tau u_{tt}^\tau, u_{tt}^\tau) + (2k(u_t^\tau)^2, u_{tt}^\tau). \tag{2.3.7}$$

Then

$$\frac{d}{dt}\left[\frac{\tau}{2}|u_{tt}^\tau|^2 + c^2(Au^\tau, u_t^\tau) + \frac{b}{2}|A^{1/2}u_t^\tau|^2\right] + |u_{tt}^\tau|^2 - c^2|A^{1/2}u_t^\tau|^2 = (2ku^\tau u_{tt}^\tau, u_{tt}^\tau) + (2k(u_t^\tau)^2, u_{tt}^\tau). \tag{2.3.8}$$

Step 2: Similarly, we take the L^2- inner product of (2.3.2) with u_t^τ then

$$\tau(u_{ttt}^\tau, u_t^\tau) + (u_{tt}^\tau, u_t^\tau) + c^2(Au^\tau, u_t^\tau) + b(Au_t^\tau, u_t^\tau) = (2ku^\tau u_{tt}^\tau, u_t^\tau) + (2k(u_t^\tau)^2, u_t^\tau). \tag{2.3.9}$$

Then

$$\frac{d}{dt}\left[\tau(u_{tt}^\tau, u_t^\tau) + \frac{1}{2}|u_t^\tau|^2 + \frac{c^2}{2}|A^{1/2}u^\tau|^2\right] - \tau|u_{tt}^\tau|^2 + b|A^{1/2}u_t^\tau|^2 = (2ku^\tau u_{tt}^\tau, u_t^\tau) + (2k(u_t^\tau)^2, u_t^\tau). \tag{2.3.10}$$

Combining (2.3.8) and $\frac{c^2}{b}$(2.3.10) gives

$$\frac{d}{dt}\Big[\frac{\tau}{2}|u_{tt}^\tau|^2+\frac{b}{2}|A^{1/2}u_t^\tau|^2+c^2(Au^\tau,u_t^\tau)+\frac{\tau c^2}{b}(u_{tt}^\tau,u_t^\tau)+\frac{c^2}{2b}|u_t^\tau|^2+\frac{c^4}{2b}|A^{1/2}u^\tau|^2\Big]+(1-\frac{\tau c^2}{b})|u_{tt}^\tau|^2$$
$$=(2ku^\tau u_{tt}^\tau,u_{tt}^\tau+\frac{c^2}{b}u_t^\tau)+(2k(u_t^\tau)^2,u_{tt}^\tau+\frac{c^2}{b}u_t^\tau). \tag{2.3.11}$$

Then by (2.3.5) and $\gamma^\tau=1-\frac{c^2\tau}{b}$, we obtain

$$\frac{d}{dt}E_1^\tau(t)+\gamma^\tau|u_{tt}^\tau|^2=(2ku^\tau u_{tt}^\tau,u_{tt}^\tau+\frac{c^2}{b}u_t^\tau)+(2k(u_t^\tau)^2,u_{tt}^\tau+\frac{c^2}{b}u_t^\tau). \tag{2.3.12}$$

Thus we obtain the energy identity. □

Notice that from (2.3.12), it implies that

$$E_1^\tau(t)+\gamma^\tau\int_0^t|u_{tt}^\tau(\sigma)|^2d\sigma$$
$$=E_1^\tau(0)+\int_0^t[(2ku^\tau(\sigma)u_{tt}^\tau(\sigma),u_{tt}^\tau(\sigma)+\frac{c^2}{b}u_t^\tau(\sigma))+(2k(u_t^\tau(\sigma))^2,u_{tt}^\tau(\sigma)+\frac{c^2}{b}u_t^\tau(\sigma))]d\sigma. \tag{2.3.13}$$

By Sobolev's embeddings, we have

$$|u_t^\tau|_{L_4}\leqslant|u_t^\tau|_{L_2}^{1/4}|u_t^\tau|_{H^1}^{3/4}.$$

$$|(u_t^\tau)^2|^2=|u_t^\tau|_{L_4}^4\leqslant|u_t^\tau|_{L_2}|u_t^\tau|_{H^1}|u_t^\tau|_{H^1}^2, \tag{2.3.14}$$
$$\leqslant|u_t^\tau|_{L_2}|u_t^\tau|_{H^1}|A^{1/2}u_t^\tau|^2. \tag{2.3.15}$$

Consider the RHS terms of (2.3.13)

$$(2ku^\tau u_{tt}^\tau,u_{tt}^\tau)\leqslant 4k^2C_\varepsilon|u^\tau|_{L_\infty}|u_{tt}^\tau|^2+\varepsilon|u_{tt}^\tau|^2.$$

$$(2ku^\tau u_{tt}^\tau,\frac{c^2}{b}u_t^\tau)\leqslant 4k^2C_\varepsilon|u^\tau|_{L_\infty}|u_{tt}^\tau|^2+\varepsilon\frac{c^2C^*}{b}|A^{1/2}u_t^\tau|^2.$$

$$(2k(u_t^\tau)^2,u_{tt}^\tau)\leqslant 4k^2C_\varepsilon|u_t^\tau|_{L_2}|u_t^\tau|_{H^1}|A^{1/2}u_t^\tau|^2+\varepsilon|u_{tt}^\tau|^2.$$

$$(2k(u_t^\tau)^2, \frac{c^2}{b}u_t^\tau) \leqslant 4k^2C_\varepsilon|u_t^\tau|_{L_2}|u_t^\tau|_{H^1}|A^{1/2}u_t^\tau|^2 + \varepsilon\frac{c^2C^*}{b}|A^{1/2}u_t^\tau|^2.$$

Then we get

$$\begin{aligned}
&\int_0^t [(2ku^\tau(\sigma)u_{tt}^\tau(\sigma), u_{tt}^\tau(\sigma) + \frac{c^2}{b}u_t^\tau(\sigma)) + (2k(u_t^\tau(\sigma))^2, u_{tt}^\tau(\sigma) + \frac{c^2}{b}u_t^\tau(\sigma))]d\sigma \\
&\quad \leqslant \int_0^t \big(8k^2C_\varepsilon|u^\tau|_{L_\infty} + 2\varepsilon\big)|u_{tt}^\tau(\sigma)|^2 d\sigma + \int_0^t \big(8k^2C_\varepsilon|u_t^\tau|_{L_2}|u_t^\tau|_{H^1} + \varepsilon\frac{2c^2C^*}{b}\big)|A^{1/2}u_t^\tau(\sigma)|^2 d\sigma.
\end{aligned}$$

Thus from (2.3.13), we obtain

$$\begin{aligned}
E_1^\tau(t) + \int_0^t \left[\gamma^\tau - \big(8k^2C_\varepsilon|u^\tau|_{L_\infty} + 2\varepsilon\big)\right]|u_{tt}^\tau(\sigma)|^2 d\sigma& \\
\leqslant E_1^\tau(0) + \int_0^t \left[8k^2C_\varepsilon|u_t^\tau|_{L_2}|u_t^\tau|_{H^1} + \varepsilon\frac{2c^2C^*}{b}\right]|A^{1/2}u_t^\tau(\sigma)|^2 d\sigma.&
\end{aligned} \tag{2.3.16}$$

We work with sufficiently smooth solutions guaranteed by the well-posedness-regularity theory. Now we want to derive the estimate on the total energy $E^\tau(t) = E_0^\tau(t) + E_1^\tau(t)$ then we need to construct $\frac{d}{dt}E_1^\tau(t) + \frac{d}{dt}E_0^\tau(t)$.

From (2.3.10) with the identity (2.3.4), we have

$$\frac{d}{dt}\tau(u_{tt}^\tau, u_t^\tau) + \frac{d}{dt}E_0^\tau(t) - \tau|u_{tt}^\tau|^2 + b|A^{1/2}u_t^\tau|^2 = (2ku^\tau u_{tt}^\tau, u_t^\tau) + (2k(u_t^\tau)^2, u_t^\tau). \tag{2.3.17}$$

Combining the identity (2.3.12) and (2.3.17) gives

$$\begin{aligned}
&\frac{d}{dt}E_1^\tau(t) + \frac{d}{dt}E_0^\tau(t) + (\gamma^\tau - \tau)|u_{tt}^\tau|^2 + b|A^{1/2}u_t^\tau|^2 \\
&\quad = -\tau\frac{d}{dt}(u_{tt}^\tau, u_t^\tau) + (2k(u_t^\tau)^2, u_{tt}^\tau + (\frac{c^2}{b}+1)u_t^\tau) + (2ku^\tau u_{tt}^\tau, u_{tt}^\tau + (\frac{c^2}{b}+1)u_t^\tau).
\end{aligned} \tag{2.3.18}$$

Integrating w.r.t time from 0 to t leads to

$$\begin{aligned}
&E_1^\tau(t) + E_0^\tau(t) + (\gamma^\tau - \tau)\int_0^t |u_{tt}^\tau(\sigma)|^2 d\sigma + b\int_0^t |A^{1/2}u_t^\tau(\sigma)|^2 d\sigma \\
&\quad \leqslant E_1^\tau(0) + E_0^\tau(0) + \tau(u_{tt}^\tau, u_t^\tau)\Big|_0^t \\
&\qquad + \int_0^t \big[(2k(u_t^\tau(\sigma))^2, u_{tt}^\tau(\sigma) + (\frac{c^2}{b}+1)u_t^\tau(\sigma)) + (2ku^\tau(\sigma)u_{tt}^\tau(\sigma), u_{tt}^\tau(\sigma) + (\frac{c^2}{b}+1)u_t^\tau(\sigma))\big]d\sigma,
\end{aligned}$$

$$
\begin{aligned}
&\leqslant E_1^\tau(0)+E_0^\tau(0)+(1+\frac{\tau b}{c^2})(E_1^\tau(t)+E_1^\tau(0))\\
&\quad+\int_0^t\big[(2k(u_t^\tau(\sigma))^2,u_{tt}^\tau(\sigma)+(\frac{c^2}{b}+1)u_t^\tau(\sigma))+(2ku^\tau(\sigma)u_{tt}^\tau(\sigma),u_{tt}^\tau(\sigma)+(\frac{c^2}{b}+1)u_t^\tau(\sigma))\big]d\sigma,\\
&\leqslant (2+\frac{\tau b}{c^2})E_1^\tau(0)+E_0^\tau(0)+(1+\frac{\tau b}{c^2})E_1^\tau(t)\\
&\quad+\int_0^t\big[(2k(u_t^\tau(\sigma))^2,u_{tt}^\tau(\sigma)+(\frac{c^2}{b}+1)u_t^\tau(\sigma))+(2ku^\tau(\sigma)u_{tt}^\tau(\sigma),u_{tt}^\tau(\sigma)+(\frac{c^2}{b}+1)u_t^\tau(\sigma))\big]d\sigma.
\end{aligned}
\tag{2.3.19}
$$

Note that From (2.3.13), we have

$$
(1+\frac{\tau b}{c^2})E_1^\tau(t)\leqslant(1+\frac{\tau b}{c^2})E_1^\tau(0)+(1+\frac{\tau b}{c^2})\int_0^t\left[8k^2C_\varepsilon|u_t^\tau|_{L_2}|u_t^\tau|_{H^1}+\varepsilon\frac{2c^2C^*}{b}\right]|A^{1/2}u_t^\tau(\sigma)|^2d\sigma,\tag{2.3.20}
$$

and

$$
\begin{aligned}
&\int_0^t\left[(2k(u_t^\tau(\sigma))^2,u_{tt}^\tau(\sigma)+(\frac{c^2}{b}+1)u_t^\tau(\sigma))+(2ku^\tau(\sigma)u_{tt}^\tau(\sigma),u_{tt}^\tau(\sigma)+(\frac{c^2}{b}+1)u_t^\tau(\sigma))\right]d\sigma\\
&\quad\leqslant\int_0^t\left[8k^2C_\varepsilon|u^\tau|_{L_\infty}+2\varepsilon\right]|u_{tt}^\tau(\sigma)|^2d\sigma+\int_0^t\left[8k^2C_\varepsilon|u_t^\tau|_{L_2}|u_t^\tau|_{H^1}+2\varepsilon C^*(\frac{c^2}{b}+1)\right]|A^{1/2}u_t^\tau(\sigma)|^2d\sigma.
\end{aligned}
\tag{2.3.21}
$$

Then from (2.3.19), we obtain

$$
\begin{aligned}
&E_1^\tau(t)+E_0^\tau(t)+(\gamma^\tau-\tau)\int_0^t|u_{tt}^\tau(\sigma)|^2d\sigma+b\int_0^t|A^{1/2}u_t^\tau(\sigma)|^2d\sigma\\
&\quad\leqslant(3+\frac{2\tau b}{c^2})E_1^\tau(0)+E_0^\tau(0)+\int_0^t\left[8k^2C_\varepsilon|u^\tau|_{L_\infty}+2\varepsilon\right]|u_{tt}^\tau(\sigma)|^2d\sigma\\
&\qquad+\int_0^t\left[(2+\frac{\tau b}{c^2})8k^2C_\varepsilon|u_t^\tau|_{L_2}|u_t^\tau|_{H^1}+2\varepsilon C^*(\frac{2c^2}{b}+1)\right]|A^{1/2}u_t^\tau(\sigma)|^2d\sigma.
\end{aligned}
\tag{2.3.22}
$$

Thus we arrive at

$$
\begin{aligned}
&E^\tau(t)+\int_0^t\left[(\gamma^\tau-\tau-8k^2C_\varepsilon|u^\tau|_{L_\infty}-2\varepsilon)\right]|u_{tt}^\tau(\sigma)|^2d\sigma\\
&\quad+\int_0^t\left[b-8k^2C_\varepsilon|u_t^\tau|_{L_2}|u_t^\tau|_{H^1}-2\varepsilon C^*(\frac{c^2}{b}+1)\right]|A^{1/2}u_t^\tau(\sigma)|^2d\sigma\\
&\qquad\leqslant(3+\frac{2\tau b}{c^2})E^\tau(0).
\end{aligned}
\tag{2.3.23}
$$

Step 3: Taking the L^2- inner product of (2.3.2) with u^τ gives

$$\tau(u^\tau_{ttt},u^\tau)+(u^\tau_{tt},u^\tau)+c^2(Au^\tau,u^\tau)+b(Au^\tau_t,u^\tau)=(2ku^\tau u^\tau_{tt},u^\tau)+(2k(u^\tau_t)^2,u^\tau). \tag{2.3.24}$$

Then

$$\frac{b}{2}\frac{d}{dt}|A^{1/2}u^\tau|^2+c^2|A^{1/2}u^\tau|^2=\frac{d}{dt}\left[\frac{\tau}{2}|u^\tau_t|^2-\tau(u^\tau_{tt},u^\tau)-(u^\tau_t,u^\tau)\right]+|u^\tau_t|^2+(2ku^\tau u^\tau_{tt},u^\tau)+(2k(u^\tau_t)^2,u^\tau). \tag{2.3.25}$$

Integrating w.r.t time from 0 to t gives

$$\begin{aligned}\frac{b}{2}|A^{1/2}u^\tau(t)|^2+c^2\int_0^t|A^{1/2}u^\tau(\sigma)|^2d\sigma=&\frac{b}{2}|A^{1/2}u^\tau(0)|^2+\left[\frac{\tau}{2}|u^\tau_t|^2-\tau(u^\tau_{tt},u^\tau)-(u^\tau_t,u^\tau)\right]_0^t+\int_0^t|u^\tau_t(\sigma)|^2d\sigma\\ &+\int_0^t\left[(2ku^\tau(\sigma)u^\tau_{tt}(\sigma),u^\tau(\sigma))+(2k(u^\tau_t(\sigma))^2,u^\tau(\sigma))\right]d\sigma. \end{aligned}\tag{2.3.26}$$

We multiply (2.3.26) with $\delta>0$ then

$$\begin{aligned}&\frac{\delta b}{2}|A^{1/2}u^\tau(t)|^2+\delta c^2\int_0^t|A^{1/2}u^\tau(\sigma)|^2d\sigma\\ &\quad=\frac{\delta b}{2}|A^{1/2}u^\tau(0)|^2+\delta\left[\frac{\tau}{2}|u^\tau_t|^2-\tau(u^\tau_{tt},u^\tau)-(u^\tau_t,u^\tau)\right]_0^t+\delta\int_0^t|u^\tau_t(\sigma)|^2d\sigma\\ &\quad+\delta\int_0^t\left[(2ku^\tau(\sigma)u^\tau_{tt}(\sigma),u^\tau(\sigma))+(2k(u^\tau_t(\sigma))^2,u^\tau(\sigma))\right]d\sigma. \end{aligned}\tag{2.3.27}$$

Consider

$$\begin{aligned}&\delta\left[\frac{\tau}{2}|u^\tau_t|^2-\tau(u^\tau_{tt},u^\tau)-(u^\tau_t,u^\tau)\right]_0^t+\delta\int_0^t|u^\tau_t(\sigma)|^2d\sigma\\ &\quad\leqslant\delta\left[\frac{\tau b}{c^2}+2C_\varepsilon(1+\tau)\right](E^\tau_1(t)+E^\tau_1(0))+\delta C^*\int_0^t|A^{1/2}u^\tau_t(\sigma)|^2d\sigma,\\ &\quad\leqslant\delta\left[\frac{\tau b}{c^2}+2C_\varepsilon(1+\tau)\right](E^\tau(t)+E^\tau(0))+\delta C^*\int_0^t|A^{1/2}u^\tau_t(\sigma)|^2d\sigma,\\ &\quad\leqslant\delta(\frac{\tau b}{c^2}+2C_\varepsilon(1+\tau))(4+\frac{2\tau b}{c^2})E^\tau(0)+\delta C^*\int_0^t|A^{1/2}u^\tau_t(\sigma)|^2d\sigma. \end{aligned}\tag{2.3.28}$$

where from (2.3.23), we have $E^\tau(t)\leqslant(3+\frac{2\tau b}{c^2})E^\tau(0)$ and

$$\delta\int_0^t\left[(2ku^\tau(\sigma)u^\tau_{tt}(\sigma),u^\tau(\sigma))+(2k(u^\tau_t(\sigma))^2,u^\tau(\sigma))\right]d\sigma$$

$$\leqslant 2\delta\varepsilon C^* \int_0^t |A^{1/2}u^\tau(\sigma)|^2 d\sigma + \int_0^t \left[4\delta k^2 C_\varepsilon |u^\tau|_{L_\infty}\right] |u_{tt}^\tau|^2 d\sigma$$
$$+ \int_0^t \left[4\delta k^2 C_\varepsilon |u_t^\tau|_{L_2} |u_t^\tau|_{H^1}\right] |A^{1/2}u_t^\tau(\sigma)|^2 d\sigma. \qquad (2.3.29)$$

Thus from (2.3.27), we have

$$\frac{\delta b}{2}|A^{1/2}u^\tau(t)|^2 + \delta[c^2 - 2\varepsilon C^*] \int_0^t |A^{1/2}u^\tau(\sigma)|^2 d\sigma$$
$$\leqslant \delta[(\frac{\tau b}{c^2} + 2C_\varepsilon(1+\tau))(4 + \frac{2\tau b}{c^2}) + \frac{b}{2}]E^\tau(0) + \int_0^t \left[4\delta k^2 C_\varepsilon |u^\tau|_{L_\infty}\right] |u_{tt}^\tau(\sigma)|^2 d\sigma$$
$$+ \int_0^t \left[4\delta k^2 C_\varepsilon |u_t^\tau|_{L_2} |u_t^\tau|_{H^1} + \delta C^*\right] |A^{1/2}u_t^\tau(\sigma)|^2 d\sigma. \qquad (2.3.30)$$

Combining (2.3.23) and (2.3.30) gives

$$E^\tau(t) + \int_0^t \left[\gamma^\tau - \tau - 4(2+\delta)k^2 C_\varepsilon |u^\tau|_{L_\infty} - 2\varepsilon\right] |u_{tt}^\tau(\sigma)|^2 d\sigma$$
$$+ \int_0^t \left[b - 4(2+\delta)k^2 C_\varepsilon |u_t^\tau|_{L_2}|u_t^\tau|_{H^1} - 2\varepsilon C^*(\frac{c^2}{b}+1) - \delta C^*\right] |A^{1/2}u_t^\tau(\sigma)|^2 d\sigma$$
$$+ \delta[c^2 - 2\varepsilon C^*] \int_0^t |A^{1/2}u^\tau(\sigma)|^2 d\sigma$$
$$\leqslant \left[\delta((\frac{\tau b}{c^2} + 2C_\varepsilon(1+\tau))(4 + \frac{2\tau b}{c^2}) + \frac{b}{2}) + 3 + \frac{2\tau b}{c^2}\right] E^\tau(0),$$
$$= M_1 E^\tau(0). \qquad (2.3.31)$$

Therefore

$$E^\tau(t) + \int_0^t C_1(u^\tau)|u_{tt}^\tau(\sigma)|^2 d\sigma + \int_0^t C_2(u^\tau)|A^{1/2}u_t^\tau(\sigma)|^2 d\sigma + \delta[c^2 - 2\varepsilon C^*] \int_0^t |A^{1/2}u^\tau(\sigma)|^2 d\sigma \leqslant M_1 E^\tau(0), \qquad (2.3.32)$$

where

$$C_1(u^\tau) = \gamma^\tau - \tau - 4(2+\delta)k^2 C_\varepsilon |u^\tau|_{L_\infty} - 2\varepsilon,$$

$$C_2(u^\tau) = b - 4(2+\delta)k^2 C_\varepsilon |u_t^\tau|_{L_2}|u_t^\tau|_{H^1} - 2\varepsilon C^*(\frac{c^2}{b}+1) - \delta C^*.$$

Recall the space $\mathbb{H}_1 \equiv \mathcal{D}(A) \times \mathcal{D}(A^{1/2}) \times L^2(\Omega)$. Consider the energy functional $\mathcal{E}^\tau(t)$ defined as

$$\mathcal{E}^\tau(t) \approx E^\tau(t) + |Au^\tau(t)|^2. \qquad (2.3.33)$$

To achieve the goal, the second term, $|Au^\tau(t)|^2$ needs to be accounted for.

Step4: To estimate the second term we employ the equality by taking the L^2-inner product of (2.3.2) with the multiplier Au^τ.

$$\tau(u^\tau_{ttt},Au^\tau)+(u^\tau_{tt},Au^\tau)+c^2(Au^\tau,Au^\tau)+b(Au^\tau_t,Au^\tau)=(2ku^\tau u^\tau_{tt},Au^\tau)+(2k(u^\tau_t)^2,Au^\tau). \tag{2.3.34}$$

Then

$$\begin{aligned}\frac{d}{dt}\big[\tau(u^\tau_{tt},Au^\tau)-\frac{\tau}{2}|A^{1/2}u^\tau_t|^2+(u^\tau_t,Au^\tau)\big]-|A^{1/2}u^\tau_t|^2+\frac{d}{dt}\frac{b}{2}|Au^\tau|^2+c^2|Au^\tau|^2\\ =(2ku^\tau u^\tau_{tt},Au^\tau)+(2k(u^\tau_t)^2,Au^\tau).\end{aligned} \tag{2.3.35}$$

Integrating w.r.t time from 0 to t gives

$$\begin{aligned}&\frac{b}{2}|Au^\tau(t)|^2+c^2\int_0^t|Au^\tau(\sigma)|^2d\sigma\\ &=\frac{b}{2}|Au^\tau(0)|^2+\big[\frac{\tau}{2}|A^{1/2}u^\tau|^2-\tau(u^\tau_{tt},Au^\tau)-(u^\tau_t,Au^\tau)\big]\Big|_0^t+\int_0^t|A^{1/2}u^\tau_t(\sigma)|^2d\sigma\\ &\quad+\int_0^t(2k(u^\tau_t(\sigma))^2,Au^\tau(\sigma))d\sigma+\int_0^t(2ku^\tau(\sigma)u^\tau_{tt}(\sigma),Au^\tau(\sigma))d\sigma.\end{aligned} \tag{2.3.36}$$

We multiply (2.3.26) with $\delta>0$ then

$$\begin{aligned}&\frac{\delta b}{2}|Au^\tau(t)(\sigma)|^2+\delta c^2\int_0^t|Au^\tau|^2d\sigma\\ &=\frac{\delta b}{2}|Au^\tau(0)|^2+\delta\big[\frac{\tau}{2}|A^{1/2}u^\tau|^2-\tau(u^\tau_{tt},Au^\tau)-(u^\tau_t,Au^\tau)\big]\Big|_0^t+\delta\int_0^t|A^{1/2}u^\tau_t(\sigma)|^2d\sigma\\ &\quad+\delta\int_0^t(2k(u^\tau_t(\sigma))^2,Au^\tau(\sigma))d\sigma+\delta\int_0^t(2ku^\tau(\sigma)u^\tau_{tt}(\sigma),Au^\tau(\sigma))d\sigma.\end{aligned} \tag{2.3.37}$$

Note:

$$\begin{aligned}\delta\big[\frac{\tau}{2}|A^{1/2}u^\tau_t|^2-\tau(u^\tau_{tt},Au^\tau)-(u^\tau_t,Au^\tau)\big]\Big|_0^t&\leqslant\delta\big[\frac{\tau}{b}|u^\tau_{tt}|^2+(\frac{\tau}{2}+\frac{1}{2})|A^{1/2}u^\tau_t|^2+\frac{1}{2}|A^{1/2}u^\tau|^2+\frac{b}{4}|Au^\tau|^2\big]\Big|_0^t,\\ &\leqslant\delta\big[(\frac{\tau}{b}+\frac{\tau}{2}+1)E^\tau(t)+\frac{b}{4}|Au^\tau|^2\big]\Big|_0^t,\\ &\leqslant\delta(\frac{\tau}{b}+\frac{\tau}{2}+1)M_1E^\tau(0)+\frac{\delta b}{4}|Au^\tau(t)|^2+\frac{\delta b}{4}|Au^\tau(0)|^2,\end{aligned} \tag{2.3.38}$$

where from (2.3.32), we have $E^\tau(t) \leqslant M_1 E^\tau(0)$ and

$$(2k(u_t^\tau)^2, Au^\tau) \leqslant 4k^2 C_\varepsilon |u_t^\tau|_{L_2} |u_t^\tau|_{H^1} |A^{1/2} u_t^\tau|^2 + \varepsilon |Au^\tau|^2.$$

$$(2ku^\tau u_{tt}^\tau, Au^\tau) \leqslant 4k^2 C_\varepsilon |u^\tau|_{L_\infty} |u_{tt}^\tau|^2 + \varepsilon |Au^\tau|^2.$$

From (2.3.37), we have

$$\begin{aligned}
\frac{\delta b}{4}|Au^\tau(t)|^2 &+ \delta\big[c^2 - 2\varepsilon\big]\int_0^t |Au^\tau(\sigma)|^2 d\sigma \\
&\leqslant \frac{3\delta b}{4}\mathcal{E}^\tau(0) + \delta(\frac{\tau}{b} + \frac{\tau}{2} + 1)M_1 E^\tau(0) + \int_0^t 4\delta k^2 C_\varepsilon |u^\tau|_{L_\infty} |u_{tt}^\tau(\sigma)|^2 d\sigma \\
&\quad + \int_0^t \left[4\delta k^2 C_\varepsilon |u_t^\tau|_{L_2} |u_t^\tau|_{H^1} + \delta\right] |A^{1/2} u_t^\tau(\sigma)|^2 d\sigma, \\
&\leqslant \big[\frac{3\delta b}{4} + \delta(\frac{\tau}{b} + \frac{\tau}{2} + 1)M_1\big]\mathcal{E}^\tau(0) + \int_0^t 4\delta k^2 C_\varepsilon |u^\tau|_{L_\infty} |u_{tt}^\tau(\sigma)|^2 d\sigma \\
&\quad + \int_0^t \left[4\delta k^2 C_\varepsilon |u_t^\tau|_{L_2} |u_t^\tau|_{H^1} + \delta\right] |A^{1/2} u_t^\tau(\sigma)|^2 d\sigma, \\
&= M_2 \mathcal{E}^\tau(0) + \int_0^t 4\delta k^2 C_\varepsilon |u^\tau|_{L_\infty} |u_{tt}^\tau(\sigma)|^2 d\sigma + \int_0^t \left[4\delta k^2 C_\varepsilon |u_t^\tau|_{L_2} |u_t^\tau|_{H^1} + \delta\right] |A^{1/2} u_t^\tau(\sigma)|^2 d\sigma.
\end{aligned} \tag{2.3.39}$$

Then

$$\begin{aligned}
\frac{\delta b}{4}|Au^\tau(t)|^2 &+ \delta\big[c^2 - 2\varepsilon\big]\int_0^t |Au^\tau(\sigma)|^2 d\sigma \\
&\leqslant M_2 \mathcal{E}^\tau(0) + \int_0^t 4\delta k^2 C_\varepsilon |u^\tau|_{L_\infty} |u_{tt}^\tau(\sigma)|^2 d\sigma + \int_0^t \left[4\delta k^2 C_\varepsilon |u_t^\tau|_{L_2} |u_t^\tau|_{H^1} + \delta\right] |A^{1/2} u_t^\tau(\sigma)|^2 d\sigma.
\end{aligned} \tag{2.3.40}$$

Combining (2.3.32) and (2.3.40) gives

$$\begin{aligned}
E^\tau(t) &+ \frac{\delta b}{4}|Au^\tau(t)|^2 \\
&+ \int_0^t C_1(u^\tau)|u_{tt}^\tau(\sigma)|^2 d\sigma + \int_0^t C_2(u^\tau)|A^{1/2} u_t^\tau(\sigma)|^2 d\sigma + C_3 \int_0^t |A^{1/2} u^\tau(\sigma)|^2 d\sigma + \big[c^2 - 2\varepsilon\big]\int_0^t |Au^\tau|^2 d\sigma \\
&\leqslant M_1 E^\tau(0) + M_2 \mathcal{E}^\tau(0) + \int_0^t 4\delta k^2 C_\varepsilon |u^\tau|_{L_\infty} |u_{tt}^\tau(\sigma)|^2 d\sigma + \int_0^t \left[4\delta k^2 C_\varepsilon |u_t^\tau|_{L_2} |u_t^\tau|_{H^1} + \delta\right] |A^{1/2} u_t^\tau(\sigma)|^2 d\sigma.
\end{aligned} \tag{2.3.41}$$

Thus

$$
\begin{aligned}
&\mathcal{E}^\tau(t)+\int_0^t\left[\gamma^\tau-\tau-8(1+\delta)k^2C_\varepsilon|u^\tau|_{L_\infty}-2\varepsilon\right]|u^\tau_{tt}(\sigma)|^2d\sigma\\
&\quad+\int_0^t\left[b-8(1+\delta)k^2C_\varepsilon|u^\tau_t|_{L_2}|u^\tau_t|_{H^1}-2\varepsilon C^*(\frac{c^2}{b}+1)-\delta(C^*+1)\right]|A^{1/2}u^\tau_t(\sigma)|^2d\sigma\\
&\quad+\delta\left[c^2-2\varepsilon C^*\right]\int_0^t|A^{1/2}u^\tau|^2d\sigma+\delta\left[c^2-2\varepsilon\right]\int_0^t|Au^\tau(\sigma)|^2d\sigma\\
&\quad=M_3\mathcal{E}^\tau(0).
\end{aligned}
\tag{2.3.42}
$$

By rescalling δ, the smallness of the initial data, $\mathcal{E}^\tau(0)\leqslant\rho$ and the Barrier's method by reconstruct the integral of the energy in terms of the initial data and superlinear terms, we obtain positive constants

$$
\begin{aligned}
\bar{C}_1&=\gamma^\tau-\tau-8(1+\delta)k^2C_\varepsilon|u^\tau|_{L_\infty}-2\varepsilon\\
&\leqslant\gamma^\tau-\tau-8(1+\delta)k^2C_\varepsilon M_3\rho-2\varepsilon\\
&\leqslant\gamma^\tau-\frac{\gamma^\tau}{2}-2\varepsilon\\
&=\frac{\gamma}{2}-2\varepsilon,
\end{aligned}
$$

where $|u^\tau|_{L_\infty}\leqslant\sqrt{M_3\mathcal{E}^\tau(0)}\leqslant\sqrt{M_3\rho}$ and by choosing $8(1+\delta)k^2C_\varepsilon\sqrt{M_3\rho}=\dfrac{\gamma}{2}$.

$$
\begin{aligned}
\bar{C}_2&=b-8(1+\delta)k^2C_\varepsilon|u^\tau_t|_{L_2}|u^\tau_t|_{H^1}-2\varepsilon C^*(\frac{c^2}{b}+1)-\delta(C^*+1),\\
&\leqslant b-8(1+\delta)k^2C_\varepsilon M_3^2\rho^2-2\varepsilon C^*(\frac{c^2}{b}+1)-\delta(C^*+1),\\
&\leqslant b-\frac{b}{2}-2\varepsilon C^*(\frac{c^2}{b}+1)-\delta(C^*+1),\\
&=\frac{b}{2}-\bar{\varepsilon},
\end{aligned}
$$

where $|u^\tau_t|^2_{L_2}|u^\tau_t|^2_{H_1}\leqslant M_3^2(\mathcal{E}^\tau(0))^2\leqslant M_3^2\rho^2$ and by choosing $8(1+\delta)k^2C_\varepsilon M_3^2\rho^2=\dfrac{b}{2}$.

$$\bar{C}_3=\delta\left[c^2-2\varepsilon C^*\right]=\bar{\varepsilon}.$$

$$\bar{C}_4=\delta\left[c^2-2\varepsilon\right]=\bar{\varepsilon}.$$

We now obtain

$$\mathcal{E}^\tau(t)+\int_0^t \bar{C}_1\tau|u_{tt}^\tau(\sigma)|^2 d\sigma+\int_0^t \bar{C}_2|A^{1/2}u_t^\tau(\sigma)|^2 d\sigma+\bar{C}_3\int_0^t |A^{1/2}u^\tau(\sigma)|^2+\bar{C}_4\int_0^t |Au^\tau(\sigma)|^2 d\sigma \leqslant M_4\mathcal{E}^\tau(0). \tag{2.3.43}$$

Hence

$$\mathcal{E}^\tau(t)+\tilde{C}\int_0^t \mathcal{E}^\tau(\sigma)d\sigma \leqslant M_4\mathcal{E}^\tau(0). \tag{2.3.44}$$

Thus by Theorem 4.1 ([40], p. 116) the rate ω can be taken as we first chose a number ρ such that $0<\rho<(\frac{M_4}{\tilde{C}})^{-1}$, then we define a number $\eta_0=\frac{M_4}{\tilde{C}}\rho^{-1}$ and choose another number η such that $\eta>\eta_0$. The rate is then given by

$$\omega=-\frac{1}{\eta}\log(\frac{M_4}{\tilde{C}}\rho)>0,$$

and is clearly independent on τ. Then the proof of theorem 2.2.1 is completed.

2.3.2 Proof of Theorem (2.2.2) - Uniform (in τ) exponential stability in $\mathbb{H}_2^\tau$

Recall the space $\mathbb{H}_2\equiv\mathcal{D}(A)\times\mathcal{D}(A)\times\mathcal{D}(A^{1/2})$. Let u^τ be the solution for (2.1.9) and consider the energy functional from $\mathfrak{E}^\tau(t)$ defined as

$$\mathfrak{E}^\tau(t)\approx\mathcal{E}^\tau(t)+|Au_t^\tau(t)|^2+|A^{1/2}u_{tt}^\tau|^2. \tag{2.3.45}$$

Step 1: Taking the L^2- inner product of (2.3.2) with Au_t^τ gives

$$(\tau u_{ttt}^\tau,Au_t^\tau)+(u_{tt}^\tau,Au_t^\tau)+(c^2Au^\tau Au_t^\tau)+(bAu_t^\tau,Au_t^\tau)=(2ku^\tau u_{tt}^\tau,Au_t^\tau)+(2k(u_t^\tau)^2,Au_t^\tau). \tag{2.3.46}$$

Then

$$\frac{d}{dt}[\tau(u_{tt}^\tau,Au_t^\tau)+|A^{1/2}u_t^\tau|^2+\frac{c^2}{2}|Au^\tau|^2]-\tau|A^{1/2}u_{tt}^\tau|^2+b|Au_t^\tau|^2=(2ku^\tau u_{tt}^\tau,Au_t^\tau)+(2k(u_t^\tau)^2,Au_t^\tau). \tag{2.3.47}$$

Integrating w.r.t time from 0 to t gives

$$\frac{c^2}{2}|Au^\tau(t)|^2+b\int_0^t |Au_t^\tau(\sigma)|^2 d\sigma-\tau\int_0^t |A^{1/2}u_{tt}^\tau(\sigma)|^2 d\sigma$$

$$= \frac{c^2}{2}|Au^\tau(0)|^2 - \tau(u^\tau_{tt}, Au^\tau_t)\Big|_0^t + \int_0^t [(2ku^\tau(\sigma)u^\tau_{tt}(\sigma), Au^\tau_t(\sigma)) + (2k(u^\tau_t(\sigma))^2, Au^\tau_t(\sigma))]d\sigma. \tag{2.3.48}$$

Step 2: Taking the L^2- inner product of (2.3.2) with Au^τ_{tt} gives

$$(\tau u^\tau_{ttt}, Au^\tau_{tt}) + (u^\tau_{tt}, Au^\tau_{tt}) + c^2(Au^\tau Au^\tau_{tt}) + b(Au^\tau_{tt}, Au^\tau_t) = (2ku^\tau u^\tau_{tt}, Au^\tau_{tt}) + (2k(u^\tau_t)^2, Au^\tau_{tt}). \tag{2.3.49}$$

Then

$$\frac{d}{dt}[\frac{\tau}{2}|A^{1/2}u^\tau_{tt}|^2 + c^2(Au^\tau_t, Au^\tau) + \frac{b}{2}|Au^\tau_t|^2] + |A^{1/2}u^\tau_{tt}|^2 - c^2|Au^\tau_t|^2 = (2ku^\tau u^\tau_{tt}, Au^\tau_{tt}) + (2k(u^\tau_t)^2, Au^\tau_{tt}). \tag{2.3.50}$$

Integrating w.r.t time from 0 to t gives

$$\begin{aligned}\int_0^t |A^{1/2}u^\tau_{tt}(\sigma)|^2 d\sigma + \frac{\tau}{2}|A^{1/2}u^\tau_{tt}(t)|^2 + \frac{b}{2}|Au^\tau_t(t)|^2 \\ = \frac{\tau}{2}|A^{1/2}u^\tau_{tt}(0)|^2 + \frac{b}{2}|Au^\tau_t(0)|^2 - c^2(Au^\tau_t, Au^\tau)\Big|_0^t \\ + c^2\int_0^t |Au^\tau_t(\sigma)|^2 d\sigma + \int_0^t [(2ku^\tau(\sigma)u^\tau_{tt}(\sigma), Au^\tau_{tt}(\sigma)) + (2k(u^\tau_t(\sigma))^2, Au^\tau_{tt}(\sigma))]d\sigma.\end{aligned} \tag{2.3.51}$$

We multiply (2.3.26) with $\delta > 0$ then

$$\begin{aligned}\delta\int_0^t |A^{1/2}u^\tau_{tt}(\sigma)|^2 d\sigma + \frac{\delta\tau}{2}|A^{1/2}u^\tau_{tt}(t)|^2 + \frac{\delta b}{2}|Au^\tau_t(t)|^2 \\ = \frac{\delta\tau}{2}|A^{1/2}u^\tau_{tt}(0)|^2 + \frac{\delta b}{2}|Au^\tau_t(0)|^2 - \delta c^2(Au^\tau_t, Au^\tau)\Big|_0^t \\ + \delta c^2\int_0^t |Au^\tau_t(\sigma)|^2 d\sigma + \delta\int_0^t [(2ku^\tau(\sigma)u^\tau_{tt}(\sigma), Au^\tau_{tt}(\sigma)) + (2k(u^\tau_t(\sigma))^2, Au^\tau_{tt}(\sigma))]d\sigma.\end{aligned} \tag{2.3.52}$$

Combining (2.3.48) and (2.3.52) leads to

$$\begin{aligned}\frac{c^2}{2}|Au^\tau(t)|^2 + \frac{\delta b}{2}|Au^\tau_t(t)|^2 + \frac{\delta\tau}{2}|A^{1/2}u^\tau_{tt}(t)|^2 + (1-\tau)\int_0^t |A^{1/2}u^\tau_{tt}(\sigma)|^2 d\sigma + (b - \delta c^2)\int_0^t |Au^\tau_t(\sigma)|^2 d\sigma \\ = \frac{c^2}{2}|Au^\tau(0)|^2 - \tau(u^\tau_{tt}, Au^\tau_t)\Big|_0^t + \frac{\delta\tau}{2}|A^{1/2}u^\tau_{tt}(0)|^2 + \frac{\delta b}{2}|Au^\tau_t(0)|^2 - \delta c^2(Au^\tau_t, Au^\tau)\Big|_0^t\end{aligned}$$

$$+\int_0^t [(2ku^\tau(\sigma)u_{tt}^\tau(\sigma), Au_t^\tau(\sigma)) + (2k(u_t^\tau(\sigma))^2, Au_t^\tau(\sigma)) + \delta(2ku^\tau(\sigma)u_{tt}^\tau(\sigma), Au_{tt}^\tau(\sigma))$$

$$+\delta(2k(u_t^\tau(\sigma))^2, Au_{tt}^\tau(\sigma))]d\sigma, \tag{2.3.53}$$

where $b > c^2$. Consider the right hand side terms

$$\begin{aligned}
&\frac{c^2}{2}|Au^\tau(0)|^2 + \frac{\delta\tau}{2}|A^{1/2}u_{tt}^\tau(0)|^2 + \frac{\delta b}{2}|Au_t^\tau(0)|^2 - [\delta c^2(Au_t^\tau, Au^\tau) + \tau(u_{tt}^\tau, Au_t^\tau)]\Big|_0^t \\
&\leqslant [\frac{c^2}{2} + \frac{\delta\tau}{2} + \frac{\delta b}{2}]\,\mathfrak{E}^\tau(0) + [\frac{\delta b}{4}|Au_t^\tau(t)|^2 + \frac{\delta c^2}{b}|Au^\tau(t)|^2 + \frac{\tau}{4}|A^{1/2}u_{tt}^\tau(t)|^2 + \tau|A^{1/2}u_t^\tau(t)|^2] \\
&\quad + [\frac{\delta b}{4}|Au_t^\tau(0)|^2 + \frac{\delta c^2}{b}|Au^\tau(0)|^2 + \frac{\tau}{4}|A^{1/2}u_{tt}^\tau(0)|^2 + \tau|A^{1/2}u_t^\tau(0)|^2], \\
&\leqslant [(\frac{c^2}{2} + \frac{\delta c^2}{b}) + \frac{1}{2}(\tau + b)(\delta + \frac{1}{2})]\,\mathfrak{E}^\tau(0) + \frac{\delta b}{4}|Au_t^\tau(t)|^2 + \frac{\delta c^2}{b}|Au^\tau(t)|^2 + \frac{\tau}{4}|A^{1/2}u_{tt}^\tau(t)|^2 + \tau|A^{1/2}u_t^\tau(t)|^2, \\
&\leqslant [(\frac{c^2}{2} + \frac{c^2}{b}) + \frac{3\tau}{4} + \frac{3b}{4} + (\frac{\delta c^2}{b} + \tau)M_4]\,\mathfrak{E}^\tau(0) + \frac{\delta b}{4}|Au_t^\tau(t)|^2 + \frac{\tau}{4}|A^{1/2}u_{tt}^\tau(t)|^2, \\
&= M_5\mathfrak{E}^\tau(0) + \frac{\delta b}{4}|Au_t^\tau(t)|^2 + \frac{\tau}{4}|A^{1/2}u_{tt}^\tau(t)|^2,
\end{aligned} \tag{2.3.54}$$

where, we apply $\mathcal{E}^\tau(t) \leqslant M_4\mathcal{E}^\tau(0)$ from (2.3.44) and $\mathcal{E}^\tau(0) \leqslant \mathfrak{E}^\tau(0)$.

Then from (2.3.53) we obtain

$$\begin{aligned}
&\frac{c^2}{2}|Au^\tau(t)|^2 + \frac{\delta b}{2}|Au_t^\tau(t)|^2 + \frac{\delta\tau}{2}|A^{1/2}u_{tt}^\tau(t)|^2 + (1-\tau)\int_0^t |A^{1/2}u_{tt}^\tau(\sigma)|^2 d\sigma + (b - \delta c^2)\int_0^t |Au_t^\tau(\sigma)|^2 d\sigma \\
&\qquad \leqslant M_5\mathfrak{E}^\tau(0) + \int_0^t [(2ku^\tau(\sigma)u_{tt}^\tau(\sigma), Au_t^\tau(\sigma)) + (2k(u_t^\tau(\sigma))^2, Au_t^\tau(\sigma)) \\
&\qquad\quad + \delta(2ku^\tau(\sigma)u_{tt}^\tau(\sigma), Au_{tt}^\tau(\sigma)) + \delta(2k(u_t^\tau(\sigma))^2, Au_{tt}^\tau(\sigma))]d\sigma.
\end{aligned} \tag{2.3.55}$$

By Sobolev's embeddings, we have

$$|u^\tau u_{tt}^\tau|^2 \leqslant |u^\tau|_{L_6}^2 |u_{tt}^\tau|_{L_3}^2 \leqslant C|u^\tau|_{L_\infty}^2 |u_{tt}^\tau|_{H^1}^2 \leqslant C|u^\tau|_{L_\infty}^2 |A^{1/2}u_{tt}^\tau|^2.$$

$$|(u_t^\tau)^2|^2 = |u_t^\tau|_{L_4}^4 \leqslant |u_t^\tau|_{H_1}^4 \leqslant C|u_t^\tau|_{H_1}^2 |Au_t^\tau|^2.$$

$$|u^\tau(A^{1/2}u_{tt}^\tau)|^2 \leqslant C|u^\tau|_{L_\infty}^2 |A^{1/2}u_{tt}^\tau|^2.$$

$$|(A^{1/2}u^\tau)u_{tt}^\tau|^2 \leqslant |A^{1/2}u^\tau|_{L_6}^2 |u_{tt}^\tau|_{L_3}^2 \leqslant C|Au^\tau|^2 |A^{1/2}u_{tt}^\tau|^2.$$

$$|u_t^\tau(A^{1/2}u_t^\tau)|^2 \leqslant |u_t^\tau|_{L_6}^2 |A^{1/2}u_t^\tau|_{L_3}^2 \leqslant C|u_t^\tau|_{H_1}^2 |Au_t^\tau|^2.$$

We now consider the nonlinear terms as follows.

$$(2ku^\tau u_{tt}^\tau, Au_t^\tau) \leqslant 4k^2C_\varepsilon|u^\tau u_{tt}^\tau|^2 + \varepsilon|Au_t^\tau|^2 \leqslant 4k^2C_\varepsilon C|u^\tau|_{L_\infty}^2|A^{1/2}u_{tt}^\tau|^2 + \varepsilon|Au_t^\tau|^2.$$

$$(2k(u_t^\tau)^2, Au_t^\tau) \leqslant 4k^2C_\varepsilon|(u_t^\tau)^2|^2 + \varepsilon|Au_t^\tau|^2 \leqslant 4k^2C_\varepsilon C|u_t^\tau|_{H_1}^2|Au_t^\tau|^2 + \varepsilon|Au_t^\tau|^2.$$

$$\begin{aligned}\delta(2ku^\tau u_{tt}^\tau, Au_{tt}^\tau) &= 2k(u^\tau(A^{1/2}u_{tt}^\tau), A^{1/2}u_{tt}^\tau) + 2k((A^{1/2}u^\tau)u_{tt}^\tau, A^{1/2}u_{tt}^\tau),\\ &\leqslant 4\delta k^2C_\varepsilon|u^\tau(A^{1/2}u_{tt}^\tau)|^2 + 4\delta k^2C_\varepsilon|(A^{1/2}u^\tau)u_{tt}^\tau|^2 + 2\delta\varepsilon|A^{1/2}u_{tt}^\tau|^2,\\ &\leqslant 4\delta k^2C_\varepsilon C|u^\tau|_{L_\infty}^2|A^{1/2}u_{tt}^\tau|^2 + 4\delta k^2C_\varepsilon C|Au^\tau|^2|A^{1/2}u_{tt}^\tau|^2 + 2\delta\varepsilon|A^{1/2}u_{tt}^\tau|^2.\end{aligned}$$

$$\begin{aligned}\delta(2k(u_t^\tau)^2, Au_{tt}^\tau) = 2\delta k(2u_t^\tau(A^{1/2}u_t^\tau), A^{1/2}u_{tt}^\tau) &\leqslant 16\delta k^2C_\varepsilon|u_t^\tau(A^{1/2}u_t^\tau)|^2 + \delta\varepsilon|A^{1/2}u_{tt}^\tau|^2,\\ &\leqslant 16\delta k^2C_\varepsilon C|u_t^\tau|_{H_1}^2|Au_t^\tau|^2 + \delta\varepsilon|A^{1/2}u_{tt}^\tau|^2.\end{aligned}$$

Then we have

$$\begin{aligned}\int_0^t [(2ku^\tau(\sigma)u_{tt}^\tau(\sigma), Au_t^\tau(\sigma)) + (2k(u_t^\tau(\sigma))^2, Au_t^\tau(\sigma)) + (2ku^\tau(\sigma)u_{tt}^\tau(\sigma), Au_{tt}^\tau(\sigma)) + (2k(u_t^\tau(\sigma))^2, Au_{tt}^\tau(\sigma))]d\sigma\\ \leqslant \int_0^t 4k^2C_\varepsilon\left[2C|u^\tau|_{L_\infty}^2 + \delta C|Au^\tau|^2 + 3\delta\varepsilon\right]|A^{1/2}u_{tt}^\tau(\sigma)|^2d\sigma\\ + \int_0^t\left[4k^2C_\varepsilon C(1+4\delta)|u_t^\tau|_{H_1}^2 + \varepsilon(1+\delta)\right]|Au_t^\tau(\sigma)|^2d\sigma.\end{aligned} \tag{2.3.56}$$

From (2.3.55), we obtain

$$\begin{aligned}&\frac{c^2}{2}|Au^\tau(t)|^2 + \frac{\delta b}{4}|Au_t^\tau(t)|^2 + \frac{\delta\tau}{4}|A^{1/2}u_{tt}^\tau(t)|^2\\ &\qquad + \int_0^t\left[1 - \tau - 4k^2C_\varepsilon(2C|u^\tau|_{L_\infty}^2 + \delta C|Au^\tau|^2 + 3\delta\varepsilon)\right]|A^{1/2}u_{tt}^\tau(\sigma)|^2d\sigma\\ &\qquad + \int_0^t\left[b - \delta c^2 - 4k^2C_\varepsilon C(1+4\delta)|u_t^\tau|_{H_1}^2 + \varepsilon(1+\delta)\right]|Au_t^\tau(\sigma)|^2d\sigma\\ &\qquad \leqslant M_5\mathfrak{E}^\tau(0).\end{aligned} \tag{2.3.57}$$

Combining (2.3.43) and (2.3.57) gives

$$\begin{aligned}
&\mathcal{E}^\tau(t)+\frac{\delta b}{4}|Au_t^\tau(t)|^2+\frac{\delta\tau}{4}|A^{1/2}u_{tt}^\tau(t)|^2\\
&+\int_0^t\left[\gamma^\tau-\tau-8(1+\delta)k^2C_\varepsilon|u^\tau|_{L_\infty}-2\varepsilon\right]\tau|u_{tt}^\tau(\sigma)|^2d\sigma\\
&+\int_0^t\left[b-8(1+\delta)k^2C_\varepsilon|u_t^\tau|_{L_2}|u_t^\tau|_{H^1}-2\varepsilon C^*(\frac{c^2}{b}+1)-\delta(C^*+1)\right]|A^{1/2}u_t^\tau(\sigma)|^2d\sigma\\
&+\delta\big[c^2-2\varepsilon C^*\big]\int_0^t|A^{1/2}u^\tau|^2+\delta\big[c^2-2\varepsilon\big]\int_0^t|Au^\tau(\sigma)|^2d\sigma\\
&+\int_0^t\left[1-\tau-4k^2C_\varepsilon(2C|u^\tau|_{L_\infty}^2+\delta C|Au^\tau|^2+3\delta\varepsilon)\right]|A^{1/2}u_{tt}^\tau(\sigma)|^2d\sigma\\
&+\int_0^t\left[b-\delta c^2-4k^2C_\varepsilon C(1+4\delta)|u_t^\tau|_{H_1}^2+\varepsilon(1+\delta)\right]|Au_t^\tau(\sigma)|^2d\sigma\\
&\leqslant M_4\mathcal{E}^\tau(0)+M_5\mathfrak{E}^\tau(0),\\
&\leqslant (M_4+M_5)\mathfrak{E}(0),\\
&=M_6\mathfrak{E}(0). \qquad (2.3.58)
\end{aligned}$$

By rescalling δ, the smallness of the initial data, $\mathfrak{E}(0)\leqslant\rho$ and the Barrier's method by reconstruct the integral of the energy in terms of the initial data and superlinear terms, we obtain positive constants

$$\begin{aligned}
\bar{\bar{\bar{C}}}_1&=\gamma^\tau-\tau-8(1+\delta)k^2C_\varepsilon|u^\tau|_{L_\infty}-2\varepsilon\\
&\leqslant\gamma^\tau-\tau-8(1+\delta)k^2C_\varepsilon M_6\rho-2\varepsilon\\
&\leqslant\gamma^\tau-\tau-\frac{\gamma}{2}-2\varepsilon\\
&\leqslant\frac{\gamma}{2}-\tau-2\varepsilon,
\end{aligned}$$

where $|u^\tau|_{L_\infty}\leqslant\sqrt{M_6\mathfrak{E}(0)}\leqslant\sqrt{M_6\rho}$ and by choosing $8(1+\delta)k^2C_\varepsilon\sqrt{M_6\rho}=\dfrac{\gamma}{2}$.

$$\begin{aligned}
\bar{\bar{\bar{C}}}_2&=b-8(1+\delta)k^2C_\varepsilon|u_t^\tau|_{L_2}|u_t^\tau|_{H^1}-2\varepsilon C^*(\frac{c^2}{b}+1)-\delta(C^*+1),\\
&\leqslant b-8(1+\delta)k^2C_\varepsilon M_6\rho-2\varepsilon C^*(\frac{c^2}{b}+1)-\delta(C^*+1),\\
&\leqslant b-\frac{b}{2}-2\varepsilon C^*(\frac{c^2}{b}+1)-\delta(C^*+1),\\
&=\frac{b}{2}-\bar{\varepsilon},
\end{aligned}$$

where $|u_t^\tau|_{L_2}|u_t^\tau|_{H^1} \leqslant M_6\mathfrak{E}(0) \leqslant M_6\rho$ and by choosing $8(1+\delta)k^2C_\varepsilon M_6\rho = \dfrac{b}{2}$.

$$\begin{aligned}\bar{\bar{C}}_3 &= 1-\tau-4k^2C_\varepsilon(2C|u^\tau|_{L_\infty}^2+\delta C|Au^\tau|^2)-12k^2C_\varepsilon\delta\varepsilon,\\ &\leqslant 1-\tau-4k^2C_\varepsilon CM_6\rho(2+\delta)-12k^2C_\varepsilon\delta\varepsilon,\\ &\leqslant 1-\tau-\frac{\tau}{2}-12k^2C_\varepsilon\delta\varepsilon,\\ &= \frac{1}{2}-\tau-\bar{\varepsilon},\end{aligned}$$

where $|u^\tau|_{L_\infty}^2+|Au^\tau|^2 \leqslant 2M_2\mathfrak{E}(0) \leqslant 2M_6\rho$ and by choosing $4k^2C_\varepsilon CM_6\rho(2+\delta) = \dfrac{\tau}{2}$.

$$\begin{aligned}\bar{\bar{C}}_4 &= b-\delta c^2-4k^2C_\varepsilon C(1+4\delta)|u_t^\tau|_{H_1}^2-\varepsilon(1+\delta),\\ &\leqslant b-\delta c^2-4k^2C_\varepsilon C(1+4\delta)M_3\rho-\varepsilon(1+\delta),\\ &\leqslant b-\delta c^2-\frac{b}{2}-\varepsilon(1+\delta),\\ &= \frac{b}{2}-\bar{\varepsilon},\end{aligned}$$

where $|u^\tau|_{H_1}^2 \leqslant M_6\mathfrak{E}(0) \leqslant M_6\rho$ and by choosing $4k^2C_\varepsilon C(1+4\delta)M_3\rho = \dfrac{b}{2}$.

Then we arrive at

$$\begin{aligned}&\mathfrak{E}^\tau(t)+\int_0^t \bar{\bar{C}}_1\tau|u_{tt}^\tau(\sigma)|^2d\sigma+\int_0^t \bar{\bar{C}}_2|A^{1/2}u_t^\tau(\sigma)|^2d\sigma\\ &+\bar{C}_1\int_0^t |A^{1/2}u^\tau(\sigma)|^2d\sigma+\bar{C}_2\int_0^t |Au^\tau(\sigma)|^2d\sigma\\ &+\int_0^t \bar{\bar{C}}_3|A^{1/2}u_{tt}^\tau(\sigma)|^2d\sigma+\int_0^t \bar{\bar{C}}_4|Au_t^\tau(\sigma)|^2d\sigma\\ &\leqslant \frac{M_6}{\min\left\{1,\dfrac{\delta b}{4},\dfrac{\delta\tau}{4}\right\}}\mathfrak{E}^\tau(0),\\ &= M_7\mathfrak{E}(0).\end{aligned} \tag{2.3.59}$$

Hence we obtain the final inequality

$$\mathfrak{E}^\tau(t)+\tilde{C}\int_0^t \mathfrak{E}^\tau(\sigma)d\sigma \leqslant \tilde{M}\mathfrak{E}^\tau(0). \tag{2.3.60}$$

Thus by Theorem 4.1 ([40], p. 116) the rate ω can be taken as we first chose a number ρ such that $0<\rho<$

$(\frac{\tilde{M}}{\tilde{C}})^{-1}$, then we define a number $\eta_0 = \frac{\tilde{M}}{\tilde{C}}\rho^{-1}$ and choose another number η such that $\eta > \eta_0$. The rate is then given by

$$\omega = -\frac{1}{\eta}\log(\frac{\tilde{M}}{\tilde{C}}\rho) > 0,$$

and is clearly independent on τ. Therefore, we complete the proof of theorem (2.2.2).

2.3.3 Proof of Theorem (2.2.3) - Convergence

Rate of Convergence

Recall the JMGT equation

$$\begin{cases} \tau u^\tau_{ttt} + (1-2ku^\tau)u^\tau_{tt} + c^2Au^\tau + bAu^\tau_t = 2k(u^\tau_t)^2, \\ u^\tau(0,\cdot) = u_0, u^\tau_t(0,\cdot) = u_1, u^\tau_{tt}(0,\cdot) = u_2. \end{cases} \tag{2.3.61}$$

The limit equation

$$\begin{cases} (1-2ku^0)u^0_{tt} + c^2Au^0 + bAu^0_t = 2k(u^0_t)^2, \\ u^0(0,\cdot) = u_0, u^0_t(0,\cdot) = u_1. \end{cases} \tag{2.3.62}$$

Let $x^\tau = u^\tau - u^0$ where u^τ and u^0 are the solutions for the problems (2.3.61) and (2.3.62) respectively with the same initial values for $u(t=0)$ and $u_t(t=0)$. By taking the difference of the two problems we can write a $x^\tau-$problem given by

$$\begin{cases} x^\tau_{tt} + c^2\mathcal{A}x^\tau + b\mathcal{A}x^\tau_t = -\tau u^\tau_{ttt} + 2k(u^\tau_t)^2 - 2k(u^0_t)^2 + 2ku^\tau u^\tau_{tt} - 2ku^0u^0_{tt}, \\ x^\tau(0) = 0, x^\tau_t(0) = 0. \end{cases} \tag{2.3.63}$$

We aim to prove that

$$|A^{1/2}x^\tau(t)|_2^2 + |x^\tau_t(t)|_2^2 \leqslant \tau\tilde{K}\mathfrak{E}(0).$$

Rewrite (2.3.63) as

$$\begin{aligned} x^\tau_{tt} + c^2\mathcal{A}x^\tau + b\mathcal{A}x^\tau_t &= -\tau u^\tau_{ttt} + 2k[(u^\tau_t - u^0_t)(u^\tau_t + u^0_t)] + 2ku^\tau(u^\tau_{tt} - u^0_{tt}) + 2ku^0_{tt}(u^\tau - u^0), \\ &= -\tau u^\tau_{ttt} + 2ku^\tau_t x^\tau_t + 2ku^0_t x^\tau_t + 2ku^\tau x^\tau_{tt} + 2ku^0_{tt}x^\tau. \end{aligned} \tag{2.3.64}$$

Step 1: Taking the L^2-inner product of (2.3.64) with the multiplier x_t^τ gives

$$(x_{tt}^\tau, x_t^\tau) + c^2(Ax^\tau, x_t^\tau) + b(Ax_t^\tau, x_t^\tau) \\ = -(\tau u_{ttt}^\tau, x_t^\tau) + (2ku_t^\tau x_t^\tau, x_t^\tau)_(2ku_t^0 x_t^\tau, x_t^\tau) + (2ku^\tau x_{tt}^\tau, x_t^\tau) + (2ku_{tt}^0 x^\tau, x_t^\tau). \tag{2.3.65}$$

Then

$$\frac{d}{dt}\frac{1}{2}|x_t^\tau|^2 + \frac{d}{dt}\frac{c^2}{2}|A^{1/2}x^\tau|^2 + b|A^{1/2}x_t^\tau|^2 \\ = -(\tau u_{ttt}^\tau, x_t^\tau) + (2ku_t^\tau x_t^\tau, x_t^\tau) + (2ku_t^0 x_t^\tau, x_t^\tau) + (2ku^\tau x_{tt}^\tau, x_t^\tau) + (2ku_{tt}^0 x^\tau, x_t^\tau). \tag{2.3.66}$$

Integrating w.r.t time from 0 to t gives

$$\frac{1}{2}|x_t^\tau(t)|^2 + \frac{c^2}{2}|A^{1/2}x^\tau(t)|^2 + b\int_0^t |A^{1/2}x_t^\tau(\sigma)|^2 d\sigma \leqslant \frac{1}{2}|x_t^\tau(0)|^2 + \frac{c^2}{2}|A^{1/2}x^\tau(0)|^2 + \int_0^t (\tau u_{ttt}^\tau(\sigma), x_t^\tau(\sigma))d\sigma \\ + \int_0^t [(2ku_t^\tau(\sigma)x_t^\tau(\sigma), x_t^\tau(\sigma)) + (2ku_t^0(\sigma)x_t^\tau(\sigma), x_t^\tau(\sigma)) \\ + (2ku^\tau(\sigma)x_{tt}^0(\sigma), x_t^\tau(\sigma)) + (2ku_{tt}^0(\sigma)x^\tau(\sigma), x_t^\tau(\sigma))]d\sigma. \tag{2.3.67}$$

With zero initial conditions of the $x^\tau-$equation, we have

$$\frac{1}{2}|x_t^\tau(t)|^2 + \frac{c^2}{2}|A^{1/2}x^\tau(t)|^2 + b\int_0^t |A^{1/2}x_t^\tau(\sigma)|^2 d\sigma \\ \leqslant \int_0^t (\tau u_{ttt}^\tau(\sigma), x_t^\tau(\sigma))d\sigma + \int_0^t [(2ku_t^\tau(\sigma)x_t^\tau(\sigma), x_t^\tau(\sigma)) + (2ku_t^0(\sigma)x_t^\tau(\sigma), x_t^\tau(\sigma)) \\ + (2ku^\tau(\sigma)x_{tt}^\tau(\sigma), x_t^\tau(\sigma)) + (2ku_{tt}^0(\sigma)x^\tau(\sigma), x_t^\tau(\sigma))]d\sigma. \tag{2.3.68}$$

Consider the RHS terms as follows

$$\int_0^t (\tau u_{ttt}^\tau(\sigma), x_t^\tau(\sigma))d\sigma \\ = \tau(u_{tt}^\tau, x_t^\tau)\big|_0^t - \tau\int_0^t (u_{tt}^\tau(\sigma), x_{tt}^\tau(\sigma))d\sigma, \\ \leqslant \left[\tau^2|u_{tt}^\tau(t)|^2 + \frac{1}{4}|x_t^\tau(t)|^2\right]\Big|_0^t + \tau\int_0^t (u_{tt}^\tau(\sigma), u_{tt}^\tau(\sigma) - u_{tt}^0(\sigma))d\sigma, \\ \leqslant \frac{\tau}{2}\mathfrak{E}(t) + \frac{\tau}{2}\mathfrak{E}(0) + \frac{1}{4}|x_t^\tau(t)|^2 + \tau\int_0^t |u_{tt}^\tau(\sigma)|^2 d\sigma + \tau\int_0^t (u_{tt}^\tau(\sigma), u_{tt}^0(\sigma))d\sigma,$$

$$
\begin{aligned}
&\leqslant \frac{\tau}{2}(\tilde{M}+1)\mathfrak{E}(0)+\frac{1}{4}|x_t^\tau(t)|^2+\tau\int_0^t|u_{tt}^\tau(\sigma)|^2d\sigma+\frac{\tau}{2}\int_0^t|u_{tt}^\tau(\sigma)|^2d\sigma+\frac{\tau}{2}\int_0^t|u_{tt}^0(\sigma)|^2d\sigma,\\
&\leqslant \frac{\tau}{2}(\tilde{M}+1)\mathfrak{E}(0)+\frac{1}{4}|x_t^\tau(t)|^2+\tau\int_0^t|u_{tt}^\tau(\sigma)|^2d\sigma+\frac{\tau}{2}\int_0^t|u_{tt}^\tau(\sigma)|^2d\sigma+\frac{\tau T}{2}\sup_{t\in[0,T]}|u_{tt}^0|^2.
\end{aligned}
\tag{2.3.69}
$$

For the estimate of $|u_{tt}^0|^2$, we apply the result of the Westervelt's equation which is obtained in Theorem 1.1 of [24] as the following. For the energy functional

$$
E_{u,0}(t)=\frac{1}{2}[|u_t|^2+|A^{1/2}u_t|^2].
$$

$$
E_{u,1}(t)=\frac{1}{2}[|u_{tt}|^2+|A^{1/2}u_t(t)|^2+|Au(t)|^2].
$$

For $t=0$, we obtain

$$
E_{u,1}(0)=\big[\frac{1}{2}|(1-2ku_0)^{-1}[c^2Au_0+bAu_1+2ku_1^2]|^2+|A^{1/2}u_1(t)|^2+|Au_0|^2\big].
$$

For $T>0$, there exist $\rho T>0$ such that $E_{u,0}(0)+E_{u,1}(0)\leqslant \rho T$, with $u^0\in C([0,T];H^2(\Omega))\cap C^1([0,T];H^1(\Omega))\cap C^2([0,T];L_2(\Omega))$ and $u_{tt}^0\in L_2((0,T);H^1(\Omega))$. Then with smallness of initial data, we obtain

$$
\frac{1}{2}|(1-2ku_0)^{-1}[c^2Au_0+bAu_1+2ku_1^2]|^2\leqslant|(1-2k|u_0|_{L_\infty})|^2[c^2|Au_0|^2+(b+4k^2C|u_1|_{H_1}^2)|Au_1|^2]\leqslant\rho T.
\tag{2.3.70}
$$

Then from (2.3.69) with Sobolev's embeddings

$$
\begin{aligned}
\int_0^t(\tau u_{ttt}^\tau(\sigma),x_t^\tau(\sigma))d\sigma &\leqslant \frac{\tau}{2}(2\tilde{M}+1)\mathfrak{E}(0)+\frac{1}{4}|x_t^\tau(t)|^2+\frac{3\tau}{2}\tilde{M}\mathfrak{E}(0)+\frac{\tau}{2}\hat{C}\mathfrak{E}(0),\\
&\leqslant \frac{\tau}{2}\big[2\tilde{M}+1+\hat{C}\big]\mathfrak{E}(0)+\frac{1}{4}|x_t^\tau(t)|^2.
\end{aligned}
\tag{2.3.71}
$$

$$
\begin{aligned}
\int_0^t(2ku_t^\tau(\sigma)x_t^\tau(\sigma),x_t^\tau(\sigma))d\sigma &\leqslant \int_0^t 4k^2C_\varepsilon|u_t^\tau|_{H^1}^2|A^{1/2}x_t^\tau(\sigma)|^2d\sigma+\int_0^t\varepsilon|x_t^\tau(\sigma)|^2d\sigma,\\
&\leqslant \int_0^t(4k^2C_\varepsilon|u_t^\tau|_{H1}^2+\varepsilon C^*)|A^{1/2}x_t^\tau(\sigma)|^2d\sigma.
\end{aligned}
$$

$$\int_0^t (2ku_t^0(\sigma)x_t^\tau(\sigma), x_t^\tau(\sigma))d\sigma \leqslant \int_0^t 4k^2 C_\varepsilon |u_t^0|_{H^1}^2 |A^{1/2}x_t^\tau(\sigma)|^2 d\sigma + \int_0^t \varepsilon |x_t^\tau(\sigma)|^2 d\sigma,$$
$$\leqslant \int_0^t (4k^2 C_\varepsilon |u_t^0|_{H_1}^2 + \varepsilon C^*)|A^{1/2}x_t^\tau(\sigma)|^2 d\sigma.$$

$$\begin{aligned}
\int_0^t (2ku^\tau(\sigma)x_{tt}^\tau(\sigma), x_t^\tau(\sigma))d\sigma &= \int_0^t (2ku^\tau(\sigma), \frac{d}{dt}(x_t^\tau)^2)d\sigma, \\
&= (2ku^\tau, (x_t^\tau)^2)\Big|_0^t - \int_0^t (2ku_t^\tau(\sigma)x_t^\tau(\sigma), x_t^\tau(\sigma))d\sigma, \\
&\leqslant 4k^2 \sup |u^\tau|_{L_\infty} |x_t^\tau(t)|^2 + \int_0^t (2ku_t^\tau(\sigma)x_t^\tau(\sigma), x_t^\tau(\sigma))d\sigma, \\
&\leqslant 4k^2 \bar{C}|x_t^\tau(t)|^2 + \int_0^t (2ku_t^\tau(\sigma)x_t^\tau(\sigma), x_t^\tau(\sigma))d\sigma, \\
&\leqslant 4k^2 \bar{C}|x_t^\tau(t)|^2 + \int_0^t (4k^2 C_\varepsilon |u_t^\tau|_{H1}^2 + \varepsilon C^*)|A^{1/2}x_t^\tau(\sigma)|^2 d\sigma.
\end{aligned}$$

where the constant $\bar{C} = \sup |u^\tau|_{L_\infty}$

$$\int_0^t (2ku_{tt}^0(\sigma)x^\tau(\sigma), x_t^\tau(\sigma))d\sigma \leqslant \int_0^t 4k^2 C_\varepsilon |u_{tt}^0|_{L_2}^2 |A^{1/2}x^\tau(\sigma)|^2 d\sigma + \int_0^t \varepsilon C^* |A^{1/2}x_t^\tau(\sigma)|^2 d\sigma.$$

Then we get

$$\begin{aligned}
\text{RHS} &\leqslant \frac{\tau}{2}\left[2\tilde{M} + 1 + \hat{C}\right]\mathfrak{E}(0) + \frac{1}{4}|x_t^\tau(t)|^2 + 4k^2\bar{C}|x_t^\tau(t)|^2 \\
&\quad + \int_0^t \left[4k^2 C_\varepsilon (2|u_t^\tau|_{H^1}^2 + |u_t^0|_{H^1}^2 + 4\varepsilon C^*)\right]|A^{1/2}x_t^\tau(\sigma)|^2 + \int_0^t 4k^2 C_\varepsilon |u_{tt}^0|_{L_2}^2 |A^{1/2}x^\tau(\sigma)|^2 d\sigma, \\
&\leqslant \frac{\tau}{2}\left[2\tilde{M} + 1 + \hat{C}\right]\mathfrak{E}(0) + (\frac{1}{4} + 4k^2\bar{C})|x_t^\tau(t)|^2 + T4k^2 C_\varepsilon |u_{tt}^0|_{L_2}^2 |A^{1/2}x^\tau|^2 \\
&\quad + \int_0^t \left[4k^2 C_\varepsilon (2|u_t^\tau|_{H^1}^2 + |u_t^0|_{H^1}^2 + 4\varepsilon C^*)\right]|A^{1/2}x_t^\tau(\sigma)|^2. \qquad (2.3.72)
\end{aligned}$$

From (2.3.68) we obtain

$$\begin{aligned}
&(\frac{1}{4} - 4k^2\bar{C})|x_t^\tau(t)|^2 + (\frac{c^2}{2} - T4k^2 C_\varepsilon |u_{tt}^0|_{L_2}^2)|A^{1/2}x^\tau(t)|^2 \\
&\qquad + \int_0^t \left[b - 4k^2 C_\varepsilon (2|u_t^\tau|_{H^1}^2 + |u_t^0|_{H^1}^2 + 4\varepsilon C^*)\right]|A^{1/2}x_t^\tau(\sigma)|^2 d\sigma \\
&\qquad \leqslant \frac{\tau}{2}\left[2\tilde{M} + 1 + \hat{C}\right]\mathfrak{E}(0). \qquad (2.3.73)
\end{aligned}$$

With smallness of initial data, $\mathfrak{E}(0) \leqslant \rho$ and (2.3.70), we have positive constants

$$\hat{C}_1 = \frac{c^2}{2} - T4k^2C_\varepsilon|u^0_{tt}|^2_{L_2} \leqslant \frac{c^2}{4}.$$

$$\hat{C}_2 = (\frac{1}{4} - 4k^2\bar{C}).$$

$$\hat{C}_3 = b - 4k^2C_\varepsilon(2|u^\tau_t|^2_{H^1} + |u^0_t|^2_{H^1} + 4\varepsilon C^*) \leqslant \frac{b}{2}.$$

Hence we arrive at

$$\hat{C}_1|A^{1/2}x^\tau(t)|^2 + \hat{C}_2|x^\tau_t(t)|^2 \leqslant \frac{\tau}{2}\left[2\tilde{M} + 1 + \hat{C}\right]\mathfrak{E}(0) = \tau\tilde{K}\mathfrak{E}(0) = \tau C(r). \tag{2.3.74}$$

Thus we obtain the rate $\tau\tilde{K}$ and this complete the proof of theorem (2.2.3) part a).

Strong Convergence

Given $\varepsilon > 0$ there exist $T > 0$ such that

$$|P(U^\tau(t,U_0)) - U^0(t,PU_0)|^2_{H_1\times L_2} \leqslant \varepsilon.$$

We aim to prove that

$$|A^{1/2}x^\tau(t)|^2 + |x^\tau_t(t)|^2 \leqslant \varepsilon,$$

for all times $t > 0$. The strategy is prove that for any given fixed time T the above inequality is true and then to choose a suitable T such that for $t > T$ the energy is still bounded above by ε.

$$\begin{aligned}|A^{1/2}x^\tau(t)|^2 + |x^\tau_t(t)|^2 &= |A^{1/2}u^\tau(t) - A^{1/2}u^0(t)|^2 + |(u^\tau_t(t) - u^0_t(t))|^2,\\ &\leqslant |A^{1/2}u^\tau(t)|^2 + |A^{1/2}u^0(t)|^2 + |u^\tau_t(t)|^2 + |u^0_t(t)|^2.\end{aligned}$$

Now observe that all the terms on the right hand side above are uniformly exponentially stable with smallness of initial data $|U_0|^2_{\mathbb{H}_2} \leqslant L_1$ by (2.3.60) and $|U_0|^2_{\mathbb{H}_2} \leqslant L_2$ by (2.3.70). Therefore, there exist positive constants $L_1, L_2, \omega, \omega_0$ such that

$$|A^{1/2}x^\tau(t)|^2 + |x^\tau_t(t)|^2 \leqslant e^{\omega t}|U_0|^2_{\mathbb{H}_2} + e^{\omega_0 t}|U_0|^2_{\mathbb{H}_2},$$

$$\leqslant \frac{L_1}{\omega} e^{-\omega t} + \frac{L_2}{\omega_0} e^{-\omega_0 t},$$

$$< \varepsilon,$$

as long as

$$t \geqslant T \equiv \max\left\{-\frac{1}{\omega}\ln\left(\frac{\varepsilon\omega}{2L_1}\right), -\frac{1}{\omega_0}\ln\left(\frac{\varepsilon\omega_0}{2L_2}\right)\right\}.$$

We obtain the convergence uniformly for all $t \geqslant 0$. This completes the proof of part (b) of Theorem 2.2.3.

2.4 Future work

In the future, I would like to continue my research and focus on several related subjects.

- Numerical analysis as related to the problem under study. Finding a suitable numerical scheme to approximate the solution to the JMGT equation with appropriate boundary and initial conditions. In particular, I am interested in using a Finite Element, Finite Volume, or Finite Difference based on space discretizations i.e., advanced time stepping schemes, approach to finding solutions to this problem [22]. In conjunction to this, I would like to engage in research with undergraduate students in applying numerical methods to PDEs in their fields of interest.

- As the theoretical work, I am interested in pursuing other nonlinear variants of the JMGT equation. For instance the third order variant of Kuznetsov equation where there is an additional dependence [nonlinear] on the gradient.

- Control problems associated with the models. For instance, [11] optimal control for nonlinear wave equations; distributed control, boundary control of semilinear equations, control problems for coupled parabolic–hyperbolic and hyperbolic–hyperbolic systems as well as for stability (with respect to perturbations in the data) of the minimizer for optimal control problems. Moreover, I would like to derive efficient schemes for the numerical solution of associated optimal control problems.

2.5 Appendix

Barrier's method: The barrier method is applied to prove that local solutions exist globally and exhibit an exponential decay rate for sufficiently small initial data and the size of the initial data does not depend on time. Please see the following argument.

$$E(t) + \gamma \int_0^t E(s)ds \leqslant C_1 E(0) + C_2 \int_0^t E^{\alpha}(s)ds,$$

for $\alpha > 1$. Then

$$E(t) + \int_0^t E(s)[\gamma - C_2 E^{\alpha-1}(s)]ds \leqslant C_1 E(0). \tag{2.5.1}$$

Step 1: Let $t = 0$ and take small initial condition so that

$$E(0) \leqslant \rho \text{ and } \gamma - C_2 E^{\alpha-1}(0) > \gamma - C_3 \rho_0 > 0.$$

Step 2: Let t increase and use the property $E(t)$ changes continuously in t since a solution is continuous in t. As long as

$$\gamma - C_2 E^{\alpha-1}(s) > 0, \tag{2.5.2}$$

for $0 \leqslant s \leqslant t$, we have from (2.5.1)

$$E(t) \leqslant C_1 E(0), \tag{2.5.3}$$

where $s \leqslant t$.

Step 3: We want to prevent that there exists $T > 0$ such that

$$\gamma - C_2 E^{\alpha-1}(T) = 0. \tag{2.5.4}$$

Step 4: Suppose such T exists, by contradiction. Then for $s \leqslant t < T$, we have by (2.5.2) and (2.5.3)

$$E^{\alpha-1}(s) \leqslant C_1^{\alpha-1} E^{\alpha-1}(0).$$

$$\begin{aligned} C_2 E^{\alpha-1}(s) &\leqslant C_2 C_1^{\alpha-1} E^{\alpha-1}(0) \\ &\leqslant C_2 C_1^{\alpha-1} \rho^{\alpha-1} \end{aligned} \tag{2.5.5}$$

Then

$$\gamma - C_2 E^{\alpha-1}(s) \geqslant C_2 C_1^{\alpha-1} \rho^{\alpha-1} > \frac{\gamma}{2} > 0. \tag{2.5.6}$$

A condition independent on T depends only on constants of problem and small radius ρ of initial energy $E(0) \leqslant \rho$. Then (2.5.6) contradicts (2.5.4) since it cannot be since $E(t)$ is continuous, that

$$\text{up to } t < T \quad \gamma - C_2 E^{\alpha-1}(t) > \frac{\gamma}{2} > 0$$

$$\text{while at } t = T \quad \gamma - C_2 E^{\alpha-1}(T) = 0.$$

So such T finite does not exists, hence by choosing ρ small as in (2.5.5), we have that (2.5.2) holds for all t, hence (2.5.3) holds for all t. Thus

$$E(t) \leqslant C_1 E(0) \quad \text{for all } t \text{ upto } T = \infty.$$

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