2021

Hermitian projections and geometric properties of Banach spaces

Priyadarshi Dey

HERNITIAN PROJECTIONS AND GEOMETRIC PROPERTIES OF BANACH SPACES

by

Priyadarshi Dey

A Dissertation

Submitted in Partial Fulfillment of the

Requirements for the Degree of

Doctor of Philosophy

Major: Mathematical Sciences

The University of Memphis

August 2021

To my family and my (late) grandfather.

ACKNOWLEDGMENTS

First and foremost, I would like to express my gratitude to my advisor, Dr. Fernanda Botelho. I consider myself to be very fortunate to have her as an advisor. I am extremely grateful to her for the invaluable supervision, continuous support and infinite amount of patience during my PhD. In every possible way whatever I have achieved is because of her guidance. Her immense knowledge and her infinite patience have always helped me to stay motivated thorughout the PhD life. Undoubtedly, I could not have imagined a better advisor for my PhD study.

I would also like to thank my committee members Dr. John Haddock, Dr. Anna Kaminska, Dr. Irena Lasiecka, Dr. Bentuo Zheng for all the support and help.

I thank all my friends and colleagues for making my stay pleasant in Memphis. Especially, I would like to thank Monika for all the stimulating discussions, for everything that you have taught me and for all the fun we have had in the past years. I also thank my friends Marcelo, Abhraneel, Niraj, Manoj, Sonal, Yogesh(bro), Sovanlal, Tattwamasi for their constant motivation and support.

I am forever thankful to my family : my parents, my sister for supporting me throughout my journey and be there all the time.

ABSTRACT

The focus of this dissertation is twofolded. First, it deals with a thorough study of hermitian operators and hermitian projections on spaces of bounded operators and Banach spaces at large. Second, it concerns extensions and reinterpretations of results dealing with the characterization of geometric aspects of Banach spaces. This includes the characterization of extreme points of the unit ball of tensor product spaces and their impact on the form of the surjective isometries supported by those spaces.

The dissertation is divided into 4 chapters. The first chapter is the introduction, the second chapter describes the hermitian operators on several settings of spaces of continuous and integrable functions and on spaces of bounded operators. We also consider operators which are both Hermitian as well as projections. Characterizations of such operators is done for spaces of vector valued continuous functions and for spaces of bounded operators. In particular, for two Banach spaces X and Y under certain conditions, the characterizations of Hermitian projections is as follows.

Let (X, Y) *be an ideal pair of Banach spaces. Let* $P\colon \mathcal{B}(X,Y) \to \mathcal{B}(X,Y)$*. Then* P *is a Hermitian projection if and only if either there exists a Hermitian projection* $Q \in \mathcal{B}(X)$ *such that* $P(A) = AQ$ *for every* $A \in \mathcal{B}(X,Y)$*, or there exists an Hermitian projection* $R \in \mathcal{B}(Y)$ *such that* $P(A) = RA$ *for every* $A \in \mathcal{B}(X,Y)$*.*

Chapter 3 deals with the study of the extreme points on spaces of tensor products. We follow the approach given in the paper by Stegall and Ruess and extend their result for the tensor product of complex Banach spaces. The main point is the characterization of the extreme points of the injective tensor product $X \otimes_\epsilon Y$ but a description for the projective tensors are also derived. The knowledge

of the extreme points is very important in the derivation of the form for the surjective isometries.

In chapter 4 we investigate the theorem due to Jasorz for the characterization of isometries on the injective tensor product space $X \otimes_\epsilon Y$ under some assumptions on the component spaces X and Y. The last point of this work deals with the characterization of generalized bicircular projection on the injective tensor product X and K with the assumption that K is strictly convex.

CONTENTS

CHAPTER 1

INTRODUCTION

In quantum mechanics, certain operators play a crucial role representing measurable quantities called observables. A physical system is associated with a Hilbert space, and an element in the space is a function representing a state of the physical system that may include quantities such as position, energy, angular momentum and so on. The approach developed by von Neumann represents a measurement on a physical system under the action of a self-adjoint operator on that Hilbert space. The dimension of the Hilbert space may be infinite, as for example the space of square-integrable functions on a line, which is used to define the quantum physics of a continuous degree of freedom.

Hermitian operators emerged as generalizations to the Banach space setting of self-adjoint operators on Hilbert spaces. The notion of self-adjoint or hermitian operators in a Hilbert space has been extended to Banach spaces by Lumer and Vidav, cf. [26] and [38]. In 1973, Berkson and Porta have shown that several classical Banach spaces only support scalar type of Hermitian operator, see [4]. Hermitian operators have two desirable properties that form the basis of quantum mechanics. First, the eigenvalues of a Hermitian operator are real. Second, the eigenfunctions of Hermitian operators are orthogonal to each other or can be made orthogonal by taking linear combinations of them. Projections are operators with a very simple spectral structure, and of central importance in the spectral representation of Hermitian operators. Questions addressing characterization of classes of projections, Hermitian operators and isometries have been investigated by several researchers, see e.g, [8], [9] and [19]. This work contributes by answering questions related to this trend of investigation. It is important to mention that these three classes of operators are interconnected. These interconnections lead to an easy transport of properties among them.

In this Chapter, we study operators that are both Hermitian and projections, called Hermitian projections. We characterize the Hermitian operators whose square is Hermitian, and Hermitian operators with Hermitian square roots. We also derive the structure of Hermitian projections and Hermitian square roots of the identity. These problems are considered for Hermitian operators on several different settings. We start by recalling relevant definitions and background results showing the interconnection among these classes of operators. We consider similar questions involving Hermitian operators on vector-valued spaces of continuous functions. We include results that hold for C^*-algebras and spaces of bounded operators between Banach spaces.

Surjective isometries between Banach spaces play an important role in operator theory and in the geometry of Banach spaces. This class of operators have instrinsic connections with projections and hermitian operators as we have shown in Chapter 2. Moreover, the type of isometries supported by a Banach space give important information on geometric aspects of the space. For example surjective isometries induce bijections on the set of extreme points of the unit ball. This often permits the derivation of the form for these operators. These include surjective isometries on spaces of continuous functions which are given as weighted composition operators. The knowledge of the set of extreme points plays a crucial role in the characterization of the surjective isometries supported by the space, see [10], [20], [33], [28], [21]. In Chapter 3, we start with a survey on tensor product spaces and on the theory of extreme points. We give a description of isometries on tensor product spaces using the form of the extreme points of the dual space. We use the techniques in [34] and extend Theorem 3.0.12 for complex Banach spaces.

We modify the proof in [34] to extend to accommodate this new field of scalars. In this Chapter 4, we use the characterization of extreme points in Theorem 4.0.6, and we derive the form for the surjective isometries on between injective tensor products. We provide a proof following the ideas in [19] for real scalar Banach spaces, redefine several points of the proof given in [19] and derive the form for the generalized bi-circular projections associated with a class of isometries.

CHAPTER 2

PROJECTIONS

Hermitian operators emerged as generalizations to the Banach space setting of self-adjoint operators on Hilbert spaces. These classes of operators are often associated with measurable physical quantities or observables and appear in the formulation of aspects of physical phenomena. Projections are operators with a very simple spectral structure, and of central importance in the spectral representation of Hermitian operators. Questions addressing characterization of classes of projections, Hermitian operators and isometries have been investigated by several researchers, see e.g, [8], [9] and [19] and this work contributes by answering questions related to this trend of investigation. It is important to mention that these three classes of operators are interconnected. These interconnections lead to an easy transport of properties among them.

In this Chapter, we include results published as joint work with F. Botelho and D. Ilisevic in [2]. We study operators that are both Hermitian and projections, called Hermitian projections. We characterize the Hermitian operators whose square is Hermitian, and Hermitian operators with Hermitian square roots. We also derive the structure of Hermitian projections and Hermitian square roots of the identity. These problems are considered for Hermitian operators on several different settings. We start by recalling relevant definitions and background results showing the interconnection among these classes of operators. In Section 2, we consider similar questions involving Hermitian operators on vector-valued spaces of continuous functions. In Sections 2 and 2, we extend previous results to include C^*-algebras and spaces of bounded operators between Banach spaces.

Definitions and Background

We will first recall the well-known definitions of linear isometry and hermitian operator [10, Theorem 5.2.6]. Unless otherwise stated, X denotes a complex Banach space and $\mathcal{B}(X)$ is the space of all bounded linear operators on X.

Definition 2.0.1. *Let X and Y be Banach spaces. A linear operator $T\colon X \to Y$ is an isometry if $\|Tx\| = \|x\|$ for every $x \in X$.*

Definition 2.0.2. *Let X be a Banach space and $T \in \mathcal{B}(X)$. An operator T is said to be Hermitian if e^{itT} is an isometry, for all $t \in \mathbb{R}$.*

Example 2.0.3. *Let X be a Banach space. For $\lambda \in \mathbb{R}$, the map $T : X \to X$ given by $Tx = \lambda x$ is a Hermitian operator. Hermitian operators of this form are designated "trivial".*

It is worthy to mention that Banach spaces possess Hermitian operators (at least the trivial ones, that is, real multiples of the identity operator). In Section 2, we list several examples of Banach spaces for which the Hermitian operators are only trivial ones.

It has been shown that Hermitian projections are the bi-circular projections, e.g. [37, 12, 25, 16]. We recall that bi-circular projections are those projections P such that for every modulus 1 complex number λ, $P + \lambda(I - P)$ is an isometry. The next proposition formulates the relation between Hermitian projections and surjective isometries.

Proposition 2.0.4. *Let X be a Banach space and $P\colon X \to X$ be a projection. Then the following statements are equivalent:*

(i) P is Hermitian.

(ii) $P + e^{it}(I - P)$ *is an isometry for all* $t \in \mathbb{R}$.

(iii) $\|P + e^{it}(I - P)\| \leq 1$ *for all* $t \in \mathbb{R}$.

Proof. The equivalence of (i) and (ii) is done in [19, Lemma 2.1]. Let us suppose (iii) holds and write $T_t = P + e^{it}(I - P)$, for every $t \in \mathbb{R}$. Then $T_t^{-1} = T_{-t}$, and

$$\|x\| = \|T_{-t}(T_t(x))\| \leq \|T_{-t}\|\|T_t(x)\| \leq \|T_t(x)\| \leq \|x\| \qquad (x \in X),$$

which implies $\|T_t(x)\| = \|x\|$, hence (ii) holds. □

Before we give more examples, we are going to recall one more definition from [14]. These are examples of hermitian projections.

Definition 2.0.5. *Let* $p \in (1, \infty)$. *A projection* P *on* X *is called an*

- *M-projection if* $\|x\|_m = \max\{\|Px\|, \|x - Px\|\}$, *for all* $x \in X$.
- *L-projection if* $\|x\|_1 = \|Px\| + \|x - Px\|$, *for all* $x \in X$.
- *p-projection if* $\|x\|_p = (\|Px\|^p + \|x - Px\|^p)^{1/p}$, *for all* $x \in X$.

It is easy to see that M, L and L_p projections are hermitian. In fact for the M-projection, we observe that, for every real number t we have

$$\begin{aligned}
\|e^{itP}x\|_m &= \| \left(P + e^{it}(I - P)\right)(x)\| \\
&= max\{\|P\left(P + \lambda(I - P)\right)(x)\|, \|(I - P)\left(P + e^{it}(I - P)\right)(x)\|\} \\
&= max\{\|Px\|, \|(I - P)x\|\} = \|x\|_m.
\end{aligned}$$

Similar considerations hold for L-projections. Given $1 < p < \infty$ then

$$\begin{aligned} \|x\|_p &= (\|Px\|^p + \|(I-P)x\|^p)^{1/p} \\ &= \left(\|e^{it}Px\|^p + \|(I-P)x\|^p\right)^{1/p} \\ &= \|(e^{it}P + (I-P))x\|_m. \end{aligned}$$

Hermitian projections are bi-contractive, i.e. they are of norm 1 and the complementary projection also has norm 1. A special class of bi-contractive and non-hermitian projections are those for which e^{itP} is an isometry for some values of t different from a multiple of 2π but not all. These projections are called generalized bi-circular and they have characterized on different settings, see[25], [**?**], [3].

Trivial Hermitian projections

As we have mentioned before, alhough, most classical Banach spaces possess non-trivial Hermitian projections (i.e. $\neq I$ and 0), we collect below several examples of spaces supporting only trivial ones. It is clear that the set of hermitian operators defined on a Banach space is always nonempty as it contains all real multiples of the identity. If the set of hermitian operators on a given Banach space reduces to just real multiples of the identity then we say it is Hermitian trivial. We list some of these spaces from [4] and [5]:

- $\text{Lip}[0,1]$, the space of all complex-valued Lipschitz functions on $[0,1]$ with $\|f\| = \|f\|_\infty + \text{ess sup}\,|f'|$;
- lip_α, $0 < \alpha < 1$, the space of all complex-valued functions f on $\mathbb{R}$ of period 1 such that $\sup_{x\in\mathbb{R}} |f(x+h) - f(x)| = o(|h|^\alpha)$, as $h \to 0$, with

$$\|f\| = \sup_{x,y,h}\{|f(x)|, |h|^{-\alpha}|f(y+h) - f(y)|\};$$

- AC$[0,1]$, the space of all absolutely continuous functions on $[0,1]$ with norm $\|f\| = \|f\|_\infty + \|f'\|_1$;
- $C^1[0,1]$, the space of continuously differentiable complex-valued functions on $[0,1]$ with $\|f\| = \|f\|_\infty + \|f'\|_\infty$;
- Hardy space $H^p(\mathbb{D})$, $1 \leq p \leq \infty$, $p \neq 2$, of all analytic functions f on the open unit disc $\mathbb{D}$ such that $f_r(\theta) = f(re^{i\theta})$, $0 < r < 1$, are uniformly bounded in the $L_p[0, 2\pi]$ norm.

Hermitian square root of identity

We start with a proposition from [2] that establishes the relation between the square root of identity, the surjective isometries and Hermitian operators.

Proposition 2.0.6. *Let X be a Banach space and let H be a Hermitian operator on X. Then H is a surjective isometry if and only if $H^2 = I$.*

Proof. Since H is Hermitian its spectrum is real (e.g. [30, Theorem 2.6.7] or [10, Theorem 5.2.6]). If H is a surjective isometry then its spectrum is contained in the unit circle (e.g. [23, p. 80]). Therefore $\sigma(H) \subseteq \{-1, 1\}$, hence $\sigma(H^2) = \{1\}$. Since H^2 is also a surjective isometry, [22, Theorem 5] implies that X can be decomposed to $Y \oplus Z$ with Y the eigenspace of 1 and $H^2(Z) \subseteq Z$. If Z is not zero then the restriction of H^2 to Z has nonempty spectrum which must be 1. Hence $X = Y$ is the eigenspace of 1, which implies $H^2 = I$.

Conversely, if $H^2 = I$ then $e^{itH} = \cos t\, I + i \sin t\, H$ is an isometry for every $t \in \mathbb{R}$. In particular for $t = \frac{\pi}{2}$, which implies that H is an isometry. Since $H^2 = I$, it is surjective. This completes the proof.

□

Hermitian surjective isometries are the square roots of the Identity, also called isometric reflections.

It is interesting to recall the existence of Banach spaces with isometry group equal to $\{Id, -Id\}$, see [27]. We may reformulate this by stating that the isometry group consists of hermitian operators.

A reflection on a Banach space X is a linear map $T\colon X \to X$ such that $T^2 = I$. A natural question is to ask whether an isometric reflection must be Hermitian. In order to prove that the answer is negative we first prove the following result.

Corollary 2.0.7. *Let X be a Banach space and let $H \in \mathcal{B}(X)$. Then the following statements are equivalent:*

(i) H is a Hermitian square root of the identity.

(ii) $(I + H)/2$ is a Hermitian projection.

Proof. Let us first note that H is a square root of the identity if and only if $P = (I + H)/2$ is a projection. If H is a Hermitian square root of the identity then, for every $t \in \mathbb{R}$,

$$e^{itH} = \cos t\, I + i \sin t\, H = (\cos t - i \sin t)\, I + 2i \sin t\, P = e^{it} P + e^{-it}(I - P).$$

This implies

$$\|e^{itH}x\| = \|e^{it}Px + e^{-it}(I - P)x\| = \|Px + e^{-2it}(I - P)x\| = \|x\| \qquad t \in \mathbb{R}, x \in X.$$

Therefore P is a Hermitian projection. If $P = (I + H)/2$ is a Hermitian projection then, Proposition 2.0.4 implies that $H = P - (I - P)$ is an isometry. This completes the proof. $\square$

Corollary 2.0.7 provides an easy tool to check whether a square root of the identity is Hermitian, as we shall see in the following example.

Recall that for a square matrix A the spectral norm of A is defined as the square root of the maximum eigenvalue of $A^T A$, i.e.,

$$\|A\|_2 = \max_{|x|_2 \neq 0} \frac{|Ax|_2}{|x|_2}.$$

Let us look at the following Example.

Example 2.0.8. *Let $M_2(\mathbb{C})$ be equipped with the spectral norm and let $H\colon M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be defined by*

$$H \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ -\gamma & \delta \end{bmatrix}.$$

Then H is an isometric square root of the identity but it is not Hermitian since $P = (I + H)/2$ is not a Hermitian projection: for $x = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ we have

$$\|(P + i(I - P))(x)\| = \sqrt{2} \neq 2 = \|x\|,$$

hence $P + i(I - P)$ is not an isometry. The conclusion follows from Corollary 2.0.7.

Remark 2.0.9. *In Proposition 2.0.6 we mention three conditions:*

(a) H is Hermitian,

(b) H *is a surjective isometry,*

(c) H *is a reflection (that is,* $H^2 = I$*),*

and prove that (a)+(b) implies (c) and (a)+(c) implies (b). What about (b)+(c) implies (a)? In order to get an affirmative answer, we recall the definition of a JB-triple.*

Definition 2.0.10. *A complex Banach space* X *is a JB*-triple if it carries a triple product* $X \times X \times X \to X$, $(x, y, z) \mapsto \{xyz\}$ *for all* $x, y, z \in X$ *and a box product* $a\Box b^* \colon X \to X$ *defined by* $z \mapsto \{abz\}$ *with the following properties.*

1. $\{x, y, z\}$ *is symmetric complex bilinear in the variable* x, z *and conjugate linear in* y.
2. $[a\Box b^*, x\Box y^*] = \{abx\}\Box y^* - x\Box\{yab\}^*$.
3. $a\Box a^*$ *as a linear operator on* X, *is Hermitian and has spectrum* ≥ 0.
4. $\|a\Box a^*\| = \|a\|^2$.

Example 2.0.11. *Most common examples of a JB*-triples are Hilbert spaces and spaces of scalar valued continuous functions* $C(\Omega)$.

- *Let* $\mathcal{H}$ *be a Hilbert space with inner product* $\langle \cdot, \cdot \rangle$*,then* $\mathcal{H}$ *endowed with the triple product*
$$\{xyz\} = \frac{\langle x, y\rangle z + \langle z, y\rangle x}{2}$$
is a JB-triple.*

- *The space* $C(\Omega)$ *of all complex valued continuous functions on a compact Hausdorff space* Ω *endowed with the triple product*
$$\{fgh\} = f\overline{g}h$$

with a triple product is a JB-triple.*

More generally, C^*-algebra with the triple product given by

$$\{xyz\} = \frac{xy^*z + zy^*x}{2}$$

also defines a JB-triple.*

The following Proposition answers the question mentioned in the Remark 2.0.9.

Proposition 2.0.12. *Let X is a JB*-triple, $H: X \to X$ is an isometric reflection such that $I + H$ has rank one. Then H is Hermitian.*

Proof. Let H is an isometric reflection such that $I + H$ has rank one. Set $P = \frac{I+H}{2}$. Then P is a rank one bicontractive projection. Then [16, Theorem 2.2] implies that P is a Hermitian projection and therefore H is Hermitian. □

Remark 2.0.13. *Example 2.0.8 shows that it is not true in general ($I + H$ in Example 2.0.8 has rank two).*

Hermitian operators and the hermitian square root of identity for Function Spaces

In this section we derive the form of the Hermitian projections on several vector valued function spaces. Let Ω denote a topological space and E a Banach space, let $\mathcal{F}(\Omega, E)$ be a space consisting of functions defined on Ω with values in E. We assume that $\mathcal{F}(\Omega, E)$ contains the constant functions and satisfies the condition: *Every Hermitian operator H, on $\mathcal{F}(\Omega, E)$, is of the following form:*

$$Hf(t) = A(t)f(t), \qquad f \in \mathcal{F}(\Omega, E),\ t \in \Omega, \tag{2.0.1}$$

where A is a function on Ω and with values in the space of Hermitian operators on E.

Proposition 2.0.14. *Let H be a Hermitian operator on $\mathcal{F}(\Omega, E)$ given by* (2.0.1)*. Then the following hold.*

(i) H has a Hermitian square root if and only if $A(t)$ has an Hermitian square root on E, for every $t \in \Omega$.

(ii) H is a square root of the identity if and only if $A(t)$ is a square root of the identity on E, for every $t \in \Omega$.

(iii) H is a n^{th} root of the identity if and only if $A(t)$ is a n^{th} root of the identity on E, for every $t \in \Omega$.

Proof. we will give a sketch of proof for (i). (ii) and (ii) follow exactly the same. For (i), let us assume that H is Hermitian such that $H^2 = I$. Then

$$H^2 f(t) = f(t) \quad \text{for every } t \in \Omega, f \in \mathcal{F}(\Omega, E)$$
$$\text{which implies } A(t)^2 f(t) = f(t)$$

Therefore $A(t)$ has a square root of identity. On the other hand the hermicity of $A(t)$ follows from the hermicity of H. The other direction is easy. □

Spaces that support Hermitian operators of the form described by equation (2.0.1) include:

- $C(\Omega, E)$, the space of all continuous functions $f : \Omega \to E$, with Ω be a compact Hausdorff space, cf. [9, Theorem 4].

- $L^p(\Omega, E)$ $(1 \le p < \infty)$, the space of all strongly measurable functions $f\colon \Omega \to X$ such that $\int_\Omega \|f\|^p d\mu < \infty$, where (Ω, Σ, μ) a finite measure space and E a separable Banach space, equipped with the norm $\|f\|_{L^p(\Omega,X)} = \Big(\int_\Omega \|f\|^p d\mu \Big)^{1/p}$, see [36, Theorem 4.2]. We recall that a function $f\colon \Omega \to X$ is strongly measurable if there exists a sequence of simple function $\{f_n\}$ such that $\lim_n \|f_n - f\| = 0$ almost everywhere.

- $L_\Phi(\mu, E)$, the space of strongly measurable functions from Ω to E for which there is $\lambda > 0$ such that

$$\int \Phi\left(\frac{|f(t)|}{\lambda}\right) d\mu < \infty,$$

  where Φ denotes a continuous strictly increasing convex function on $[0, \infty)$ with $\Phi(0) = 0$, $\Phi(1) = 1$, $\lim_{t\to\infty} \Phi(t) = \infty$ and $\Phi(t) \neq t^2$. This space endowed with the Luxemburg norm,

$$\|f\| = \inf\left\{\lambda \,\middle|\, \Phi\left(\frac{|f(t)|}{\lambda}\right) \le 1\right\},$$

  is a Banach space, see [18, Theorem 2.5].

- For a compact 2-connected metric space X and for a complex Banach space E endowed with a norm $\|\cdot\|_E$, $\mathrm{Lip}(X, E)$ denotes the space of all functions $f\colon X \to E$ such that

$$L(f)\colon = \sup_{x\neq y} \frac{\|f(x) - f(y)\|_E}{d(x, y)} < \infty$$

  with the norm $\|f\| = \max\{L(f), \|f\|_\infty\}$.

The next proposition describes the Hermitian square roots of the identity on $C(\Omega)$, with Ω a compact Hausdorff space.

Proposition 2.0.15. *Let Ω be a compact Hausdorff space. Then H is a Hermitian square root of the identity on $C(\Omega)$ if and only if there exists Ω_0, a clopen subset of Ω, such that $H(f) = f \cdot \chi_{\Omega_0} - f \cdot \chi_{\Omega \setminus \Omega_0}$, for every $f \in C(\Omega)$.*

Proof. Let H be a Hermitian square root of the identity operator. Then $e^{itH} = \cos t\, I + i \sin t\, H$. Since H and e^{itH} are surjective isometries on $C(\Omega)$, by the Banach-Stone theorem there exist modulus 1 continuous functions τ and μ_t and homeomorphisms of Ω, φ and ψ_t such that

$$\cos t\, f(x) + i \sin t\, \tau(x) f(\varphi(x)) = \mu_t(x)\, f(\psi_t(x)), \qquad x \in \Omega, f \in C(\Omega), t \in \mathbb{R}.$$

Setting f equal to the constant function equal to 1, we obtain $\cos t + i \sin t\, \tau(x) = \mu_t(x)$. Since μ_t is a unimodular function, and setting $\tau(x) = a(x) + ib(x)$, where a and b are real valued functions, we have $(\cos t - b(x) \sin t)^2 + a(x)^2 \sin^2 t = 1$. Therefore, for all $t \in \mathbb{R}$, $-2b(x) \cos t \sin t = 0$. This implies $b(x) = 0$. Thus $\tau(x) = 1$ or -1.

If there exist $t \in \mathbb{R}$ and $x_0 \in \Omega$ such that $\psi_t(x_0) \notin \{x_0, \varphi(x_0)\}$, then setting f to be a Urysohn's function with $f(x_0) = f(\varphi(x_0)) = 0$ and $f(\psi_t(x_0)) = 1$ we arrive at a contradiction. Thus for every $t \in \mathbb{R}$ and every $x \in \Omega$ we have $\psi_t(x) = x$ or $\psi_t(x) = \varphi(x)$. Suppose that there exists $x_0 \in \Omega$ such that $\varphi(x_0) \neq x_0$. For $t = \frac{\pi}{4}$ let f be a Urysohn's function with $f(\varphi(x_0)) = 0$ and $f(x_0) = 1$. Then $f(\psi_t(x_0))$ can always be chosen equal to either 0 or 1, and we have $\frac{\sqrt{2}}{2} = \mu_t(x_0)\, f(\psi_t(x_0))$, which is impossible. Hence $\psi_t(x) = x = \varphi(x)$ for every $t \in \mathbb{R}$ and $x \in \Omega$. Then $\mu_t(x) = e^{it}$ or $\mu_t(x) = e^{-it}$, depending on the sign of τ. Let

$$\Omega_0 = \{x \in \Omega : \tau(x) = 1\},\ \Omega \setminus \Omega_0 = \{x \in \Omega :\ \tau(x) = -1\},\ \text{and}\ \ H(f) = f \cdot \chi_{\Omega_0} - f \cdot \chi_{\Omega \setminus \Omega_0}.$$

It follows that both sets Ω_0 and $\Omega \setminus \Omega_0$ are closed. This completes the proof. □

Remark 2.0.16. *It follows from Proposition 2.0.15 and Corollary 2.0.7 that P is a Hermitian projection on $C(\Omega)$ if and only if there exists a clopen subset, Ω_0, of Ω such that $P(f) = f \cdot \chi_{\Omega_0}$. The Hermitian projections on $C(\Omega)$ are the M-projections.*

Hermitian square roots of the identity, not associated with an M-projection, exist. Just consider H on ℓ_2 given by

$$H(x_1, x_2, x_3, \cdots) = (-x_1, x_2, x_3, \cdots).$$

The associated projection $P : \ell_2 \to \ell_2$ is given by $P(x_1, x_2, x_3, \cdots) = (0, x_2, x_3, \cdots)$. Let $x = (1, 1, 1, 0, \cdots)$, then $\|x\|_2 = \sqrt{3}$, $\|Px\|_2 = \sqrt{2}$, and $\|(I - P)x\|_2 = 1$, thus $\|x\|_2 \neq \max\{\|Px\|_2, \|(I - P)x\|_2\}$. Hence P is not an M-projection. It is well known that the Hermitian projections on a Hilbert space setting are the orthogonal projections.

C^*-algebras

Let A be a C^*-algebra, that is, a complex Banach $*$-algebra $(A, \| \, . \, \|)$ such that $\|x^*x\| = \|x\|^2$ for all $x \in A$. Let us recall that $x \in A$ is self-adjoint if $x^* = x$, positive if $x = y^*y$ for some $y \in A$, a projection if $x = x^* = x^2$, and (if A has unit 1) unitary if $x^*x = xx^* = 1$. Every positive element of A has a unique positive square root. If $x \in A$ is such that $xAx = 0$ then $xx^*x = 0$ and $(x^*x)^2 = 0$. Hence $\|x\|^4 = \|x^*x\|^2 = \|(x^*x)^2\| = 0$ and $x = 0$. This proves that all C^*-algebras are semiprime. The C^*-algebra of all bounded operators, as well as the C^*-algebra of all compact operators, on some Hilbert space, is prime (for $x, y \in A$, $xAy = 0$ implies $x = 0$ or $y = 0$). An ideal I of A is said to be essential if its annihilator $I^\perp = \{x \in A : xI = Ix = 0\}$ is zero. We write $I^{\perp\perp}$ for the annihilator of $I^\perp$. For any ideal I, the ideal $I \oplus I^\perp$ is an essential ideal of A. The set

$M(A) = \{x \in A^{**} : xA \subseteq A, \, Ax \subseteq A\}$, where A^{**} is the double dual of A, can be equipped with addition, multiplication, involution and norm, so that it becomes a C^*-algebra which is called the multiplier algebra of A. A popular construction of the multiplier of A, involves the notion of double centralizer. This consists of pairs of operators on A, (f, g) satisfying the property:

$$xf(y) = g(x)y,$$

for all $x, y \in A$. It is possible to define an algebra structure on the space of such pairs. This algebra has unit and contains A as an essential ideal. The algebra $M(A)$ is the maximal unitization of A in the category of C^*-algebras such that A is an essential ideal of $M(A)$. For more details we refer the reader to [1].

By [1, Theorems 4.1.27 and 4.1.28], a bounded linear operator $T \colon A \to A$ is Hermitian if and only if there exist self-adjoint $a, b \in M(A)$ such that $T(x) = ax + xb$ for every $x \in A$.

We give an example of a Hermitian operator whose square is not Hermitian. We define $H \colon M_2(\mathbb{R}) \to M_2(\mathbb{R})$ as follows

$$H \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x & y \\ z & w \end{bmatrix}}_{W} + \underbrace{\begin{bmatrix} x & y \\ z & w \end{bmatrix}}_{W} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{B} = \begin{bmatrix} x+z & w \\ x+z & y \end{bmatrix}$$

Clearly H is Hermitian on the C^*-algebra $M_2(\mathbb{R})$. Also

$$H^2 \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 2(x+z) & y \\ 2(x+z) & w \end{bmatrix} = A^2W + 2AWB + WB^2 = CW + WD$$

If H^2 were Hermitian, then the operator T given by

$$T\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

was Hermitian, then $TW = \frac{1}{2}[A^2 - C]W + \frac{1}{2}W[B^2 - D]$ for some Hermitian operators C and D. This is not possible since

$$TW = \begin{bmatrix} z & w \\ x & y \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} z & 0 \\ x & 0 \end{bmatrix}, T^2W = \begin{bmatrix} z & 0 \\ x & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence

$$e^{itT}W = W + it\begin{bmatrix} z & 0 \\ x & 0 \end{bmatrix} + \frac{(it)^2}{2!}\begin{bmatrix} z & 0 \\ x & 0 \end{bmatrix} + \cdots = e^{it}\begin{bmatrix} z & 0 \\ x & 0 \end{bmatrix} + \begin{bmatrix} x-z & y \\ z-x & w \end{bmatrix}$$

Setting $W = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, we have $\|W\| = 1$. We have

$$e^{itT}W = \begin{bmatrix} \sqrt{2} + \frac{e^{it}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\sqrt{2} - \frac{e^{it}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Taking $t = 0$, $\left\| \begin{bmatrix} \sqrt{2} + \frac{e^{it}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\sqrt{2} - \frac{e^{it}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \right\| > 1$. Therefore, H^2 is not Hermitian.

We now state a theorem from [2] and we include its proof for completeness of exposition.

Theorem 2.0.17. *Let A be a C^*-algebra. If $H \in \mathcal{B}(A)$ is such that both H and H^2 are Hermitian, then there exist an ideal $\mathcal{I}$ of A and a self-adjoint $v \in M(A)$ such that $H(x) = vx$ for every $x \in I^{\perp}$ and $H(x) = xv$ for every $x \in I^{\perp\perp}$. In particular, if*

$H \in \mathcal{B}(A)$ is Hermitian with a Hermitian square root then such v is positive.

Proof. Since H is Hermitian there exist self-adjoint $a, b \in M(A)$ such that $H(x) = ax + xb$ for every $x \in A$. Since H^2 is also Hermitian there exist self-adjoint $c, d \in M(A)$ such that $H^2(x) = cx + xd$ for every $x \in A$. Then

$$cx + xd = H(H(x)) = aH(x) + H(x)b = a^2x + 2axb + xb^2, \qquad x \in A,$$

that is,

$$(c - a^2)x + x(d - b^2) = 2axb, \qquad x \in A. \tag{2.0.2}$$

Using a standard method, as in the proof of [37, Lemma 3.1], we get

$$[a, x]z[b, y] = 0, \qquad x, y, z \in A, \tag{2.0.3}$$

where we write $[u, v]$ for $uv - vu$. Let $v = a + b \in M(A)$. Let $I = A[A, b]A$. Then $[I^\perp, b] = 0$, hence $H(x) = vx$ for every $x \in I^\perp$. By (2.0.3), $[a, A]I = 0$, hence $[a, A] \subseteq I^\perp$. Then $[a, I^{\perp\perp}] = 0$, which implies $H(x) = xv$ for every $x \in I^{\perp\perp}$. □

Remark 2.0.18. *If H^2 is the identity in the first statement of Theorem 2.0.17, then v is a self-adjoint unitary. Indeed, $v^2x = x$ for every $x \in I^\perp$ and $xv^2 = x$ for every $x \in I^{\perp\perp}$. Since $I^{\perp\perp}$ is a norm closed ideal, it is self-adjoint, and then we also have $v^2x = x$ for every $x \in I^{\perp\perp}$. Thus $(v^2 - 1)(I^\perp \oplus I^{\perp\perp}) = 0$ implies $v^2 = 1$.*

As an immediate consequence of Theorem 2.0.17 we get the following result (c.f. [12, Theorem 3.3]).

Corollary 2.0.19. *Let A be a C^*-algebra. If $P \in \mathcal{B}(A)$ is a Hermitian projection then there exist an ideal I of A and a projection $p \in M(A)$ such that $P(x) = px$ for every $x \in I^\perp$ and $P(x) = xp$ for every $x \in I^{\perp\perp}$.*

Remark 2.0.20. *By [12, Example 3.1] the converse of Corollary 2.0.19 is true in the case when* $I^{\perp} \oplus I^{\perp\perp} = A$*. It would be interesting to characterize those* C^**-algebras in which* $I^{\perp} \oplus I^{\perp\perp} = A$ *for every ideal* I *of* A*. It is obviously true if* A *is prime (in this case either* $I^{\perp} = 0$ *or* $I^{\perp\perp} = 0$*). It is also true if* A *is a* C^**-algebra of (not necessarily all) compact operators on some Hilbert space (see [12, Corollary 3.8]) or a von Neumann algebra (see [12, Corollary 3.9]).*

Furthermore, let us recall that an AW^**-algebra is a* C^**-algebra* A *with the property that the left annihilator of each right ideal is of the form* Ap *for some projection* $p \in A$*. Then* $I^{\perp} = Ap = pA$ *and* $I^{\perp\perp} = A(1-p) = (1-p)A$*, which implies* $I^{\perp} \oplus I^{\perp\perp} = A$*.*

Remark 2.0.21. *In case of prime or commutative* C^**-algebras or* C^**-algebras of compact operators or von Neumann algebras, if both* H *and* H^2 *are Hermitian, then* H^n *is Hermitian for every positive integer* n*, and if a Hermitian* H *has a Hermitian square root then it has a Hermitian* n*th root for every positive integer* n*.*

Characterization of Hermitian projections for spaces of bounded operators

In this section we consider Hermitian projections on spaces of bounded operators. We use the interdependence of Hermitian projections and surjective isometries to derive the form of these projections on spaces of bounded operators between two Banach spaces. We show that Hermitian projections in such a setting, are either a multiplication on the left by a Hermitian projection or a multiplication on the right by a Hermitian projection.

The following proposition collects equivalent conditions for the hermicity of projections on spaces of bounded operators.

Proposition 2.0.22. *Let X and Y be Banach spaces. Let $Q \in \mathcal{B}(X)$ be a projection. Then the following statements are equivalent:*

(i) *Q is a Hermitian projection on $\mathcal{B}(X)$.*

(ii) *$2Q - I_X$ is a Hermitian square root of the identity of X.*

(iii) *$Q + \lambda(I_X - Q)$ is a surjective isometry of $\mathcal{B}(X)$ for every modulus one $\lambda \in \mathbb{C}$.*

(iv) *$A \mapsto AQ$ is a Hermitian projection on $\mathcal{B}(X, Y)$.*

(v) *$A \mapsto A(2Q - I_X)$ is a Hermitian square root of the identity of $\mathcal{B}(X, Y)$.*

(vi) *$A \mapsto A(Q + \lambda(I_X - Q))$ is a surjective isometry of $\mathcal{B}(X, Y)$ for every modulus one $\lambda \in \mathbb{C}$.*

Proof. We start by showing that (i) and (ii) are equivalent. We observe that

$$e^{it(2Q-I)} = e^{-it}e^{2itQ},$$

then $e^{it(2Q-I)}$ is an isometry, for every t, if and only if e^{itQ} is an isometry, for all $t \in \mathbb{R}$. Further, Q is a projection if and only if $2Q - I$ is a square root of the identity. The equivalence between (i) and (ii) is established.

Since Q is a hermitian projection then $e^{itQ} = (I - Q) + e^{it}Q$ is a surjective isometry and the equivalence between (i) and (iii) follows.

Let $S : \mathcal{B}(X, Y) \to \mathcal{B}(X, Y)$ be given by $S(A) = AQ$. The equivalence between (i) and (iv) can be shown from the relation

$$e^{itS}A = A((I - Q) + e^{it}Q).$$

Hence the norm preserving condition of e^{itS} is equivalent to that of $(I-Q)+e^{it}Q$. Moreover, Q being a projection implies that $S^2(A) = AQQ = AQ = S(A)$ and S is also a projection. Conversely, $S^2 = S$ implies that $AQ = AQ^2$ for all A then $Q^2 = Q$. Similar reasoning applies to prove the remaining two statements. □

Characterization of hermitian projections on spaces of bounded operators

We give a characterization for the hermitian projections on spaces of bounded operators $T : X \to Y$, with X and Y are Banach spaces satisfying some additional conditions. We start by recalling the staement of the Fong-Sourour Theorem, which is the main tool for the derivation of the form for the hermitian projections.

Fong-Sourour Theorem in [11, Theorem 1] is formulated for operators from a Banach space into itself but it also holds for operators between two (possibly different) Banach spaces.

Let X and Y be Banach spaces. Let $\{A_i\}_{i=1,\ldots,m}$ and $\{B_i\}_{i=1,\ldots,m}$ be bounded operators in $\mathcal{B}(Y)$ and in $\mathcal{B}(X)$, respectively. Let Φ be an elementary operator on $\mathcal{B}(X,Y)$ given by:

$$\Phi(T) = A_1TB_1 + A_2TB_2 + \cdots + A_mTB_m.$$

Theorem 2.0.23. *(cf. [11, Theorem 1]) If $\Phi(T) = 0$, for all $T \in \mathcal{B}(X,Y)$, then either $\{B_1, B_2, \ldots B_m\}$ is linearly independent and $A_1 = A_2 = \cdots = A_m = 0$ or $\{B_1, B_2, \ldots, B_m\}$ is linearly dependent. In the latter case, let $\{B_1, B_2, ..., B_n\}$, $n < m$, be linearly independent, and (c_{kj}) denote constants for which*

$$B_j = \sum_{k=1}^{n} c_{kj} B_k, \qquad n+1 \le j \le m.$$

Then $\Phi(T) = 0$*, for all* $T \in \mathcal{B}(X, Y)$*, if and only if*

$$A_k = -\sum_{j=n+1}^{m} c_{kj} A_j, \qquad 1 \leq k \leq n.$$

Our derivation of the form for the Hermitian projections on $\mathcal{B}(X, Y)$ we rely on the form of the surjective isometries supported on these spaces. We employ a characterization due to Khalil and Saleh for surjective isometries on $\mathcal{B}(X, Y)$ for "ideal" pairs of Banach spaces (X, Y).

Characterization of hermitian projection on $\mathcal{B}(X, Y)$, with (X, Y) an ideal pair

First, we review the definition of an "ideal" pair from [21] and we recall the definition of M-ideal.

A subspace J of a Banach space X is said to be an M-ideal in X if and only if the annihilator of J (i.e. $J^{\perp}$ of all functionals in X^* vanishing on J) is an L-summand in X^*, $X^* = J^* \oplus_1 J^{\perp}$.

Definition 2.0.24. *A pair of Banach spaces* (X, Y) *is called an ideal pair if*

1. X *and* Y *are reflexive.*
2. X *and* Y^* *are strictly convex.*
3. X^* *has the metric approximation property.*
4. $\mathcal{K}(X, Y)$ *is an M-ideal in the space* $\mathcal{B}(X, Y)$*.*

Example 2.0.25. *We give some examples of "ideal" pairs:*

- *For* $1 < p \leq q < \infty, (\ell^p, \ell^q)$ *is an ideal pair, see [7].*

- *For two non-isometric Hilbert Spaces $\mathcal{H}$ and $\mathcal{K}$, $(\mathcal{H}, \mathcal{K})$ is an ideal pair.*

We refer the reader to the paper [21] for several other examples of ideal pairs of Banach spaces.

For ideal pairs of Banach spaces, a surjective isometry T of $\mathcal{B}(X, Y)$ is of one of the following forms:

- $T : A \mapsto UAV$, for some surjective isometries $U \in \mathcal{B}(Y)$ and $V \in \mathcal{B}(X)$, see [21, Theorem 1.2], and also [35, Theorem 12], or
- $T : A \mapsto UA^*V$, for some surjective isometries $U\colon X^* \to Y$ and $V\colon X \to Y^*$.

Theorem 2.0.26. *[2, Theorem 5.3] Let (X, Y) be an ideal pair of Banach spaces. Let $P\colon \mathcal{B}(X, Y) \to \mathcal{B}(X, Y)$. Then P is a Hermitian projection if and only if either there exists a Hermitian projection $Q \in \mathcal{B}(X)$ such that $P(A) = AQ$ for every $A \in \mathcal{B}(X, Y)$, or there exists an Hermitian projection $R \in \mathcal{B}(Y)$ such that $P(A) = RA$ for every $A \in \mathcal{B}(X, Y)$.*

Proof. Let P be an Hermitian projection. Let us fix a modulus one complex number $\lambda \neq \pm 1$ and set $T = P + \lambda(I - P)$. Then T is a (surjective) isometry satisfying $T^2 - (\lambda + 1)T + \lambda I = 0$. If T is of the standard form, i.e. $T\colon A \mapsto UAV$ for some surjective isometries $U \in \mathcal{B}(Y)$ and $V \in \mathcal{B}(X)$, then we have

$$U^2AV^2 - (\lambda + 1)UAV + \lambda A = 0, \qquad A \in \mathcal{B}(X, Y). \tag{2.0.4}$$

If V^2 and V are linearly independent then, there exist scalars α and β such that $I = \alpha V + \beta V^2$. Then (2.0.4) becomes $(U^2 + \lambda\beta I)AV^2 + (-(\lambda + 1)U + \lambda\alpha I)AV = 0$. Then $-(\lambda + 1)U + \lambda\alpha I = 0$ and $U = \mu I_Y$ for some modulus one $\mu \in \mathbb{C}$ ($\mu = \frac{\lambda\alpha}{\lambda+1}$). We enter this information in (2.0.4) to obtain $\mu^2V^2 - (\lambda + 1)\mu V + \lambda I_X = 0$. Hence

$T\colon A \mapsto \mu AV$. This implies that $Q \in \mathcal{B}(Y)$ defined by $Q = \frac{\mu V - \lambda I_X}{1-\lambda}$ is a projection and $P(A) = AQ$ for every $A \in \mathcal{B}(X,Y)$. Straightforward calculations show that Q is an Hermitian projection. If V^2 and V are linearly dependent then $V = \mu I_X$, for some modulus one $\mu \in \mathbb{C}$, and similar reasoning applies.

If X and Y^* are isometric, and the surjective isometry T is of the form $T\colon A \mapsto UA^*V$, for some surjective isometries $U\colon X^* \to Y$ and $V\colon X \to Y^*$, we shall show that leads to no solutions. Since X is reflexive and $U^*\colon Y^* \to X^{**}$ we also denote by U^* the surjective isometry from Y^* onto X; similar notation is used for V^*. Moreover, the reflexivity of both X and Y allows the identification of any $A \in \mathcal{B}(X,Y)$ with A^{**}. We prove that this case yields no Hermitian projections. Let $P\colon \mathcal{B}(X,Y) \to \mathcal{B}(X,Y)$ be a Hermitian projection induced by such a nonstandard isometry. Let us fix any modulus one $\lambda \neq \pm 1$ and define $T = P + \lambda(I - P)$. Suppose that $T\colon A \to UA^*V$. Then $T^2 - (\lambda+1)T + \lambda I = 0$ can be written as

$$UV^*AU^*V - (\lambda+1)UA^*V + \lambda A = 0, \qquad A \in \mathcal{B}(X,Y). \tag{2.0.5}$$

This first implies

$$UA^*V = \frac{UV^*AU^*V + \lambda A}{\lambda+1},$$

and then, after taking the adjoints and multiplying both the numerator and denominator by λ,

$$V^*AU^* = \frac{\lambda V^*UA^*VU^* + A^*}{\lambda+1}. \tag{2.0.6}$$

Inserting (2.0.6) in the first summand of (2.0.5) we arrive at

$$A = \frac{\lambda+2}{\lambda+1}UA^*V - \frac{1}{\lambda+1}UV^*UA^*VU^*V,$$

which implies

$$A^* = \frac{2\lambda+1}{\lambda+1}V^*AU^* - \frac{\lambda}{\lambda+1}V^*UV^*AU^*VU^*. \tag{2.0.7}$$

We insert (2.0.7) in (2.0.5) to obtain

$$(UV^*)^2A(U^*V)^2 - 2UV^*AU^*V + A = 0, \qquad A \in \mathcal{B}(X,Y).$$

If we define $S(A) = T^2(A) = UV^*AU^*V$, then

$$S^2 - 2S + I = 0. \tag{2.0.8}$$

Since $T = P + \lambda(I-P)$ we have $S = P + \lambda^2(I-P)$ and $S^2 = P + \lambda^4(I-P)$. Inserting this into (2.0.8) we get

$$(\lambda^2-1)^2(I-P) = 0.$$

Since $\lambda \neq \pm 1$ we get $P = I$ which implies $T = I$. This is impossible.

□

Remark 2.0.27. *Proposition 2.0.6 implies that given an ideal pair of Banach spaces (X,Y), any Hermitian square root H of the identity on $\mathcal{B}(X,Y)$ is a surjective isometry. Therefore, either:*

1. *$H(A) = UAV$, for all $A \in \mathcal{B}(X,Y)$, with U and V surjective isometries on Y and X respectively, and λ a unimodular complex number, such that U is a square root of λI_Y and V a square root of $\overline{\lambda} I_X$, or*
2. *X^* is isometrically isomorphic to Y and $H(A) = UA^*V$, with $U\colon X^* \to Y$ and $V\colon X \to Y^*$ surjective isometries such that $UV^* = \lambda I_Y$ and $U^*V = \lambda I_X$, for some unimodular complex number λ.*

Hermitian projections on $\mathcal{B}(X, Y)$: Additional Cases

We study hermitian projections on $\mathcal{B}(X, Y)$, where the pair (X, Y) is not an ideal pair. We derive the form from the projections from the description for the surjective isometries. Khalil and Saleh in [21] have described the surjective isometries on $\mathcal{B}(c_0)$. For completeness of exposition we include the following proposition.

Proposition 2.0.28. *Let $J: \mathcal{B}(c_0) \to \mathcal{B}(c_0)$ be a bounded linear operator. then J is a surjective isometry if and only if there are two surjective isometries U and V of c_0 such that*

$$J(T) = UTV$$

for every $T \in \mathcal{B}(c_0)$

The form for the surjective isometries described above implies the next theorem.

Theorem 2.0.29 (Hermitian Projections on $\mathcal{B}(c_0)$)**.** *A projection P on $\mathcal{B}(c_0)$ is a Hermitian projection if and only if there exists a Hermitian projection $Q \in \mathcal{B}(c_0)$ such that $P(A) = AQ$ for every $A \in \mathcal{B}(c_0)$ or there exists a Hermitian projection $R \in \mathcal{B}(c_0)$ such that $P(A) = RA$ for every $A \in \mathcal{B}(c_0)$.*

Proof. Let P be a Hermitian projection on $\mathcal{B}(c_0)$. Then for every modulus 1 scalar λ, $T = P + \lambda(I - P)$ is a surjective isometry. It is easy to see that T is a surjective isometry satisfying

$$T_\lambda^2 - (\lambda + 1)T_\lambda + \lambda I = 0.$$

By Proposition (2.0.28) using the form of isometries we get for some surjective isometries U_λ and V_λ on c_0,

$$U_\lambda^2 A V_\lambda^2 - (\lambda+1)U_\lambda A V_\lambda + \lambda A = 0$$

for every $A \in \mathcal{B}(c_0)$. Following with an application of Fong-Sourour Theorem we conclude that either $V_\lambda = \alpha I$ and $U_\lambda^2 - \overline{\alpha}(\lambda+1)U_\lambda + \lambda\overline{\alpha}^2 = 0$ or $U_\lambda = \alpha I$ and $V_\lambda^2 - \overline{\alpha}(\lambda+1)V_\lambda + \lambda\overline{\alpha}^2 = 0$, for some modulus 1 complex number α. For the first case there exist E_λ and F_λ, closed subspaces of c_0 such that $c_0 = E_\lambda \oplus F_\lambda$ and projections Q_λ and R_λ with ranges E_λ and F_λ respectively such that $U_\lambda = \lambda\overline{\alpha}Q_\lambda + \overline{\alpha}R_\lambda$. Therefore we have

$$PA = \frac{T_\lambda A - \lambda A}{1-\lambda} = Q_\lambda A, \ \ \forall\ A \in \mathcal{B}(c_0).$$

For the second case there exists a projection, also denoted Q_λ, such that for every bounded operator on c_0 we have $PA = AQ_\lambda$. We define two subsets of $\mathbb{S}^1 \setminus \{1\}$, S_l and S_r consisting of all values of λ that yield a left multiplication (i.e. $PA = Q_\lambda A$) and those that yield a right multiplication, respectively. It is clear that one of the sets S_1 or S_2 must be empty and the map that assigns to $\lambda \in \mathbb{S}^1$ the projection Q_λ is constant. We assume that the projection P is given by $A \to QA$ and $e^{itP}A = e^{itQ}A$ for all $A \in \mathcal{B}(c_0)$. This implies that Q is a hermitian projection. $\square$

We now discuss the hermitian projections on spaces of compact operators where a characterization of the surjective isometries has been established. We strt by considering the space of compact operators , $\mathcal{K}(\ell^1)$, compact operators on ℓ^1. Let A be a compact operator from ℓ^1 to ℓ^1 i.e., $A \in \mathcal{K}(\ell^1)$.

Proposition 2.0.30. *[21, Theorem 2.2] Let $J\colon \mathcal{K}(\ell^1) \to \mathcal{K}(\ell^1)$ be a bounded linear operator. Then J is a surjrctive isometry if and only if there are surjective* *isometries* $S_n\colon \ell^1 \to \ell^1, n \in \mathbb{N}$, *and a permutation* $\pi\colon \mathbb{N} \to \mathbb{N}$ *such that for* $f = (f_n)$,

$$J(f) = \left(S_n f_{\pi(n)}\right)$$

As done before, the form of Hermitian projections on $\mathcal{K}(\ell^1)$ can be derived from the form of the isometries. Recall thet P on a Banach space X is said to be a Hermitian projection if e^{itP} is a surjective isometry for every $t \in \mathbb{R}$. It can be shown that P is a Hermitian projection if and only if for a modulus one scalar λ, $P + \lambda (I - P)$ is a surjective isometry, say T_λ with defining symbols $\pi(\lambda)$, $\varphi(\lambda)$ and S_λ.

Let P be a hermitian projection on $\mathcal{K}(\ell^1)$, then $T_\lambda^2 - (\lambda+1)T_\lambda + \lambda I = 0$. We assume that $\pi(\lambda)^2 \neq id$ then there exists $i \in \mathbb{N}$ such that $\pi(\lambda)^2(i) \neq i$ and therefore $\pi(\lambda)(i) \neq i$. We choose A defined by $Ae_n = 0$ for $n \neq i$ and $Ae_i = e_i$. The operator A is finite rank then compact. Therefore,

$$T_\lambda^2 A - (\lambda+1)T_\lambda A + \lambda A = 0 \tag{2.0.9}$$

leads to a contradiction. This implies that $\pi(\lambda)^2 = id$ if $\pi(\lambda) \neq id$ then applying equation 2.0.9 at A defined above implies that $\lambda = -1$. This is also impossible hence $\pi(\lambda) = id$. Therefore equation 2.0.9 becomes

$$S_{\lambda,j}^2 - (\lambda+1)S_{\lambda,j} + \lambda I = 0, \ \forall\ \lambda \in \mathbb{S}^1 \ \text{ and } \ j \in \mathbb{N}.$$

Following a similar approach we show that $\varphi(\lambda, j) = id$ and $S_{\lambda,j} : \ell_1 \to \ell_1$ is of the form $S_{\lambda,j}(x_1, x_2, \ldots) = (e^{it_{1,j_\lambda}} x_1, e^{it_{2,j_\lambda}} m x_2, \ldots)$. Since $P = \frac{T_\lambda - \lambda I}{1-\lambda}$ then $T_\lambda = I$ or $T = \lambda I$. Therefore the only hermitian projections are trivial. We summary this in the following statement.

Proposition 2.0.31. *$\mathcal{K}(\ell^1)$, the space of all compact operators on ℓ^1 supports only trivial projections.*

CHAPTER 3

EXTREME POINTS AND TENSOR PRODUCTS

Surjective isometries between Banach spaces play an important role in operator theory and in the geometry of Banach spaces. This class of operators have instrinsic connections with projections and hermitian operators as we have shown in Chapter 2. Moreover, the type of isometries supported by a Banach space give important information on geometric aspects of the space. For example surjective isometries induce bijections on the set of extreme points of the unit ball. This often permits the derivation of the form for these operators. Here we recall the Banach Stone theorem stating that surjective isometries on spaces of continuous functions are weighted composition operators. The knowledge of the set of extreme points has played a crucial role in the characterization of the surjective isometries supported by the space, see [10], [20], [33], [28], [21]. In this Chapter, we start with a survey on tensor product spaces and on the theory of extreme points. We give a description of isometries on tensor product spaces using the form of the extreme points of the dual space. We use the techniques in [34] and extend theorem 3.0.12 for complex Banach spaces. We modify the proof in [34] to extend to this new setting.

Tensor products on Banach spaces

Let X, Y, Z be vector spaces. Consider

$$A\colon X \times Y \to Z,$$

a bilinear mapping from $X \times Y$, the Cartesian product of X and Y.

Definition 3.0.1 (Bilinear map)**.** *A map A is said to be bilinear if it is linear in each variable, i.e.,*

$$A(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 A(x_1, y) + \alpha_2 A(x_2, y)$$
$$A(x, \beta_1 y_1 + \beta_2 y_2) = \beta_1 A(x, y_1) + \beta_2 A(x, y_2)$$

for all $x_i, x \in X, y_i, y \in Y$ and α_i, β_i are scalars. We denote by $B(X \times Y, Z)$ the vector space of all the bilinear mappings.

If Z is the scalar field, then it is denoted by $B(X \times Y)$. Recall that $B(X \times Y)'$ denotes the set of all linear functionals on $B(X \times Y)$.

For given $x \in X$ and $y \in Y$ we define a linear functional $x \otimes y \in B(X \times Y)'$ by the formula

$$(x \otimes y)(A) = A(x, y),$$

for every $A \in B(X \times Y)$.

Definition 3.0.2 (Tensor product)**.** *The* ***tensor product*** *$X \times Y$ is the subspace of $B(X \times Y)'$ spanned by all elements $x \otimes y$, $x \in X$ and $y \in Y$, i.e.,*

$$X \otimes Y = span\{x \otimes y \colon x \in X, y \in Y\}$$

Thus, an element in $X \otimes Y$ is of the form

$$u = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i$$

where $n \in \mathbb{N}, \lambda_i \in \mathbb{K}, x_i \in X, y_i \in Y, i = 1, 2, \cdots n$

Remark 3.0.3. *Note that the representation is not* ***unique!***

Before we introduce a norm on the algebraic tensor product $X \otimes Y$, let us recall the definiton of norm.

Definition 3.0.4. *Let X be a vector space. A function $\|\cdot\|: X \to [0, \infty)$ is called a norm if*

(i) *If $\|x\| = 0$ then $x = 0$ for any $x \in X$.*

(ii) *$\|\alpha x\| = |\alpha|\|x\|$ for any $x \in X$ and $\alpha \in \mathbb{K}$.*

(iii) *$\|x + y\| \leq \|x\| + \|y\|$ for any $x, y \in X$.*

Let X and Y be Banach spaces. We ask the natural question. What kind of norm should we define on the tensor product $X \otimes Y$? We will consider the following condition:

Definition 3.0.5. *A tensor norm on $X \otimes Y$ is a norm $\|\cdot\|$ on $X \otimes Y$ such that*

$$\|x \otimes y\| \leq \|x\|\|y\|$$

Injective tensor product

We define a norm on $X \otimes Y$ that satisfies the condition on definition 3.0.5. An approach to define such a norm starts with a n interpretation of the elements in $X \otimes Y$ as bilinear forms on the product $X' \times Y'$. More precisely, if $u = \sum_{i=1}^{n} x_i \otimes y_i$ then the associated bilinear form is given by

$$B_u(\phi, \psi) = \sum_{i=1}^{n} \phi(x_i)\psi(y_i),$$

for all $\phi \in X', \psi \in Y'$.

The restriction of B_u to $X^* \times Y^*$ is a bounded bilinear form. Therefore, we get the following embedding

$$X \otimes Y \hookrightarrow B(X^* \times Y^*)$$

and the injective norm of u is set to be equal to the norm of the bilinear operator B_u.

Definition 3.0.6 (Injective norm)**.** *The injective norm $\|\cdot\|_\epsilon$ on $X \otimes Y$ is defined by*

$$\|u\|_\epsilon = sup \left\{ \left| \sum_{i=1}^{n} \phi(x_i)\psi(y_i) \right| : \phi \in B_{X^*}, \psi \in B_{Y^*} \right\}$$

where $\sum_{i=1}^{n} x_i \otimes y_i$ is a representation of $u \in X \otimes Y$.

The completion of $X \otimes Y$ with respect to this norm is denoted by $X \otimes_\epsilon Y$ and it is called the injective tensor product of X and Y.

Previous considerations imply that $X \hat{\otimes}_\epsilon Y$ is isometrically embedded in $B(X^* \times Y^*)$.

We also note that for $u = \sum_{i=1}^{n} x_i \otimes y_i$ we can define operators $L_u \colon X^* \to Y$ and $R_u \colon Y^* \to X$ by

$$L_u(\phi) = \sum_{i=1}^{n} \phi(x_i) y_i, \quad R_u(\psi) = \sum_{i=1}^{n} \psi(y_i) x_i$$

. Thus, we have the two isometric embeddings

$$X \hat{\otimes}_\epsilon Y \subset \mathcal{L}(X^*, Y), \quad X \hat{\otimes}_\epsilon Y \subset \mathcal{L}(Y^*, X).$$

Hence

$$\|u\|_\epsilon = \sup\left\{\left\|\sum_{i=1}^{n}\phi(x_i)y_i\right\| : \phi \in B_{X^*}\right\} \tag{3.0.1}$$

$$= \sup\left\{\left\|\sum_{i=1}^{n}\psi(y_i)x_i\right\| : \psi \in B_{Y^*}\right\} \tag{3.0.2}$$

We now give some examples of injective tensor products from [35].

Example 3.0.7. *For any Banach space X and a compact topological space Ω, we define the space $C(\Omega, X)$ as the space of continuous functions with range in X:*

$$C(\Omega, X) = \{f : \Omega \to X : f \text{ is continuous on } \Omega\}$$
$$\|f\|_\infty = sup\{\|f(t)\|_X : t \in \Omega\}$$

Then

$$C(\Omega)\hat{\otimes}_\epsilon X \simeq C(\Omega, X)$$

Extreme points

The extreme points of a subset $S \subseteq X$ are the points which do not lie in the interior of a line segment between two points of S.

Definition 3.0.8. *Let X be a Banach space and $S \subseteq X$. An extreme point $x \in S$ is a point with the property that if $x = ty + (1-t)z$ with $y.z \in S$ and $0 < t < 1$, then $y = x$ or $z = x$. The set of all extreme points of S will be denoted by ext S.*

Example 3.0.9.

- *If $a, b \in \mathbb{R}$ with $a < b$, then a, b are extreme points of the interval $[a, b]$.*
- *The extreme points of the closed unit disk of $\mathbb{R}^2$ are those points in the unit*

circle.

- *The extreme points of the closed unit ball of* $\ell^2_\infty(\mathbb{R})$ *are* $(\pm 1, \pm 1)$.

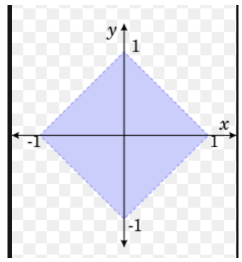

Figure 3.1: Extreme points of $\ell^2_\infty(\mathbb{R})$

- *The closed unit ball of* c_0 *has* ***no*** *extrerme points. [7, Example 7.7, p 143]*

- *The closed unit ball of* L^1 *has* ***no*** *extrerme points. [7, Example 7.2(f), p 141]*

The following well-known result plays an important role in our discussion.The proof is elementary, so it is omitted the proof. The closed unit ball of a Banach space X is denoted by B_X.

Proposition 3.0.10. *Let* X *and* Y *be Banach spaces and* $T\colon X \to Y$ *is a surjective linear isometry. Then* T *preserves extreme points, namely* $T(x)$ *is an extreme point of* B_Y *if and only if* x *is an extreme point of* B_X.

We observe that this statement does not hold if the isometry is not surjective. We consider the composition operator $T : C[0,1] \to C[0,1]$ given by

$$Tf(t) = f(2t), \text{ for } t \in [0, 1/2]$$

and

$$Tf(t) = f(1), \text{ for } t \in [1/2, 1].$$

This operator is an isometry and $T^*(\delta_t) = \delta_1$, restricted to the range of T, for $t > \frac{1}{2}$. Where δ_t denotes the point evaluation functional.

Extreme points of Function Spaces

In this section, we study the extreme points of spaces of scalar valued continuous functions over the $\mathbb{R}$ or $\mathbb{C}$, respectively. For a Banach Space X, we denote the spaces of continuous functions over $\mathbb{R}$ and $\mathbb{C}$ by $C_{\mathbb{R}}(X)$ and $C(X)$ respectively. The next proposition characterizes the extreme points of the dual of the unit ball of $C(X)$.

Proposition 3.0.11. *[7, Theorem 8.4] Let X be a compact Hausdorff space, then*

$$extB_{C(X)^*} = \{\alpha\delta_x : |\alpha| = 1, x \in X\}$$

where $\delta_x : C(X) \to \mathbb{R}$ is a map given by $\delta_x(f) := f(x)$.

It is worth mentioning that for a compact connected Hausdorff space X the only extreme points of the unit ball of $C_{\mathbb{R}}(X)$ are $f = 1$ and $f = -1$. For compact Hausdorff spaces, extreme points of the unit ball of $C_{\mathbb{C}}(X)$ are those continuous functions with values in the unit circle.

Extreme points of the Injective tensor product

We extend the characterization for the extreme points of the dual unit ball of injective tensor products of real Banach spaces, due to Ruess and Stegall ([34]) to include the complex case. We assume that X and Y are Banach spaces over the complex numbers. We follow closely the technique employed in the aforementioned paper and explain the steps needed to to include the scalar field of complex numbers. We start with some notation to be used throughout. For a Banach space X, B_X denotes the closed unit ball of X. Further, $X \otimes_\epsilon Y$ denotes the completion of the algebraic tensor space $X \otimes Y$ relative to the injective norm. We recall that given

$u \in X \otimes Y$, of the form $u = \sum_{j=1}^{n} x_j \otimes y_j$, the injective norm is defined as follows:

$$\epsilon(u) = \sup \left\{ \left| \sum_{j=1}^{n} \varphi(x_j)\psi(y_j) \right| : \ (\varphi, \psi) \in B_X \times B_Y \right\}.$$

It is important to observe that $X \otimes_\epsilon Y$ is isometrically embedded in $K_{w^*}(X^*, Y)$, the space of all compact operators relative to the weak-* topology on X^* and the weak topology on Y. The embedding is defined as follows, given $u = \sum_{j=1}^{n} x_j \otimes y_j$ in $X \otimes Y$, we define $\tilde{u}$ as follows: $\tilde{u}(\varphi) = \sum_{j=1}^{n} \varphi(x_j)y_j$. This assignment preserves the norm and the extension to $X \otimes_\epsilon Y$ yields an isometric embedding. This embedding may not be surjective. For example, $\ell_2 \otimes_\epsilon \ell_2$ is embedded isometrically in $\mathcal{K}_{w^*}(\ell_2)$, the identity operator is in $\mathcal{K}_{w^*}(\ell_2)$, but is not in the range of the identity operator.

We prove that $\text{ext}B_{H^*} \subset \text{ext}B_{X^*} \otimes \text{ext}B_{Y^*}$. The next theorem formulates a stronger statement.

Theorem 3.0.12. *(cf. [34]) Let X, Y be complex Banach spaces, $x_0^* \in extB_{X^*}$, $y_0^* \in extB_{Y^*}$ and let H be any linear subspace of the space $K_{w^*}(X^*, Y)$ of compact weak*-weakly continuous linear operators from X^* into Y, containing $X \otimes Y$. Then $extB_{H^*} = extB_{X^*} \otimes extB_{Y^*}$.*

We first observe that $T : \mathcal{K}_{w^*}(X^*, Y) \to C(B_{X^*} \times B_{Y^*})$ defined by $T(A)(x^*, y^*) = y^*(Ax^*)$ is an isometry.

Lemma 3.0.13. *Given $\tau \in extB_{H^*}$, there exists $x_0^* \in extB_{X^*}$ and $y_0^* \in extB_{Y^*}$ such that $\tau = \delta_{(x_0^*, y_0^*)}$.*

Proof. Since H is a subspace of $K_{w^*}(X^*, Y)$, then it is isometrically embedded in

$C(B_{X^*} \times B_{Y^*})$. We set

$$E_\tau = \{\tilde{\tau} : \tilde{\tau} \text{ is a Hahn Banach extension of } \tau \text{ to } C(B_{X^*} \times B_{Y^*})\}.$$

It is standard to check that E_τ is convex, closed and an extremal subset of $B_{C(B_{X^*} \times B_{Y^*})^*}$. An application of Krein-Milman Theorem implies that E_τ has an extreme point, and Arens-Kelley Theorem ensures that an extreme point is a point evaluation functional: $\delta_{(x_0^*, y_0^*)}$ with $x_0^* \in extB_{X^*}$ and $y_0^* \in extB_{Y^*}$. □

The previous lemma shows that $extB_{H^*} \subset extB_{X^*} \times extB_{Y^*}$. Towards the converse inclusion we prove that every functional $x^* \otimes y^*$, $x^* \in \text{ext}B_{X^*}$ and $y^* \in \text{ext}B_{Y^*}$, is an extreme point of B_{H^*}. This relies on the repesentation of extreme functionals as probability measures. This leads to the characterization of support of certain Randon measures defined on the Borel sets of the unit ball of the dual space of X. This point is explained in the next lemma.

We start with a result that establishes the existence of barycenter for a probability positive measure on the Borel subsets of the unit ball of the dual space B_{X^*}. This important result is due to Choquet and the proof for real Banach spaces can be found in [31]. We now review the result for complex Banach spaces, some minor adjustment is required and for completeness we include its proof.

Definition 3.0.14. *Let $x^* \in B_{X^*}$ and let μ be a positive probability measure on the Borel subsets of B_{X^*}. Then x^* is said to represent μ if and only if for every $f \in X^{**}$ $f(x^*) = \int_{B_{X^*}} f d\mu$.*

Lemma 3.0.15. *Let X be a complex Banach space and let μ be a positive probability measure on the Borel subsets of B_{X^*}, then there exists a unique $x^* \in B_{X^*}$ such that x^* represents μ.*

Proof. Given $f \in X^{**}$ we set

$$H_f = \left\{ x^* : f(x^*) = \int_{B_{X^*}} f d\mu \right\}.$$

The set H_f is a closed hyperplane in X^*. We define $\Phi : X^* \to \mathbb{C}$ given by $\Phi(x^*) = f(x^*)$. We show that $(\cap_{f \in X^{**}} H_f) \cap B_{X^*}$ is nonempty. Since B_{X^*} is weak-* compact, it is sufficient to show that finite intersections

$$H_{f_1} \cap H_{f_2} \cap \ldots \cap H_{f_k} \cap B_{X^*}$$

are nonempty. Let $\Psi : X^* \to \mathbb{C}^k$ given by $\Phi(x^*) = (f_1(x^*), f_2(x^*), \ldots, f_k(x^*))$. The map Ψ is linear and bounded, hence $\Psi(X_1^*)$ is compact and convex. We show that $P = (\int_{B_{X^*}} f_1 d\mu, \int_{B_{X^*}} f_2 d\mu, \ldots \int_{B_{X^*}} f_k d\mu,) \in \Psi(B_{X^*})$. Suppose otherwise, $P \notin \Psi(B_{X^*})$. The separation theorem implies the existence of a function $\lambda : \mathbb{C}^k \to \mathbb{C}$ such that

$$Re\lambda(P) > \sup\{Re\lambda(\Psi(x^*)) : \ x^* \in B_{X^*}\}. \tag{3.0.3}$$

Let $a_j = \lambda e_j$, with $e_j = (0, 0, \ldots 1, 0, \ldots 0)$, then $\lambda(z_1, \ldots z_k) = \sum_{i=1}^k a_j z_j$. Let $g = \sum_{j=1}^k a_j f_j$, $g = g_1 + ig_2$ and

$$Re\lambda(P) = Re \int_{B_{X^*}} g d\mu = \int_{B_{X^*}} g_1 d\mu.$$

We also have

$$\sup\{Re(\lambda\Psi(x^*)) : x^* \in B_{X^*}\} = Re\lambda(\Psi(x_0^*)),$$

for some $x_0^* \in B_{X^*}$. Since $\Psi(x_0^*) = (f_1(x_0^*), f_2(x_0^*), \ldots, f_k(x_0^*))$, we have

$$\lambda(\Psi(x_0^*)) = \sum_{j=1}^k a_j f_j(x_0^*) = g(x_0^*).$$

From (3.0.3) we have $\|g_1\|_\infty \geq \int_{B_{X^*}} g_1 d\mu > g_1(x_0^*)$. This contradiction shows that $H_{f_1} \cap H_{f_2} \cap \ldots \cap H_{f_k} \cap B_{X^*}$ is nonempty and compactness of B_{X^*} implies that $(\cap_{f \in X^{**}} H_f) \cap B_{X^*}$ is nonempty. □

Lemma 3.0.16. *Let X be a complex Banach space, $K = B_{X^*}$ and $x_0^* \in extB_{X^*}$. Also, let μ be a positive Radon measure on K, $\|\mu\| \leq 1$ such that*

$$|x_0^* x| \leq \int_K |x^* x| d\mu$$

$x \in X$. Then supp $\mu \subset \{x_0^*, -x_0^*, ix_0^*, -ix_0^*\}$

Proof. Let us consider the sequence of injections

$$X \xrightarrow{I} C(K) \xrightarrow{J} L^1(K, \mu).$$

Now, define $T : JI(X) \to \mathbb{C}$ by $T(JI(x)) = x_0^* x$. Note that, T is a continuous linear functional on $JIX \subset L^1(K, \mu)$ with $\|T\| \leq 1$, therefore by Hahn-Banach theorem T has an extension of the same norm to $L^1(K, \mu)$. Hence, there exists $h : K \to \mathbb{C}$, $\|h\| \leq 1$ such that

$$|x_0^* x| = \int_{X_1^*} x^*(x) h(x^*) d\mu \text{ for all } x \in X. \tag{3.0.4}$$

Let $\nu = h d\mu$ and let $\nu = \nu_r^+ - \nu_r^- + i\nu_i^+ - i\nu_i^-$ be the Hahn decomposition of ν. Assume $0 < \nu_r^+, \nu_r^-, \nu_i^+, \nu_i^- < 1$. Given any $x \in X$, we have:

$$\begin{aligned}
x_0^* x = &\int x^* h(x^*) d\mu = \int x^* x d\nu_r^+ + \int (-x^*(x)) d\nu_r^- + i \int (x^*(x)) d\nu_i^+ + i \int (-x^*(x)) d\nu_i^- \\
=&\|\nu_r^+\| \int x^*(x) \|\nu_r^+\|^{-1} d\nu_r^+ + \|\nu_r^-\| \int (-x^*(x)) \|\nu_r^-\|^{-1} d\nu_r^- \\
&+ i\|\nu_i^+\| \int (x^*(x)) \|\nu_i^+\|^{-1} d\nu_i^+ + i\|\nu_i^-\| \int (-x^*(x)) \|\nu_i^-\|^{-1} d\nu_i^-
\end{aligned}$$

Now using the barycenter result (Lemma (3.0.15)) we obtain:

$$x_0^* x = \|\nu_r^+\| x_1^*(x) + \|\nu_r^-\|(-x_2^*(x)) + i\|\nu_i^+\| x_3^*(x) + i\|\nu_i^-\|((-x_4^*)(x))$$

where $x_1^*, x_2^*, x_3^*, x_4^* \in K$ are the barycenters. Since $x_0^* \in \text{ext}B_{X^*}$, then $x_0^* = x_1^* = -x_2^* = ix_3^* = -ix_4^*$, and by uniqueness, $\nu_r^+ = \|\nu_r^+\|\delta_{x_0^*}, \nu_r^- = \|\nu_r^-\|\delta_{-x_0^*}, \nu_i^+ = \|\nu_i^+\|\delta_{-ix_0^*}, \nu_i^- = \|\nu_i^-\|\delta_{-x_4^*}$. Thus, we get:

$$\text{supp } \mu \subset \{x_0^*, -x_0^*, ix_0^*, -ix_0^*\} \cup \{|h| = 0\}$$

. On the other hand, let $(x_n) \subset X$ be a sequence such that $\|x_n\| = 1$ and $x_0^*(x_n) \to 1$. Then

$$\begin{aligned} |x_0^* x| &= \int x^*(x)h(x^*)d\mu \leq \int |x^*(x_n)||h(x^*)|d\mu \\ &\leq \int \sup_n |x^*(x_n)||h(x^*)|d\mu \leq \mu(X) \leq 1 \end{aligned}$$

Hence $\int \sup_n |x^*(x_n)||h(x^*)|d\mu = 1$. Which implies $\mu(\{|h| > 0\}) = 1$. Therefore supp $\mu \subset \{x_0^*, -x_0^*, ix_0^*, -ix_0^*\}$ □

Lemma 3.0.17. *Let X, Y be complex Banach spaces, $x_0^* \in extB_{X^*}, y_0^* \in extB_{Y^*}$ and let H be a linear subspace of $K_{w^*}(X^*, Y)$, containing $X \otimes Y$. Let $h^* \in extB_{H^*}$ with $\|h^*\| = 1$ such that $h^*|_{X\otimes Y} = \delta_{(x_0^*, y_0^*)}\big|_{X\otimes Y}$. Then $h^* \in extB_{H^*}$ and $h^* = \delta_{(x_0^*, y_0^*)}\big|_H$.*

We set

$$A = \{(x^*, y^*) : \, x^* = \eta x_0^*, \, y^* = \gamma y_0^*, \text{ with } \eta, \gamma, \in \{1, -1, i, -i\} \text{ and } \eta\gamma = 1\}$$

and $\mathcal{C} = \text{co}(A)$.

- **Claim 1.** $\mathcal{C}$ is the set of norm-one extensions of h^* to $C(B_{X^*} \times B_{Y^*})$.

- **Claim 2.** $\mathcal{C}$ is an extremal subset of $B_{M(B_{X^*}\times B_{Y^*})}$

Towards the proof of Claim 2, we start with the lemma.

Lemma 3.0.18. *Let K be a compact, Hausdorff space, $p_1, p_2, ..., p_n$ are distinct points in K and $\epsilon_j = e^{i\alpha_j}$, with $\alpha_j \in \mathbb{R}$. Let $E = co\{\epsilon_1\delta_{p_1}, ..., \epsilon_n\delta_{p_n}\}$. Then E is an extremal subset of $B_{M(K)}$.*

Proof. Let $f \in B_{C(K)}$ such that $f(p_i) = \overline{\epsilon_i}$. Given any
$\sigma = \sum_{i=1}^n t_i\epsilon_i\delta_{p_i}$with $\sum_{i=1}^n t_i = 1 \in E$, we have:

1. $\int f d\sigma = \sum_{i=1}^n f(p_i)t_i\epsilon_i = \sum_{i=1}^n t_i = 1.$
2. Now assume $\mu_1, \mu_2 \in B_{M(K)}$ and $0 < t < 1$ such that $t\mu_1 + (1-t)\mu_2 \in E$.

   Then we have:

$$\begin{aligned} 1 &= \int f d(t\mu_1 + (1-t)\mu_2) \\ &= \left| \int f d|t\mu_1 + (1-t)\mu_2 \right| \\ &\leq t \int |f| d|\mu_1| + (1-t)\int |f|d|\mu_2| \leq 1 \end{aligned}$$

   So

$$t\int |f|d|\mu_1| + (1-t)\int |f|d|\mu_2| = 1.$$

3. From the last step $\int |f|d|\mu_k| = 1$ for $k = 1, 2$. Further, $\int f d\mu_k = 1$. Indeed, let $\int f d\mu_k = a + ib$, with a and b real numbers, then $1 = ta_k + i(1-t)b_k$ implies $(1-t)b_k = 0$ and $a_k = 1$.

   Thus supp $\mu \subset \{p_1, p_2, ..., p_n\}$.

4. $\sum_{i=1}^{n} |\mu_1(p_i)| = \sum_{i=1}^{n} |\mu_2(p_i)| = 1$.

5. For $k = 1, 2, |\mu_k(p_i)| = f(p_i)\mu_k(p_i) = \overline{\epsilon_i}\mu_k(p_i)$. Therefore

$$\mu_k = \sum_{i=1}^{n} \mu_k(p_i)\epsilon_i\delta_{p_i} = \sum_{i=1}^{n} |\mu_k(p_i)|\epsilon_i\delta_{p_i}.$$

   Since $\sum_{j=1}^{n} |\mu_k(p_j)| = 1$, we have $\mu_k \in E$.

$\square$

Taking $K = B_{X^*} \times B_{Y^*}$ and $\{\alpha(\eta x_0^*, \gamma y_0^*) : \alpha, \lambda, \gamma \in \{\pm 1, \pm i\}\}$, with $(\eta x_0^*, \gamma y_0^*)$ the distinct points in Lemma 3.0.18. Therefore an application of Lemma 3.0.18 proves Claim 2. To prove Claim 1, we show that a norm extension of h^* , satisfying the hypotheses in Lemma 3.0.17 has finite support and then it can be written as a combination of finitely many point evaluation functionals. We then show that the coefficients are nonnegative real numbers with sum equals to 1. Towards these, we need to show the following folklore lemma. We include its proof for completeness of exposition.

Lemma 3.0.19. *Let $a_1, a_2, ..., a_n$ be complex numbers such that $\sum_{i=1}^{n} a_i = 1 = \sum_{i=1}^{n} |a_i|$. Then $a_i \in \mathbb{R}$ and $a_i \geq 0$.*

Proof. Since $a_1 + (a_2 + ... + a_n) = 1$ and $|a_1| + |a_2 + ... + a_n| = 1$, letting $a_1 = x + iy$ and $a_2 + ... + a_n = z + i\omega$ we see that $a_1 = x \in \mathbb{R}$ and $a_2 + .. + a_n = z \in \mathbb{R}$. The given conditions also imply $x + z = 1$ and $|x| + |z| = 1$. This system leads to the following cases: 1. $x = 1$ and $z = 0$; 2. $z = 1$ and $x = 0$; and 3. $0 < z < 1$ and $0 < x < 1$. In case 1. it is apparent that $a_1 = 1$ and $a_i = 0$ for all $i \neq 1$. Cases 2 and 3 imply the result by an induction argument. We illustrate this with case 3. Since $0 < z < 1$ and $0 < x < 1$, we have $a_2 + ... + a_n = 1 - a_1$ and $|a_2| + ... + |a_n| = 1 - a_1$.

Therefore, $\frac{a_2}{1-a_1} + ... + \frac{a_n}{1-a_1} = 1$ and $\frac{|a_2|}{1-a_1} + ... + \frac{|a_n|}{1-a_1} = 1$. The induction hypothesis applies since it ensures the statement is true for a sum with less than n terms, then $a_i \geq 0$ with sum equals 1. $\square$

Finally we prove claim 1 to conclude the proof of Theorem 4.0.6.

Proof of Claim 1. Note that any element in $\mathcal{C}$ is a norm-one extension of h^* to $C(B_{X^*} \times B_{Y^*})$, since $h^*|_{X \otimes Y} = \delta_{(x_0^*, y_0^*)}|_{X \otimes Y}$. Let λ be a norm-one extension of h^* to $C(B_{X^*} \times B_{Y^*})$, i.e., $\lambda \in M(B_{X^*} \times B_{Y^*})$ such that

$$\lambda\big|_{X \otimes Y} = \delta_{(x_0^*, y_0^*)}\big|_{X \otimes Y}, \|\lambda\| = 1$$

Now $\lambda(x \otimes y) = \delta_{(x_0^*, y_0^*)}(x \otimes y)$ then
$|\lambda(x \otimes y)| = |x_0^*(x) y_0^*(y)| \leq \int_{B_{X^*} \times B_{Y^*}} |x^*(x) y^*(y)| d|\lambda|$

Let $\mu_1 = |\lambda|\big|_{C(B_{X^*})}$ and $\mu_2 = |\lambda|\big|_{C(B_{Y^*})}$. Choose $y_n \in Y$ such that $\|y_n\| = 1$, which implies there exists $y_0^* \in Y^*$ such that $y_0^*(y_n) \to 1$. Then

$$\begin{aligned} |x_0^* x| &\leq \int_{B_{X^*} \times B_{Y^*}} |x^*(x) y^*(y_n)| d|\lambda| \\ &\leq \int_{B_{X^*} \times B_{Y^*}} |x^*(x)| d\lambda = \int_{B_{X^*}} |x^*(x)| d\mu_1 \end{aligned}$$

By Lemma 3.0.16, supp $\mu_1 \subset \{x_0^*, -x_0^*, ix_0^*, -ix_0^*\}$ and similarly supp
$\mu_2 \subset \{y_0^*, -y_0^*, iy_0^*, -iy_0^*\}$. Consequently, supp
$|\lambda| \subset B = \{(x^*, y^*) : \ x^* = \alpha x_0^*, \ y^* = \gamma y_0^*, \ \text{ with } \ \lambda, \gamma \in \{1, -1, i, -i\}\}$ and
$\lambda = \sum_{i=1}^{16} s_j \delta_{(x_j^*, y_j^*)}$ with $(x_j^*, y_j^*) \in B$. We set $x_j^* = \alpha_j x_0^*$ and $y^* = \gamma_j y_0^*$. Since

$$|\lambda|(B) = 1 = \sum_{i=1}^{16} |s_i|, \tag{3.0.5}$$

It is left to be shown that $s_j \in \mathbb{R}$ and $s_j \geq 0$. Evaluating λ at $(x_0 \otimes y_0)$ we get $\sum_{j=1}^{16} s_j = 1$. Choosing f, of norm 1, such that $f(x_0^*, y_0^*) = \frac{\overline{s_1}}{|s_1|}, ..., f(-ix_0^*, -iy_o^*) = -\frac{\overline{s_{16}}}{|s_{16}|}$ we get $\sum_{i=1}^{16} |s_i| = 1$. Lemma 3.0.19 implies that $s_j \in \mathbb{R}$ and $s_j \geq 0$. Therefore $\lambda \in \mathcal{C}$.

Proof of Theorem 4.0.6. By Claim 1 & Claim 2, we conclude that $extB_{X^*} \otimes extB_{Y^*} \subset extB_{H^*}$. That completes the proof.

$\square$

CHAPTER 4

ISOMETRIES ON INJECTIVE TENSOR PRODUCTS

As we have discussed in Chapter 3, one of the well-known techniques to find the form of isometries is to use the knowledge of extreme points of the unit ball of the dual space. In this Chapter, using the characterization of extreme points in Theorem 4.0.6, we will describe the form of isometry. Although Theorem 4.0.6 holds for complex scalars as well, but we will be restricting for Banach spaces with real scalars only.

The study done in this chapter is based on the proof given by K. Jarosz for the characterization of cetain surjective isometries on injective tensor products, see [19]. We follow the same scheme while providing several omitted details and also adapting some proofs for an easier reading. In addition, we also characterized a class of contractive projections associated with the aforementioned isometries. We start with the statement of the main Theorem from [19].

The main Theorem

Theorem 4.0.1. *Let X, Y, H, K be real Banach spaces and assume H^*, K^* are strictly convex. Let $T\colon X \otimes_\epsilon K \to Y \hat{\otimes}_\epsilon H$ be an onto isometry. Then there are Banach spaces Z and X_2 such that*

$$X \simeq (Z \hat{\otimes}_\epsilon H) \bigoplus_\infty X_2 \text{ and } Y \simeq (Z \hat{\otimes}_\epsilon K) \bigoplus_\infty X_2$$

and upto this identification, the operator T has the following form

$$\begin{aligned} T((z \otimes h, x) \otimes k) &= T(z \otimes h \otimes k, x \otimes k) \\ &= (z \otimes k \otimes h, T_2(x \otimes k)) \end{aligned} \tag{4.0.1}$$

where $T_2 \colon X_2 \otimes K \to X_2 \otimes H$ is given by

$$T_2(x \otimes k)(x^* \otimes h^*) = x^*(x) \cdot h^* (\Phi(x^*)(k))$$

and $\Phi \colon ext\ B(X_2^*) \to \mathcal{I}(K, H)$ is an operator with values in the surjective isometries from K onto H.

Proof of the main Theorem

The proof of this theorem relies on a series of results describing the action of the adjoint of T on special elements of the dual space. Crucial facts are that a surjective isometry induces a bijection between the set of the extreme points of the unit ball of the dual spaces. We start by reviewing preliminary results to be used repeatedly throughout the paper. First, we recall the description for the extreme points of the unit ball of the dual space of the injective tensor product $X \otimes_\epsilon K$, due to Ruess and Stegall (see [34]). We denote by the unit ball of a Banach space X by $B(X)$.

Lemma 4.0.2. *Let X and K be real Banach spaces. Then*

$$ext\ B((X \otimes_\epsilon K)^*) = ext\ B(X^*) \otimes ext\ B(K^*).$$

The next lemma considers two elementary tensors, with sum also an elementary tensor. It describes the dependence among the vectors involved. We use the notation between two vectors $u_1 \parallel u_2$ to represent that $u_1 = au_2$ or $u_2 = au_1$, for some real scalar a.

Lemma 4.0.3. *Let U and V be linear spaces. If $u_1, u_2, u_3 \in U$ and $v_1, v_2, v_3 \in V$ such that $u_1 \otimes v_1 + u_2 \otimes v_2 = u_3 \otimes v_3$, then either $u_1 \parallel u_2 \parallel u_3$ or $v_1 \parallel v_2 \parallel v_3$.*

Proof. For every $v^* \in V^*$, $u_1 \otimes v_1 + u_2 \otimes v_2 = u_3 \otimes v_3$ implies

$$u_1 v^*(v_1) + u_2 v^*(v_2) = u_3 v^*(v_3). \tag{4.0.2}$$

Suppose $v_1 \nparallel v_2 \nparallel v_3 \nparallel v_1$. We select v^* such that $v^*(v_3) \neq 0$, $v^*(v_1) \neq 0$, and $v^*(v_2) = 0$, from (4.0.2) we get

$$\frac{v^*(v_1)}{v^*(v_3)} u_1 = u_3$$

or $u_1 \parallel u_3$. Similarly, we show that $u_1 \parallel u_2$ and $u_2 \parallel u_3$. If $v_1 \nparallel v_2 \nparallel v_3 \nparallel v_1$ does not hold then without loss of generality we may assume that $v_1 = av_2$, for some scalar a. The equation displayed in (4.0.2) becomes $(u_1 + au_2) \otimes v_2 = u_3 \otimes v_3$. If $v_2 \parallel v_3$, then the statement in lemma follows. Otherwise, we have $u_3 = 0$ and $u_1 + au_2 = 0$. Therefore $u_1 \parallel u_2 \parallel u_3$. This completes the proof. □

Lemma 4.0.4. *Let X and Y be Banach spaces and suppose $x_\alpha^* \in X^*$ and $y_\alpha^* \in Y^*$ are weak-* converging nets to x_0^* and y_0^*, respectively. Then*

$$x_\alpha^* \otimes y_\alpha^* \to x_0^* \otimes y_0^*,$$

in the weak- topology.*

Further, for a fixed $y^* \in Y^*$ (or $x^* \in X^*$),

$$x_\alpha^* \otimes y^* \to x_0^* \otimes y^*, \ \ (or\ x^* \otimes y_\alpha^* \to x^* \otimes y_\alpha^*),$$

in the weak- topology, if and only if x_α^* converges to x_0^* (or y_α^* converges to y_0^*) in* *the weak-* topology. The proof for the second statement is easier, so it is omitted.*

Proof. The uniform boundedness principle implies that $\{x_\alpha^*\}$ and $\{y_\alpha^*\}$ are bounded, i.e. there exists positive numbers M and N such that $\|x_\alpha^*\| \leq M$ and $\|y_\alpha^*\| \leq N$, for every α. Given a nontrivial element $z = \sum_{i=1}^k x_i \otimes y_i$, we set
$A = \max\{k(M+N)\max\{\|x_i\| : i = 1, \cdots k\}, k(M+N)\max\{\|y_i\| : i = 1, \cdots k\}\}$.
For $\delta > 0$, there exits $\alpha_{\delta,z}$ such that for every $\alpha > \alpha_{\delta,z}$ we have

$$|y_\alpha^*(y_i) - y_0^*(y_i)| < \frac{\delta}{3A} \text{ and } |x_\alpha^*(x_i) - x_0^*(x_i)| < \frac{\delta}{3A}.$$

Therefore,

$$\begin{aligned}
|(x_\alpha^* \otimes y_\alpha^*)(z) - (x_0^* \otimes y_0^*)(z)| &= |\sum_{i=1}^k (x_\alpha^*(x_i)y_\alpha^*(y_i) - x_0^*(x_i)y_0^*(y_i))\,| \\
&\leq M\max\{\|x_i\|\} \sum_{i=1}^k |y_\alpha^*(y_i) - y_0^*(y_i)| + N\max\{\|y_i\|\} \sum_{i=1}^k |x_\alpha^*(x_i) - x_0^*(x_i)| \\
&< 2\frac{\delta}{3}, \text{ for every } \alpha > \alpha_{\delta,u}.
\end{aligned}$$

Given $u \in X^*\hat{\otimes}_\epsilon Y^*$, there exists $z = \sum_{i=1}^k x_i \otimes y_i$ such that $\epsilon(u - z) < \frac{\epsilon}{8MN}$. We have, for every α,

$$|\frac{x_\alpha^*}{M} \otimes \frac{y_\alpha^*}{N}(u - z)| \leq \epsilon(u - z) \leq \frac{\delta}{8MN}$$

and

$$|\frac{x_0^*}{M} \otimes \frac{y_0^*}{N}(u - z)| \leq \epsilon(u - z) \leq \frac{\delta}{8MN}.$$

Therefore, $|(x_\alpha^* \otimes y_\alpha^*)(u) - (x_0^* \otimes y_0^*)(u)| < \delta$, for $\alpha > \alpha_{\delta,u}$. This completes the proof. □

Throughout this paper X, Y, H, and K are real Banach spaces with K^* and H^* strictly convex. The operator T is a surjective isometry from the injective tensor product $X\hat{\otimes}_\epsilon K$ onto the injective tensor product $Y\hat{\otimes}_\epsilon H$. It is easy to see that T^* maps the set of extreme points of the unit ball of the dual space $(Y \otimes_\epsilon H)^*$ onto the set of extreme points of the unit ball of the dual space $(X \otimes_\epsilon K)^*$. The adjoint operator, T^* defines a bijection from

$\{y^* \otimes h^* \in Y^* \otimes H^* : \ y^* \in \text{ext}B(Y^*), \text{ and } \|h^*\| = 1, \}$ onto

$\{x^* \otimes k^* \in X^* \otimes K^* : \ x^* \in \text{ext}B(X^*), \text{ and } \|k^*\| = 1\}$.

Proposition 4.0.5. *Let $y_0^* \in ext\, B(Y^*)$, then one of the following statements holds:*

1. *There exist $k^* \in ext\, B(K^*)$ and a linear isometry $\Phi : H^* \to X^*$ such that $T^*(y_0^* \otimes h^*) = \Phi(h^*) \otimes k^*$, for every $h^* \in H^*$.*
2. *There exist $x^* \in ext\, B(X^*)$ and a linear isometry $\Psi : H^* \to K^*$ such that $T^*(y_0^* \otimes h^*) = x^* \otimes \Psi(h^*)$, for every $h^* \in H^*$.*

Proof. Suppose the first statement does not hold. Then there exist two sets of linearly independent norm 1 functionals, $\{h_1^*, h_2^*\}$ and $\{k_1^*, k_2^*\}$ such that

$$T^*(y_0^* \otimes h_1^*) = x_1^* \otimes k_1^* \text{ and } T^*(y_0^* \otimes h_2^*) = x_2^* \otimes k_2^*.$$

Since H^* is strictly convex then $T^*\left(y_0^* \otimes \frac{h_1^*-h_2^*}{\|h_1^*-h_2^*\|}\right)$ is an extreme point of $(X \otimes_\epsilon K)_1^*$. Hence

$$T^*\left(y_0^* \otimes \frac{h_1^* - h_2^*}{\|h_1^* - h_2^*\|}\right) = \frac{1}{\|h_1^* - h_2^*\|}\left(x_1^* \otimes k_1^* - x_2^* \otimes k_2^*\right) = x_0^* \otimes k_0^*.$$

Lemma 4.0.3 implies that $x_0^* \parallel x_1^* \parallel x_2^*$, since k_1^* and k_2^* are linearly independent. Then $x_2^* = x_1^*$ or $x_2^* = -x_1^*$. This implies the second statement with $\Psi(h_1^*) = k_1^*$ and $\Psi(h_2^*) = k_2^*$ (or $= -k_2^*$, respectively). Given a functional h_3^* such that $\{h_1^*, h_2^*, h_3^*\}$ is a linearly independent set, and repeating the same argument we conclude that $T^*\,(y_0^* \otimes h_3^*) = x_1^* \otimes \Psi(h_3^*)$. Therefore we claim that the representation described in the second statement holds. To see this it remains to check whether it is possible for two functionals $\{h_1^*, h_2^*\}$ yield the relations

$$T^*(y_0^* \otimes h_1^*) = x_1^* \otimes \Psi(h_1^*), \quad T^*(y_0^* \otimes h_2^*) = x_1^* \otimes \Psi(h_2^*),$$

and $T^*((y_0^* \otimes (t\, h_1^* + s\, h_2^*)) = \Phi(t\, h_1^* + s\, h_2^*) \otimes k^*$, with t and s scalars. In such case, we have

$$\Phi(t\, h_1^* + s\, h_2^*) \otimes k^* = tx_1^* \otimes \Psi(h_1^*) + sx_1^* \otimes \Psi(h_2^*).$$

Lemma 4.0.3 implies that $\Phi(t\, h_1^* + s\, h_2^*) = ax_1^*$ for some scalar a. Then $\Psi(t\, h_1^* + s\, h_2^*) = t\Psi(h_1^*) + s\,\Psi(h_2^*)$. The norm preserving follows from the isometric assumption on T.

We also have that Φ and Ψ are linear isometries. We observe that the two statements in the proposition cannot coexist. Otherwise, for every $h^* \in H^*$, we would have $\Phi(h^*) \otimes k^* = x^* \otimes \Psi(h^*)$ for every $h^* \in H^*$. This implies that $\Phi(h^*) = \lambda(h^*)x^*$, for some scalar valued linear functional $\lambda(h^*)$. Unless $dim(H^*) = 1$ there exist a nonzero element in H^*, say h^* such that $\lambda(h^*) = 0$ and thus $\Phi(h^*) = 0$. Then $T^*(y_0^* \otimes h^*) = 0$. This is impossible because T^* is an isometry. □

Proposition 4.0.5 determines the sets $S_1 = \{y^* \in \mathrm{ext}B(Y^*) : T^*(y_0^* \otimes h^*) = \Phi_1(y_0^*, h^*) \otimes \Psi_1(y_0^*),\ \text{for all}\ h^* \in H^* \text{ and } \Psi_1(y_0^*) \in \mathrm{ext}B(K^*)\}$ and $S_2 = \mathrm{ext}B(Y^*) \setminus S_1$. If $y_0^* \in S_2$ then $T^*(y_0^* \otimes h^*) = \Phi_2(y_0^*) \otimes \Psi_2(y_0^*, h^*)$, for every $h^* \in H^*$ and $\Phi_2(y_0^*) \in \mathrm{ext}B(X^*)$.

Proposition 4.0.6. *There exist maps $\Phi_1 : S_1 \times H^* \to X^*$, $\Psi_1 : S_1 \to extB(K^*)$, $\Phi_2 : S_2 \to extB(X^*)$ and $\Psi_2 : S_2 \times H^* \to K^*$ such that:*

1. *$T^*(y^* \otimes h^*) = \Phi_1(y^*, h^*) \otimes \Psi_1(y^*)$, for every $y^* \in S_1$ and for every $h^* \in H^*$. For each $y^* \in S_1$, $\Phi_1(y^*, \cdot) : H^* \to X^*$ is a weak-$*$-continuous linear isometry.*

2. *$T^*(y^* \otimes h^*) = \Phi_2(y^*) \otimes \Psi_2(y^*, h^*)$, for every $y^* \in S_2$ and for every $h^* \in H^*$. For each $y^* \in S_2$, $\Psi_2(y^*, \cdot) : H^* \to K^*$ is a weak-$*$-continuous linear isometry.*

Proof. The existence of $\Phi_1 : S_1 \times H^* \to X^*$, $\Psi_1 : S_1 \to \text{ext}B(K^*)$, $\Phi_2 : S_2 \to \text{ext}B(X^*)$ and $\Psi_2 : S_2 \times H^* \to K^*$ is established in Proposition 4.0.5. For $y_1^* \in S_1$ and $y_2^* \in S_2$, the maps $\Phi_1(y_1^*, \cdot) : H^* \to X^*$ and $\Psi_2(y_2^*, \cdot) : H^* \to K^*$ are linear isometries, this also follows from Proposition 4.0.5. It remains to prove the weak-$*$ continuity of these maps. We show the argument for $\Phi_1(y_1^*, \cdot)$, the other case is similar. Let $h_\alpha^* \to h_0^*$ in the weak-* topology. Since T^* is continuous, then $\{T^*(y_1^* \otimes h_\alpha^*\}$ converges in the weak-* topology to $T^*(y_1^* \otimes h_0^*)$. Therefore

$$\Phi_1(y_1^*, h_\alpha^*) \otimes \Psi_1(y_1^*) \to \Phi_1(y_1^*, h_0^*) \otimes \Psi_1(y_1^*).$$

Lemma 4.0.4 implies that $\Phi_1(y_1^*, h_\alpha^*) \to \Phi_1(y_1^*, h_0^*)$, in the weak-* topology. □

Similar arguments apply to T^{-1}. The set of extreme points of $B(X^*)$ is partitioned into two subsets: $\tilde{S}_1$ and $\tilde{S}_2$, defined as follows:

For every $x^* \in \tilde{S}_1$, we have

$$(T^{-1})^*(x^* \otimes k^*) = \tilde{\Phi}_1(x^*, k^*) \otimes \tilde{\Psi}_1(x^*) \text{ for every } k^* \in K^* \tag{4.0.3}$$

and for every $x^* \in \tilde{S}_2$, we have

$$(T^{-1})^*(x^* \otimes k^*) = \tilde{\Phi}_2(x^*) \otimes \tilde{\Psi}_2(x^*, k^*), \text{ for every } k^* \in K^*, \tag{4.0.4}$$

with

$$\tilde{\Phi}_1 \colon \tilde{S}_1 \times K^* \to Y^*, \tilde{\Psi}_1 \colon \tilde{S}_1 \to \text{ext } B(H^*)$$
$$\tilde{\Phi}_2 \colon \tilde{S}_2 \to \text{ext } B(Y^*), \tilde{\Psi}_2 \colon \tilde{S}_2 \times K^* \to H^*.$$

Furthermore, For every $x_1^* \in \tilde{S}_1$ (and $x_2^* \in \tilde{S}_2$), the function the function $\tilde{\Phi}_1(x_1^*, \cdot) : K^* \to Y^*$ (and $\tilde{\Psi}_2(x_2^*, \cdot) : K^* \to H^*$, respectively) is a linear isometry.

Lemma 4.0.7. *Let S_2 and Φ_2 be as described before. Then $\Phi_2(S_2) \subset \tilde{S}_2$.*

Proof. Let $y_0^* \in S_2$ and $h^* \in H^*$ be of norm 1. Then there exist $\Phi_2(y_0^*) \in \text{ext } B(X^*)$ and $\Psi_2(y_0^*, h^*) \in \text{ext } B(K^*)$ such that

$$T^*(y_0^* \otimes h^*) = \Phi_2(y_0^*) \otimes \Psi_2(y_0^*, h^*)$$

Suppose $\Phi_2(y_0^*) \in \tilde{S}_1$. Then

$$\begin{aligned} y_0^* \otimes h^* &= (T^*)^{-1} \left(T^*(y_0^* \otimes h^*)\right) \\ &= (T^*)^{-1} \left(\Phi_2(y_0^*) \otimes \Psi_2(y_0^*, h^*)\right) \\ &= \tilde{\Phi}_1 \left(\Phi_2(y_0^*), \Psi_2(y_0^*, h^*)\right) \otimes \widetilde{\Psi}_1(\Phi_2(y_0^*)) \end{aligned}$$

Hence

$$\widetilde{\Psi_1}(\Phi_2(y_0^*)) = \pm h^*.$$

Since dim $H > 1$ this is impossible. Therefore $\Phi_2(S_2) \subset \tilde{S}_2$.

This completes the proof. □

Following a similar argument we show that $\tilde{\Phi}_2(\tilde{S}_2) \subset S_2$. In the next lemma we revisit Proposition 4.0.6 to prove the surjective of $\Psi_2(y_0^*, \cdot)$.

Lemma 4.0.8. *For every $y_0^* \in S_2$, $\Psi_2(y_0^*, \cdot) : H^* \to K^*$ is surjective.*

Proof. Let $k^* \in K^*$ be a norm 1 functional. Since $\Phi_2(S_2) \subset \tilde{S_2}$, we have $\Phi_2(y_0^*) \in \tilde{S_2}$ and thus

$$(T^{-1})^* \left(\Phi_2(y_0^*) \otimes k^*\right) = \tilde{\Phi}_2(\Phi_2(y_0^*)) \otimes \tilde{\Psi}_2(\Phi_2(y_0^*), k^*). \tag{4.0.5}$$

For every unit functional $h^* \in H^*$ we have

$$(T^{-1})^* T^*(y_0^* \otimes h^*) = \tilde{\Phi}_2(\Phi_2(y_0^*)) \otimes \tilde{\Psi}_2(\Phi_2(y_0^*), \psi_2(y_0^*, h^*)).$$

Therefore $\tilde{\Phi}_2(\Phi_2(y_0^*)) = \pm y_0^*$. Applying T^* to (4.0.5) we get

$$\begin{aligned} T^*\left(\pm y_0^* \otimes \tilde{\Psi}_2(\Phi_2(y_0^*), k^*)\right) &= T^*\left(y_0^* \otimes \tilde{\Psi}_2(\Phi_2(y_0^*), \pm k^*)\right) && \text{(by the linearity of } T^*\text{)} \\ &= \Phi_2(y_0^*) \otimes \Psi_2\left(y_0^*, \tilde{\Psi_2}(\Phi_2(y_0^*), \pm k^*)\right) && \text{(by Proposition (4.0.6))} \end{aligned}$$

From (4.0.5) we get

$$k^* = \Psi_2\left(y_0^*, \tilde{\Psi_2}(\Phi_2(y_0^*), \pm k^*)\right).$$

This implies the surjectivity of $\Psi_2(y^*, \cdot)$. □

The next lemma states that any two subspaces in $\{\Phi_1(y^*, \cdot)(H^*) : y^* \in S_1\}$ are either equal or with $\{0\}$ intersection. It also establishes that when two such subspaces are equal then the corresponding linear isometries are either equal or with sum equal to zero.

Lemma 4.0.9. *Let H^* and K^* be strictly convex Banach spaces.*

1. *For two extreme points* $y_1^*, y_2^* \in S_1$ *we have*

$$\Phi_1(y_1^*, \cdot)(H^*) = \Phi_1(y_2^*, \cdot)(H^*)$$

or

$$\Phi_1(y_1^*, \cdot)(H^*) \cap \Phi_1(y_2^*, \cdot)(H^*) = \{0\}$$

2. *If* $\Phi_1(y_1^*, \cdot)(H^*) = \Phi_1(y_2^*, \cdot)(H^*)$, *then*

$$\Phi_1(y_1^*, \cdot) = \lambda \Phi_1(y_2^*, \cdot)$$

where $\lambda = \pm 1$.

Proof. We start by proving that

$$\Phi_1(y_1^*, h_1^*) = \Phi_1(y_2^*, h_2^*), \text{ implies } y_1^* \parallel y_2^* \text{ or } h_1^* \parallel h_2^*. \tag{4.0.6}$$

Since T^* is onto there are $y_3^* \in S_1, h_3^* \in H^*$ such that

$$T^*(y_3^* \otimes h_3^*) = \Phi_1(y_1^*, h_1^*) \otimes (\Psi_1(y_1^*) + \Psi_1(y_2^*)).$$

This holds because $T^*(y^* \otimes h_1^*)$ is an extreme point, when h_1^* has norm 1. Therefore, $\Phi_1(y_1^*, h_1^*) \otimes \Psi_1(y_1^*)$ is an extreme point, where $\Phi_1(y_1^*, h_1^*)$ is an extreme point of $B(X^*)$. Using the strict convexity of K^* we have that $\frac{\Psi_1(y_1^*)+\Psi_1(y_2^*)}{\|\Psi_1(y_1^*)+\Psi_1(y_2^*\|}$ is also an extreme point, then

$$(T^*)^{-1} (\Phi_1(y_1^*, h_1^*) \otimes (\Psi_1(y_1^*) + \Psi_1(y_2^*))) = y_3^* \otimes h_3^*.$$

Hence,

$$T^*(y_1^* \otimes h_1^* + y_2^* \otimes h_2^*) = \Phi_1(y_1^*, h_1^*) \otimes (\Psi_1(y_1^*) + \Psi_1(y_2^*)) \qquad (\text{since } \ \Phi_1(y_1^*, h_1^*) = \Phi_1(y_2^*, h_2^*)) \tag{4.0.7}$$

$$= T^*(y_3^* \otimes h_3^*). \tag{4.0.8}$$

Lemma 4.0.3 implies $y_1^* \parallel y_2^* \parallel y_3^*$ or $h_1^* \parallel h_2^* \parallel h_3^*$. Therefore y_3^* is either equal to y_1^* or equal to y_2^*, hence $y_3^* \in S_1$.

Now, we assume that $\Phi_1(y_1^*, \cdot)(H^*) \neq \Phi_1(y_2^*, \cdot)(H^*)$ and $\Phi_1(y_1^*, \cdot)(H^*) \cap \Phi_1(y_2^*, \cdot)(H^*) \neq \{0\}$. Let $y_1^*, y_2^* \in S_1$, and $h_0^*, h_1^*, h_2^* \in H^* \setminus \{0\}$, with h_0^* of norm 1, be such that

$$\Phi_1(y_1^*, h_1^*) = \Phi_1(y_2^*, h_2^*), \ \text{ since } \Phi_1(y_1^*, \cdot)(H^*) \neq \Phi_1(y_2^*, \cdot)(H^*) \tag{4.0.9}$$

$$\text{and } \ \Phi_1(y_1^*, h_0^*) \notin \Phi_1(y_2^*, H^*), \ \text{ since } \Phi_1(y_1^*, \cdot)(H^*) \cap \Phi_1(y_2^*, \cdot)(H^*) \neq \{0\}. \tag{4.0.10}$$

The functionals $\Phi_1(y_1^*, h_0^*)$ and $\Phi_1(y_1^*, \frac{h_1^*+h_0^*}{\|h_1^*+h_0^*\|})$ are extreme points of $B(X^*)$, then there exist $y_4^*, y_5^* \in S_1$ and $h_4^*, h_5^* \in H^*$ such that

$$\begin{aligned} T^*(y_4^* \otimes h_4^*) &= \Phi_1(y_1^*, h_0^*) \otimes \Psi_1(y_2^*) \\ T^*(y_5^* \otimes h_5^*) &= \Phi_1(y_1^*, h_1^* + h_0^*) \otimes \Psi_1(y_2^*). \end{aligned} \tag{4.0.11}$$

Therefore

$$T^*(y_2^* \otimes h_2^* + y_4^* \otimes h_4^*) = \Phi_1(y_2^*, h_2^*) \otimes \Psi_1(y_2^*) + \Phi_1(y_1^*, h_0^*) \otimes \Psi_1(y_2^*) \tag{4.0.12}$$

$$= \Phi_1(y_1^*, h_1^* + h_0^*) \otimes \Psi_1(y_2^*) \quad (\text{using the linearity of } \Phi \text{ and (4.0.9)}) \tag{4.0.13}$$

$$= T^*(y_5^* \otimes h_5^*). \tag{4.0.14}$$

From (4.0.6) and (4.0.12) we have $h_2^* \parallel h_4^*$ or $y_2^* \parallel y_4^*$. These relations together with

$h_1^* \parallel h_2^*$ or $y_1^* \parallel y_2^*$ yield the following list of possibilities:

1. $h_1^* \parallel h_2^*$ and $h_2^* \parallel h_4^*$ (which implies $h_1^* \parallel h_4^*$).
2. $y_1^* \parallel y_2^*$
3. $y_2^* \parallel y_4^*$

We show that each of these three possibilities leads to a contradiction. We assume (1). There exist $y_6^* \in S_1$ and $h_6^* \in H^*$ such that

$$\begin{aligned} T^*(y_4^* \otimes h_4^* + y_1^* \otimes h_0^*) &= \Phi_1(y_1^*, h_0^*) \otimes \Psi_1(y_2^*) + \Phi_1(y_1^*, h_0^*) \otimes \Psi_1(y_1^*) \\ &= \Phi_1(y_1^*, h_0^*) \otimes (\Psi_1(y_2^*) + \Psi_1(y_1^*)) \\ &= T^*(y_6^* \otimes h_6^*). \end{aligned}$$

Lemma (4.0.3) implies that $h_4^* \parallel h_0^*$. Then $h_0^* \parallel h_1^*$ and $h_1^* \parallel h_4^*$. Since $h_0^* \parallel h_1^*$, the linearity of $\Phi_1(y_1^*, \cdot)$ implies

$$\Phi_1(y_1^*, h_0^*) \parallel \Phi_1(y_1^*, h_1^*).$$

The assumption displayed in (4.0.9), $\Phi_1(y_1^*, h_1^*) = \Phi_1(y_2^*, h_2^*)$, implies that

$$\Phi_1(y_2^*, h_2^*) \in \Phi_1(y_2^*, H^*).$$

Thus

$$\Phi_1(y_1^*, h_0^*) \in \Phi_1(y_2^*, H^*).$$

This contradicts (4.0.10). If (2) holds, then $y_1^* \parallel y_2^*$ and $y_1^* = -y_2^*$. Therefore

$$\begin{aligned} T^*(y_1^* \otimes h^*) &= \Phi_1(y_1^*, h^*) \otimes \Psi_1(y^*) \\ &= T^*(-y_2^* \otimes h^*) \\ &= -\Phi_1(y_2^*, h^*) \otimes \Psi_1(y_2^*) \end{aligned}$$

and $\Psi_1(y_2^*) = \pm\Psi_1(y_1^*)$. For every $h^* \in H^*$, $\Phi_1(y_1^*, h^*) = \mp\Phi_1(y_2^*, h^*)$. In particular, for $h_0^* \in H^*$, we have

$$\begin{aligned} \Phi_1(y_1^*, h_0^*) &= \mp\Phi_1(y_2^*, h_0^*) \\ &= \Phi_1(y_2^*, \mp h^*) \\ &\in \Phi_1(y_2^*, H^*). \end{aligned}$$

This contradicts (4.0.10). If (3) holds, then $y_2^* = \pm y_4^*$ and
$\Phi_1(y_2^*, H^*) = \pm\Phi_1(y_4^*, H^*) = \Phi_1(y_4^*, H^*)$. From (4.0.11) we obtain
$\Phi_1(y_1^*, h_0^*) = \Phi_1(y_4^*, h_4^*) \in \Phi_1(y_4^*, H^*) = \Phi_1(y_2^*, H^*)$, which contradicts (4.0.9) and proves the statement.

In order to prove (ii) we assume that $\Phi_1(y_1^*, \cdot)(H^*) = \Phi_1(y_2^*, \cdot)(H^*)$. We claim that for every $h^* \in H^*$, there is a modulus 1 number λ_{h^*} such that

$$\Phi_1(y_1^*, h^*) = \lambda_{h^*}\Phi_1(y_2^*, h^*). \tag{4.0.15}$$

We fix $h_1^* \in H^*$, since $\Phi_1(y_1^*, \cdot)(H^*) = \Phi_1(y_2^*, \cdot)(H^*)$, there exists $h_2^* \in H^*$ such that

$$\Phi_1(y_1^*, h_1^*) = \Phi_1(y_2^*, h_2^*). \tag{4.0.16}$$

The statement in (4.0.6) implies that either $y_1^* \parallel y_2^*$ or $h_1^* \parallel h_2^*$. If $y_1^* \parallel y_2^*$, then

$y_1^* = -y_2^*$ and

$$T^*(y_1^* \otimes h^*) = -T^*(y_2^* \otimes h^*)$$
$$\Phi_1(y_1^*, h^*) \otimes \Psi_1(y_1^*) = -\Phi_1(y_2^*, h^*) \otimes \Psi_1(y_2^*),$$

for every $h^* \in H^*$. Hence $\Phi_1(y_1^*, h^*) = \pm\Phi_1(y_2^*, h^*)$ and then (4.0.15) follows with $\lambda_{h^*} = \pm 1$.

Now we consider the sets.

$$A = \{h^* \in \text{ext } B(H^*) : \Phi_1(y_1^*, h^*) = \Phi_1(y_2^*, h^*)\}$$
$$B = \{h^* \in \text{ext } B(H^*) : \Phi_1(y_1^*, h^*) = -\Phi_1(y_2^*, h^*)\}$$

Note that A and B are disjoint. Suppose not, then there is a $h^* \in \text{ext}B(H^*)$ such that $\Phi_1(y_2^*, h^*) = 0$ and $\|h^*\| = 0$. This is impossible. It can also be seen that both A and B are weak*-closed. We consider a net $\{h_\alpha\}$ converging in the weak-* topology to h_0^*. The continuity of T^* implies

$$\Phi_1(y_1^*, h_\alpha^*)\hat{\otimes}_\epsilon \Psi_1(y_1^*) = T^*(y_1^* \hat{\otimes}_\epsilon h_\alpha^*) \to T^*(y_1^* \otimes h_0^*) = \Phi_1(y_1^*, h_0^*) \otimes \Psi_1(y_1^*)$$

Therefore $\Phi_1(y_1^*, h_\alpha^*) \to \Phi_1(y_1^*, h_0^*)$, and $\Phi_1(y_2^*, h_\alpha^*) \to \Phi_1(y_1^*, h_0^*)$ in the weak-* sense. Similarly we can prove $\Phi_1(y_2^*, h_\alpha^*) \to \Phi_1(y_2^*, h_0^*)$. Hence $\Phi_1(y_1^*, h_0^*) = \Phi_1(y_2^*, h_0^*)$, so $h_0^* \in A$. Therefore A is weak* closed. Similarly we show that B is weak*-closed. Since $A \cup B = \text{ext } B(H^*)$ and ext $B(H^*)$ is connected, then either A or B is empty. This concludes the proof. □

Remark 4.0.10. *The representation derived in Proposition 4.0.6 can be extended to the weak-* closure of S_1 and S_2. Similar arguments apply to the corresponding representations for the closures of $\tilde{S}_1$ and $\tilde{S}_2$.*

The previous lemma allows the representations for T^* and $(T^{-1})^*$ given in Proposition 4.0.6 and 4.0.4 to be extended to the weak-* closures of S_1, S_2, $\tilde{S}_1$ and $\tilde{S}_2$.

Let $0 \neq y_0^* \in \bar{S}_1$. We consider a net with functionals in S_1, weak-* convergent to y_0^*. We may assume that $(\Psi_1(y_\alpha^*)) \to k_0^* \in K^*$, since $\{\Psi(y_\alpha^*\}$ is in $B(X^*)$, which is weak-* compact. We observe that $k_0^* \neq 0$. Notice that $\psi_1(y_\alpha^*)$ converging to 0 implies that for every norm 1 vector $u \in X^* \otimes K^*$, $|(y_\alpha^* \otimes h^*)T(u)| \to 0$. This is impossible because T is a surjective isometry and $\|y_\alpha^* \otimes h^*)\| = 1$.

Therefore, for every $h^* \in H^*$, the net $\{\Phi_1(y_\alpha^*, h^*)\}$ is convergent. Moreover, for every $h^* \in H^*$,

$$\begin{aligned} T^*(y_0^* \otimes h^*) &= \lim_\alpha T^*\left(y_\alpha^* \otimes h^*\right) \\ &= \lim_\alpha \Phi_1(y_\alpha^*, h^*) \otimes \lim_\alpha \Psi(y_\alpha^*), \end{aligned}$$

the limits in the equations displayed above are in the weak-* sense.

Following a similar argument presented before, we extend the functions $\Phi_i, \tilde{\Phi}_i, \Psi_i, \tilde{\Psi}_i$ to the closures of the corresponding sets of extreme points. Therefore we have the representations for T^* and $(T^{-1})^*$:

$$\begin{aligned} T^*(y^* \otimes h^*) &= \Phi_1(y^*, h^*) \otimes \Psi_1(y^*), \quad \text{for all } y^* \in \bar{S_1}^{w^*}, \text{ and } h^* \in H^* \\ T^*(y^* \otimes h^*) &= \Phi_2(y^*) \otimes \Psi_2(y^*, h^*), \quad \text{for all } y^* \in \bar{S_2}^{w^*}, \text{ and } h^* \in H^* \\ (T^{-1})^*(x^* \otimes k^*) &= \tilde{\Phi}_1(x^*, k^*) \otimes \tilde{\Psi}_1(x^*), \quad \text{for all } x^* \in \overline{\tilde{S}_1}^{w^*}, \text{ and } k^* \in K^* \\ (T^{-1})^*(x^* \otimes k^*) &= \tilde{\Phi}_2(x^*) \otimes \tilde{\Psi}_2(x^*, k^*), \quad \text{for all } x^* \in \overline{\tilde{S}_2}^{w^*}, \text{ and } k^* \in K^*. \end{aligned}$$

For the remaining of the paper we use $\overline{S}_i$ and $\overline{\tilde{S}}_i$ to denote the weak-* closures of the sets S_i and $\tilde{S}_i$, with $i = 1$ or 2. Next, we show that Φ_2 and $\tilde{\Phi}_2$ appearing in the representations of T^* and $(T^{-1})^*$, can be modified such that $\Phi_2^{-1} = \tilde{\Phi}_2$. For all $h^* \in H^*$ and $y^* \in \bar{S}_2$, we get

$$\begin{aligned} \|y^*\|\|h^*\| = \|y^* \otimes h^*\| &= \|T^*(y^* \otimes h^*)\| \\ &= \|\Phi_2(y^*)\|\|\Psi_2(y^*, h^*)\| \end{aligned}$$

For every $y^* \in \bar{S}_2 \setminus \{0\}$, there is $\lambda_{y^*} \in \mathbb{R} \setminus \{0\}$ such that $\|\lambda_{y^*}\Phi_2(y^*)\| = \|y^*\|$. Thus $\|\frac{1}{\lambda_{y^*}}\Psi_2(y^*, h^*)\| = \|h^*\|$. Without loss of generality we assume that $\|\Phi_2(y^*)\| = \|y^*\|$ and $\|\Psi_2(y^*, h^*)\| = \|h^*\|$, for every $y^* \in \overline{S}_2 \setminus \{0\}$ and $h^* \in H^*$. Similar conditions are assumed for $\tilde{\Phi}_2(x^*)$ and $\tilde{\Psi}_2(x^*, \cdot)$, for every $x^* \in \overline{\tilde{S}}_2$.

We define an equivalence relation on $\bar{S}_2 \setminus \{0\}$ by

$$y_1^* \sim y_2^* \quad \text{if} \quad \Psi_2(y_1^*, \cdot) = \lambda\Psi_2(y_2^*, \cdot), \qquad \text{for some } \lambda \in \mathbb{R}. \tag{4.0.17}$$

It is worthy noticing that given $h^* \in H^*$, y_1^* and y_2^* be in $\bar{S}_2 \setminus \{0\}$ such that

$$\Psi_2(y_1^*, h^*) = \lambda\Psi_2(y_2^*, h^*)$$

for some nonzero number λ. Then λ is equal to 1 or -1, because $\Psi_2(y_1^*, \cdot)$ and $\Psi_2(y_2^*, \cdot)$ are isometries.

Lemma 4.0.11. *Let y_1^* and y_2^* be in $\bar{S}_2 \setminus \{0\}$. The following statements are equivalent:*

(i) For every $h^* \in H^*$ there exists $\lambda(h^*) = \pm 1$ such that $\Psi_2(y_1^*, h^*) = \lambda(h^*)\Psi_2(y_2^*, h^*)$.

(ii) $\Psi_2(y_1^*, \cdot) = \Psi_2(y_2^*, \cdot)$ *or* $\Psi_2(y_1^*, \cdot) = -\Psi_2(y_2^*, \cdot)$.

(iii) $y_1^* \sim y_2^*$.

Proof. We prove that (i) implies (ii), the reversed implication is clear. Since $\lambda(h^*) = 1$ or -1, a connectedness argument shows that it must be constant. The equivalence between (ii) and (iii) is also clear. □

We may assume that $y^* \mapsto \Psi_2(y^*, \cdot)$ is constant in each equivalence class defined by the equivalence relation in (4.0.17), up to a multiplication by -1.

The proof given for Lemma 4.0.7 and Remark 4.0.10 imply that for $y^* \in \overline{S}_2$, $\Phi_2(y^*) \in \bar{\tilde{S}}_2$. Thus, for any $y^* \in \bar{S}_2$ and all $h^* \in H^*$, we get

$$\begin{aligned} y^* \otimes h^* &= (T^*)^{-1} \left(\Phi_2(y^*) \otimes \Psi_2(y^*, h^*) \right) \\ &= \widetilde{\Phi_2}(\Phi_2(y^*)) \otimes \widetilde{\Psi_2} \left(\Phi_2(y^*), \Psi_2(y^*, h^*) \right) \end{aligned}$$

Therefore $y^* \parallel \widetilde{\Phi_2}(\Phi_2(y^*))$, meaning that there is a modulus 1 real number ϵ such that $y^* = \epsilon \widetilde{\Phi_2}(\Phi_2(y^*))$. Similarly, for any $\lambda \in \mathbb{R}$ such that $\lambda y^* \in \bar{S}_2$ we have

$$\lambda y^* = \epsilon_\lambda \widetilde{\Phi_2}(\Phi_2(\lambda y^*)).$$

We show that $y^* \sim \lambda y^*$. To see this we use the linearity of T^* as follows:

$$\lambda T^*(y^* \otimes h^*) = \lambda \Phi_2(y^*) \otimes \Psi_2(y^*, h^*) \tag{4.0.18}$$

$$T^*(\lambda y^* \otimes h^*) = \Phi_2(\lambda y^*) \otimes \Psi_2(\lambda y^*, h^*). \tag{4.0.19}$$

Thus, for every $h^* \in H^*$ we have $\Psi_2(y^*, h^*) = \alpha \Psi_2(\lambda y^*, h^*)$, for some constant α. Lemma (4.0.11) implies $y^* \sim \lambda y^*$. Therefore $\Psi_2(y^*, \cdot) = \Psi_2(\lambda y^*, \cdot)$. It is also easy to show that $\Phi_2(y^*) \sim \Phi_2(\lambda y^*)$. First , we notice that

$$(T^{-1})^*(\Phi_2(y^*) \otimes \Psi_2(y^*, \lambda\, h^*)) = \tilde{\Phi}_2(\Phi_2(y^*)) \otimes \lambda \tilde{\Psi}_2(\Phi_2(y^*), \Psi_2(y^*, h^*)).$$

Then applying $(T^{-1})^*$ to equations 4.0.18 and 4.0.19 we get

$$\tilde{\Phi}_2(\Phi_2(y^*)) \otimes \lambda \tilde{\Psi}_2(\Phi_2(y^*), \Psi_2(y^*, h^*)) = \tilde{\Phi}_2(\Phi_2(\lambda y^*)) \otimes \tilde{\Psi}_2(\Phi_2(\lambda y^*)), \Psi_2(\lambda y^*, h^*)).$$

Since $\Psi_2(y^*, \cdot) = c\Psi_2(\lambda y^*, \cdot)$, for some constant c, and $\Psi_2(y^*, \cdot)$ is onto we have $\tilde{\Psi}_2(y^*, \cdot) = \beta \tilde{\Psi}_2(\Phi_2(\lambda y^*), \cdot)$. Therefore,

$$\tilde{\Psi_2}(\Phi_2(y^*), \Psi_2(y^*, \cdot)) = \tilde{\Psi}_2(\Phi_2(\lambda y^*), \Psi_2(\lambda y^*, \cdot)).$$

On the other hand, we have

$$y^* \otimes h^* = \widetilde{\Phi_2}(\Phi_2(y^*)) \otimes \widetilde{\Psi_2}\,(\Phi_2(y^*), \Psi_2(y^*, h^*))$$

and

$$\lambda y^* \otimes h^* = \widetilde{\Phi_2}(\Phi_2(\lambda y^*)) \otimes \widetilde{\Psi_2}\,(\Phi_2(\lambda y^*), \Psi_2(\lambda y^*, h^*))$$

Then $\epsilon_\lambda = \epsilon$.

This implies that over each set $(\{\mathbb{R} \setminus \{0\}) \times \bar{S}_2 \cup \bar{S}_2$, we have that $\tilde{\Phi}_2 \Phi_2 = Id$ or $= -Id$. At those functionals $y^* \in \overline{S}_2 \setminus \{0\}$ such that $\tilde{\Phi}_2\Phi_2(y^*) = -y^*$ we replace $\tilde{\Phi}_2$ over $\lambda \Phi_2(y^*)$ with $-\tilde{\Phi}_2$, and also $\tilde{\Psi}_2$ with $-\tilde{\Psi}_2$. Therefore, we may assume that $\tilde{\Phi}_2\Phi_2(y^*) = y^*$, for every $y^* \in \overline{S}_2 \setminus \{0\}$.

We now define

$$X_i = \left\{ f\Big|_{\tilde{S}_i} : f \in X \subset C(\tilde{S}_1 \cup \tilde{S}_2) \right\}$$
$$Y_i = \left\{ g\Big|_{S_i} : g \in Y \subset C(S_1 \cup S_2) \right\}$$

for $i = 1, 2$. We will show the following.

1. X_i, Y_i are Banach spaces for $i = 1, 2$.

2. $X = X_1 \bigoplus_\infty X_2$ and $Y = Y_1 \bigoplus_\infty Y_2$.

Lemma 4.0.12. *For any $y^* \in S_1$, $(\Phi_1(y^*, \cdot))^*$ maps X onto H.*

Proof. For $y^* \in S_1$, $\Phi_1(y^*, \cdot)$ is an isometry, and $T^*(y^* \otimes h^*) = \Phi_1(y^*, h^*) \otimes \psi_1(y^*)$, where $\psi_1(y^*) \in ext(K_1^*)$. Since $Ker\Phi_1(y^*, \cdot) = \{0\}$ and $\Phi_1(y^*, \cdot)$ is weak-* continuous, then

$$H^{**} = [Ker\Phi_1(y^*, \cdot)]^\perp = \overline{Range[\Phi_1(y^*, \cdot)^*]}^{w*}.$$

The weak-* continuity implies $Range(\Phi_1(y^*, \cdot)^*) = H^{**}$. Therefore $\Phi_1^*(y^*, \cdot)$ is a surjective isometry.

We have that $T^{**} : X^{**} \otimes K^{**} \to Y^{**} \otimes H^{**}$ is given by

$$\begin{aligned} T^{**}(x^{**} \otimes k^{**})(y^* \otimes h^*) &= (x^{**} \otimes k^{**})(\Phi_1(y^*, h^*) \otimes \psi_1(y^*)) \\ &= x^{**}(\Phi_1(y^*, h^*))\, k^{**}(\psi_1(y^*)) \\ &= (\Phi_1(y^*, h^*))^*(x^{**})\, k^{**}(\psi_1(y^*)). \end{aligned}$$

We show that $\Phi_1^*(y^*,\cdot)(x) \in H$. For $x^{**} \in X$ and $k^{**} \in K$ we have

$$T^{**}(x^{**} \otimes k^{**})(y^* \otimes \cdot) = \Phi_1^*(y^*,\cdot)(x^{**})k^{**}\left(\Psi_1(y^*)\right) \in H.$$

Since T^{**} restricted to $X \otimes K$ coincides with T and T is a surjective isometry onto $H \otimes Y$ then $(\Phi_1(y^*,\cdot))^*(X) = H$. □

We consider $(\Phi_1(y^*,\cdot))^* |_X$, denoted by $\phi_1^*(y^*,\cdot)$ for simplicity of notation. We set

$$\Omega = \{\lambda\Phi_1^*(y^*,\cdot) : y^* \in \bar{S}_1, \lambda = \pm 1\},$$

a subset of $\mathcal{L}(X, H)$ equipped with the topology Γ given by the family of seminorms :

$$\mathcal{A} = \{\rho : R \mapsto |h^*(Rx)| : x \in X, h_\alpha^* \in H^*\}.$$

We say that given a net $\{S_i\}_{i\in\Lambda}$ with $S_i \in \Omega$,

$$S_i \to_\Gamma S \iff \forall \rho \in \mathcal{A}, \rho(S - S_i) \to 0.$$

In other words S_i converges to S in the topology Γ if and only if for every $\epsilon > 0$, $h^* \in H^*$ and $x \in X$, there exists α_0 such that for every $\alpha > \alpha_0$,

$$|h^*(S - S_i)(x)| < \epsilon.$$

Lemma 4.0.13. *The set Ω is compact.*

Proof. We show that every net in Ω has a convergent subnet. Let $\{\lambda_\alpha\Phi_1^*(y_\alpha^*,\cdot)\}_{\alpha\in\Lambda}$, with $y_\alpha^* \in \bar{S}_1$ and $\lambda_\alpha = \pm 1$. Without loss of generality we can assume $\lambda_\alpha = 1$,

$y_\alpha^* \to y_0^*$ and $\Psi_1^*(y_\alpha^*)$ converges to k^* . First, we assume that $k^* \neq 0$. Then for every $h^* \in H^*$ and $x \otimes k \in X \otimes K$ we have

$$|(T^*(y_\alpha^* \otimes h^*) - T^*(y_0^* \otimes h^*)(x \otimes k)| = |(y_\alpha^* - y_0^*) \otimes h^*(T(x \otimes k)| \to 0.$$

Hence

$$|\Phi_1(y_\alpha^*, h^*) \otimes \psi_1(y_\alpha^*) - \Phi_1(y_0^*, h^*) \otimes k^*| \to 0,$$

in the weak-* topology. Let $\epsilon > 0$, for an arbitrary unit elementary tensor $x \otimes k \in X \otimes K$ such that $k^*(k) > a > 0$, we have $|\Psi_1(y_\alpha^*)(k) - k^*(k)| < \epsilon/4$ and

$$|T^*(y_\alpha^* \otimes h^*) - T^*(y_0^* \otimes h^*)(x \otimes k)| < \epsilon/4,$$

for $\alpha > \alpha_0$. Therefore,

$$\begin{aligned}
\frac{\epsilon}{4} \geq & |T^*(y_\alpha^* \otimes h^*) - T^*(y_0^* \otimes h^*)(x \otimes k)| \\
& \geq |x^{**}(\Phi_1(y_\alpha^*, h^*) - x^{**}\Phi_1(y_0^*, h^*)||k^*k| - |x^{**}\Phi_1(y_\alpha^*, h^*)||\Psi_1^*(y_\alpha^*)(k) - k^*(k)| \\
& \geq a|x^{**}(\Phi_1(y_\alpha^*, h^*) - x^{**}\Phi_1(y_0^*, h^*)| - \frac{\epsilon}{4}
\end{aligned}$$

Thus

$$\begin{aligned}
|h^*\left(\Phi_1^*(y_\alpha^*, \cdot) - \Phi_1^*(y_0^*, \cdot)\right)(x)| &= |x^{**}\left(\Phi_1(y_\alpha, h^*) - \Phi_1^*(y_0, h^*)\right)| \\
&< \epsilon/4a \quad \text{for every } \alpha > \alpha_0.
\end{aligned}$$

This proves that $\Phi_1^*(y_\alpha^*, \cdot) \to \Phi_1^*(y_0^*, \cdot)$ in the Γ topology.

We consider the case $\Psi_1(y_\alpha^*) \to 0$. Then for every h^*, $T^*(y_\alpha^* \otimes h^*) \to 0$. This implies that $T^*(y_0^* \otimes h^*) = 0$ and $y_0^* = 0$. We set $\Phi_1(y_0^*, \cdot) = 0$. This completes the proof. □

For $y^* \in \overline{S}_1$, the operator $\Phi_1(y^*, \cdot)$ is weak-* continuous. We define $Q : S_1 \to \Omega \otimes extB(K^*)$ as follows:

$$Q(y^*) = \Phi_1^*(y^*, \cdot) \otimes \Psi_1(y^*).$$

We also define

$Q_1 = Q \otimes Id_{extB(H^*)} : S_1 \otimes extB(H^*) \to \Omega \otimes extB(K^*) \otimes extB(H^*)$, given $Q_1(x^* \otimes h^*) = Q(y^*) \otimes h^*$.

We show that Q and Q_1 are continuous surjections. For this, it is sufficient to show the continuity and surjectivity of Q. The strict convexity of H^* implies that the every norm 1 functional $h^* \in H^*$ is an extreme point of the unit ball $B(H^*)$.

Lemma 4.0.14. *For every $y^* \in S_1$*

$$\Phi_1(y^*, \cdot)(extB(H^*)) \subset ext\ B(X^*)$$

Proof. Let $y_0^* \in S_1$ and $\|h^*\| = 1$ be such that $\Phi_1(y_0^*, h^*)$ is not an extreme point of $B(X^*)$. Then

$$\Phi_1(y_0^*, h^*) = \frac{x_1^* + x_2^*}{2}$$

for some $x_1^* \neq x_2^* \in B(X^*)$. Then

$$\begin{aligned} \Phi_1(y_0^*, \cdot) \otimes \Psi_1(y_0^*) &= \frac{x_1^* \otimes \Psi_1(y_0^*)}{2} + \frac{x_2^* \otimes \Psi_1(y_0^*)}{2} \\ T^*(y_0^* \otimes h_0^*) &= \frac{x_1^* \otimes \Psi_1(y_0^*)}{2} + \frac{x_2^* \otimes \Psi_1(y_0^*)}{2} \end{aligned}$$

Since $T^*(y_0^* \otimes h_0^*)$ is an extreme point, this implies $x_1^* = x_2^*$. This contradiction proves the statement. □

We recall that $-y^* \in S_1$ if $y^* \in S_1$. We may assume without loss of generality that $\Omega \otimes ext(K_1^*) = \{\Phi_1^*(y^*, \cdot) \otimes \Psi_1(y_1^*): \ y^* \text{ and } y_1^* \in S_1\}$. The topology on $\Omega \otimes ext(K_1^*)$, is defined as follows: A net $\{\Phi_1^*(y_\alpha^*, \cdot) \otimes \Psi_1(y_\alpha^*)\}$ converges to $\Phi_1^*(y_0^*, \cdot) \otimes \Psi_1(y_0^*)$ if and only if for every $h^* \in H^*$, $x \in X$ and $k \in K$ we have

$$\Phi_1(y_\alpha^*, h^*)(x)\Psi_1(y_\alpha^*)(k) \to \Phi_1(y_0^*, h^*)(x)\Psi_1(y_0^*)(k).$$

Lemma 4.0.15. *The function Q is a continuous bijection.*

Proof. We consider a net in S_1, $\{y_\alpha^*\}$, that converges to $y_0^* \in S_1$. We prove that $Q(y_\alpha^*)$ converges to $Q(y_0^*)$. Let $h^* \in H^*$ and $x \in X$. Since K_1^* is weak-* compact we may assume that $\Psi_1(y_\alpha^*)$ converges to k^*. First , we consider $k^* \neq 0$. Since $T^*(y_\alpha^* \otimes h^*) \to T^*(y_0^* \otimes h^*)$, we have

$$T^*(y_\alpha^* \otimes h^*)(x \otimes k) = \Phi_1(y_\alpha^*, h^*)(x)\Psi_1(y_\alpha^*)(k) \to \Phi_1(y_0^*, h^*)(x)k^*(k) = T^*(y_0^* \otimes h^*)(x \otimes k).$$

To prove the injectivity of Q, let y_1^* and y_2^* be such that

$$\Phi_1^*(y_1^*, \cdot) \otimes \Psi_1(y_1^*) = \Phi_1^*(y_2^*, \cdot) \otimes \Psi_1(y_2^*). \tag{4.0.20}$$

For $h^* \in H^*$ and $x \otimes k \in X \otimes K$, equation (4.0.20) implies that

$$\Phi_1(y_1^*, h^*)(x)\Psi_1(y_1^*)(k) = \Phi_1(y_2^*, h^*)(x)\Psi_1(y_2^*)(k)$$

or $T^*(y_1^* \otimes h^*) = T^*(y_2^* \otimes h^*)$. Therefore $y_1^* = y_2^*$.

To prove that Q is onto, it is sufficient to show that, for any $y_0^* \in S_1$ and $k_0^* \in \text{ext } B(K^*)$, there is a $y^* \in S_1$ such that $\Phi_1(y_0^*, \cdot) = \epsilon\Phi_1(y^*, \cdot)$ and $k_0^* = \epsilon\Psi_1(y^*)$ where $|\epsilon| = 1$.

If $\Phi_1^*(y_0^*, \cdot) = \epsilon\Phi_1^*(y^*, \cdot)$ and $k_0^* = \epsilon\Psi_1(y^*)$ then

$$\begin{aligned}\Phi_1^*(y_0^*, h^*) \otimes k_0^* &= \epsilon\Phi_1^*(y^*, \cdot) \otimes \epsilon\Psi_1(y^*) \\ &= \Phi_1^*(y^*, \cdot) \otimes \Psi_1(y^*) = Q(y^*).\end{aligned}$$

Let $h_0^* \in \text{ext } B(H^*)$. Since T^* is onto, there is $y^* \in S_1$ and $h^* \in \text{ext } B(H^*)$ such that

$$T^*(y^* \otimes h^*) = \Phi_1(y_0^*, h_0^*) \otimes k_0^* \qquad (4.0.21)$$

From (4.0.5) we have

$$T^*(y^* \otimes h^*) = \Phi_1(y^*, h^*) \otimes \Psi_1(y^*). \qquad (4.0.22)$$

Therefore $\Phi_1(y_0^*, h_0^*) = \alpha\Phi_1(y^*, h^*)$ for some real scalar α. This implies that $\Phi_1(y_0^*, H^*) \cap \Phi_1(y^*, H^*) \neq 0$. Lemma (4.0.9) implies that $\Phi_1(y_0^*, H^*) = \Phi_1(y^*, H^*)$ and hence there is $\epsilon = \pm 1$ such that $\Phi_1(y_0^*, \cdot) = \epsilon\Phi_1(y^*, \cdot)$. From (4.0.21) and (4.0.22) we get $k_0^* = \epsilon\Psi_1(y^*)$.

$\square$

We define $Q_1 = Q \otimes Id_{extB(H^*)} : S_1 \otimes extB(H^*) \to \Omega \otimes extB(H^*) \otimes extB(H^*)$ as follows:

$$Q_1(y^* \otimes h^*) = Q(y^*) \otimes h^*.$$

Hence,

$$Q_1^{-1}(w \otimes k^* \otimes h^*) = y_1^* \otimes h^*,$$

where $Q(y_1^*) = w \otimes k^*$, then $k^* = \Psi_1^*(y_1^*)$. Furthermore,

$T^*(y_1^* \otimes h^*) = \Phi_1(y_1^*, h^*) \otimes \Psi_1(y_1^*)$. We set $w(h^*) = \Phi_1(y_1^*, h^*)$. Therefore,

$$Q_1^{-1}(w \otimes k^* \otimes h^*) = (T^{-1})^*(w(h^*) \otimes k^*).$$

Lemma 4.0.16. *Q^{-1} and Q_1^{-1} are continuous.*

Proof. We prove that Q^{-1} is continuous. Suppose $\omega_\alpha \otimes k_\alpha^* \to \omega_0 \otimes k_0^*$, $Q^{-1}(\omega_\alpha \otimes k_\alpha^*) = z_\alpha^*$ and $Q^{-1}(\omega_0 \otimes k_0^*) = z_0^*$. We prove $z_\alpha^* \xrightarrow{w^*} z_0^*$. Since we have $\Phi_1^*(z_\alpha^*, \cdot) \otimes \Psi_1(z_\alpha^*) \to \Phi_1^*(z_0^*, \cdot) \otimes \Psi_1(z_0^*)$, for every $h^* \in H^*, x \in X$ and $k \in K$,

$$\Phi_1(z_\alpha^*, h^*)(x)\Psi_1(z_\alpha^*)(k) \to \Phi_1(z_0^*, h^*)(x)\Psi_1(z_0^*)(k).$$

Equivalently, $T^*(z_\alpha^* \otimes h^*)(x \otimes k) \to T^*(z_0^* \otimes h^*)(x \otimes k)$. Applying $(T^{-1})^*$ on both the sides, we get

$$(z_\alpha^* \otimes h^*)(x \otimes k) \to (z_0^* \otimes h^*)(x \otimes k)$$

Then $z_\alpha^* \otimes h^* \xrightarrow{w^*} z_0^* \otimes h^*$. Lemma (4.0.4) implies $z_\alpha^* \xrightarrow{w^*} z_0^*$, which completes the proof. □

As done before, we define $\tilde{\Phi}_1^*(x^*, \cdot) \in \mathcal{L}(Y, K)$ for $x^* \in \tilde{S}_1$ we set

$$\tilde{\Omega} = \{\lambda \Phi_1^*(x^*, \cdot) : x^* \in \tilde{S}_1,\ \lambda = \pm 1\}$$

and the homeomorphisms

$$P : \tilde{S}_1 \to \tilde{\Omega} \otimes extB(H^*), \ \ P(x^*) = \tilde{\Phi}_1^*(x^*, \cdot) \otimes \tilde{\Psi}_1(x^*),$$

and $P_1 = P \otimes Id_{extB(K^*)}$. The homeomorphisms Q, Q_1, P and P_1 define the isometric embeddings:

$$Q^0 : Y_1 \to C(\Omega \otimes extB(K^*))$$

$$Q_1^0 : Y_1 \otimes H \to C(\Omega \otimes extB(K^*)) \otimes B(H^*),$$

$$P^0 : X_1 \to C(\tilde{\Omega} \otimes extB(H^*)),$$

and

$$P^0 : X_1 \otimes K \to C(\tilde{\Omega} \otimes extB(H^*) \otimes extB(K^*)).$$

To be more precise, we set $Q^0(f_y)Q^{-1}(w \otimes k^*) = y_1^*(y)$, with f_y representing the point evaluation at y of all functionals in S_1, and y_1^* to be the functional in S_1 such that $Q(y_1^*) = w \otimes k^*$. It is clear that they are isometries, $\|f_y\| = \sup_{\alpha \in \Omega \otimes ext(K_1^*)} |Q^{-1}(\alpha)(y)| = \|Q^0(f_y)\|_\infty$. The other maps are defined similarly and it is also apparent that they are isometries.

We notice that

$$\begin{array}{rcl} P_1 \circ T^* \circ Q_1^{-1} : \Omega \otimes extB(K^*) \otimes extB(H^*) & \longrightarrow & \tilde{\Omega} \otimes extB(H^*) \otimes extB(K^*) \\ w \;\otimes\; k^* \;\otimes\; h^* & \longrightarrow & \varphi(w) \;\otimes\; h^* \;\otimes\; k^* \end{array}$$

is a homeomorphism and φ defines a homeomorphism from Ω onto $\tilde{\Omega}$. Given $w \otimes k^* \otimes h^*$, applying Q_1^{-1}, we get $y^* \otimes h^*$ with $y^* \in S_1$ such that $\Phi_1^*(y^*, \cdot) = w$ and $\Psi_1(y^*) = k^*$. Then $T^*(y^* \otimes h^*) = \Phi_1(y^*, h^*) \otimes k^*$. We set $w(h^*) = \Phi_1(y^*, h^*) \in \tilde{S}_1$, then $P_1(\Phi_1(y^*, h^*) \otimes k^*) = (\tilde{\Phi}_1^*(w(h^*), \cdot) \otimes \tilde{\Psi}_1(w(h^*)) \otimes k^*)$. Since

$$(T^{-1})^* T^*(y^* \otimes h^*) = y^* \otimes h^* = \tilde{\Phi}_1^*(w(h^*), \cdot) \otimes \tilde{\Psi}_1(w(h^*))$$

we have $\tilde{\Psi}_1(w(h^*) = h^*$. This explains the form for $P_1 \circ T^* \circ Q_1^{-1}$ described above.

Remark 4.0.17. *For every $h^* \in extB(H^*)$ and $k^* \in extB(K^*)$ we have*

$$ImQ^0|_{\Omega\otimes k^*} = ImQ_1^0|_{\Omega\otimes k^*\otimes h^*} \simeq ImP_1^0|_{\tilde{\Omega}\otimes h^*\otimes k^*} = ImP^0|_{\tilde{\Omega}\otimes h^*}.$$

First, we observe that $ImQ^0|_{\Omega\otimes k^*}$ does not depend on k^*. We set $Z = ImQ^0|_{\Omega\otimes k^*}$.

Therefore we have

$$Y_1 \simeq ImQ^0 \simeq ImP_1^0|_{\tilde{\Omega}\otimes h^*\otimes extB(K^*)} \subset Z \otimes_\epsilon K.$$

This holds because

$$ImP_1^0|_{\tilde{\Omega}\otimes h^*\otimes extB(K^*)} \simeq X_1 \otimes K$$

and

$$X_1 \simeq ImP^0 \simeq Q^0|_{\Omega\otimes k^*}.$$

Similarly can be shown that

$$X_1 \simeq ImP^0 \simeq ImQ_1^0|_{\Omega\otimes k^*\otimes extB(H^*)} \subset Z \otimes_\epsilon H.$$

Then Y_1 and X_1 are isometric to a subspace of $Z \otimes_\epsilon K$ and $Z \otimes_\epsilon H$, respectively. In fact, $Z \subset ImQ^0 \simeq Y_1$, which is isometric to a subspace of $Z \otimes_\epsilon H$. Similar reasoning applies to X_1.

We prove that $Y_1 \simeq Z \otimes_\epsilon K$ and $X_1 \simeq Z \otimes_\epsilon H$. It is sufficient to prove Y_1 and X_1 are complete. We note that $Z \otimes H \subset X_1 \subseteq Z \otimes_\epsilon H$, $Z \otimes K \subset Y_1 \subseteq Z \otimes_\epsilon K$. This can be seen as follows. Since $Z \simeq Y_1$, therefore

$$Z \otimes H \simeq Y_1 \otimes H \simeq ImQ_1^0|_{\Omega \otimes k^* \otimes extB(H^*)} \simeq X_1.$$

A similar argument applies for the case $Z \otimes K \subset Y_1$. Without loss of generality, we assume that $S_1 = \Omega \otimes \text{ext } B(K^*)$, $\tilde{S}_1 = \tilde{\Omega} \otimes \text{ext } B(H^*), \varphi = \text{id}_\Omega$. Therefore

$$T^*(\omega \otimes k^* \otimes h^*) = \omega \otimes h^* \otimes k^*, \quad \text{for any } \omega \otimes k^* \in S_1 \text{ and } h^* \in \text{ext } B(H^*). \tag{4.0.23}$$

For any $h^* \in H^*$ and $k \in K$ we define a continuous, linear operator $S_{h^*,k}\colon X \to Y$ such that

$$y^*(S_{h^*,k}(x)) = y^* \otimes h^*(T(x \otimes k)) = T^*(y^* \otimes h^*)(x \otimes k) \text{ for any } y^* \in \text{ext } B(Y^*)$$

Similarly, for any $k^* \in K^*$ and $h \in H$ we define a continuous, linear operator $\tilde{S}_{k^*,h}\colon Y \to X$ such that

$$x^*(\tilde{S}_{k^*,h}(y)) = x^* \otimes k^*(T^{-1}(y \otimes h)) = (T^{-1})^*(x^* \otimes k^*)(y \otimes h) \text{ for any } x^* \in \text{ext } B(X^*)$$

Now using (4.0.29) and representation of T^*, we get

$$y^*(S_{h^*,k}(x)) = \begin{cases} \omega \otimes h^*(x)k^*(k) \text{ for } y^* = \omega \otimes k^* \in S_1 \\ \Phi_2(y^*)(x)\Psi_2(y^*, h^*)(k) \text{ for } y^* \in \bar{S}_2 \end{cases} \tag{4.0.24}$$

and similarly

$$x^*(\tilde{S}_{k^*,h}(y)) = \begin{cases} \omega \otimes k^*(y)h^*(h) \text{ for } x^* = \omega \otimes h^* \in \tilde{S}_1 \\ \Phi_2(x^*)(y)\Psi_2(x^*, k^*)(h) \text{ for } x^* \in \overline{\tilde{S}_2}. \end{cases} \tag{4.0.25}$$

Equations (4.0.24) and (4.0.25) imply for any $x_0^* = \omega_0 \otimes h_0^* \in \tilde{S}_1$ the following

$$\begin{aligned} x_0^* \left(\widetilde{S_{k^*,h}} \circ S_{h^*,k}(x) \right) &= \omega_0 \otimes k^*(S_{h^*,k}(x)) h_0^*(h) \\ &= \omega_0 \otimes h^*(x) k^*(k) h_0^*(h) \end{aligned} \tag{4.0.26}$$

and

$$\begin{aligned} x_0^* \left(\tilde{S}_{k^*,h} \circ S_{h^*,k}(x) \right) &= x_0^* \left(\tilde{S}_{k^*,h}(S_{h^*,k}(x)) \right) \\ &= \tilde{\Psi}_2(x_0^*)(S_{h^*,k}(x)) \tilde{\Psi}_2(x_0^*, k^*)(h) \quad \text{(using (4.0.24))} \\ &= \Phi_2(\tilde{\Phi}_2(x_0^*))(x) \Psi_2(\tilde{\Phi}_2(x_0^*), h^*)(k) \tilde{\Psi}_2(x_0^*, k^*)(h) \\ &= x_0^*(x) \Psi_2(\tilde{\Phi}_2(x_0^*), h^*)(k) \tilde{\Psi}_2(x_0^*, k^*)(h) \quad \text{(using } \Phi_2 \circ \tilde{\Phi}_2 = Id_{\overline{\tilde{S}_2}}\text{)}, \\ &\text{(for any } x_0^* \in \overline{\tilde{S}_2}\text{).} \end{aligned} \tag{4.0.27}$$

Before we proceed further we start with the lemma.

Lemma 4.0.18. *Let X, X_1, X_2 be described as above. Then $X \simeq X_1 \bigoplus_\infty X_2$.*

Proof. Let us define a map $\phi\colon X \to C(\tilde{S}_1 \cup \tilde{S}_2)$ by the point evaluation $x \mapsto f_x$. It is easy to see that ϕ is an isometry. Moreover,

$$\begin{aligned} \text{Range } \phi &= \{f_x \colon x \in X\} \\ &= \{\left(f_x^1, f_x^2\right) : x \in X\}, \text{where} f_x^1, f_x^2 \text{ are elements from } X_1, X_2 \text{ respectively} \end{aligned}$$

Therefore by the First Isomorphism Theorem we get $X \simeq X_1 \bigoplus_\infty X_2$. □

We prove that X_1 is complete. This follows from the claim.

Claim. *If $x = (x_1, x_2) \in X$, then $(x_1, 0) \in X$.*

Assuming this claim, the completeness of X_1 is clear. Given a Cauchy sequence in X_1, $\{x_n\}$, the sequence $\{(x_n, 0)\}_n$ is in X and Cauchy. The limit is also of the form $(x, 0)$ which implies the convergence of $\{x_n\}$ in X_1.

We first notice that the map $X \ni (x_1, x_2) \mapsto (x_1, 0) \in X$ is linear and continuous and $Z \otimes H \subset X_1$ is dense, it is sufficient to consider x_1 to be an elementary tensor $z_0 \otimes h_0 \in Z \otimes H$. The next lemma is pivotal in the proof of the Claim.

Lemma 4.0.19. *For every $x_0 = (z_0 \otimes h_0, x_2) \in X$ with $\|z_0\| = \|h_0\| = 1$, and $\epsilon > 0$ there exists a continuous operator $A\colon X \to X$ such that $Ax_0 = (z_0 \otimes h_0, x_2')$ with $\|x_2'\| \leq \epsilon$.*

Proof. Let $x_0^* \in \overline{\tilde{S}_2}$ and let $h_1^* \in \text{ext}B(H^*)$ be such that $h_1^*(h_0) = 1$. This is possible because H^* is strictly convex. Furthermore, if dim $K \geq 2$, let $k_1 \in K$ be an element of norm 1 such that $\Psi_2\left(\tilde{\Phi}_2(x_0^*), h_1^*\right)(k_1) = 0$. We denote by $k_1^* \in extB(K^*)$, a functional attaining its norm in k_1. Applying equation 4.0.26 at $x^* = w \otimes h^* \in \tilde{S}_1$ we have

$$\begin{aligned} w \otimes h^* \left[\tilde{S}_{k_1^*, h_0} \circ S_{h_1^*, k_1}(x_0)\right] &= w \otimes h_1^*(z_0 \otimes h_1^*(z_0 \otimes h_0, x_2) k_1^*(k_1) h^*(h_0) \\ &= w(z_0) h_1^*(h_0) h^*(h_0) = w(z_0) h^*(h_0) \\ &= w \otimes h^*(z_0 \otimes h_0) = w \otimes h^*(x_0). \end{aligned}$$

Therefore

$$\tilde{S}_{k_1^*, h_0} \circ S_{h_1^*, k_1}(x_0)|_{\tilde{S}_1} = x_0.$$

For every $x^* = w \otimes h^* \in \tilde{S}_2$ we have

$$x^* \left[\tilde{S}_{k_1^*,h_0} \circ S_{h_1^*,k_1}(x_0) \right] = (x^*)(x_0)\Psi_2(\tilde{\Phi}_2(x^*), h_1^*)(k_1)\tilde{\Psi}_2(x^*, k_1^*)(h_0)$$
$$= x^* \left(f_{x_0^*}(x^*) \cdot x_0 \right), \tag{4.0.28}$$

with $f_{x_0^*}(x^*) = \Psi_2(\tilde{\Phi}_2(x^*), h_1^*)(k_1)\tilde{\Psi}_2(x^*, k_1^*)(h_0)$. It is easy to see that $\tilde{S}_{k_1^*,h_0} \circ S_{h_1^*,k_1}(x_0)|_{\overline{\tilde{S}_2}} = f_{x_0^*} \cdot x_0$. Therefore $f_{x_0^*}$ is continuous over $\overline{\tilde{S}_2}$. Further, $\|f_{x_0^*}\|_\infty \leq 1$ and $f_{x_0^*}(x_0^*) = 0$. For a neighborhood around x_0^* in $\overline{\tilde{S}_2}$ we have that $x' = \tilde{S}_{k_1^*,h_0} \circ S_{h_1^*,k_1}(x_0)$ at every point of the aforementioned neighborhood has absolute value less than ϵ. Repeat the procedure for every point $x_0^* \in \overline{\tilde{S}_2}$. Each such functional determines an operator $\tilde{S}_{k_1^*,h_0} \circ S_{h_1^*,k_1}$ and a neighborhood N over which $\tilde{S}_{k_1^*,h_0} \circ S_{h_1^*,k_1}(x_0)$ and has absolute value less or equal to ϵ. These neighborhoods cover the entire $\overline{\tilde{S}_2}$. We apply the compactness of $\overline{\tilde{S}_2}$ to select a finite subset, enough to cover the entire $\overline{\tilde{S}_2}$. In this process we define a finite set of operators

$\tilde{S}_{k_{1,j}^*,h_0} \circ S_{h_{1,j}^*,k_{1,j}}$, with $j = 1, \cdots n$ such that

$$\tilde{S}_{k_{1,j}^*,h_0} \circ S_{h_{1,j}^*,k_{1,j}}(x_0)|_{\tilde{S}_1} = x_0 \text{ and } \tilde{S}_{k_{1,j}^*,h_0} \circ S_{h_{1,j}^*,k_{1,j}}(x_0)|_{\overline{\tilde{S}_2}} = f_{x_j^*} \cdot x_0.$$

We set $A = \Pi_{j=1}^n \tilde{S}_{k_{1,j}^*,h_0} \circ S_{h_{1,j}^*,k_{1,j}}$. We have $A(x_0) = x_0$ restricted to those extreme points in $\tilde{S}_1$, equivalently we say that for every $x^* \in \tilde{S}_1$, $x^*(A(x_0)) = x^*(x_0)$. Also for $x^* \in \overline{\tilde{S}_2}$ we have $x^*(A(x_0)) = f_{x_1^*}(x^*) \cdot f_{x_2^*}(x^*) \cdots f_{x_n^*}(x^*)x^*(x_0)$. The functional x^* belongs to some neighborhood N, hence $|x^*(A(x_0))| \leq \epsilon$. This completes the proof. □

Proof of the Claim. Given $x_0 = (z_0 \otimes h_0, x_2)$ and $\epsilon = 1/k$, Lemma 4.0.19 implies the existence of A_k such that $A_k(x_0)|_{\tilde{S}_1} = x_0$ and $|x^*(A_k(x_0))| < 1/k$, for every $x^* \in \overline{\tilde{S}_2}$. We show that $\{A_k(x_0)\}_k$ is a Cauchy sequence in X. Towards this, we just observe that

$$\|A_k(x_0) - A_{k+j}(x_0)\| \le \|A_k(x_0) - x_0\| + \|x_0 - A_{k+j}(x_0)\|$$
$$= sup_{x^* \in ext(B(X^*))}|x^*(A_k(x_0) - x_0)| + sup_{x^* \in ext(B(X^*))}|x^*(A_{k+j}(x_0) - x_0)|$$
$$\le \frac{1}{k} + \frac{1}{k+j}.$$

Hence $\{A_k(x_0)\}_k$ converges to $x \in X$. We show that $x = (z_0 \otimes h_0, 0)$. This follows because for every $x^* \in \tilde{S}_1$ we have

$$x^*(A_k(x_0)) = x^*(z_0 \otimes h_0).$$

For every $x^* \in \overline{\tilde{S}_2}$ we have

$$x^*(A_k(x_0)) = \lambda x^*(x_0), \text{ with } \lambda < 1/k.$$

Then the limit of $\{A_k(x_0)\}_k$ restricted to $\overline{\tilde{S}_2}$ is equal to zero. This proves that $(z_0 \otimes h_,0) \in X$. Since an element in the algebraic tensor product $Z \otimes H$ is a finite sum of elementary tensors and $Z \otimes H$ is dense in X_1 we derive the claim. □

Function Module Representation of a Banach Space

In this section we review the definition of function module and the representation of a Banach space as a functional module. We discuss some insightful examples and lastly we give an overview of the meaning of the representation of a Banach space in a maximal function module. Most definitions and theorems can be found in [**?**].

For a family of Banach spaces (X_k), the space $\prod_{k \in K}^{\infty} X_k$ denotes the functions

x in the product space endowed with the norm

$$\|x\|_\infty = \sup\{\|x(k)\| : k \in K\} < \infty$$

Definition 4.0.20. *(Function module) A Banach funciton module, or function module is a triple $(K, (X_k)_{k\in K}, X)$, where K is a non-empty compact Hausdorff space (the base space), $(X_k)_{k\in K}$ a family of Banach spaces (the component spaces), and X is a closed subspace of $\prod_{k\in K}^{\infty} X_k$ such that the following conditions are satisfied.*

1. *$hx \in X$ for $x \in X$ and $h \in C(K)$ where $(hx)(k) := h(k)x(k)$.*
2. *$k \mapsto \|x(k)\|$ is an upper semi continuous function for every $x \in X$.*
3. *$X_k = \{x(k) : x \in X\}$ for every $k \in K$.*
4. *$\overline{\{k : k \in K, X_k \neq \{0\}\}} = K$*

An example of a function module is the space $C(K, E)$, where K is a compact Hausdorff topological space and E is a Banach space. We set $X_k = E$ for all $k \in K$. If K is locally compact, instead of K we take the Stone-Cech compactifiction of K, βK and $X_k = E$ for $t \in K$ and $X_k = \{0\}$ if $t \in \beta K \setminus K$. It can be shown that any Banach space X can be thought of as a function module. Before explaining this further, we start with the following definitions.

Definition 4.0.21. *Let T be a bounded linear operator on a Banach space X.*

1. *The operator T is called a multiplier of X if every element of ext B_{X^*} is an eigenvector for T^*. That is, for each $x^* \in$ ext B_{X^*}, there is a scalar $a_T(x^*)$ such that*

$$T^*x^* = a_T(x^*)x^*.$$

2. *For a multiplier T on X, we say that a multiplier S on X is an adjoint for T if $a_S = \overline{a_T}$.*

3. *The centralizer of X, denoted by $Z(X)$, is the set of all multipliers for which an adjoint exists.*

Definition 4.0.22. *(Function module representation) Let X be a Banach space. A function module representation $[\rho, (K, (X_k)_{k\in K}, \tilde{X})]$ of X is a function module $(K, (X_k)_{k\in K}, \widetilde{X})$ together with an isometric isomorphism $\rho\colon X \to \widetilde{X}$.*

The following Proposition says that any Banach space X can be represented as a function module. Recall that for an algebra A, $\mathcal{M}(A)$ denotes the set of all non-zero homomorphisms from A to $\mathbb{C}$.

Proposition 4.0.23. *Let X be a Banach space and $K_X = \mathcal{M}(Z(X))$. Then there exists a function module representation $[\rho, (K_X, (X_k)_{k\in K}, \tilde{X})]$ of X such that $Z_\rho(X) = Z(X)$. Thus every Banach space can be regarded as a function module in a suitable product $\prod_{k\in K_X}^{\infty} X_k$ such that the operators in the centralizer of X are precisely the multiplications by a continuous function, M_h for $h \in C(K_X)$.*

We now prove Theorem (4.0.1). We have shown before that $X_1 \simeq Z \otimes_\epsilon H$ and $Y_1 \simeq Z \otimes_\epsilon K$, see [?]. Furthermore, it was also shown that X_1, Y_1, X_2, Y_2 are complete spaces, $X \simeq X_1 \bigoplus_\infty X_2$ and $Y \simeq Y_1 \bigoplus_\infty Y_2$. In order to derive the form for the surjective isometry $T : X \otimes_\epsilon K \to Y \otimes_\epsilon H$, it is sufficient to investigate the restriction of T to $X_2 \otimes K$.

We notice that from the equation displayed in (4.0.29) and identifying the spaces corresponding to setting Q_1 and P_1 equal to the identity, T^* maps $S_1 \otimes extB(H^*)$ onto $\tilde{S}_1 \otimes extB(K^*)$. Therefore $T^{**}(X_1 \otimes K) = T(X_1 \otimes K) = Y_1 \otimes H$. From the isomorphisms, $X_1 \simeq Z \otimes H$ and $Y_1 \simeq Z \otimes K$, we assume (wlog) the equalities: $X_1 = Z \otimes H$ and $Y_1 = Z \otimes K$. The information displayed in (4.0.29),

$$T^*(\omega \otimes k^* \otimes h^*) = \omega \otimes h^* \otimes k^*, \quad \text{for any } \omega \otimes k^* \in S_1 \text{ and } h^* \in \text{ext } B(H^*), \tag{4.0.29}$$

it also follows from previous considerations that $T^*(Y_1^* \otimes H^*) = X_1^* \otimes K^*$ with $Y_1 = Z \otimes H$ and $X_1 = Z \otimes K$. Furthermore

$$T^{**}(z^{**} \otimes k^{**} \otimes h^{**}) = (z^{**} \otimes h^{**} \otimes k^{**}).$$

The restriction to $Z \otimes H \otimes K$ yields $T(z \otimes h \otimes k) = z \otimes k \otimes h$.

Since

$$T : ((Z \otimes_\epsilon H) \otimes_\epsilon K) \oplus_\infty (X_2 \otimes_\epsilon K) \to ((Z \otimes_\epsilon K) \otimes_\epsilon H) \oplus_\infty (Y_2 \otimes_\epsilon H)$$

is such that $T = (T_1, T_2)$, with $T_1 : ((Z \otimes_\epsilon H) \otimes_\epsilon K) \to ((Z \otimes_\epsilon K) \otimes_\epsilon H)$ and $T_2 : (X_2 \otimes_\epsilon K) \to (Y_2 \otimes_\epsilon H)$ are surjective isometries, we have $T(\alpha, \beta) = (T_1(\alpha), T_2(\beta))$, and $T_1(z \otimes h \otimes k) = z \otimes k \otimes h$. We are left to investigate the form of T_2, or the restrictedof T to $X_2 \otimes K$, for simplicity also denoted by T, i.e. $T : X_2 \otimes K \to Y_2 \otimes H$ and we use X and Y for X_2 and Y_2 respectively. Without loss of generality, we can assume that X and Y are subspaces of function modules, $\prod_{\alpha \in \tilde{\Gamma}} X_\alpha$ and $\prod_{\alpha \in \Gamma} Y_\alpha$ respectively, and the identity embeddings $\pi_X : X \to \prod_{\alpha \in \tilde{\Gamma}} X_\alpha$ and $\pi_Y : Y \to \prod_{\alpha \in \Gamma} Y_\alpha$ are the maximal function representations. We recall that Γ denotes the space of all continuous and multiplicative norm 1 functionals defined on the commutative C^*- algebra $Z(X)$ ($\equiv Z(Y)$), the centralizer of X (Y, respectively). Now since $\prod_{\alpha \in \Gamma} Y_\alpha = \{f : \Gamma \to \cup_{\alpha \in \Gamma} Y_\alpha : \forall \alpha \in \Gamma, f(\alpha) \in Y_\alpha, f \text{ is continuous}\}$, therefore we can identify Y as a subspace of $C(\Gamma, \cup_{\alpha \in \Gamma} Y_\alpha)$ and since the extreme points of the dual of $C(\Gamma, \cup_{\alpha \in \Gamma} Y_\alpha)$ is the set of point evaluations, then an extreme point $y^* \in S_2$ is such that

$$\prod_{\alpha\in\Gamma} Y_\alpha \supset Y \ni y \xrightarrow{\delta_\alpha\otimes y_\alpha^*} y_\alpha^*(y(\alpha)),$$

for some $y_\alpha^* \in \text{ext}B(Y_\alpha^*)$ and $\alpha \in \Gamma$.

Let $k \in K, h \in H, k^* \in K^*, h^* \in H^*$. Then by (4.0.28), the operator $\tilde{S}_{k^*,h} \circ S_{h^*,k} \colon X \to X$ is of the form

$$\delta_\alpha \otimes x_\alpha^* \left(\tilde{S}_{k^*,h} \circ S_{h^*,k}((x_\alpha)_{\alpha\in\Gamma})\right) = f(\delta_\alpha \otimes x_\alpha^*) \cdot x_\alpha^*(x_\alpha), \tag{4.0.30}$$

where $f(\delta_\alpha \otimes x_\alpha^*) = \Psi_2(\tilde{\Phi}_2(\delta_\alpha \otimes x_\alpha^*), h^*)(k)\tilde{\Psi}_2(\delta_\alpha \otimes x_\alpha^*, k^*)(h)$.

We can see that the operator $U := \tilde{S}_{k^*,h} \circ S_{h^*,k} \colon X \to X$ is a multiplier so that the adjoint always exists, therefore the operator $U \in Z(X)$. The maximal function module representation implies that the operator U is a multiplication by a continuous function, i.e. M_h for some $h \in C(\Gamma)$. Therefore, the function f will only depend on $\alpha \in \Gamma$, but not on x_α^* and consequently the functions $\tilde{S}_2 \ni \delta_\alpha \otimes x_\alpha^* \mapsto \tilde{\Phi}_2(\delta_\alpha \otimes x_\alpha^*)$ and $\tilde{S}_2 \ni \delta_\alpha \otimes x_\alpha^* \mapsto \tilde{\Psi}_2(\delta_\alpha \otimes x_\alpha^*, \cdot)$ also do not depend on x_α^* but only on $\alpha \in \Gamma$. Hence, from Equation (4.0.4), the operator $(T^{-1})^*$ is of the form

$$\begin{aligned}(T^{-1})^*(\delta_\alpha \otimes x_\alpha^* \otimes k^*) &= \tilde{\Phi}_2(\delta_\alpha \otimes x_\alpha^*) \otimes \tilde{\Psi}_2(\delta_\alpha \otimes x_\alpha^*, k^*)\\ &= \delta_{\phi(\alpha)} \otimes \tilde{\Phi}_\alpha(x_\alpha^*) \otimes \tilde{\Psi}_\alpha(k^*),\end{aligned} \tag{4.0.31}$$

where $\phi \colon \Gamma \to \tilde{\Gamma}$ and $\tilde{\Phi}_\alpha \colon X_\alpha^* \to Y_\alpha^*, \tilde{\Psi}_\alpha \colon K^* \to H^*$ are weak*-continuous onto isometries. Composing the above formula with T^*, we get

$$\delta_\alpha \otimes x_\alpha^* \otimes k^* = \delta_{\psi\circ\phi(\alpha)} \otimes \left(\Phi_{\phi(\alpha)} \circ \tilde{\Phi}_\alpha(x_\alpha^*)\right) \otimes \left(\Psi_{\phi(\alpha)} \circ \tilde{\Psi}_\alpha(k^*)\right).$$

Therefore, $\psi \circ \phi(\alpha) = \alpha$, which implies ϕ is a bijection between Γ and $\tilde{\Gamma}$ and hence we assume $\Gamma = \tilde{\Gamma}$, $\phi = id_\Gamma$, and hence T^* is of the following form

$$T^*(\delta_\alpha \otimes y_\alpha^* \otimes h^*) = \delta_\alpha \otimes \phi_\alpha(y_\alpha^*) \otimes \psi_\alpha(y_h^*),$$

where $\Phi_\alpha \colon Y_\alpha^* \to X_\alpha^*, \Psi_\alpha \colon H^* \to K^*$ are weak*-continuous onto isometries.

Put

$$A = \prod_{\alpha \in \Gamma} \Phi_\alpha^* \colon \prod_{\alpha \in \Gamma} X_\alpha \to \prod_{\alpha \in \Gamma} Y_\alpha.$$

Where $A((x_\alpha)_{\alpha\in\Gamma}) = (\Phi_\alpha^*(x_\alpha))_{\alpha\in\Gamma}$. The operator A is an onto isometry, and we claim that $A(X) = Y$. For any $h^* \in \text{ext}B(H^*)$, $k \in K$, the operator $\tilde{S}_{k^*,h} \circ A \colon A^{-1}(Y) \to X$, assuming that $\tilde{\Phi}_2$ is the identity, we have the following:

$$\begin{aligned} x_\alpha^* \left(\tilde{S}_{k^*,h} \circ A(\omega)(\alpha) \right) &= \tilde{\Phi}_2(x_\alpha^*)(A(\omega)_\alpha)\tilde{\Psi}_2(x_\alpha^*, k^*)(h) \\ &= x_\alpha^*(\omega_\alpha)\tilde{\Psi}_2(x_\alpha^*, k^*)(h) \\ &= x_\alpha^* \left(\omega(\alpha) \cdot \tilde{\Psi}_\alpha(k^*)(h) \right). \end{aligned}$$

Therefore the function $\Gamma \ni \alpha \mapsto \tilde{\Psi}_\alpha(k^*)(h)$ is continuous, and since $A^{-1}(Y) \subset \prod_{\alpha\in\Gamma} X_\alpha$ is a function module, we get,

$$\tilde{S}_{k^*,h} \circ A(\omega) \in X \cap A^{-1}(Y)$$

for any $\omega \in A^{-1}(Y), k^* \in \text{ext}B(K^*), h \in H$.

In order to complete the proof of $A(X) = Y$, by symmetry, it is enough to prove $A^{-1}(Y) \subset X$. In order to prove this it is sufficient to show that the set

$$B := \text{span}\{\tilde{S}_{k^*,h} \circ A(\omega) \colon \omega \in A^{-1}(Y), k^* \in \text{ext}B(K*), h \in H\}$$

is dense in $A^{-1}(Y)$. Indeed, if we can prove that $B \subset X$ is dense in $A^{-1}(Y)$, then taking closures on both the sides, we get $A^{-1}(Y) \subset X$. So let us try to prove the denseness of the set B.

Suppose if possible the set B is not dense in $A^{-1}(Y)$. Then there exists $\omega \in A^{-1}(Y)$ such that $\omega \notin \overline{B}$. By using Hahn-Banach separation Theorem, choose an $x^* \in X^*$ such that

$$x^*(\omega) \neq 0 \text{ and } x^*|_{\overline{B}} = 0.$$

Now for every $k^* \in K^*$, we have

$$\begin{aligned} x^* \otimes k^* \left(T^{-1} \left(\sum_{j=1}^{n} y_j \otimes h_j \right) \right) &= x^* \left(\sum_{j=1}^{n} \tilde{S}_{k^*,h_j}(y_j) \right) \\ &= 0, \end{aligned}$$

for every $x^*|_{\overline{B}}$. Hence $x^* \otimes k^*(X \otimes_\epsilon K) = 0$, because $x^* \left(\sum_{j=1}^{n} \tilde{S}_{k^*,h_j}(y_j) \right) = 0$, continuity implies that $x^*(x) = 0$ for all $x \in \overline{B}$. Therefore $x^* \otimes k^*(X \otimes_\epsilon K) = 0$, since T^{-1} is surjective. This leads to a contradiction which implies that the set B is dense in $A^{-1}(Y)$ and the proof is complete.

Thus, it has been shown that X_2 is isomorphic to Y_2 and T induces a surjective isometry from $X_2 \otimes K \to X_2 \otimes H$.

Proposition 4.0.24. *Let T_2 be the restriction of T to $X_2 \otimes K$. Then $T_2 : X_2 \otimes K \to X_2 \otimes H$ is a surjective isometry. There exists an operator Φ from the set of extreme points of the unit ball of the dual X_2^* into the isometries from K onto H such that*

$$T_2(x_2 \otimes k_2)(x^* \otimes h^*) = x^*(x_2) \cdot h^*(\Phi(x^*)(k_2)).$$

Proof. Since X_2 is isometric to Y_2, therefore $x^* \in S_2$. For $x^* \in X_2^*, h^* \in H^*$ and

$k_2 \in K$, we have

$$\begin{aligned} T_2(x_2 \otimes k_2)(x^* \otimes h^*) &= (x_2^* \otimes k_2^*) T_2^*(x^* \otimes h^*) \\ &= (x_2^* \otimes k_2^*) \left[\Phi_2(x^*) \otimes \Psi_2(x^*, h^*)\right] \\ &= \left[\Phi_2(x^*)(x_2^*)\right] \cdot \left[\Psi_2(x^*, h^*)(k_2^*)\right] \\ &= \left[\Phi_2(x^*)(x_2^*)\right] \cdot h^* \left[\Psi_2^*(x^*, \cdot)(k_2)\right] \end{aligned}$$

Since the map $\Psi_2^*(x^*, \cdot) : K \to H$ is an isometry, that proves the proposition. □

Generalized bicircular projections for special case

In Chapter 1, we have introduced the concept of bicircular projections on Banach spaces. It has been shown that bicircular projections are the hermitian projections. In [13], the author proposed a generalization of bicircular projections, known in the lit by generalized bicircular projections, see. In this Section we are going to derive th eform of generalized bicircular projection for the space $X\hat{\otimes}_\epsilon K$ associated with a class of isometries. We start with the definion.

Definition 4.0.25. *Let X be a Banach space and $\mathcal{I}$ be a class of surjective isometries on X. A projection $P\colon X \to X$ is a generalized bicircular projection associated with $\mathcal{I}$ iff there exists $\lambda \in \mathbb{C} \setminus \{1\}$ of modulus 1 such that $P + \lambda(I - P) \in \mathcal{I}$.*

Remark 4.0.26. *We observe every bicircular projection is a generalized bicircular projection. Also, we recall that the bicircular projections are hermitian projections.*

Let us recall some basic facts about generalized bicircular projections.

Proposition 4.0.27. *[25, Corollary 2] Every generalized bicircular projection is bicontractive.*

Generalized bicircular projection on $X\hat{\otimes}_\epsilon K$

Let us consider the isometry $T\colon (Z\otimes K\otimes K)\bigoplus_\infty(X_2\otimes K)\to(Z\otimes K\otimes K)\bigoplus_\infty(X_2\otimes K)$ of the form $T=(T_1,T_2)$ where $T_1\colon Z\otimes K\otimes K\to Z\otimes K\otimes K$ given by $T_1(z\otimes k_1\otimes k_2)=z\otimes k_2\otimes k_1$ and $T_2\colon X_2\otimes K\to X_2\otimes K$ given by $T_2(x\otimes k)=S(x)\otimes\Phi(x)(k)$ where $S\colon X_2\to X_2$ and $\Phi(x)\colon K\to K$ are surjective isometries respectively. Under this isometry, the following Theorem describes the form of generalized bicircular projections for the injective tensor product of X and K.

Theorem 4.0.28. *Let P be a generalized bircircular projection on $X\otimes_\epsilon K$. Then there exists isometries $S\colon X_2\to X_2$ and $\Phi(x)\colon K\to H$ with $S^2(x)=\alpha(x)x$ and $\Phi(x)^2(k)=\overline{\alpha(x)}k$ where $\alpha(x)$ is a scalar of modulus 1 such that*

$$P(z\otimes k_1\otimes k_2,x\otimes k)=\left(\frac{z\otimes k_1\otimes k_2+z\otimes k_2\otimes k_1}{2},\frac{S(x)\otimes\Phi(x)(k)+x\otimes k}{2}\right).$$

Proof. Assume $P=(P_1,P_2)$ is a generalized bicircular projection. Then $P+\lambda(I-P)$ is an isometry for some $\lambda\in\mathbb{R}$. For $i=1,2$, solving for P_i, we get $P_i=\frac{T_i-\lambda I}{1-\lambda}$. Since P_i is a projection for $i=1,2$, therefore, we get $T_i^2-(\lambda+1)T_i+\lambda I=0$. Using the form of isometry described above, we get $(1+\lambda)(z\otimes k_1\otimes k_2-z\otimes k_2\otimes k_1)=0$. Since the dimension of $K>2$, that implies $\lambda=-1$. Hence $S^2(x)\otimes\Phi(x)^2(k)=x\otimes k$, which implies $S^2(x)=\alpha(x)x$ and $\Phi(x)^2(k)=\overline{\alpha(x)}k$ for some modulus one scalar α. Therefore by the form of projection P_i above, we get

$$P(z\otimes k_1\otimes k_2,x\otimes k)=\left(\frac{z\otimes k_1\otimes k_2+z\otimes k_2\otimes k_1}{2},\frac{S(x)\otimes\Phi(x)(k)+x\otimes k}{2}\right).$$

□

Remark 4.0.29. *There is no bicircular projections associated with the group of isometries described above.*

REFERENCES

BIBLIOGRAPHY

[1] P. Ara, M. Mathieu, *Local multipliers of C^*-algebras,* Springer-Verlag, London, 2003.

[2] F. Botelho, P. Dey, D. Ilišević, *Hermitian projections on some Banach spaces and related topics*, Linear algebra and applications 598(2020), 92-104.

[3] F. Botelho, J. Jamison, *Generalized bi-circular projections on minimal ideals of operators*, Proc. Amer. Math. Soc. 136 (2008), 1397–1402.

[4] E. Berkson, H. Porta, *Hermitian operators and one-parameter groups of isometries in Hardy spaces*, Trans. Amer. Math. Soc. 185 (1973), 331–344.

[5] E. Berkson, A. Sourour, *The Hermitian operators on some Banach spaces*, Studia Math. 52 (1974), 33–41.

[6] J.S.Cohen, *Absolutely p-summing, p-nuclear operators and their conjugates,* Mathematische Annalen, volume 201, pages 177–200(1973).

[7] J.B. Conway, *A Course in Functional Analysis*, Germany Springer New York.

[8] I. Doust, W.J. Ricker, *Spectral properties for Hermitian operators*, Linear Algebra Appl. 175 (1992), 75-96.

[9] R.J. Fleming, J.E. Jamison *Hermitian operators on $C(X, E)$ and the Banach-Stone theorem*, Math. Z. 170 (1980), 77–84.

[10] R.J. Fleming, J.E. Jamison, *Isometries on Banach spaces: function spaces,* Chapman & Hall / CRC, Boca Raton, FL, 2003.

[11] C.K. Fong, A.R. Sourour, *On the operator identity* $\sum A_k X B_k = 0$, Canad. J. Math. 31 (1979), 845–857.

[12] M. Fošner, D. Ilišević, *On a class of projections on ∗-rings*, Comm. Algebra 33 (2005), 3293—3310.

[13] M. Fosner, D. Ilisevic, C. LI, *G-invariant norms and bicircular projections,* Linear Algebra and its Applications 420 (2007) 596–60.

[14] P. Harmand, D. Werner, W. Werner, *M-ideals in Banach Spaces and Banach Algebras*, Lecture Notes in Mathematics.

[15] D. Ilišević, *Generalized bicircular projections on JB*-triples*, Linear algebra and applications, 432(2010), 1267-276.

[16] D. Ilišević, *Generalized bicircular projections on JB*-triples*, Linear Algebra Appl. 432 (2010), 1267–1276.

[17] J.E. Jamison, *Bicircular projections on some Banach spaces*, Linear Algebra Appl. 420 (2007), 29—33.

[18] J.E. Jamison, I. Loomis, *Isometries of Orlicz spaces of vector valued functions*, Math. Z. 193 (1986), 363–371.

[19] K. Jarosz, *Isometries between injective tensor products of Banach spaces*, Pacific J. Math. 121(2): 383-396 (1986).

[20] K. Jarosz, V. Pathak, *Isometries between function spaces,* Trans. Amer. Math. Soc. 305 (1988) 193–206.

[21] R. Khalil, A. Saleh, *Isometries of certain operator spaces*, Proc. Amer. Math. Soc. 132 (2004), 1473–1481.

[22] D. Koehler and P. Rosenthal, *On isometries of normed linear spaces*, Studia Math. 36 (1970), 213—216.

[23] K.B. Laursen and M.M. Neumann, *An introduction to local spectral theory*, The Clarendon Press, Oxford University Press, New York, 2000.

[24] P.K. Lin, *Generalized bi-circular projections*, J. Math. Anal. Appl. 340 (2008), 1–4.

[25] P.K. Lin, *The isometries of* $L^2(\Omega, X)$, Illinois J of Math. (13) 1989, 621-630.

[26] G. Lumer, Semi-inner product spaces, Trans. Am. Math. Soc., 100, 29-43.

[27] M. Martin, *The group of isometries of a Banach space and duality*, Journal of Functional Analysis 255 (2008) 2966–2976.

[28] E. Mayer-Wolf, *Isometries between Banach spaces of Lipschitz functions,* Israel J. Math. 38 (1981) 58–74.

[29] R. E. Megginson,, *An Introduction to Banach Space Theory*, United States: Springer New York (2012).

[30] T.W. Palmer, *Banach algebras and the general theory of* $*$*-algebras.* Vol. I. Algebras and Banach algebras, Cambridge University Press, Cambridge, 1994.

[31] R. R. Phelps, *Lectures on Choquet's Theorem*, Springer, 2001.

[32] T.S.S.R.K. Rao, *Some generalizations of Kadison's theorem: a survey*, Extracta Math. 19 (2004), 319—334.

[33] A.K. Roy, *Extreme points and linear isometries of the Banach space of Lipschitz functions,* Canad. J. Math. 20 (1968) 1150–1164.

[34] W.M. Ruess and Ch. P. Stegall, *Extreme points in duals of operator spaces*, Mathematische Annalen, 261, pages 535–546(1982).

[35] R. A. Ryan, *Introduction to Tensor Products of Banach Spaces,* United Kingdom: Springer London (2013).

[36] A.R. Sourour, *The isometries of* $L^p(\Omega, X)$, J. Funct. Anal. 30 (1978), 276–285.

[37] L.L. Stachó, B. Zalar, *Bicircular projections on some matrix and operator spaces*, Linear Algebra Appl. 384 (2004), 9–20.

[38] I. Vidav, Einemetrische Kenn zeichung der selbstadjun gierton operatoren, Math. Z., 66, 121-28.