2019

Topological Properties of the Standard Operations on Spaces of Continuous Functions and Integrable Functions

Holly Michelle Renaud

TOPOLOGICAL PROPERTIES OF THE STANDARD OPERATIONS ON SPACES OF CONTINUOUS FUNCTIONS AND INTEGRABLE FUNCTIONS

by

Holly Renaud

A Dissertation

Submitted in Partial Fulfillment of the

Requirements for the Degree of

Doctor of Philosophy

Major: Mathematical Sciences

The University of Memphis

May 2019

DEDICATION

I wish to dedicate this dissertation to several individuals, without whom I would not have succeeded.

First, I dedicate this work to my family. My husband, Luis Herrera, encouraged me and supported me at every step of this process, even when I considered leaving this program "A.B.D." He took over most of the housework and childcare so that I could research and write in peace, and I am eternally grateful to him for this. My son, Oliver Herrera, attended multiple advising meetings, even as a newborn baby, and managed to make me smile during times of great stress. Mr. Kenneth Marlin, Diane Renaud, my mother, and Amy Renaud, my twin sister, all prayed for and encouraged me throughout my entire time in graduate school.

Second, I dedicate this work to my dear friend Matthew Smith. When I began this program, I was severely under-prepared. Matthew was readily available by phone and allowed me to call him with general questions from my lecture notes, which allowed me to save more complicated questions from my notes for my professors' office hours.

Finally, most of all, I dedicate this dissertation to my Lord and Savior, Jesus Christ. It is by His grace and strength that I completed this program, and I consider this success to be truly His, not mine.

ACKNOWLEDGMENTS

I wish to thank my advisor, Dr. Fernanda Botelho. She was very patient with me throughout the process of researching and writing my dissertation, during which I went through a difficult pregnancy, had a child, and began a new job. I could not have finished this dissertation without her patience and encouragement during those stressful life changes.

I also wish to thank my other committee members, Drs. Ben McCarty, John Haddock, Irena Lasiecka, and Anna Kaminska, for their time, patience, and feedback, which have been invaluable to me.

PREFACE

I decided to consider the topic of openness properties of maps for my dissertation after taking a class on C^* algebras. Initially, I was not planning to work in Functional Analysis, but I read a paper, entitled "Multiplying balls in the space of continuous functions on $C[0, 1]$" ([6]), and I was immediately intrigued and decided to switch to Functional Analysis.

Throughout this work, research from two of my publications will appear. The first publication, which is joint-work with Dr. Fernanda Botelho ([15]), is entitled "Topological properties of operations on spaces of differentiable functions" and appears in the Advances in Operator Theory journal. Research from this paper appears in several places, primarily in Chapters 1-4. The second publication, which is a solo paper ([27]), is entitled "Results on topological properties of operations on function spaces" and has been accepted for publication in the Contemporary Mathematics (CONM) book series. Several results from this paper appear in Chapters 1, 5, and 6.

ABSTRACT

In this work, we consider different notions of openness for several operations on a large collection of different settings. We start with the scalar multiplication on sequence spaces, spaces of continuous functions, and integrable functions. We apply different techniques to derive weak-openness of multiplication on spaces of differentiable functions, endowed with a large collection of quasi-algebra norms.

We consider the openness property of the dense-defined standard multiplication on spaces of integrable functions, as multiplication on these settings is only defined on a dense set. We adapt the definition of openness for the multiplication to include dense-defined products and then prove that the multiplication on these spaces, restricted to their corresponding domains, is uniformly open.

We investigate sufficient and necessary conditions for pairs of functions to be points of local openness for the multiplication on normed spaces of differentiable functions. Next, we establish openness for other kinds of operations, more precisely, the maximum and minimum operations on spaces of scalar-valued integrable functions from a topological space into a finite measure on an algebra of subsets of that space. The last section of this work deals with the connection between openness of the multiplication on spaces of continuous functions and topological properties of the domain of those functions.

CONTENTS

CHAPTER 1

INTRODUCTION

This work concerns the study of openness of surjective maps on topological spaces. The notions we consider are:

- The standard open condition for a map, which asserts that the image of any open set is open;
- The dense openness of a map, which means that at every point in the domain of the map, the closure of the image of an open set containing that point contains a neighborhood of the image of the point; and
- The weak openness of a map, which asserts that the image of any open set has nonempty interior.

The interest on openness properties of maps may be traced to the fact that the open mapping theorem does not hold for bilinear maps (see [29]). This leads to questions on when certain collections of maps are open or which elements in the domain of those maps are points of local openness (see Definition 2.0.6). Several operations on classical Banach spaces have been studied from this point of view; for example, see [4], [5] , [11], and references therein. In this chapter, we make the three main definitions precise for maps between topological spaces. We also consider a notion of uniform openness when the map is defined on a metric space and the contained neighborhood can be defined independently of the point given. We first illustrate some of the questions we address throughout this dissertation for the easy case of addition on a topological vector space. We see that addition on a normed space is uniformly open. We also give a brief explanation of the main differences among the openness conditions under study, and we give examples illustrating some of the questions we address throughout this dissertation. We introduce the spaces of our study. We shall prove that, on a normed space, addition is uniformly open. We also show that addition is open on the more general setting of a topological vector space. Some of these results come from [15].

In Chapter 2, we consider openness questions for the scalar product on normed spaces. We derive conditions under which the scalar product is open. In some specific cases, we identify the set of points where the product is open, respectively weak-open and dense-open. We establish openness properties for the scalar product on classes of sequence spaces over the same field $\mathbb{R}$ or $\mathbb{C}$. We establish necessary and sufficient conditions for openness of scalar multiplication on these spaces. In the process, we identify the points of local openness for the scalar product on sequence spaces, spaces of continuous functions, and spaces of integrable functions.

In Chapter 3, we study similar properties for multiplication. We first review existing results, and then we present new extensions. We review openness properties for the multiplication on $C([0,1],\mathbb{R})$, due to Balcerzak, Wachowicz, and Wilczyński (see [6]), and then we prove an extension for the multiplication on $C(\Omega,\mathbb{R})$, with Ω a compact, connected Hausdorff space. We also give conditions under which we can show weak-openness of multiplication in $C([0,\ 1],\mathbb{C})$. Then, we proceed by showing similar results for $C(\Omega, E)$, with E a Banach algebra and Ω a compact Hausdorff space.

In Chapter 4, we extend techniques from [6] and [31] to the study of openness properties for the multiplication on a collection of new spaces. These are spaces of continuously n-differentiable functions defined on the unit interval. They shall be called KKM spaces, since they are generalizations of a class of spaces consisting of continuously differentiable functions endowed with a variety of norms. These spaces were first introduced by Kawamura, Koshimizu, and Miura (in [23]). We also follow an approach developed by Behrends in [10] to investigate conditions under which the product of two maps is in the interior of the product of two open sets containing the maps, or, equivalently, conditions under which a pair of points is a point of local openness for multiplication.

In Chapter 5, we study a collection of binary maps. More precisely, we consider the "max", denoted by $\vee$, and the "min", denoted by $\wedge$, both defined on several appropriate spaces. We first consider spaces of scalar-valued continuous functions, as in [6], and then we present a study for spaces of vector-valued continuous functions when the range space has a lattice structure. We also consider spaces of Lipschitz functions and spaces of integrable functions. We start with a survey of results and describe all relevant techniques for our study. Then we define dense-defined multiplication on spaces of integrable functions. Here, multiplication is not defined on the entire space, only on a dense subset. We derive results on the openness of these maps.

In Chapter 6, we study a relation between openness and topological features. We collect results that relate the openness of the multiplication on spaces of scalar-valued continuous functions with dimension of the domain of those functions. We present extensions of these results to the vector-valued counterparts.

In Chapter 7, we list some problems and questions for further investigation, and, as needed, we offer to an appendix (see Chapter 8) background material for the development of some of these chapters.

We now provide precise definitions for the aforementioned notions of openness.

Definition 1.0.1. *Let E and F be topological spaces and $f : E \to F$ a surjective map. Let $x \in E$.*

1. *f is open at x if and only if f maps every neighborhood of x onto a neighborhood of $f(x)$.*

2. *f is dense-open (d-open) at x if and only if f maps every neighborhood of x onto a dense subset of a neighborhood of $f(x)$.*

3. *f is weak-open (w-open) at x if and only if f maps every neighborhood of x onto a set with nonempty interior.*

Then f is open (d-open or w-open) if f is open (respectively, d-open or w-open) at x, for every $x \in E$.

This definition can be rephrased for maps between metric spaces. Then a map is open, provided that for every $x \in E$ and every $\epsilon > 0$, there exists $\delta > 0$, independent of x, such that

$$f\left(\mathcal{B}(x, \epsilon)\right) \supset \mathcal{B}(f(x), \delta).$$

Similarly, we can define d-open and w-open, where $\mathcal{B}(f(x), \delta) \subseteq \overline{f(\mathcal{B}(x, \epsilon))}$ and $int(f(\mathcal{B}(x, \epsilon))) \neq \emptyset$, respectively.

It is clear that "open" implies "d-open" and also "w-open". However, neither "d-open" implies "w-open" nor "w-open" implies "d-open". We give two examples to support the last claim. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x$ for $x \in \mathbb{Q}$, otherwise $f(x) = -x$. The function f is d-open but not w-open. Let $g : \mathbb{R} \to \mathbb{R}$, given by $g(x) = |x|$ for $x \leq 2$ and $g(x) = 4 - x$ for $x > 2$. The function g is w-open but not d-open (g is not d-open at $x = 2$).

We now adapt Definition 1.0.1 to metric spaces. In a metric space, an open ball centered at point x and of radius $r > 0$ is denoted by $\mathcal{B}(x, r)$.

Definition 1.0.2. *Let (E, d) and (F, D) be two metric spaces and $f : E \to F$ a surjective map. Then*

1. *f is open (d-open or w-open) if and only if for every $x \in E$ and $\epsilon > 0$ there exists $\delta > 0$ such that*

$$f(\mathcal{B}(x,\epsilon)) \supseteq \mathcal{B}(f(x),\delta), \;\; \left(\overline{f(\mathcal{B}(x,\epsilon))} \supseteq \mathcal{B}(f(x),\delta) \text{ or } int(f(\mathcal{B}(x,\epsilon))) \neq \emptyset, \text{ resp.}\right).$$

2. *f is uniformly open if for every $\epsilon > 0$ there exists $\delta > 0$, independent of x, such that*

$$f(\mathcal{B}(x,\epsilon)) \supseteq \mathcal{B}(f(x),\delta), \quad \forall\, x \in E.$$

3. *f is dense-weak-open (d-w-open) if $int\left(\overline{f(\mathcal{B}(x,\epsilon))}\right) \neq \emptyset$, for all x.*

We now consider openness of the addition, and we show that addition on a topological vector space is open and, on a normed space, is uniformly open. We first review the definition of topological vector space.

Definition 1.0.3. *A topological vector space is a vector space such that addition and scalar multiplication are continuous. We require that the topology be Hausdorff.*

We first revisit a proposition from [6] that formulates that the sum of two balls is again a ball, centered at the sum of the centers of the given balls, in the right setting.

Proposition 1.0.4. *(cf. [6])*

For any normed space X,

$$B(x_1, r_1) + B(x_2, r_2) = B(x_1 + x_2, r_1 + r_2)$$

where $B(x_i, r_i)$ denotes the open ball centered at x_i with radius r_i for $i = 1, 2$.

Proof. We begin by showing the ($\subseteq$) inclusion. Let $\tilde{x} + \tilde{y} \in B(x_1, r_1) + B(x_2, r_2)$ for

$\tilde{x} \in B(x_1, r_1)$ and $\tilde{y} \in B(x_2, r_2)$. We aim to show that $\tilde{x} + \tilde{y} \in B(x_1 + x_2, r_1 + r_2)$. Observe that

$$\begin{aligned}\|\tilde{x} + \tilde{y} - (x_1 + x_2)\| &= \|\tilde{x} - x_1 + \tilde{y} - x_2\| \\ &\leq \|\tilde{x} - x_1\| + \|\tilde{y} - y_1\| \\ &< r_1 + r_2\end{aligned}$$

since $\tilde{x} \in B(x_1, r_1)$ and $\tilde{y} \in B(x_2, r_2)$. Thus, $\tilde{x} + \tilde{y} \in B(x_1 + x_2, r_1 + r_2)$. Now we show the $(\supseteq)$ inclusion. In order to show this, we will apply the following strategy. Let $z \in B(x_1 + x_2, r_1 + r_2)$. Then

$$\|z - (x_1 + x_2)\| < r_1 + r_2.$$

We want to find z_1 and z_2 such that $z = z_1 + z_2$, with $z_1 \in B(x_1, r_1)$ and $z_2 \in B(x_2, r_2)$. Let $z_i = x_i + r_i \left(\frac{z - (x_1 + x_2)}{r_1 + r_2} \right)$. Observe that, for $i = 1, 2$,

$$\begin{aligned}\|z_1 - x_1\| &= \left\| x_1 + r_1 \left(\frac{z - x_1 - x_2}{r_1 + r_2} \right) - x_1 \right\| \\ &= \frac{r_1}{r_1 + r_2} \|z - x_1 - x_2\| \\ &< \frac{r_1}{r_1 + r_2} (r_1 + r_2) \\ &= r_1.\end{aligned}$$

A similar argument holds for showing that $\|z_2 - x_2\| < r_2$. Thus, $z_1 \in B(x_1, r_1)$ and

$z_2 \in B(x_2, r_2)$. It remains to show that $z = z_1 + z_2$. Observe that

$$\begin{aligned}
z_1 + z_2 &= x_1 + r_1 \left(\frac{z - (x_1 + x_2)}{r_1 + r_2} \right) + x_2 + r_2 \left(\frac{z - (x_1 + x_2)}{r_1 + r_2} \right) \\
&= \frac{(r_1 + r_2)(x_1)}{r_1 + r_2} + r_1 \left(\frac{z - (x_1 + x_2)}{r_1 + r_2} \right) + \frac{(r_1 + r_2)(x_2)}{r_1 + r_2} + r_2 \left(\frac{z - (x_1 + x_2)}{r_1 + r_2} \right) \\
&= \frac{r_1 x_1 + r_2 x_1 + r_1 z - r_1 x_1 - r_1 x_2 + r_1 x_2 + r_2 x_2 + r_2 z - r_2 x_1 - r_2 x_2}{r_1 + r_2} \\
&= \frac{z(r_1 + r_2)}{r_1 + r_2} = z.
\end{aligned}$$

□

Remark 1.0.5. *This proposition implies that the addition,* $+ : X \times X \to X$*, is a uniformly open mapping.*

Topological vector spaces (TVS)

We observe that a weaker statement holds true for the addition operation on a topological vector space. We recall that V is a topological vector space over a field, $\mathbb{R}$ or $\mathbb{C}$, if V is vector space with both the addition and the scalar multiplication continuous maps. We discuss this further in this subsection.

Lemma 1.0.6. *Let* (V, Γ) *be a topological vector space. Then addition is an open mapping.*

Proof. We show that $+ : V + V \to V$ is open (i.e. O_1 and O_2 open implies $O_1 + O_2$ is open). We notice that

$$O_1 + O_2 = \{x + y : x \in O_1, y \in O_2\} = \cup_{y \in O_2} O_1 + \{y\}.$$

For a fixed $y \in O_2$, the map $T : V \to V$ defined by $T(u) = u + y$ is a homeomorphism. The result now follows, since the union of open sets is open. □

Examples of topological vector spaces are ubiquitous. We now give an example that shall be recalled later, for further analysis. Let V be a vector space over either $\mathbb{R}$ or $\mathbb{C}$, and let V' denote the set of all functionals on V. We consider the weak topology on V; this means the smallest topology that makes all elements in V' continuous. More precisely, an open neighborhood of $v \in V$ is a set of the form

$$W(v, \tau_1, \ldots, \tau_k, \epsilon) = \{w \in V : |\tau_i(w - v)| < \epsilon\}$$

with $\epsilon > 0$ and $\tau_i \in V'$. We denote by Γ_w the topology on V generated by these sets. It is clear from the definition that

$$W(v, \tau_1, \ldots, \tau_k, \epsilon) = W(0, \tau_1, \ldots, \tau_k, \epsilon) + \{v\},$$

and it is straightforward that $W(v, \tau_1, \ldots, \tau_k, \epsilon)$ is convex. The vector space V endowed with Γ_w is an example of a topological vector space. Moreover, this is an example of a locally convex topological vector space, since each point has a system of convex neighborhoods.

We now define spaces of functions that we shall consider in forthcoming chapters. These are normed spaces, and, hence, the uniform openness of the addition holds.

Remark 1.0.7. *First, we describe a collection of spaces of continuous functions, defined on a compact Hausdorff space Ω, with values in $\mathbb{R}$ or $\mathbb{C}$, or in a normed space E. These spaces are denoted by $C(\Omega, \mathbb{R})$ and $C(\Omega, \mathbb{C})$, endowed with the norm $\|f\|_\infty = \max_{x\in\Omega} |f(x)|$, and, for $C(\Omega, E)$, $\|f\|_\infty = \max_{x\in\Omega} \|f(x)\|_E$. We also consider spaces of Lipschitz functions, denoted by $Lip[0,1]$ or $Lip(\Omega)$, with Ω a compact metric space and endowed with one of the following metrics:*
$\|f\|_1 = \|f\|_\infty + L(f)$ *or* $\|f\|_\infty = \max\{\|f\|_\infty, L(f)\}$, *where* $L(f) = \sup_{x\neq y} \frac{|f(x)-f(y)|}{d(x,y)}$.

We shall also work with spaces of continuously n-differentiable functions defined on *the interval* $[0, 1]$*. These spaces are denoted* $C([0, 1], \mathbb{R})$*,* $C([0, 1], \mathbb{C})$*, and* $C([0, 1], E)$ *and are endowed with a norm defined from a connected and compact subset of the hypercube* $[0, 1]^{n+1}$*. For details, we refer the reader to the Chapter 4.*

The third large collection of spaces is the space of p-integrable functions ($p \geq 1$)*, defined on a finite measure space* Ω*. These spaces are denoted* $L^p(\Omega, \mathbb{R})$*,* $L^p(\Omega, \mathbb{C})$*, and* $L^p(\Omega, E)$*, and they are endowed with the standard p-norm. We also consider* $L^\infty(\Omega, \mathbb{R})$*, denoted by* $L^\infty(\Omega)$*, which is the space of all essentially bounded measurable functions from* Ω *into* $\mathbb{R}$*, endowed with the following norm:* $\|f\|_\infty = \inf\{M > 0 : \ \lambda\{x \in \Omega : |f(x)| \geq M\} = 0\}$*. It follows from considerations presented above that the addition on each of these spaces is uniformly open (just take* $\delta = \frac{\epsilon}{2}$ *in Part 2 of Definition 1.0.2).*

CHAPTER 2

THE SCALAR PRODUCT

In this chapter, we investigate the openness of the scalar product on several spaces of sequences, spaces of continuous functions, and spaces of integrable functions. We give necessary and sufficient conditions for openness, w-openness and d-openness of the scalar product. In particular, we show that the scalar product on a topological vector space is weak-open.

Openness of the Scalar Product

We recall from [29] (problem 11 on p. 54) the product: $\cdot : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ given by $\cdot\,(t,(x,y)) = t \cdot (x,y) = (tx,ty)$. It is easy to check that $\cdot$ is not open at $(0,(1,0))$. Set $\epsilon = 1/2$. For every $\delta > 0$, we have that $(0,\delta/2) \in \mathcal{B}((0,0),\delta)$, yet $(0,\delta/2) \notin \mathcal{B}(0,\epsilon) \cdot \mathcal{B}((1,0),\epsilon)$. Suppose otherwise. Then there exists $t \in \mathcal{B}(0,\epsilon)$ and $(x,y) \in \mathcal{B}((1,0),\epsilon)$ such that $(tx,ty) = (0,\frac{\delta}{2})$. Since $x \neq 0$, then $t = 0$. Therefore $ty = 0 \neq \frac{\delta}{2}$. A similar argument proves that $\cdot$ is not open at every point of the form $(0,(a,b)) \in \mathbb{R} \times (\mathbb{R}^2 \setminus (0,0))$.

We revisit the previous example for several new settings. We consider a sequence of normed spaces A_k, over the same field $\mathbb{F}$, with corresponding norms $\|\cdot\|_k$ $(k \in \mathbb{N})$.

Let A denote one of the following sequence spaces:

- $c(\{A_k\}_k) = \{\{a_k\}_k : a_k \in A_k$ and $\{\|a_k\|_k\}$ converges$\}$,
- $c_0(\{A_k\}_k) = \{\{a_k\}_k : a_k \in A_k$ and $\{\|a_k\|_k\} \to 0\}$,
- (For $1 \leq p < \infty$) $\ell_p(\{A_k\}_k) = \{\{a_k\}_k : a_k \in A_k$ and $\{\|a_k\|_k\} \in \ell_p\}$,

- $\ell_\infty(\{A_k\}_k) = \{\{a_k\}_k : a_k \in A_k \text{ and } \{\|a_k\|_k\} \in \ell_\infty\}$,

endowed with the standard norms. We denote by $\overline{\mathbf{a}}$ the sequence $(a_1, a_2, \ldots) \in \Pi_n A^n$. We denote by $\overline{\mathbf{0}}$ the sequence of all zeros.

Proposition 2.0.1. *Let A be a sequence space with norm denoted by $\|\cdot\|$. Let $T : \mathbb{F} \times A \to A$ be given by $T(t, \overline{\mathbf{a}}) = t \cdot \overline{\mathbf{a}} = (ta_i)_{i=1,\ldots n}$. Then the following statements are equivalent:*

(a) $t \neq 0$ *or* $(t, \overline{\mathbf{a}}) = (0, \overline{\mathbf{0}})$.

(b) T *is open at* $(t, \overline{\mathbf{a}})$.

(c) T *is d-open at* $(t, \overline{\mathbf{a}})$.

Proof. We show that *(a)* implies *(b)*. We assume that $t \neq 0$. For every $\epsilon > 0$, we set $\delta = |t|\epsilon$. We claim that

$$\mathcal{B}(t\overline{\mathbf{a}}, \delta) \subset \{\lambda\overline{\mathbf{y}} : \lambda \in \mathcal{B}(t, \epsilon) \textit{ and } \overline{\mathbf{y}} \in \mathcal{B}(\overline{\mathbf{a}}, \epsilon)\}.$$

Suppose $\overline{\mathbf{x}} \in \mathcal{B}(t\overline{\mathbf{a}}, \delta)$; then $\|t\overline{\mathbf{a}} - \overline{\mathbf{x}}\| < \delta = |t|\epsilon$. We set $\overline{\mathbf{y}} = \dfrac{1}{t}\overline{\mathbf{x}}$ and $\lambda = t$. Clearly $\overline{\mathbf{y}} = (y_i)_{i=1\ldots n} \in \mathcal{B}(\overline{\mathbf{a}}, \epsilon)$.

If $t = 0$, $a_i = 0$ for all i, and $\epsilon > 0$, then we set $\delta = \dfrac{\epsilon}{k}$, with k such that $\dfrac{1}{k} < \epsilon$. We show that

$$\mathcal{B}(\overline{\mathbf{0}}, \delta) \subset \mathcal{B}(0, \epsilon) \cdot \mathcal{B}(\overline{\mathbf{0}}, \epsilon) = \{(\lambda\overline{\mathbf{y}}) : \lambda \in \mathcal{B}(0, \epsilon) \text{ and } \overline{\mathbf{y}} \in \mathcal{B}(\overline{\mathbf{0}}, \epsilon)\}.$$

Given $\overline{\mathbf{x}} \in \mathcal{B}(\overline{\mathbf{0}}, \delta)$, we set $\lambda = \frac{1}{k} \in \mathcal{B}(0, \epsilon)$, and, hence, $\overline{\mathbf{y}} = k\overline{\mathbf{x}} \in \mathcal{B}(\overline{\mathbf{0}}, \epsilon)$. This shows that T is open at $(0, \overline{\mathbf{0}})$.

We show that T is not open at $(0, \overline{\mathbf{a}})$, provided that some a_i is not equal to zero. We assume that $a_1 \neq 0$. We set $\epsilon = \dfrac{\|a_1\|}{2}$, and, for every $\delta > 0$, we consider $\overline{\mathbf{b}} \in \mathcal{B}(\overline{\mathbf{0}}, \delta)$, given by $b_2 = \frac{\delta a_1}{2\|a_1\|}$ and $b_i = 0$ with $i \neq 2$. We claim that $\overline{\mathbf{b}}$ is not in $\mathcal{B}(0, \epsilon) \cdot \mathcal{B}(\overline{\mathbf{a}}, \epsilon)$. Suppose otherwise. For some $t \in \mathcal{B}(0, \epsilon)$ and $\overline{\mathbf{x}} \in \mathcal{B}(\overline{\mathbf{a}}, \epsilon)$, we have $tx_i = 0$ with $i \neq 2$. Since $x_1 \neq 0$ and $tx_1 = 0$, we obtain $t = 0$. Hence $0 \cdot \overline{\mathbf{x}} = \overline{\mathbf{0}} \neq \overline{\mathbf{a}}$. This shows that *(b)* implies *(a)*.

It is clear that "open" implies "d-open". It remains to show that *(c)* implies *(a)*. We show that T is not d-open at every point of the form $(0, \overline{\mathbf{a}})$, with $\overline{\mathbf{a}} \neq \overline{\mathbf{0}}$. Without loss of generality, we may assume that $\overline{\mathbf{a}} = (0, \ldots, 0, a_i, \ldots a_n, \ldots)$ with $a_i \neq 0$ and $i \neq 1$. We choose $\epsilon = \dfrac{\|a_i\|}{2}$. Then for every $\delta > 0$,

$$\overline{\mathbf{b}} = \left(\frac{\delta\, a_i}{2\|a_i\|}, 0, 0, \ldots \right) \notin \overline{\mathcal{B}(0, \epsilon) \cdot \mathcal{B}(\overline{\mathbf{a}}, \epsilon)}.$$

Suppose that there exist sequences $\{t_k\}$ and $\{\overline{\mathbf{c}}^k\}$, with $t_k \in \mathcal{B}(0, \epsilon)$ and $\overline{\mathbf{c}}^k \in \mathcal{B}(\overline{\mathbf{a}}, \epsilon)$, such that $t_k \overline{\mathbf{c}}^k \to \overline{\mathbf{b}}$. Since $\overline{\mathbf{c}}_i^k \neq 0$, then $t_k \to 0$. Therefore, the sequence $\{t_k \overline{\mathbf{c}}_1^k\}_k$ must converge to zero. This is impossible because $\overline{\mathbf{b}}_1 = \dfrac{\delta\, a_i}{2\|a_i\|} \neq 0$. This completes the proof. □

Remark 2.0.2. *We observe that (a) implies (b) holds for an arbitrary normed space. Furthermore, a variation of the proof given for the aforementioned Proposition, together with the Tietze Extension Theorem ([18, 30]), imply the equivalence of the three statements in the Proposition 2.0.1 for spaces of vector-valued continuous functions, $C(\Omega, E)$, with Ω a compact metric space and E a normed space.*

Definition 2.0.3. *We say that a measure space (Ω, μ) is non-atomic, provided that every positive measurable subset Ω_1 has a partition into two positive measurable sets (i.e. For $\Omega_1 \subset \Omega$ such that $\mu(\Omega_1) > 0$, there exist A and B disjoint measurable* *subsets of Ω_1 such that $\Omega_1 = A \cup B$, $\mu(A) > 0$, and $\mu(B) > 0$).*

Corollary 2.0.4. *Proposition 2.0.1 holds for $A = L_p(\Omega, \mathbb{C})$ with $1 \leq p < \infty$ and $T : \mathbb{F} \times L_p(\Omega, \mathbb{C}) \longrightarrow L_p(\Omega, \mathbb{C})$, where $T(\lambda, f) = \lambda \cdot f$ and Ω is a non-atomic measure space.*

Proof. We use a similar strategy to the one given for Proposition 2.0.1. We start with $(a) \implies (b)$: Suppose $\lambda \neq 0$. We claim that $\mathcal{B}(\lambda \cdot f, |\lambda| \cdot \epsilon) \subseteq \mathcal{B}(\lambda, \epsilon) \cdot \mathcal{B}(f, \epsilon)$. Let $h \in \mathcal{B}(\lambda \cdot f, |\lambda| \cdot \epsilon)$. Then

$$\begin{aligned} \|h - \lambda \cdot f\|_p &= |\lambda| \cdot \left\| \frac{h}{\lambda} - f \right\|_p \\ &< |\lambda| \cdot \epsilon. \end{aligned}$$

Hence, $\left\| \frac{h}{\lambda} - f \right\|_p < \epsilon$, so choose $\lambda \in \mathcal{B}(\lambda, \epsilon)$ and $\frac{h}{\lambda} \in \mathcal{B}(f, \epsilon)$. For $\lambda = 0$ and $f = 0$ almost everywhere, we claim that $\mathcal{B}\left(\overline{\mathbf{0}}, \frac{\epsilon^2}{2}\right) \subseteq \mathcal{B}(0, \epsilon) \cdot \mathcal{B}(\overline{\mathbf{0}}, \epsilon)$, with $\overline{\mathbf{0}}$ denoting the measurable function that is equal to zero almost everywhere. Take $h \in \mathcal{B}\left(\overline{\mathbf{0}}, \frac{\epsilon^2}{2}\right)$, $\frac{\epsilon}{2} \in \mathcal{B}(0, \epsilon)$. Note that $\left\| \frac{2}{\epsilon} \cdot h \right\|_p \leq \frac{2}{\epsilon} \cdot \frac{\epsilon^2}{2} < \epsilon$, so $\frac{2}{\epsilon} \cdot h \in \mathcal{B}(\overline{\mathbf{0}}, \epsilon)$. Now we show $(b) \implies (a)$: We prove the contrapositive. Consider $(0, f)$ with $f \neq 0$ and $\Omega_1 \subseteq \Omega$ with $\mu(\Omega_1) > 0$ such that $|f(x)| > 0$ for all $x \in \Omega_1$. Since μ is non-atomic, there exist disjoint subsets of Ω_1, A and B, with positive measure such that $\|f|_A\|_p > 0$ and $\|f|_B\|_p \geq |f|_A\|_p > 0$. We set $\epsilon = \frac{\|f|_A\|_p}{2}$. For $\delta > 0$, we consider $g \in \mathcal{B}(\overline{\mathbf{0}}, \delta)$ with $g|_A = \frac{\delta \cdot f|_A}{2 \cdot \|f|_A\|_p}$ and $g|_{\Omega \setminus A} = 0$. Suppose for a contradiction that $g \in \mathcal{B}(0, \epsilon) \cdot \mathcal{B}(f, \epsilon)$. Then, for $\lambda \in \mathcal{B}(0, \epsilon)$ and $h \in \mathcal{B}(f, \epsilon)$, we have that $\lambda \cdot h|_A = 0$, so $\lambda = 0$. But $0 \cdot h|_A = \overline{\mathbf{0}} \neq f$. The proof that $(b) \implies (c)$ follows immediately from the definition of openness, so it remains to show that $(c) \implies (a)$. We proceed by contrapositive. We show that T is not d-open at every pair $(0, f)$ with $f \in L_p(\Omega)$ and $f \neq \overline{\mathbf{0}}$. Since f is not the zero function and (Ω, μ) is non-atomic, there exists a positive measurable subset of Ω, say B, such that for every $x \in B$, we have that $|f(x)| > 0$

and $\Omega \setminus B$ is also of positive measure. We may assume without loss of generality that $|f(x)| > L > 0$ for $x \in B$. We set $g|_{\Omega \setminus B} = 1$ and $g|_B = 0$. It is clear that $g \in L_p(\Omega)$. We show that $g \notin \overline{\mathcal{B}(0,\epsilon) \cdot \mathcal{B}(f,\epsilon)}$. Suppose that there exist $\{\lambda_n\}$, with $\lambda_n \in \mathcal{B}(0,\epsilon)$, and $\{f_n\}$, with $f_n \in \mathcal{B}(f,\epsilon)$, such that $\|g - \lambda_n \cdot f_n\|_p \longrightarrow 0$. Observe $\|g|_B - \lambda_n \cdot f_n|_B\|_p \longrightarrow 0$, so $\lambda_n \cdot f_n|_B \longrightarrow 0$. But

$$\mu(B) \cdot |\lambda_n|^p \cdot \left(\frac{L}{2}\right)^p \leq \int_B |\lambda_n|^p \cdot |f_n|^p d\mu \longrightarrow 0,$$

for all n after a certain order. Thus, $\lambda_n \longrightarrow 0$. On the other hand, we also have

$$\mu(\Omega \setminus B) = \left[\int_{\Omega \setminus B} |g - \lambda_n f_n|^p d\mu\right]^{1/p} \leq |\lambda_n| \cdot \|f_n\|_p \leq |\lambda_n| \cdot (\epsilon + ||f\|_p).$$

Since $\mu(\Omega \setminus B) > 0$ and $|\lambda_n| \to 0$ we reach a contradiction. This proves the claim. □

Remark 2.0.5. *Observe that a similar statement to the one in Corollary 2.0.4 holds for $p = \infty$, using an appropriate adaptation of the proof we just gave.*

Definition 2.0.6. *Let X be a normed algebra. Let x and y be two elements in X. We say that the pair (x, y) has the property $(*)$ (i.e. is a point of local openness for the multiplication) if, for every $\epsilon > 0$, $x \cdot y \in int\left(\mathcal{B}(x,\epsilon) \cdot \mathcal{B}(y,\epsilon)\right)$.*

Corollary 2.0.7. *Let A and T be as in Proposition 2.0.1. Then T is w-open.*

Proof. First, observe that we characterize the set of points of local openness of T as:

$$O(T) = \{(t, \overline{\mathbf{a}}) : t \neq 0\} \cup \{(0, \overline{\mathbf{0}})\}.$$

Since $\mathcal{O}(T)$ is dense in $\mathbb{F} \times A$, we obtain the following containment:

$$B(0,\epsilon) \cdot B(\overline{\mathbf{a}},\epsilon) \supseteq B(\epsilon/2, \epsilon/4) \cdot B(\overline{\mathbf{a}}, \epsilon/4) \supseteq B(\epsilon/2 \cdot \overline{\mathbf{a}}\,, \epsilon^2/8),$$

by simply noting that

$$\left\|\overline{\mathbf{x}} - \frac{\epsilon}{2} \cdot \overline{\mathbf{a}}\right\| < \frac{\epsilon^2}{8} \text{ implies } \left\|\frac{2}{\epsilon} \cdot \overline{\mathbf{x}} - \overline{\mathbf{a}}\right\| < \frac{\epsilon}{4}, \text{ where } \frac{\epsilon}{2} \cdot \left(\frac{2}{\epsilon}\overline{\mathbf{x}}\right) \in \mathcal{B}\left(\frac{\epsilon}{2}, \frac{\epsilon}{4}\right) \cdot \mathcal{B}\left(\overline{\mathbf{a}}, \frac{\epsilon}{4}\right).$$

□

Proposition 2.0.8. *For any normed space X,*

$$\lambda B(x_1, r_1) = B(\lambda x_1, |\lambda| r_1)$$

where $B(x_1, r_1)$ denotes the open ball centered at x_1 with radius $r_1 > 0$ and $\lambda \neq 0$ a scalar.

Proof. We begin with the ($\subseteq$) containment. Let $y \in B(x_1, r_1)$. Then, clearly, $\lambda y \in \lambda B(x_1, r_1)$ Observe that

$$\|\lambda y - \lambda x_1\| = |\lambda| \|y - x_1\| < |\lambda| r_1.$$

Thus, $\lambda y \in B(\lambda x_1, |\lambda r_1|)$. Now we show the ($\supseteq$) containment. Let $y \in B(\lambda x_1, \lambda r_1)$. Observe that

$$\begin{aligned} \|y - \lambda x_1\| &= |\lambda| \left\|\frac{y}{\lambda} - x_1\right\| \\ &< |\lambda| r_1. \end{aligned}$$

Since $\lambda \neq 0$, we may cancel the $|\lambda|$ term and are left with the inequality $\left\|\frac{y}{\lambda} - x_1\right\| < r_1$. Thus, $\frac{y}{\lambda} \in B(x_1, r_1)$, and, multiplying by λ, we have that $y \in \lambda B(x_1, r_1)$ and $\lambda B(x_1, r_1) = B(\lambda x_1, |\lambda| r_1)$. □

Previous observations can be extended to topological vector spaces.

Lemma 2.0.9. *Let V be a topological vector space over the field $\mathbb{F}$. Then $T : \mathbb{F} \setminus \{0\} \times V \to V$, given by $T(\lambda, v) = \lambda \cdot v$, is an open map.*

Proof. We show that, for O_1, O_2 open, where $O_1 \subset \mathbb{F}$ and $O_2 \subset V$, $O_1 \cdot O_2 = \{\lambda \cdot y : \lambda \in O_1, y \in O_2\}$ is open. We notice that for a fixed $\lambda \in \mathbb{F} \setminus \{0\}$, the scalar-multiplication, $M_\lambda : V \to V$, given by $M_\lambda(v) = \lambda \cdot v$, is a homeomorphism and is, thus, open. Therefore $O_1 \cdot O_2 = \cup_{\lambda \in O_1} \lambda \cdot O_2$ is open. This completes the proof. □

Remark 2.0.10. *The map $T : \mathbb{F} \times V \to V$ is open at $(0, \overline{\mathbf{0}})$. Given W, a neighborhood of $\overline{\mathbf{0}}$, and $\mathcal{B}(0, \epsilon)$, an open ball in $\mathbb{F}$, we have*

$$\frac{\epsilon}{2} \cdot W \subset \mathcal{B}(0, \epsilon) \cdot W.$$

Since $M_{\epsilon/2}$ is a homeomorphism, $\frac{\epsilon}{2} \cdot W$ is a neighborhood of $\overline{\mathbf{0}}$.

Proposition 2.0.11. *Let V be a topological vector space over the field $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). Then $T : \mathbb{F} \times V \to V$, given by $T(\lambda, \overline{\mathbf{x}}) = \lambda \cdot \overline{\mathbf{x}}$, is w-open.*

Proof. From Lemma 2.0.9 and Remark 2.0.10, it remains to show that T is w-open at every point of the form $(0, \overline{\mathbf{x}})$ with $\overline{\mathbf{x}} \neq \overline{\mathbf{0}}$. Given $\epsilon > 0$ and W, a neighborhood of $\overline{\mathbf{x}}$, we select $t_0 \in \mathcal{B}(0, \epsilon) \setminus \{0\}$. Then there exists $\delta > 0$ such that $\mathcal{B}(t_0, \delta) \subset \mathcal{B}(0, \epsilon)$. Since T is open at $(t_0, \overline{\mathbf{x}})$, there exists a neighborhood of $\overline{\mathbf{0}}$, U, such that

$$t_0 \overline{\mathbf{x}} + U \subset \mathcal{B}(t_0, \delta) \cdot W \subset \mathcal{B}(0, \epsilon) \cdot W.$$

This completes the proof. □

It is interesting to observe that $T : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $T : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ are open maps. In fact, they are uniformly open. The proof for the real case is given in [4]. For completeness, we present an adapted argument for the complex case.

Proposition 2.0.12. *For $z, w \in \mathbb{C} \times \mathbb{C}$ and $\epsilon > 0$,*

$$\mathcal{B}\left(zw, \frac{\epsilon^2}{4^2}\right) \subset \mathcal{B}(z, \epsilon) \cdot \mathcal{B}(w, \epsilon).$$

Proof. Let $u \in \mathcal{B}\left(zw, \frac{\epsilon^2}{4^2}\right)$.

1. If $|z| \geq \frac{\epsilon}{4}$, we set $u = x \cdot y$, with $x = z$ and $y = \frac{u}{z}$. Clearly $x \in \mathcal{B}(z, \epsilon)$, so we simply need to show that $y \in \mathcal{B}(w, \epsilon)$: we have

$$\left|\frac{u}{z} - w\right| = |u - wz|\frac{1}{|z|} \leq \frac{\epsilon}{4}.$$

   A similar argument holds for $|w| \geq \frac{\epsilon}{4}$ (just set $x = w$ and $y = \frac{u}{w}$).

2. Let $|z| < \frac{\epsilon}{4}$ and $|w| < \frac{\epsilon}{4}$. We set $x = \sqrt{|u|}e^{i\theta}$ and $y = \sqrt{|u|}$ with θ such that $u = |u|e^{i\theta}$. We need to show that $x \in \mathcal{B}(z, \epsilon)$ and $y \in \mathcal{B}(w, \epsilon)$. Observe that

$$|x - z| = |\sqrt{|u|}e^{i\theta} - z| \leq \sqrt{|u|} + |z| \leq \sqrt{|u - zw| + |zw|} + \frac{\epsilon}{4} \leq \sqrt{\frac{\epsilon^2}{4^2} + \frac{\epsilon^2}{4^2}} + \frac{\epsilon}{4} < \epsilon.$$

   Similarly, we show that $|y - w| < \epsilon$.

□

These two cases are examples of scalar multiplications that behave well relative to openness, contrarily to many previous examples (see Proposition 2.0.1).

Given a normed space A over the field $\mathbb{F}$, either $\mathbb{R}$ or $\mathbb{C}$, then the scalar product $\mathbb{F} \times A \to A$ is uniformly open if and only if the dimension of A is equal to 1. Previous considerations show that if the dimension of A is 1, then the product is uniformly open. Suppose the dimension of A is greater than 1. There exist two linearly independent unit vectors $\{v_1, v_2\}$. We denote by $\langle v_1 \rangle$ the span of v_1. Let $d = inf\{\|v_2 - tv_1\| : \ t \in \mathbb{F}\}$. We set $\epsilon = \frac{d}{2}$; then, for every $\delta < \frac{d}{2}$ and every vector sv_2 with $|s| = \frac{\delta}{2}$, we have

$$\frac{s}{2} v_2 \notin (-\epsilon, \epsilon) \times \mathcal{B}(v_1, \epsilon).$$

To show this, let $\delta = \frac{d}{4}$. Then $\pm\frac{d}{4} \cdot v_2 = t_0 \cdot w$, where $|t_o| < \frac{d}{2}$ and $\|w - v_1\| < \frac{d}{2}$. We proceed by contradiction. Observe, then, that

$$\begin{aligned} d = d(v_2, \langle v_1 \rangle) &= \inf_t \{\|v_2 - t \cdot v_1\|\} \\ &= \inf_t \left\{ \left\| \frac{4}{d} \cdot t_0 \cdot w - t \cdot v_1 \right\| \right\} \\ &= \frac{4 \cdot |t_0|}{d} \cdot d(w, \langle v_1 \rangle) \\ &\leq \frac{4 \cdot t_0}{d} \cdot \frac{d}{2} \\ &= 2 \cdot |t_0| < d, \end{aligned}$$

which is impossible. This proves the following proposition.

Proposition 2.0.13. *Let A be a normed space over $\mathbb{R}$ or $\mathbb{C}$, then the scalar product is open if and only if A is 1-dimensional.*

CHAPTER 3

MULTIPLICATION ON SPACES OF CONTINUOUS FUNCTIONS

There are several different notions of openness that we consider, applied to the multiplication operation, as reviewed in Chapter 1. The work in [6] motivates this section. Since different algebras have different degrees of openness for the multiplication, we consider several of these questions for each individual algebra in our list. In this chapter, we provide results for the openness properties of the multiplication on algebras of continuous functions.

The Scalar-Valued Case

We turn our attention to scalar-valued continuous functions.

The Space of Real-Valued Continuous Functions on a Compact Set with Infinite Norm

We consider the standard pointwise multiplication of functions $P : C[0, 1] \times C[0, 1] \to C[0, 1]$, where $P(f, g) = f \cdot g$.

Theorem 3.0.1. *Let Ω be a compact Hausdorff topological space. Then $C(\Omega, \mathbb{R})$ does not have the property that multiplication is open.*

Remark 3.0.2. *The Fremlin function, $f(x) = x - \dfrac{1}{2}$, is an example that shows that P is not open. This follows because $f^2 \notin int\,(\mathcal{B}(f, 1/2) \cdot \mathcal{B}(f, 1/2))$, and P is not open at (f, f) (see [6]). It is interesting to observe that $(f, f) \in \overline{\mathcal{B}(f, 1/2) \cdot \mathcal{B}(f, 1/2)}$, since $f^2 = \lim_n \left(f + \dfrac{1}{n}\right)^2$. We extend this example further. Let $f \in C[0, 1]$ be such* *that there exists* $a < b \in [0,1]$ *with* $f(a) \cdot f(b) < 0$. *Then* $f^2 \notin Int(B(f, \min\{|f(a)|, |f(b)|\}) \cdot B(f, \min\{|f(a)|, |f(b)|\}))$. *As before,* $f^2 + \frac{\delta}{2}$ *has no zeros, but any function given as the product of two functions in* $(B(f, \min\{|f(a)|, |f(b)|\})$ *must vanish at some point in* $[0,1]$.

We now review a result from [6], which we state without proof. Then we will provide a proof in a more general space.

Theorem 3.0.3. *Let* $f \in C([0,1], \mathbb{R})$. *Then* $f^2 \in Int(B^2(f,r))$ *for all* $r > 0$ *if and only if either* $f \geq 0$ *on* $[0,1]$ *or* $f \leq 0$ *on* $[0,1]$.

Remark 3.0.4. *In [6] (cf. Theorem 5), we see that the multiplication* $P : C([0,1], \mathbb{R}) \times C([0,1], \mathbb{R}) \longrightarrow C([0,1], \mathbb{R})$ *is weak-open.*

We now consider the following generalization of Theorem 3.0.3:

Theorem 3.0.5. *Let* Ω *be a compact, connected Hausdorff space. Let* $f \in C(\Omega, \mathbb{R})$ *Then* $f^2 \in Int(B^2(f,r))$ *for all* $r > 0$ *if and only if either* $f \geq 0$ *on* Ω *or* $f \leq 0$ *on* Ω.

Proof. We begin with the ($\Rightarrow$) direction. We proceed by contradiction: Suppose that there exist $x, y \in \Omega$ such that $f(x) < 0$ and $f(y) > 0$. Let $r = min\{|f(x)|, |f(y)|\}$, and let $g \in B(f,r)$. Then $g(x) < 0$ and $g(y) > 0$. The connectedness assumption on Ω implies that $g(\Omega)$ is connected. Then there exists $z_0 \in \Omega$ such that $g(z_0) = 0$. Thus, for every $h \in B^2(f,r)$, there exists $z_0 \in \Omega$ such that

$$h(z_0) = 0. \tag{3.0.1}$$

Since $f^2 \in Int(B^2(f,r))$, it follows that $f^2 + \epsilon \in B^2(f,r)$ for $\epsilon > 0$ small. Clearly, $f^2 + \epsilon > 0$. Since $f^2 + \epsilon \in B^2(f,r)$, $f^2 + \epsilon$ must have a zero. This leads to a contradiction and proves the statement.

Now we show the ($\Leftarrow$) direction. The proof follows closely to the one provided in [6]. For completeness of exposition, we provide the entire argument. Without loss of generality, we suppose that $f \geq 0$ (if $f \leq 0$, then simply replace f with $-f$, and, since $B^2(f,r) = B^2(-f,r)$, replace $B^2(f,r)$ with $B^2(-f,r)$). Let $r > 0$, and, since we shall deal with square roots, we set $\epsilon = \lambda^2 r^2$, for some $\lambda > 0$. We will show that we may choose $\lambda^2 = \dfrac{1}{9}$. We show that $B(f^2, \epsilon) \subseteq B^2(f,r)$. Observe that for $g \in B(f^2, \epsilon)$, then

$$f^2(x) - \epsilon < g(x) < f^2(x) + \epsilon \tag{3.0.2}$$

Let $g = g_1 \cdot g_2$ with $g_1 = \sqrt{|g|}$ and $g_2 = \sqrt{|g|} \cdot sgn(g)$; they are clearly continuous on Ω. First, we will show that $g_1 \in B(f,r)$. From the rightmost inequality, we see that $|g(x)| < f^2(x) + \epsilon$. Thus, it follows that

$$\sqrt{|g(x)|} < \sqrt{f^2(x) + \epsilon} < f(x) + \sqrt{\epsilon}. \tag{3.0.3}$$

On the other hand, since

$$\begin{aligned}
(\sqrt{|g(x)|} + \sqrt{\epsilon})^2 &= |g(x)| + 2\sqrt{|g(x)|\epsilon} + \epsilon \\
&\geq |g(x)| + \epsilon \\
&> f^2(x),
\end{aligned}$$

by the rightmost inequality, we have $\sqrt{|g(x)|} + \sqrt{\epsilon} > f(x)$. Therefore $\sqrt{|g(x)|} > f(x) - \sqrt{\epsilon}$, and

$$f(x) - \sqrt{\epsilon} < \sqrt{|g(x)|} < f(x) + \sqrt{\epsilon}.$$

Thus, $g_1 = \sqrt{|g|} \in B(f, \sqrt{\epsilon})$. Recall that we let $\epsilon = \lambda^2 r^2$, so $\sqrt{\epsilon} = \lambda r$. We find a value for λ such that $B(f, \sqrt{\epsilon}) = B(f, \lambda r) \subseteq B(f, r)$. Note that for $\lambda^2 < 1$, this will be true (we will show why we specifically want to choose $\lambda^2 = \dfrac{1}{9}$ later). Thus,

$$g_1 = \sqrt{|g|} \in B(f, \sqrt{\epsilon}) = B(f, \lambda r) \subseteq B(f, r).$$

Now we want to show that $g_2 \in B(f, r)$. We need only consider $g(x) < 0$ since, if $g(x) \geq 0$, we have the same situation as we did with g_1. By the leftmost inequality and since $g(x) < 0$, we have that

$$f^2(x) - \epsilon < g(x) < 0.$$

In order for us to show that $g_2 \in B(f, r)$, we prove that, for an appropriate choice of $\lambda < 1$,

$$f(x) - r < -f(x) - \sqrt{\epsilon} < g_2(x) < f(x) + \sqrt{\epsilon} < f(x) + r.$$

Since $\lambda < 1$, then $\sqrt{\epsilon} < r$ and

$$f(x) - \sqrt{\epsilon} > f(x) - r. \tag{3.0.4}$$

From $f^2(x) - \epsilon < g(x) < 0$, we have that $f^2(x) < \epsilon$. Thus, $f(x) < \sqrt{\epsilon}$. For $\lambda \leq \dfrac{1}{2}$, we have

$$\begin{aligned} 2f(x) &< 2\sqrt{\epsilon} \\ &\leq \frac{2r}{2} = r, \end{aligned}$$

and then $f(x) < \frac{r}{2}$. Hence, we set $\lambda = \frac{1}{3}$. Continuing from above, observe that

$$\begin{aligned} 2f(x) &< 2\sqrt{\epsilon} \\ &= \frac{2r}{3} \\ &= r - \sqrt{\epsilon}. \end{aligned}$$

We break $2f(x)$ into $f(x) + f(x)$ and rearrange the inequality we just obtained (i.e. $f(x) - r < -f(x) - \sqrt{\epsilon}$). Since $\sqrt{|g(x)|} < f(x) + \sqrt{\epsilon}$, it follows that $-f(x) - \sqrt{\epsilon} < -\sqrt{|g(x)|}$. Thus,

$$-\sqrt{|g(x)|} > -f(x) - \sqrt{\epsilon} > f(x) - r.$$

But

$$-\sqrt{|g(x)|} < \sqrt{|g(x)|} < f(x) + \sqrt{\epsilon}.$$

Hence,

$$f(x) - r < -f(x) - \sqrt{\epsilon} = -f(x) - \frac{r}{3} < -\sqrt{|g(x)|} < f(x) + \sqrt{\epsilon} = f(x) + \frac{r}{3} < f(x) + r,$$

and we have that $g_2 = \sqrt{|g|} \cdot sgn(g) \in B(f, r)$. Since both g_1 and g_2 are elements of $B(f, r)$, it follows that $g = g_1 \cdot g_2 \in B^2(f, r)$. $\square$

Remark 3.0.6. *Let Ω be a compact, connected Hausdorff space with more than one point. The multiplication $C(\Omega, \mathbb{R}) \times C(\Omega, \mathbb{R}) \longrightarrow C(\Omega, \mathbb{R})$ is not an open mapping.*

Proof. We provide a similar example to the one given in Remark 3.0.2. Let x, y be distinct points in Ω. Observe that the set $F = \{x, y\}$ is closed in Ω, as Ω is Hausdorff. Consider the continuous function $f : F \longrightarrow [-1, 1]$, via $x \longmapsto -1$ and

$y \longmapsto 1$. Since every compact Hausdorff space is normal, then the Tietze Extension Theorem implies the existence of a continuous function $\tilde{f} : \Omega \longrightarrow [-1, 1]$ with $\tilde{f}(x) = f(x) = -1 < 0$ and $\tilde{f}(y) = f(y) = 1 > 0$. For $h \in B(\tilde{f}, 1) \cdot B(\tilde{f}, 1)$, $h(t) = 0$ for some $t \in \Omega$ because $h(x) > 0$ and $h(y) < 0$. But $\tilde{f}^2 + \epsilon > 0$, so h cannot be an element of $B^2(\tilde{f}, f)$. By Theorem 3.0.5, we have that $\tilde{f}^2 \notin Int(B^2(\tilde{f}, r))$, for all $r > 0$. □

Corollary 3.0.7. *Let Ω be a compact, connected Hausdorff space. Then*

$$A = \{f \in C(\Omega, \mathbb{R}) : f^2 \in Int(B^2(f, r)) \ \forall r > 0\}$$

is closed.

Proof. Let $\{f_n\} \in A$, and suppose that $f_n \longrightarrow f$ in $\|\cdot\|_\infty$. Therefore, for all $x \in \Omega$, $f_n(x) \longrightarrow f(x)$. It remains to show that $f \in A$. Since $f_n \in A$, we have that $f_n^2 \in Int(B^2(f_n, r))$. By Theorem 3.0.5, we have that $f_n(x) \leq 0$ for all x or $f_n(x) \geq 0$ for all x. Without loss of generality, we may assume that $f_n(x) \geq 0$ for all n. Then, since the limit of a nonnegative convergent sequence is nonnegative (a similar result holds for $f_n(x) \leq 0$), we must have that $f(x) \geq 0$. But, applying Theorem 3.0.5 again, this implies that $f^2 \in Int(B^2(f, r))$ for $r > 0$ and $f \in C(\Omega, \mathbb{R})$. Thus, $f \in A$, and we have that A is closed. □

Multiplication in $C([0, 1], \mathbb{C})$

Now we turn our attention to $C([0, 1], \mathbb{C})$. First, we provide a brief review of some results from complex analysis and introduce some notation. Then we state and prove the main result of this subsection.

We shall assume that f is complex-valued. We recall that any nonzero complex number $z = a + ib$ has a polar representation as $\rho e^{i\eta}$, with $\rho = \sqrt{a^2 + b^2}$ and $\eta(z)$ the unique angle in the interval $[0,\, 2\pi)$ such that $\rho e^{i\eta} = \rho\,(\cos(\eta) + i\sin(\eta)) = a + ib$ (we dropped the argument z for simplicity). We shall call this angle η the principal argument of the complex number $a + ib$. This angle is measured counterclockwise from the positive real axis. The only point that leads to an ambiguous situation is when $z = 0$.

Let f be a continuous function on $[0,\, 1]$ with values in $\mathbb{C} \setminus \{0\}$. We set $|f| : [0,\, 1] \to (0,\, \infty)$ defined as $|f|(x) = |f(x)|$, and we set $\eta_f(x)$ to denote the principal argument of $f(x)$. Thus, $\eta_f : [0, 1] \to [0,\, 2\pi)$, is given by $x \to \eta(f(x))$ such that

$$f(x) = |f|(x)\, e^{i\,\eta_f(x)}.$$

The continuity of f implies the continuity of $|f|$, which follows from the inequality

$$|\; |f|(x) - |f|(y)\; | \leq |f(x) - f(y)|.$$

This also implies that $e^{i\eta_f}$ is continuous, since $e^{i\eta_f}$ is the quotient of two continuous maps (i.e. $e^{i\eta_f} = \frac{f}{|f|}$).

Theorem 3.0.8. *Let $f \in C[0,1]$ such that $f(x) \notin \mathbb{R}$ for all x in $[0,1]$. Then $f^2 \in Int(B^2(f,r))$ for all $r > 0$.*

Proof. Let

$$m = \min\{d_x = \min\{|f(x) - a|,\, a \in \mathbb{R}\} : x \in [0,1]\} > 0 \text{ and fix } r = \min\left\{1, \frac{m}{2}\right\}.$$

Let $\epsilon < \frac{m}{(4+\sqrt{m})^2} r^2$. We show that $B(f^2, \epsilon) \subseteq B^2(f, r)$. Let $g \in B(f^2, \epsilon)$. Then

$$|g - f^2| < \epsilon. \tag{3.0.5}$$

Hence, we have that $|g(x)| < |f^2(x)| + \epsilon$ for each $x \in [0,1]$. Let $g = g_1 \cdot g_2$, where $g_1 = \sqrt{|g(x)|} \cdot e^{(i\eta_{f^2}(x))/2}$ and $g_2 = \sqrt{|g(x)|} \cdot e^{(i\eta_{f^2}(x))/2} \cdot e^{i(\eta_g(x) - \eta_{f^2}(x))}$. We observe that $g_1, g_2 \in C[0,1]$. As $|g(x)| < |f^2(x)| + \epsilon$, it follows that

$$\sqrt{|g(x)|} < \sqrt{|f^2(x)| + \epsilon} < |f(x)| + \sqrt{\epsilon}.$$

Since

$$\left(\sqrt{|g(x)|} + \sqrt{\epsilon}\right)^2 = |g(x)| + 2\sqrt{|g(x)|\epsilon} + \epsilon > |f^2(x)|,$$

it follows that $\sqrt{|g(x)|} + \sqrt{\epsilon} > \sqrt{|f^2(x)|} = |f(x)|$. Thus, $|g_1(x)| > |f(x)| - \sqrt{\epsilon}$, so $|g_1| \in B(|f|, r)$. Without loss of generality, we may assume that $Im(f(x)) > 0$. Since $f(x) = |f(x)| \cdot e^{(i\eta_{f^2}(x))/2}$, we have that

$$g_1(x) = |g_1(x)| \cdot e^{(i\eta_{f^2}(x))/2} \in B(|f(x)| \cdot e^{(i\eta_{f^2}(x)/2}, \sqrt{\epsilon}),$$

where $0 < \eta_f(x) < \pi$. Therefore, $g_1 \in B(f, \sqrt{\epsilon}) \subseteq B(f, r)$. It remains to show that $g_2 \in B(f, r)$. Since g is complex-valued, we cannot use the same strategy that was used in [6] for showing that $g_2 \in B(f, r)$. However, if we can show the angles for f and g_2 are close together and that $|f(x)|$ and $|g_2(x)|$ are also close, then the result follows. From the fact that f^2 and g are within ϵ, we have

$$\begin{aligned} |g_2(x) - f(x)| &\leq |g_2(x) - g_1(x)| + |g_1(x) - f(x)| \\ &\leq |g_1(x)| \cdot \left| e^{i\eta_{f^2}(x)/2 + i(\eta_g(x) - \eta_{f^2}(x))} - e^{i\eta_f(x)} \right| + \sqrt{\epsilon} \end{aligned}$$

We observe that

$$\begin{aligned}\left|e^{i\eta_{f^2}(x)/2+i(\eta_g(x)-\eta_{f^2}(x))} - e^{i\eta_f(x)}\right| &= \left|e^{i\eta_g(x)-i\eta_{f^2}(x)/2} - e^{i\eta_f(x)}\right| \\ &= \left|e^{i\eta_g(x)-i\eta_f(x)} - e^{i\eta_f(x)}\right| \\ &= \left|e^{-i\eta_f(x)}\right| \cdot \left|e^{i\eta_g(x)} - e^{2i\eta_f(x)}\right| \\ &= \left|e^{i\eta_g(x)} - e^{i\eta_{f^2}(x)}\right|\end{aligned}$$

Since $f^2 \neq 0$, the function g is ϵ-close to f^2 and $|g(x)| = |g_1^2(x)|$, we have:

$$\begin{aligned}|g_1(x)| \cdot \left|e^{i\eta_g(x)} - e^{i\eta_{f^2}(x)}\right| &= \frac{|g_1^2(x)|}{|g_1(x)|} \cdot \left|e^{i\eta_g(x)} - e^{i\eta_{f^2}(x)}\right| \\ &= \frac{|\ |g(x)| \cdot e^{i\eta_g(x)} - |g(x)|e^{i\eta_{f^2}(x)}\ |}{\cdot}\sqrt{|g(x)|} \\ &\leq \frac{|\ |g(x)|e^{i\eta_g(x)} - |f^2(x)|e^{i\eta_{f^2}(x)}\ | + |\ |f^2(x)|e^{i\eta_{f^2}(x)} - |g(x)|e^{i\eta_{f^2}(x)}\ |}{\sqrt{|g(x)|}} \\ &= \frac{|g(x) - f^2(x)| + |\ |f^2(x)| - |g(x)|\ |}{\sqrt{|g(x)|}}.\end{aligned}$$

Since $|\ |g(x)| - |f^2(x)|\ | \leq |g(x) - f^2(x)| < \epsilon$, as $g \in B(f^2, \epsilon)$, we have that

$$-\epsilon + m \leq -\epsilon + |f(x)|^2 < |g(x)| < \epsilon + |f(x)|^2.$$

Moreover, $\epsilon < 1$ and $0 < \epsilon < \dfrac{m}{4}$ implies that

$$\begin{aligned}|g_2(x) - f(x)| &\leq \frac{|g(x) - f^2(x)| + |\ |f^2(x)| - |g(x)| \cdot |e^{i\eta_{f^2}(x)}\ |}{\sqrt{|g(x)|}} + \sqrt{\epsilon} \\ &< \frac{\epsilon + \epsilon}{\sqrt{|g(x)|}} + \sqrt{\epsilon} < \frac{4\epsilon}{\sqrt{m}} + \sqrt{\epsilon} \\ &< \left(\frac{4 + \sqrt{m}}{\sqrt{m}}\right) \cdot \sqrt{\epsilon} < \left(\frac{4 + \sqrt{m}}{\sqrt{m}}\right) \cdot \left(\frac{\sqrt{m}}{4 + \sqrt{m}}\right) < r.\end{aligned}$$

Note that if $\pi < \eta_f(x) < 2\pi$, we just use $-f$. Then $-B(-f, r) \cdot -B(-f, r)$. This completes the proof. $\square$

Remark 3.0.9. *Under the conditions given in Theorem 3.0.8, we see that* $C([0,1],\mathbb{C})$ *has w-open multiplication. We do not yet have a proof for when* $f \cdot g \in Int(B(f,r) \cdot B(g,r))$ *for* $f, g \in C([0,1],\mathbb{C})$*, but we do discuss a somewhat similar result in Chapter 4.*

The Vector-Valued Case

We now turn our attention to vector-valued continuous functions. An algebra with the multiplication openness property (M-Op) satisfies the property that multiplication is open. Similarly, an algebra with the multiplication-uniform openness property (M-UOp) satisfies the property that multiplication is uniformly open. Notice that an algebra with the M-UOp also has the M-Op. Moreover, a subalgebra of an algebra with the M-UOp (or M-Op) also has the M-UOp (respectively M-Op). Both the M-Op and M-UOp are clearly topologically invariant, as homeomorphisms map open sets to open sets.

Proposition 3.0.10. *Let* Ω *be a compact Hausdorff topological space and* E *an algebra. The following holds:*

1. *If* $C(\Omega, E)$ *has the M-Op (or M-UOp), then* E *has the M-Op (or M-UOp, repectively), and*
2. *If* $C(\Omega, E)$ *has the M-Op (or M-UOp) and* E *is unital, then* $C(\Omega)$ *has the M-Op (or M-UOp, respectively).*

Proof. We just notice that E is embedded as a subalgebra of $C(\Omega, E)$ via the constant functions (i.e. $T : E \longrightarrow C(\Omega, E)$ such that Tu is the constant function equal to u). As a subalgebra of an algebra with the M-Op (or M-UOp) has the M-Op (or M-UOp respectively), the first statement holds. Let e be the unit in E. We define $S : C(\Omega) \longrightarrow C(\Omega, E)$, given by $Sf = f \cdot e$. It is clear that S is an isometric algebra isomorphism. This implies the statement and completes the proof. □

Now we recall a useful result from [7] (c.f. page 331):

Theorem 3.0.11. *Let X be a normal Hausdorff space and A a closed subset of X. Let $f \in C(A)$, and let P and Q be nonempty, equicontinuous subsets of $C(X)$. Suppose that for each $p \in P$, $q \in Q$, and $x \in A$,*

$$p(x) \leq f(x) \leq q(x).$$

Then f has an extension $F \in C(X)$ such that for each $x \in X$ and each $p \in P$ and $q \in Q$,

$$p(x) \leq F(x) \leq q(x)$$

if and only if

$$r(x) = \sup\{p(x) : p \in P\} \leq \inf\{q(x) : q \in Q\} = s(x).$$

We use Theorem 3.0.11 to prove the following proposition:

Proposition 3.0.12. *Let Ω be a compact metric space and Ω_1 a closed subspace of Ω. If $C(\Omega)$ has the M-Op, then $C(\Omega_1)$ has the M-Op.*

Proof. Let $f, g \in C(\Omega_1)$ and $\epsilon > 0$. By Theorem 3.0.11, there exist extensions $\tilde{f}$ and $\tilde{g}$ such that $\tilde{f}|_{\Omega_1} = f$ and $\tilde{g}|_{\Omega_1} = g$. Since $C(\Omega, E)$ has the M-Op, there exists $\delta > 0$

such that $B(\tilde{f} \cdot \tilde{g}, \delta) \subseteq B(\tilde{f}, \epsilon) \cdot B(\tilde{g}, \epsilon)$. We claim that

$$B(f \cdot g, \delta) \subseteq B(\tilde{f}, \epsilon) \cdot B(\tilde{g}, \epsilon).$$

Let $\phi \in B(f \cdot g, \delta)$, where, for $d(\phi, f \cdot g) = \delta_0 < \delta$, we have $f \cdot g - \delta_0 \leq \phi \leq f \cdot g + \delta_0$. Let $P = \{\tilde{f} \cdot \tilde{g} - \delta\}$ and $Q = \{\tilde{f} \cdot \tilde{g} - \delta\}$. As $\tilde{f}(x) \cdot \tilde{g}(x) - \delta \leq \phi \leq \tilde{f}(x) \cdot \tilde{g}(x) + \delta$ for all $x \in \Omega_1$ (recall $\tilde{f}|_{\Omega_1} = f$ and $\tilde{g}|_{\Omega_1} = g$), we may apply Theorem 3.0.11 to see that, for the extension $\tilde{\phi}$,

$$\tilde{f}(x) \cdot \tilde{g}(x) - \delta \leq \tilde{\phi} \leq \tilde{f}(x) \cdot \tilde{g}(x) + \delta$$

(the requirement that $\sup\{p(x) : p \in P\} \leq \inf\{q(x) : q \in Q\}$ holds trivially), where $\tilde{\phi} = \tilde{f}_1 \cdot \tilde{g}_1$ with $\tilde{f}_1 \in B(\tilde{f}, \epsilon)$ and $\tilde{g}_1 \in B(\tilde{g}, \epsilon)$. Since, for all $x \in \Omega_1$, we have that

$$\tilde{\phi}(x) = \phi(x) = \tilde{f}_1(x) \cdot \tilde{g}_1(x),$$

it follows that $\tilde{f}_1|_{\Omega_1} \in B(f, \epsilon)$ and $\tilde{g}_1|_{\Omega_1} \in B(g, \epsilon)$. □

Proposition 3.0.13. *Let Ω be a compact metric space, and F a closed subalgebra of an algebra E. If $C(\Omega, E)$ has the M-Op (M-UOp, respectively), then $C(\Omega, F)$ has the M-Op (M-UOp, respectively).*

Proof. Since $C(\Omega, E)$ has the M-Op, and as F is a subspace of E, we have that $C(\Omega, F)$ is embedded in $C(\Omega, E)$ and, hence, also has the M-Op. □

CHAPTER 4

MULTIPLICATION ON SPACES OF DIFFERENTIABLE FUNCTIONS

We have seen several results about openness properties of $C([0,1],\mathbb{R})$ (e.g. multiplication is w-open but not open). A natural question is whether any of these results extend to spaces of differentiable functions. This chapter will contain several results about openness properties of multiplication in spaces of all functions from $[0,1]$ into $\mathbb{R}$ with the first n derivatives continuous. We first consider this space, denoted by $C^{(n)}[0,\,1]$, endowed with the following norm:

$$\|f\|_m = \max_{0\leq i\leq n}\max_{x\in[0,\,1]}|f^{(i)}(x)|.$$

We first review some existing results for $C^{(n)}_m[0,\,1]$, and then we extend some of these results to $C^{(n)}[0,\,1]$ under a broader class of norms. Similarly to earlier, we consider the standard pointwise multiplication of functions

$$P : C^{(n)}_m[0,\,1]\times C^{(n)}_m[0,\,1]\to C^{(n)}_m[0,\,1],$$

where $P(f,g)=f\cdot g$. From Remark 3.0.2, we see that multiplication on $(C^{(n)}[0,1],\,\|\cdot\|_m)$ is not open. We also have the following result:

Lemma 4.0.1. *P on $C^{(n)}_m[0,\,1]$ is not dense-open.*

Proof. Let $\delta>0$, and let f be the Fremlin function, $f(x)=x-\frac{1}{2}$. We show that

$$\mathcal{B}(f^2,\delta)\not\subset\overline{\mathcal{B}(f,1/4)\cdot\mathcal{B}(f,1/4)}.$$

We consider the function $f^2 + \delta/2$. We show that, for every sequence $\{f_n\}$ and $\{g_n\} \in \mathcal{B}(f, 1/4)$,

$$\lim_n f_n \cdot g_n \neq f^2 + \delta/2.$$

We observe that $\delta/2 \leq \min_{x\in[0,\,1]} \{f^2(x) + \delta/2\}$ and that, for every n, there exists x_n such that $f_n(x_n) \cdot g_n(x_n) = 0$. Hence $\|f^2 + \frac{\delta}{2} - f_n \cdot g_n\|_m \geq \frac{\delta}{2}$. This completes the proof. □

However, we recall the following theorem from [31]:

Theorem 4.0.2. *The multiplication on* $(C^{(n)}[0,1], \|\cdot\|_m)$ *is weak-open.*

It is an easy observation that openness properties are invariant under homeomorphisms. We conclude that the same results hold for $C_1^{(n)}[0,\,1]$, endowed with the norm $\|f\|_1 = \sum_{0\leq i\leq n} \max_{x\in[0,\,1]} |f^{(i)}(x)|$, since $\|f\|_m \leq \|f\|_1 \leq (n+1)\|f\|_m$, for every $f \in C^{(n)}[0,\,1]$. We extend these two cases, further, to include a large class of n-continuously differentiable, real functions on $[0,1]$ (see [23] for $n = 1$). In [23], a new norm is defined for $C^1[0,1]$. We first extend this norm to $C^{(n)}[0,1]$.

Proposition 4.0.3. *Let* D *be a compact, connected subset of* $[0,1]^{n+1}$ *and* $\|f\|_{<D>} = \sup_{r\in<D>}\{|f(r_0)| + \Sigma_{i=1}^n |f^{(i)}(r_i)|\}$ *for* $r = (r_0, r_1, r_2, ..., r_n)$ *with* $f \in C^{(n)}[0,1]$*. Then* $\|\cdot\|_{<D>}$ *is a norm on* $C^{(n)}[0,1]$ *if and only if* $\cup_{i=0}^n \pi_i(D) = [0,1]$.

Proof. We begin with the forward direction, proceeding in a similar manner to the proof provided in [23]. We prove the contrapositive. Suppose that $\cup_{i=1}^n \pi_i(D) \neq [0,1]$. We show that $\|f\|_{<D>}$ is not a norm. Since $\cup_{i=0}^n \pi_i(D) \neq [0,1]$, there exists some $x \in [0,1]$ such that $x \notin \cup_{i=1}^n \pi_i(D)$. Thus, there exists $f \in C^{(n)}[0,1]$ and a neighborhood O containing $\cup_{i=0}^n \pi_i(D)$ and $x \notin O$ such that $\|f\|_\infty \neq 0$ and $f|_O \equiv 0$. Thus, by definition of $\|f\|_{<D>}$, we have that $\|f\|_{<D>} = 0$ but $f \neq 0$. Thus, $\|f\|_{<D>}$ is not a norm.

Now we prove the reverse direction. Suppose $\cup_{i=0}^{n}\pi_i(D) = [0,1]$. First, we show that $\|f\|_{<D>} = 0$ if and only if $f \equiv 0$. Suppose that $\|f\|_{<D>} = 0$. Then $\sup_{r\in<D>}\{|f(r_0)| + \Sigma_{i=1}^{n}|f^{(i)}(r_i)|\} = 0$, where $D \subseteq \prod_{i=0}^{n}\pi_i(D)$. This implies that $f^{(i)}|_{\pi_i(D)} = 0$, with $i \in \{0,1,\ldots,n\}$. Since D is compact and connected, then $\pi_i(D)$ is a closed subinterval of [0, 1], possibly a singleton. We set $\pi_j(D) = I_j$, where I_j represents a closed interval. We first show that, given two nontrivial intervals I_p and I_q with nonempty intersection with $p < q \leq n$, then f restricted to the union $I_p \cup I_q$ is a polynomial of degree $p-1$. This is clear if the intersection of the two intervals is a non-degenerate interval. We claim that there exists $x_0 \in I_p \cap I_q$ for the following reason: Suppose that any two I_j have empty intersection. We know that $0 \in I_k$ for some k. Then $(\cup_{i\neq k}\ I_i) \cap I_k = \emptyset$. Define $m = \min\{x : x \in I_i, i \neq k\}$ (we know there is a minimum because $(\cup_{i\neq k}\ I_i) \cap I_k$ is closed as a finite union of closed sets. Since $m \notin I_k$, we have that for $y < m$, (y, m) is not in $\cup_{j=0}^{n-1}I_j$. But we have that $I_0 \cup \ldots \cup I_{n-1} = [0,1]$, which is a contradiction. Thus, we may assume that there exists $x_0 \in I_p \cap I_q$. This assumption contains both possible cases of whether the intersection is a non-degenerate interval or a singleton. Without loss of generality, we may assume that $x_0 \neq 0$ (since both intervals are non-degenerate and zero is the leftmost point in [0, 1]). Since $\|f\|_{\langle D\rangle} = 0$, $f^{(p)}|_{I_p} = 0$ implies $f|_{I_p} = \sum_{i=0}^{p-1} a_i\, x^i$, and $f^{(q)}|_{I_q} = 0$ implies $f|_{I_q} = \sum_{i=0}^{q-1} b_i x^i$. Since $f \in C^{(n)}[0,\ 1]$, the values of these two polynomials and all their derivatives, up to order n, computed at x_0, must coincide. Therefore, we have

$$
\begin{cases}
\sum_{j=0}^{p-1}(a_j - b_j)x_0^j - \sum_{k=p}^{q-1} b_k x_0^k = 0 \\
\sum_{j=1}^{p-1} j(a_j - b_j)x_0^{j-1} - \sum_{k=p}^{q-1} b_k k x_0^{k-1} = 0 \\
\vdots \\
(p-1)!(a_{p-1} - b_{p-1}) - p! b_p x_0 - \ldots - (q-1)(q-2)\cdots(q-p+1)\, b_{q-1} x_0^{q-p} = 0 \\
-p! b_p - (p+1)! b_p x_0 - \ldots - (q-1)(q-2)\cdots(q-p)\, b_{q-1}\, x_0^{q-p-1} = 0 \\
\vdots \\
-(q-1)!\, b_{q-1} = 0.
\end{cases}
$$

The solution set of this homogeneous system consists of sequences (a_i) and (b_i) such that $a_i = b_i$, for $i \in \{0, 1, \ldots, p-1\}$, and $b_i = 0$, for $i \in \{p, \ldots, q-1\}$. Note that if $q = 1$, then $f(x) = a$ for $x \in I_1$. Since $p < q$, we have that $p = 0$, and, thus, $a = 0$. This proves our original claim that f restricted to the union of the two intervals is a polynomial of degree $p - 1$, with p being the smaller of the two indices. Therefore, we conclude that f must be a polynomial on $[0, 1]$ of degree less than n. We assume that $f(x) = a_0 + a_1 x + ... + a_{n-1}x^{n-1}$. We choose points $x_i \in [0, 1]$, such that $f^{(i)}(x_i) = 0$ with $i \in \{0, 1, \ldots, n-1\}$. The system that translates this is as follows:

$$
\begin{bmatrix}
1 & x_0 & x_0^2 & \cdots & x_0^{n-1} \\
0 & 1 & 2x_1 & \cdots & (n-1)x_1^{n-2} \\
0 & 0 & 2 & \cdots & (n-1)(n-2)x_2^{n-3} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & \cdots & (n-1)!
\end{bmatrix}
\begin{bmatrix}
a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1}
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ \vdots \\ 0
\end{bmatrix}.
$$

This system has the zero solution, which implies that $f = 0$.

If $f = 0$, then $\|f\|_{<D>} = \sup_{r \in <D>}\{0 + 0 + ... + 0\} = 0$. Next, we note that $\|f\|_{<D>} \geq 0$, since $\|f\|_{<D>}$ is defined to be the supremum of absolute values. Now we need to show that $\|\alpha \cdot f\|_{<D>} = |\alpha| \cdot \|f\|_{<D>}$. Observe that

$$\begin{aligned}
\|\alpha \cdot f\|_{<D>} &= \sup_{r \in <D>} \{|\alpha \cdot f(r)| + \Sigma_{i=1}^{n} |\alpha \cdot f^{(i)}(r_i)|\} \\
&= \sup_{r \in <D>} \{|\alpha| \cdot |f(r)| + \Sigma_{i=1}^{n} |\alpha| \cdot |f^{(i)}(r_i)|\} \\
&= \sup_{r \in <D>} \{|\alpha| \cdot (|f(r)| + \Sigma_{i=1}^{n} |f^{(i)}(r_i)|)\} \\
&= |\alpha| \cdot \sup_{r \in <D>} \{|f(r)| + \Sigma_{i=1}^{n} |f^{(i)}(r_i)|\} \\
&= |\alpha| \cdot \|f\|_{<D>}.
\end{aligned}$$

Finally, we need to check that $\|f + g\|_{<D>} \leq \|f\|_{<D>} + \|g\|_{<D>}$. Observe that

$$\|f\|_{<D>} + \|g\|_{<D>} = \sup_{r \in <D>} \{|f(r)| + \Sigma_{i=1}^{n} |f^{(i)}(r_i)|\} + \sup_{r \in <D>} \{|g(r)| + \Sigma_{i=1}^{n} |g^{(i)}(r_i)|\} \tag{4.0.1}$$

and

$$\|f + g\|_{<D>} = \sup_{r \in <D>} \{|(f + g)(r)| + \Sigma_{i=1}^{n} |(f + g)^{(i)}(r_i)|\} \tag{4.0.2}$$

Note that

$$\sup_{r \in <D>} \{|(f+g)(r)| + \Sigma_{i=1}^{n} |(f+g)^{(i)}(r_i)|\} \leq \sup_{r \in <D>} \{|f(r)+g(r)| + \Sigma_{i=1}^{n} |f^{(i)}(r_i)+g^{(i)}(r_i)|\}.$$

As $|f(r)| + \Sigma_{i=1}^{n} |f^{(i)}(r_i)| \leq \sup_{r \in <D>} \{|f(r)| + \Sigma_{i=1}^{n} |f^{(i)}(r_i)|\}$ (and similarly for g), we have that

$$\begin{aligned}
|f(r) + g(r)| + \Sigma_{i=1}^{n} |f^{(i)}(r_i) + g^{(i)}(r_i)| \leq &\sup_{r \in <D>} \{|f(r)| + \Sigma_{i=1}^{n} |f^{(i)}(r_i)|\} \\
&+ \sup_{r \in <D>} \{|g(r)| + \Sigma_{i=1}^{n} |g^{(i)}(r_i)|\}.
\end{aligned}$$

As $\sup_{r \in <D>} \{|f(r) + g(r)| + \Sigma_{i=1}^{n} |f^{(i)}(r_i) + g^{(i)}(r_i)|\}$ is a least upper bound for $\|f + g\|_{<D>}$, we have that $\|f + g\|_{<D>} \leq \|f\|_{<D>} + \|g\|_{<D>}$, which shows that

$\|f\|_{<D>}$ is a norm. □

Remark 4.0.4. *Let $f \in C^1([0,1])$ with norm $\|f\|_1$. If $f^2 \in Int(B^2(f,r))$ for all $r > 0$, then either $f(x) \geq 0$ for all $x \in [0,1]$ or $f(x) \leq 0$ for all $x \in [0,1]$. The exact same proof for the forward implication in Theorem 3.0.3 works for $C^1[0,1]$.*

Remark 4.0.5. *The space $C^{(n)}[0,\,1]$ with a norm $\|\cdot\|_{\langle D\rangle}$ is denoted by $C^{(n)}_{\langle D\rangle}[0,\,1]$. If D has surjective projections along each component (i.e. $\pi_j(D) = [0,\,1]$ for every $j = 0,\dots,n$), then $\|\cdot\|_{\langle D\rangle}$ is denoted by $\|\cdot\|_1$ and the corresponding space $C^{(n)}_1[0,\,1]$. We denote by $C^{(n)}_m[0,\,1]$ the space $C^{(n)}[0,\,1]$, equipped with $\|f\|_m = \max\{\|f^{(i)}\|_\infty : i = 0,\dots,n\}$. We observe that, in general, $C^{(n)}_{\langle D\rangle}[0,\,1]$ is not complete. We just consider $n = 1$ and $D = [0,1]\times\{0\}$, and we let $f_n(x) = \frac{x}{1+nx^2}$. This sequence is Cauchy in this norm. We assume that $m > n$; then*

$$\|f_n - f_m\|_{\langle D\rangle} = \|f_n - f_m\|_\infty + |f_n'(0) - f_m'(0)|$$

Observe

$$\begin{aligned} f_n(x) - f_m(x) &= \frac{x^3\cdot(m-n)}{(1+nx^2)\cdot(1+mx^2)} \\ &\leq \frac{x}{(1+nx^2)}\cdot\frac{x^2 m}{(1+mx^2)} \leq \frac{x}{(1+nx^2)} \leq \frac{1}{2\sqrt{n}}. \end{aligned}$$

This sequence converges to the constant function equal to zero and $f_n'(0) = 1$, for every n. Therefore $\{f_n\}$ does not converge to the zero function in $C^{(n)}_{\langle D\rangle}[0,\,1]$.

Proposition 4.0.6. *$C^{(n)}_{\langle D\rangle}[0,\,1]$ is complete whenever $D = \prod_{1\leq i\leq n-1} I_i \times [0,1]$, where each I_i is an interval.*

Proof. We provide the proof for $n = 1$ (so for $C^1_{\langle D\rangle}[0,1]$), as the general case follows similarly. Suppose that $\{f_n\}$ is Cauchy on $C^1_{\langle D\rangle}[0,1]$, where $D = I_1\times[0,1]$ with

$I_1 = [a_1, b_1] \subseteq [0, 1]$. Observe that

$$\|f_n - f_m\|_{\langle D\rangle} = \|f_n - f_m\|_{x\in I_1} + \|f_n' - f_m'\|_\infty.$$

Since $\{f_n\}$ is Cauchy on $C^1_{\langle D\rangle}[0, 1]$, we must have that $\|f_n - f_m\|_{x\in I_1} \longrightarrow 0$ and $\|f_n' - f_m'\|_\infty \longrightarrow 0$. Thus, $\{f_n'\}$ is also Cauchy and has a uniform limit $\tilde{h}_1$. Similarly, $f_n(a_1) \to \tilde{a_1}$ for some $\tilde{a}$. Define $H = \tilde{a_1} + \int_{a_1}^x \tilde{h}_1(t)dt$ and $f_n(x) = f_n(a_1) + \int_{a_1}^x f_n'(t)dt$. Then

$$\begin{aligned}
\|f_n - H\|_{\langle D\rangle} &= \left\| f_n(a_1) + \int_{a_1}^x f_n'(t)dt - \left(\tilde{a} + \int_{a_1}^x \tilde{h}_1(t)dt\right)\right\| + \|f_n' - \tilde{h}_1\|_\infty \\
&\leq |f_n(a_1) - \tilde{a_1}| + \int_{a_1}^x |f_n'(t) - \tilde{h}_1(t)|dt + \|f_n' - \tilde{h}_1\|_\infty \\
&\leq b_1 \cdot \epsilon.
\end{aligned}$$

□

We now give some examples.

Example 4.0.7. *Let $n = 1$, $D_1 = [0, 1] \times \{0\}$, and $D_2 = [0, 1] \times \{1\}$. The corresponding norms on $C^{(n)}[0, 1]$ are denoted $\|\cdot\|_{\langle D_1\rangle}$ and $\|\cdot\|_{\langle D_2\rangle}$. Then $f_k(x) = x^k$ and $g_k(x) = (x - 1)^k$ (with k an integer greater than 1) show that these norms are not equivalent: We have $\|f_k\|_{\langle D_1\rangle} = 1$ and $\|f_k\|_{\langle D_2\rangle} = 1 + k$. Also $\|g_k\|_{\langle D_1\rangle} = 1 + k$ and $\|g_k\|_{\langle D_2\rangle} = 1$. There exists $\alpha > 0$ such that*

$$1 = \|f_k\|_{\langle D_1\rangle} \geq \alpha\|f_k\|_{\langle D_2\rangle} = \alpha(k + 1).$$

Sine k can grow arbitrarily large, this is impossible. Let $D = \{0\} \times [0, 1]$ and $f(x) = x$. Then $\|f\|_{\langle D\rangle} = 1$, and $\|f^2\|_{\langle D\rangle} = 2$. This shows that $\|\cdot\|_{\langle D\rangle}$ is not an algebra norm in general.

Definition 4.0.8. *(cf. [1] and [33]) If $(X, \|\cdot\|)$ is a normed space and X is an algebra, then we say that X is a quasi-normed algebra if there exists a constant $C > 0$ such that $\|x \cdot y\| \leq C\|x\| \cdot \|y\|$ for all $x, y \in X$.*

In the next lemma, we identify a collection of quasi-normed algebras.

Lemma 4.0.9. *Let D be a connected and compact subset of $[0,\, 1]^{n+1}$ such that $[0,\, 1] = \pi_0(D) \supseteq \pi_1(D) \supseteq \cdots \supseteq \pi_n(D)$. Then, for every $f, g \in C^{(n)}[0,\, 1]$, we have $\|f \cdot g\|_{\langle D\rangle} \leq (n+1)\cdot 2^n \cdot \|f\|_{\langle D\rangle} \cdot \|g\|_{\langle D\rangle}$.*

Proof. Let $f \in C^{(n)}[0,\, 1]$ be given. For $n = 0$, we have $\|f \cdot g\|_\infty \leq \|f\|_\infty \cdot \|g\|_\infty$, and, for $n = 1$, we have $\|f \cdot g\|_{\langle D\rangle} = \|f \cdot g\|_\infty + \|f \cdot g' + g' \cdot f\|_1 \leq \|f\|_{\langle D\rangle} \cdot \|g\|_{\langle D\rangle}$. For $D \subset [0,\, 1]^{n+1}$, we set $\|f^{(j)}\|_k = \max\{|f^{(j)}(x)|,\ \text{ with } x \in \pi_k(D)\}$. Then

$$
\begin{aligned}
\|f \cdot g\|_{\langle D\rangle} &= \|f \cdot g\|_0 + \|(f \cdot g)'\|_1 + \cdots + \|(f \cdot g)^{(n)}\|_n \\
&\leq \|f\|_0 \cdot \|g\|_0 + (\|f'\|_1 \cdot \|g\|_1 + \|f\|_1 \cdot \|g'\|_1) + \cdots + \sum_{k=0}^{n} \binom{n}{k} \cdot \|f^{(n-k)}\|_n \cdot \|g^{(k)}\|_n \\
&= \sum_{j=0}^{n} \sum_{k=j}^{n} \binom{k}{j} \cdot \|f^{(k-j)}\|_k \cdot \|g^{(j)}\|_k \\
&\leq \sum_{j=0}^{n} \left(\sum_{k=j}^{n} \binom{k}{j} \cdot \|f^{(k-j)}\|_k \right) \cdot \|g^{(j)}\|_j \\
&\text{(since } k \geq j, \text{ we have } ||g^{(j)}||_k \leq ||g^{(j)}||_j) \\
&\leq \sum_{j=0}^{n} \|g^{(j)}\|_j \cdot \sum_{i=0}^{n-j} \binom{i+j}{j} \cdot \|f^{(i)}\|_i \\
&\text{(by setting } i = k - j \text{ and using that } ||f^{(i)}||_{i+j} \leq ||f^{(i)}||_i) \\
&\leq (n+1) \cdot \sum_{j=0}^{n} \|g^{(j)}\|_j \cdot \sum_{j=0}^{n} \binom{n}{j} \cdot \sum_{i=0}^{n} \|f^{(i)}\|_i = (n+1) \cdot 2^n \cdot \|f\|_{\langle D\rangle} \cdot \|g\|_{\langle D\rangle},
\end{aligned}
$$

since $\binom{i+j}{j} \leq \binom{n}{j}$ and $\sum_{i=0}^{n-j} \binom{i+j}{j} \leq (n+1)\binom{n}{j}$. This completes the proof. □

Remark 4.0.10. *A challenging problem seems to be a characterization of the* *compact and connected subsets of* $[0,\ 1]^{n+1}$ *that determine a quasi-algebra norm on* $C^{(n)}[0,\ 1]$.

If D is such that $\pi_k(D) = I_k$, an interval in $[0,\ 1]$ such that $\cup_{i=1,\ldots n}\pi_k(D) = [0,\ 1]$, and f is a function with n-continuous derivatives, then we have

$$\|f\|_{\langle D\rangle} = \sum_{i=0}^{n} \|f^{(i)}\|_i \leq n\|f\|_m,$$

with $\|f^{(i)}\|_i = \max_{x\in I_i} |f^{(i)}(x)|$.

Lemma 4.0.11. $P_* : C^{(n)}_{\langle D\rangle}[0,\ 1] \times C^{(n)}_{\langle D\rangle}[0,\ 1] \to C^{(n)}_m[0,\ 1]$ *is weak-open.*

Proof. Given $\epsilon > 0$, we have

$$P_*\left(\mathcal{B}_{\langle D\rangle}(f,\epsilon) \times \mathcal{B}_{\langle D\rangle}(g,\epsilon)\right) \supset P_*\left(\mathcal{B}_m\left(f, \frac{\epsilon}{n+1}\right) \times \mathcal{B}_m\left(g, \frac{\epsilon}{n+1}\right)\right).$$

Since the multiplication in $(C^{(n)}[0,\ 1], \|\cdot\|_m)$ is weak-open, there exists $h \in C^{(n)}([0,\ 1])$ and $\delta > 0$ such that

$$\mathcal{B}_m(h,\delta) \subset P_*\left(\mathcal{B}_m\left(f, \frac{\epsilon}{n+1}\right) \times \mathcal{B}_m\left(g, \frac{\epsilon}{n+1}\right)\right) \subset P_*\left(\mathcal{B}_{\langle D\rangle}(f,\epsilon) \times \mathcal{B}_{\langle D\rangle}(g,\epsilon)\right).$$

This completes the proof of the lemma. □

Example 4.0.12. *We now give an example of a compact and connected set* D *such that the identity map* $\mathrm{id} : (C^{(n)}[0,\ 1], \|\cdot\|_{\langle D\rangle}) \to (C^{(n)}[0,\ 1], \|\cdot\|_m)$ *is not continuous. As expected, these two norms are not equivalent. Let* $0 < \delta < 1$ *and* $D = [0,\ 1] \times \{0\} \times \ldots \times \{0\}$. *Then* $\mathcal{B}_{\langle D\rangle}(h,\eta)$ *is not contained in* $\mathcal{B}_m(h,\delta)$, *for every* $\eta > 0$ *and* $h \in C^{(n)}[0,\ 1]$. *Let* k *be a positive integer, and let* $f(x) = \dfrac{1}{k+1}x^{n+k}$. *Then* $\|f\|_{\langle D\rangle} = \dfrac{1}{k+1}$, *and* $f \in \mathcal{B}_{\langle D\rangle}\left(\mathbf{\bar{0}}, \dfrac{1}{k}\right)$. *The* m*-norm of* f *is equal to*

$\max\left\{\frac{1}{k+1}, \frac{n+k}{k+1}, \cdots, \frac{(n+k)\cdot(n+k-1)\cdots(k+1)}{k+1}\right\} > 1$. *Therefore,* $f \notin \mathcal{B}_m(\overline{\mathbf{0}}, \delta)$.

Fremlin's example shows that multiplication in $C[0,\,1]$ is not open, but Wachowicz, in [31], showed that the multiplication on $C_m^{(n)}[0,\,1]$ is weak-open. In this section, we study this property for the multiplication on $C_{\langle D\rangle}^{(n)}[0,\,1]$. We recall the statement of Wachowicz's Theorem. We also refer the reader to [2].

Theorem 4.0.13. *(cf. [31], Theorem 1) The multiplication on* $C_m^{(n)}[0,\,1]$ *is weak-open.*

Adapting the proof presented in [31] shows the following theorem.

Theorem 4.0.14. *(cf. [31]) Let D be a connected and compact subset of* $[0,\,1]^{n+1}$ *such that* $[0,\,1] = \pi_0(D) \supseteq \pi_1(D) \supseteq \cdots \supseteq \pi_n(D)$*, and let* $C_{\langle D\rangle}^{(n)}[0,\,1]$ *denote the space of all n-continuously differentiable functions endowed with the* $\|\cdot\|_{\langle D\rangle}$*. Then the multiplication in* $C_{\langle D\rangle}^{(n)}[0,\,1]$ *is weak-open.*

We use the steps of the proof by Wachowicz in [31], which we outline next.

Step 1. Let f and g be functions in $C^{(n)}[0,\,1]$. Bernstein polynomials uniformly approximate the functions and their derivatives, up to order n (see [17] and references therein). Each polynomial has a decomposition into a product of irreducible factors, either quadratic polynomials or $x-\alpha$. Then a polynomial $p(x)$ can be written as $p(x) = \Pi_{i=1}^{n}(x^2+a_ix+b_i)^{k_i}\cdot\Pi_{j=1}^{m}(x-\alpha_j)^{l_j}$. If p is $\epsilon/2$-close to f, and p has multiple roots, say $(x-\alpha)^l$, then a perturbation of the roots defines an arbitrarily close polynomial with simple roots: $p_1(x) = (x-\alpha)\cdot(x-(\alpha+\epsilon_0))\cdots(x-(\alpha+(l-1)\epsilon_0))$, for a conveniently small ϵ_0, so that $\|f-p_1\|_1 < \epsilon$, for ϵ a given positive number. For simplicity of notation, we also denote by f and g the polynomials with simple zeros and disjoint non-empty zero sets ϵ-close to the original functions. We denote the zero sets of f and g by $Z(f)$ and $Z(g)$, respectively.

Step 2. Define a partition of [0, 1], $0 = x_0 < x_1 < x_2 < \cdots < x_m = 1$ such that

$$Z(f) \subset \cup_{k \in \{0,2,\cdots\}} [x_k,\, x_{k+1}] \text{ and } Z(g) \subset \cup_{k \in \{1,3,\cdots\}} [x_k,\, x_{k+1}].$$

Step 3. Extension Lemma. Let φ and h be functions in $C^{(n)}[0,\, 1]$. Let $\eta > 0$ and $x_0 \in [0,\, 1]$ such that $|\varphi^{(j)}(x_0) - h^{(j)}(x_0)| < \eta$, with $j = 0, \cdots, n$. For every $x \in [0,\, 1]$, we set

$$k(x) = h(x) + \sum_{j=0}^{n} \left(\varphi^{(j)}(x_0) - h^{(j)}(x_0)\right) \frac{(x - x_0)^j}{j!}.$$

Then for every $j \in \{0, 1, \ldots, n\}$ we have $k^{(j)}(x_0) = \varphi^{(j)}(x_0)$ and $k \in \mathcal{B}(h, e\eta)$. The function k is an extension of φ to the interval [0, 1].

Step 4. Given φ, δ-close to the product $f \cdot g$, the construction of $\tilde{f}$ and $\tilde{g}$, ϵ-close to f and g, respectively, is done as follows: Since $f|_{[0,x_1]} \neq 0$, we set $f_1 = f$ and $g_1 = \frac{\varphi}{f}$ over the interval $[x_0, x_1]$. Then, applying the Extension Lemma, there exists g_2 that extends g_1 to the interval $[x_1, x_2]$ so that $Z(g_2) \cap [x_1, x_2] = \emptyset$ and $g_2 \in \mathcal{B}(g, e\epsilon)$. Now extend f_1 to the interval $[x_1, x_2]$ by setting $f_2 = \frac{\varphi}{g_2}$ on $[x_1, x_2]$. This procedure repeats until we reach $[x_{m-1},\, x_m]$. At each step of the construction, we decrease the value of δ in order to get the extended function and the ratio in the corresponding ϵ-ball (i.e. $\mathcal{B}(f, \epsilon)$ and $\mathcal{B}(g, \epsilon)$).

The distances $\|f - f_i\|_m$ and $\|g - g_i\|_m$ are bounded by a constant c times δ. The constant c depends only on f and g. Hence, the value of δ can be adjusted so that $c\delta < \epsilon$. In this proof, it is crucial that $\| \cdot \|_m$ be a quasi-algebra norm, more precisely, that $\|f \cdot g\|_m \leq 2^n \cdot \|f\|_m \cdot \|g\|_m$.

Proof of Theorem 4.0.14. Lemma 4.0.9 asserts that $\| \cdot \|_{\langle D \rangle}$ is a quasi-algebra norm. Given $\epsilon > 0$ and $f, g \in C^{(n)}[0, 1]$, we apply Step 1 to approximate f and g with polynomials with simple zeros and disjoints zero sets. We apply Step 2 to define a partition that includes all the endpoints of the interval $\pi_i(D) = I_i = [a_i, b_i]$. Given $\varphi \in \mathcal{B}(f \cdot g, \delta)$, we define $\tilde{f}_n$ and $\tilde{g_n}$ so that $\|\tilde{f}_n - f|_{I_n}\|_{\langle D \rangle} < \epsilon$ and $\|\tilde{g_n} - g|_{I_n}\|_{\langle D \rangle} < \epsilon$. We then pursue with the construction to extend to $[a_{n-1}, a_n]$ and $[b_n, b_{n-1}]$ by applying Steps 3 and 4. Hence, we extend $\tilde{f}_n$ and $\tilde{g_n}$ to these two intervals. We continue with the procedure until we cover the entire interval $[0, 1]$. By finitely many adjustments of δ, we have that $\varphi = \tilde{f} \cdot \tilde{g}$, with these two functions within ϵ of f and g, respectively. $\square$

Openness of Multiplication on $C^{(n)}_{\langle D \rangle}[0, 1]$

In [10], Behrends considered pairs of functions $(f, g) \in C[0, 1] \times C[0, 1]$ with the property that for every $\epsilon > 0$, the product $f \cdot g$ is in the interior of the product of the balls $\mathcal{B}(f, \epsilon) \cdot \mathcal{B}(f, \epsilon)$. Pairs of continuous functions satisfying this condition are said to have the property $(*)$. This property can be rephrased in terms of local openness for the multiplication. Indeed, a pair (f, g) satisfying $(*)$ is a point of local openness for the multiplication. Characterizations of points of local openness of various non-open bilinear and multilinear maps have been obtained by Behrends in [11] and [10], see also [9] and [12]. Behrends, following an interesting approach, characterized those pairs of functions $(f, g) \in C[0, 1] \times C[0, 1]$ with the property $(*)$.

Given f and g in $C[0, 1]$, $\gamma(t) = (f(t), g(t))$ with $t \in [0, 1]$ describes a path in $\mathbb{R}^2$. The question can now be formulated as under what conditions on f and g, a continuous perturbation of the product $f \cdot g$, say $f \cdot g + p$, with p continuous such that $\|p\|_\infty < \delta$, does there exist a small perturbation of γ, $(\gamma+\tau)(t) = (f_1(t), g_1(t))$, such that $f \cdot g + p = f_1 g_1$. Equivalently, we ask the following question: Under what conditions on f and g is (f, g) a point of local openness? In [10], it was shown that the pairs (f, g), such that the corresponding path does not cross the origin, have the $(*)$ property. This is a consequence of the Implicit Function Theorem (see Lemma 2.1 in [10]). The difficulty relies on those paths that pass through the origin. In that case, there are two essentially distinct possibilities. One can be labeled as an acceptable crossing (AC) and the other an unacceptable crossing (UC). A UC is a crossing where the path crosses the origin using the first and third quadrants or using the second and fourth. All the other crossing are acceptable (i.e. AC). More precisely, $\gamma(t) = (f(t), g(t))$ has a UC if for some $t_0 \in (0, 1)$, $\gamma(t_0) = (0, 0)$, and there exists $\epsilon > 0$ such that $f(t) \cdot g(t) \geq 0$, for every $t \in (t_0 - \epsilon, t_0 + \epsilon)$, or $f(t) \cdot g(t) \leq 0$ for every $t \in (t_0 - \epsilon, t_0 + \epsilon)$. In [10], Behrends proved the following amazing result.

Theorem 4.0.15. *Consider f and g in $C[0, 1]$. Then for every $\epsilon > 0$ the product $f \cdot g$ is in the interior of $\mathcal{B}(f \cdot g, \epsilon)$ if and only if γ has only acceptable crossings.*

Our goal is to extend some of these ideas to the new class of spaces $C^{(n)}_{\langle D\rangle}[0, 1]$. We start by recalling Definition 2.0.6, or local openness for the multiplication:

Definition 4.0.16. *Let X be a quasi-normed algebra. Let x and y be two elements in X. We say that the pair (x, y) has the property $(*)$ (i.e. is a point of local openness for the multiplication) if, for every $\epsilon > 0$, $x \cdot y \in int\,(\mathcal{B}(x, \epsilon) \cdot \mathcal{B}(y, \epsilon))$.*

We consider the space $C^{(n)}_{\langle D\rangle}[0, 1]$. Then, we denote by $f \cdot g$ the product of f and g. We consider the function $H : C^{(n)}_{\langle D\rangle}[0, 1] \times C^{(n)}_{\langle D\rangle}[0, 1] \to C^{(n)}_{\langle D\rangle}[0, 1]$, given by $H(f, g) = f \cdot g$. For simplicity of notation, the symbols E and F denote, respectively, the spaces $C^{(n)}_{\langle D\rangle}[0, 1] \times C^{(n)}_{\langle D\rangle}[0, 1]$ and $C^{(n)}_{\langle D\rangle}[0, 1]$.

Given $(f_0, g_0) \in E$, we consider the operator $dH|_{(f_0,g_0)} : E \to F$, given by $dH|_{(f_0,g_0)}(f, g) = f_0 \cdot g + g_0 \cdot f$.

Proposition 4.0.17. *If D is a compact, connected subset of $[0, 1]^{n+1}$ such that $[0, 1] = \pi_0(D) \supseteq \pi_1(D) \supseteq \cdots \supseteq \pi_n(D)$, then, for every $(f_0, g_0) \in E$, $dH|_{(f_0,g_0)}$ is the Fréchet derivative of H at (f_0, g_0).*

Proof. Given ω_0 and ω_1 in $C^{(n)}_{\langle D\rangle}[0, 1]$, we have

$$\begin{aligned}
&\lim_{\|\omega\|\to 0} \frac{\|H(f_0+\omega_0, g_0+\omega_1) - H(f_0, g_0) - dH|_{(f_0,g_0)}(\omega_0, \omega_1)\|_{\langle D\rangle}}{\|(\omega_0, \omega_1)\|} \\
&\qquad = \lim_{\|\omega\|\to 0} \frac{\|(f_0+\omega_0)\cdot(g_0+\omega_1) - f_0\cdot g_0 - \omega_0\cdot g_0 - \omega_1\cdot f_0\|_{\langle D\rangle}}{\max\{\|\omega_0\|_{\langle D\rangle}, \|\omega_1\|_{\langle D\rangle}\}} \\
&\qquad = \lim_{\|\omega\|\to 0} \frac{\|\omega_0\cdot\omega_1\|_{\langle D\rangle}}{\max\{\|\omega_0\|_{\langle D\rangle}, \|\omega_1\|_{\langle D\rangle}\}}.
\end{aligned}$$

Since the D-norm is a quasi-algebra norm with constant $2^n(n+1)$, we have

$$\begin{aligned}
&\lim_{\|\omega\|\to 0} \frac{\|H(f_0+\omega_0, g_0+\omega_1) - H(f_0, g_0) - dH|_{(f_0,g_0)}(\omega_0, \omega_1)\|_{\langle D\rangle}}{\|(\omega_0, \omega_1)\|} \\
&\qquad \le \lim_{\|\omega\|\to 0} \frac{2^n(n+1)\cdot\|\omega_0\|_{\langle D\rangle}\cdot\|\omega_1\|_{\langle D\rangle}}{\|\omega_0\|_{\langle D\rangle}} \\
&\qquad \longrightarrow 0.
\end{aligned}$$

This completes the proof. □

The next proposition characterizes those pairs (f_0, g_0) that yield a surjective Fréchet derivative $dH|_{(f_0,g_0)}$. We denote the zero set of a map $f \in C^{(n)}[0, 1]$ by $\mathcal{Z}(f)$ (i.e. $\mathcal{Z}(f) = \{t \in [0, 1] : f(t) = 0\}$).

Proposition 4.0.18. *The Fréchet derivative of H, $dH|_{(f_0,g_0)}$, is surjective if and only if $\mathcal{Z}(f_0) \cap \mathcal{Z}(g_0) = \emptyset$.*

Proof. We first prove that the condition on the zero sets is necessary. Suppose there exists t_0 such that $f_0(t_0) = g_0(t_0) = 0$. Then the range of $dH|_{(f_0,g_0)}$ is contained in the space of all functions that vanish at t_0. This implies that $dH|_{(f_0,g_0)}$ is not surjective.

Conversely, we first assume that $f_0(0) \neq 0$ and $\mathcal{Z}(f_0) \cup \mathcal{Z}(g_0)$ is finite. Then there exists a (finite) partition of [0, 1], $0 = a_0 < b_0 < a_1 < b_1 < \ldots < 1$, such that $f_0|_{[a_i,\, b_i]} \neq 0$ and $g_0|_{[b_i, a_{i+1}]} \neq 0$. Let $L > 0$ such that

$$L \leq \min\{|f_0(t)|,\, |g_0(s)| :\; t \in \cup_i[a_i, b_i],\text{ and } s \in \cup_i[b_i, a_{i+1}]\}.$$

We choose $\epsilon_0 > 0$ such that $\epsilon_0 < \min\{\frac{|a_0-b_0|}{3}, \frac{|a_1-b_0|}{3}, \cdots\}$,

$$\min\{|f_0(t)| :\; t \in \cup_i(a_i - \epsilon_0,\, b_i + \epsilon_0)\} \geq \frac{L}{2},$$

and

$$\min\{|g_0(s)| :\; s \in \cup_i(b_i - \epsilon_0,\, a_{i+1} + \epsilon_0)\} \geq \frac{L}{2}.$$

For each i, we define a $C^\infty[0, 1]$ bump function $\varphi_i|_{[a_i,b_i]} \equiv 1$ and

$$\varphi_i|_{[0,\,1]\setminus(a_i-\epsilon_0,b_i+\epsilon_0)} \equiv 0.$$

Let $\varphi = \sum_i \varphi_i$. Similarly we define ψ as the sum of ψ_i, where $\psi_i|_{[b_i,\, a_{i+1}]} = 1$ and $\psi_i|_{[0,\,1]\setminus(b_i-\epsilon_0,a_{i+1}+\epsilon_0)} \equiv 0$. We notice that $\varphi + \psi$ is never equal to zero.

In order to prove surjectivity of $dH|_{(f_0,g_0)}$, we consider Z a function in F, and we need to construct $(X, Y) \in E \times E$ such that $X \cdot f_0 + Y \cdot g_0 = Z$. We set $X(t) = \frac{Z(t)\cdot\varphi(t)}{f_0(t)\cdot(\varphi(t)+\psi(t))}$ for $t \notin \mathcal{Z}(f_0)$, $X(t) = 0$ for $t \in \mathcal{Z}(f_0)$, $Y(t) = \frac{Z(t)\cdot\psi(t)}{g_0(t)\cdot(\varphi(t)+\psi(t))}$ for $t \notin \mathcal{Z}(g_0)$, and $Y(t) = 0$ for $t \in \mathcal{Z}(g_0)$. We notice that $f_0 \cdot X + g_0 \cdot Y = Z$. Moreover, if $\mathcal{Z}(f_0) = \emptyset$ (or $\mathcal{Z}(g_0) = \emptyset$), then we just set $X(t) = \frac{Z(t)}{f_0(t)}$ and $Y(t) = 0$ (respectively, $X(t) = 0$ and $Y(t) = \frac{Z(t)}{g_0(t)}$).

It remains to show that, for arbitrary zero sets with empty intersection, we can construct a partition with the property described above. We have assumed that $f_0(0) \neq 0$ (a similar argument works if we assume that $g_0(0) \neq 0$). We set $a_0 = 0$. Let $x_0 = \sup\{t : \mathcal{Z}(f_0) \cap [0, t] = \emptyset\}$. It is clear that, if $f_0(x_0) = 0$, then $g_0(x_0) \neq 0$. We choose $0 < \epsilon_0 < \frac{x_0}{2}$ such that $\mathcal{Z}(g_0) \cap [x_0 - \epsilon_0, x_0 + \epsilon_0] = \emptyset$. We set $b_0 = x_0 - \frac{\epsilon_0}{2}$. Let $y_0 = \sup\{t : \mathcal{Z}(g_0) \cap [x_0, t] = \emptyset\}$. Clearly, $g_0(y_0) = 0$, and then $f_0(y_0) \neq 0$. There exists $\epsilon_1 > 0$ such that $\epsilon_1 < \frac{y_0 - x_0}{2}$ and $\mathcal{Z}(f_0) \cap [y_0 - \epsilon_1, y_0 + \epsilon_1] = \emptyset$. We set $a_1 = y_0 - \frac{\epsilon_1}{2}$. We continue this process to find a sequence $a_0 = 0 < b_0 < a_1 < b_1 < \cdots$ with the desirable property. We claim that this sequence is finite, since, otherwise, a_i and b_i would converge to some point t in [0, 1]; then either $f_0(t) \neq 0$ or $g_0(t) \neq 0$. We assume that $f_0(t) \neq 0$. Then f_0 does not vanish in a small neighborhood of t. This neighborhood contains all the intervals $[a_i, b_i]$ and $[b_i, a_{i+1}]$, after a certain order. This is impossible because each interval $[b_i, a_{i+1}]$ must contain a zero of f_0. This contradiction shows that $a_0 = 0 < b_0 < a_1 < b_1 < \cdots < (a_n \text{ or } b_n =) \, 1$. We observe that the argument above handles the case when $\mathcal{Z}(f_0) \cup \mathcal{Z}(g_0)$ is infinite: Since the intersection of these two zero sets is empty, it allows us to define the partition as explained above, and the argument follows. This completes the proof. □

Definition 4.0.19. *We say that a subspace M of a normed space X is complemented if there exists a subspace N such that $X = M \oplus N$.*

Lemma 4.0.20. *Let D be a compact connected subset of $[0,\, 1]^{n+1}$ such that $[0,\, 1] = \pi_0(D) \supseteq \pi_1(D) \supseteq \cdots \supseteq \pi_n(D)$. Then, for every $(f_0, g_0) \in E$, such that $\mathcal{Z}(f_0) = \emptyset$ or $\mathcal{Z}(g_0) = \emptyset$, the kernel of the Fréchet derivative $dH_{|(f_0,g_0)}$ is complemented in E.*

Proof. We assume that $\mathcal{Z}(f_0) = \emptyset$. We show that $W = \ker dH|_{(f_0,g_0)}$ is a complemented subspace. To this end, we show that $P : (f, g) \longrightarrow (f, -\frac{g_0}{f_0} \cdot f)$ is a bounded projection such that $W \oplus Ran(P) = E$. It is clear that P is a projection. We show that P is bounded:

$$\begin{aligned}\left\|\left(f, -\frac{g_0}{f_0} \cdot f\right)\right\| &= \max\left\{\|f\|_{\langle D\rangle}, \left\|\frac{g_0}{f_0} \cdot f\right\|_{\langle D\rangle}\right\} \\ &\leq \max\left\{\|f\|_{\langle D\rangle}, 2^n(n+1)\left\|\frac{g_0}{f_0}\right\|_{\langle D\rangle} \cdot \|f\|_{\langle D\rangle}\right\} \\ &\leq 2^n(n+1) \cdot \max\left\{1, \left\|\frac{g_0}{f_0}\right\|_{\langle D\rangle}\right\} \cdot \max\{\|f\|_{\langle D\rangle} \cdot \|g\|_{\langle D\rangle}\}.\end{aligned}$$

Since $(f, g) - P(f, g) \in W$ and $P(f, g) \in Ran(P)$, it follows that $W \oplus Ran(P) = E$. □

Remark 4.0.21. *We observe that the proof provided for the Lemma 4.0.20 also implies that the kernel of the Fréchet derivative $dH|_{(f_0,g_0)}$ is complemented in E if either the ratio $\frac{f_0}{g_0}$ or $\frac{g_0}{f_0}$ has continuous extension to the interval $[0, 1]$.*

We now state a version of the Submersion Theorem for normed spaces, which we apply later.

Theorem 4.0.22. *(cf. [22] and [21]) Let E, F be normed spaces, $U \subset E$ open, and let $\phi \in C^{(k)}(U, F)$ with $k \geq 1$. Assume that there exist $a \in U$ and a subspace E_1 of E such that $d\phi|_a$ is an isomorphism between E_1 and F. Moreover, assume also that $E = E_1 \oplus \ker d\phi|_a$ (i.e. $\ker d\phi|_a$ is a complemented subspace in E). Then there exist $U' \subset U$, an open set containing a, $W \subset F$, an open set containing $\phi(a)$, and $\tilde{U} \subset \ker d\phi|_a$, an open set containing 0, such that the map*

$$\begin{aligned} g : & U' \longrightarrow W \times \tilde{U} \\ & x \longmapsto (\phi(x), \pi(x - a)) \end{aligned}$$

is a $C^{(k)}$*-diffeomorphism from* U' *onto* $g(U')$*, where* $\pi := E_1 \oplus \ker d\phi|_a \to \ker d\phi|_a$ *denotes the projection onto* $\ker d\phi|_a$*.*

This allows us to derive the following result.

Proposition 4.0.23. *Let* D *be a compact, connected subset of* $[0,\,1]^{n+1}$ *such that* $[0,\,1] = \pi_0(D) \supseteq \pi_1(D) \supseteq \cdots \supseteq \pi_n(D)$*. If* $(f_0, g_0) \in E \times E$ *such that* $\mathcal{Z}(f_0) = \emptyset$ *or* $\mathcal{Z}(g_0) = \emptyset$*, then* (f_0, g_0) *has the* $(*)$ *property.*

Proof. The statement follows from an application of Theorem 4.0.22. Lemma 4.0.20 and Proposition 4.0.18 imply the hypotheses of the proposition. Then H is a submersion by Theorem 4.0.22, and there exist open neighborhoods of f_0 and g_0 in $C^{(n)}_{\langle D\rangle}[0,\,1]$, U and V, and a neighborhood in $C^{(n)}_{\langle D\rangle}[0,\,1]$ of 0, W, such that W is $C^{(n)}$-diffeomorphic to a subset of $U \times V$. This implies the statement. □

We now give a necessary condition to assure that a pair of functions in $C^{(n)}_{\langle D\rangle}[0,\,1]$ has the property $(*)$.

Proposition 4.0.24. *Let* D *be a connected and compact subset of* $[0,\,1]^{n+1}$ *such that* $\pi_0(D) = [0,\,1]$*. Let* f_0 *and* g_0 *be functions in* $C^{(n)}_{\langle D\rangle}[0,\,1]$*. If* (f_0, g_0) *is a point of local openness for the multiplication, then* $\gamma(t) = (f_0(t), g_0(t))$ *has only acceptable crossings.*

Proof. We assume that γ has a positive crossing. This means that there exist $t_0 \in (0,1)$ and a small interval $(t_0 - \epsilon_0,\, t_0 + \epsilon_0)$ around t_0 such that $f_0(t)$ and $g_0(t)$ are both positive over the interval $(t_0,\, t_0 + \epsilon_0)$ and both negative over the interval $(t_0 - \epsilon_0,\, t_0)$. Therefore, $(f \cdot g)|_{(t_0-\epsilon_0,\, t_0+\epsilon_0)}$ is nonnegative. Given $\delta > 0$, the function $f \cdot g + \frac{\delta}{2}$ is in $\mathcal{B}_{\langle D\rangle}(f \cdot g, \delta)$ and strictly positive over the open interval $(t_0 - \epsilon_0,\, t_0 + \epsilon_0)$. Let $\epsilon = \frac{1}{2}\min\{|f(t_0 - \epsilon_0)|,\, |f(t_0 + \epsilon_0)|\}$. Given h in $\mathcal{B}_{\langle D\rangle}(f_0, \epsilon)$, we have $\|f_0 - h\|_\infty \leq \|f_0 - h\|_{\langle D \rangle} < \epsilon$. Therefore, h must vanish at some point in the interval $(t_0 - \epsilon_0,\, t_0 + \epsilon_0)$. This implies that every function in the product $\mathcal{B}(f_0, \epsilon) \cdot \mathcal{B}(g_0, \epsilon)$ must vanish at some point in the interval $(t_0 - \epsilon,\, t_0 + \epsilon)$. Therefore $f \cdot g + \frac{\delta}{2}$ is not in $\mathcal{B}(f_0, \epsilon) \cdot \mathcal{B}(g_0, \epsilon)$. This completes the proof. $\square$

CHAPTER 5

BINARY OPERATIONS ON SPACES OF INTEGRABLE FUNCTIONS

In this chapter, we establish openness of the maximum and minimum operations on spaces of scalar-valued integrable functions, $L^p(\Omega,\ \mathcal{A},\ \lambda),\ 1 \leq p \leq \infty$, with Ω a topological space and λ a finite measure on a σ-algebra $\mathcal{A}$ of subsets of Ω. We consider the openness property of the dense-defined standard multiplication on $L^p(\Omega),\ 1 \leq p < \infty$; we observe that multiplication on these settings is only defined on a dense set. We adapt the definition of openness for the multiplication to include dense-defined products and then prove that the multiplication on $L^p(\Omega),\ 1 \leq p < \infty$, restricted to its domain, is uniformly open. We also establish the openness for the multiplication on $L^\infty(\Omega)$. Many of these results are also in [27].

The $\vee$ and $\wedge$ operations

In [15], we investigate conditions for openness of multiplication in a variety of spaces of continuous and differentiable functions. Let Ω be a topological space and λ a finite measure on a σ-algebra of subsets of Ω. The space $L^p(\Omega)$ $(1 \leq p < \infty)$ consists of all real valued p-integrable functions defined on Ω. We define the operations maximum and minimum, which we denote by $\vee$ and $\wedge$, respectively. Given f and g in $L^p(\Omega)$, $(f \vee g)(x) = \max\{f(x), g(x)\}$, for λ-almost every $x \in \Omega$. Similarly $(f \wedge g)(x) = \min\{f(x), g(x)\}$, or λ-almost every $x \in \Omega$. If $\vee : L^p(\Omega) \times L^p(\Omega) \to L^p(\Omega)$ (or $\wedge : L^p(\Omega) \times L^p(\Omega) \to L^p(\Omega)$) is open ($\wedge : L^p(\Omega) \times L^p(\Omega) \to L^p(\Omega)$ is open) then we say that $L^p(\Omega)$ has the $\vee$-open property ($L^p(\Omega)$ has the $\wedge$-open property, respectively).

Proposition 5.0.1. *Let $1 \leq p < \infty$. $L^p(\Omega)$ has the $\vee$-uniform openness and the*

$\wedge$-*uniform openness properties.*

Proof. Given f and g in L^p and $\epsilon > 0$ we show that there exists δ such that

$$\mathcal{B}(f \vee g, \delta) \subset \mathcal{B}(f, \epsilon) \vee \mathcal{B}(g, \epsilon).$$

Let $A = \{x \in \Omega : f(x) > g(x) + \frac{\epsilon}{6}\}$ and $B = \{x \in \Omega : g(x) > f(x) + \frac{\epsilon}{6}\}$. Then let $f \vee g = f \cdot \chi_A + g \cdot \chi_B + f \vee g \cdot \chi_C$, with $C = (A \cup B)^c$. Let $h \in \mathcal{B}(f \vee g, \delta)$ with $0 < \delta < \frac{\epsilon}{6}$. Since

$$\|(h - f \vee g) \cdot \chi_A\|_p = \|(h - f) \cdot \chi_A\|_p < \delta < \frac{\epsilon}{6}$$

$$\|(h - f \vee g) \cdot \chi_B\|_p = \|(h - g) \cdot \chi_B\|_p < \delta < \frac{\epsilon}{6}$$

$$\|(h - f) \cdot \chi_C\|_p \leq \|(h - f \vee g) \cdot \chi_C\|_p + \|(f \vee g - f) \cdot \chi_C\|_p < 2\delta < \frac{\epsilon}{3}$$

and

$$\|(h - g) \cdot \chi_C\|_p \leq \|(h - f \vee g) \cdot \chi_C\|_p + \|(f \vee g - g) \cdot \chi_C\|_p < \frac{\epsilon}{3},$$

we set $f_1 = h \cdot \chi_{A \cup C} + f \cdot \chi_B$ and $g_1 = h \cdot \chi_{B \cup C} + g \cdot \chi_A$. Then we have $\|f - f_1\|_p = \|(f - h) \cdot \chi_C \ + (f - h) \cdot \chi_A\| < \frac{\epsilon}{3} + \frac{\epsilon}{6} < \frac{\epsilon}{2}$. Similarly, $\|g - g_1\|_p < \frac{\epsilon}{2}$. It is left to check that $h = f_1 \vee g_1$. We see that

$$\begin{aligned} f_1 \vee g_1 \, (x) &= (h \cdot \chi_{A \cup C} + f \cdot \chi_B) \vee (h \cdot \chi_{B \cup C} + g \cdot \chi_A) \, (x) \\ &= \begin{cases} h(x) \vee g(x) & \text{if} \quad x \in A \\ f(x) \vee h(x) & \text{if} \quad x \in B \\ h(x) & \text{if} \quad x \in C \end{cases} \\ &= h(x), \end{aligned}$$

since $g(x) > f(x) + \dfrac{\epsilon}{6}$ on B and $f(x) > g(x) + \dfrac{\epsilon}{6}$ on A. A similar argument is used to show that $L^p(\Omega)$ has the $\wedge$-open property. $\square$

We recall that $L^\infty(\Omega)$ consists of all measurable functions $f : \Omega \to \mathbb{R}$ for which there exists $M > 0$ such that $\lambda\{x \in \Omega : |f(x)| \geq M\} = 0$. Each function is a representative of an equivalence class of all functions that coincide with the given one except possibly on a set of λ-measure zero. This space is endowed with $\|f\|_\infty = \inf\{M > 0 : \ \lambda\{x \in \Omega : |f(x)| \geq M\} = 0\}$. We observe that the proof given for Proposition 5.0.1 also shows the next corollary.

Corollary 5.0.2. *$L^\infty(\Omega)$ has the $\vee$- and $\wedge$-uniform openness properties.*

We observe that the $\vee$- and $\wedge$-uniform openness properties also hold in $C(\Omega, \mathbb{R})$, which we formalize in the following proposition.

Proposition 5.0.3. *For all $F, G \in C(\Omega)$ with Ω a compact Hausdorff topological space and $r > 0$,*

$$\vee\{B(F,r), B(G,r)\} = B(\vee\{F,G\}, r) \tag{5.0.1}$$

$$\wedge\{B(F,r), B(G,r)\} = B(\wedge\{F,G\}, r). \tag{5.0.2}$$

Proof. Since the proofs for both cases are nearly identical, we will show only the second case. We begin by showing the $(\subseteq)$ containment. Let $h = \wedge\{f, g\}$, where $f \in B(F,r)$ and $g \in B(G,r)$. Then, clearly, $F(x) - r < f(x) < F(x) + r$ and $G(x) - r < g(x) < G(x) + r$. Since $h = \wedge\{f, g\}$, it follows that

$$\wedge\{F,G\}(x) - r < h(x) < \wedge\{F,G\}(x) + r. \tag{5.0.3}$$

Since $\wedge\{F, G\}$ and h are continuous on Ω and Ω is compact, we have that

$$\begin{aligned}\|h - \wedge\{F, G\}\|_\infty &= \sup_{x\in\Omega}\{|h(x) - \wedge\{F, G\}(x)|\} \\ &< r.\end{aligned}$$

Thus, we have that $h \in B(\wedge\{F, G\}, r)$.

Now we show the $(\supseteq)$ containment. Let $h \in B(\wedge\{F, G\}, r)$. Let $\alpha(x) = h(x) - \wedge\{F, G\}(x)$, and define $f(x) = F(x) + \alpha(x)$ and $g(x) = G(x) + \alpha(x)$. Since $h \in B(\wedge\{F, G\}, r)$, we have that $\|\alpha\|_\infty < r$. We claim that $f \in B(F, r)$ and $g \in B(G, r)$. Observe that

$$\begin{aligned}\|F - f\|_\infty &= \|F - F - \alpha\|_\infty \\ &= \|\alpha\|_\infty < r.\end{aligned}$$

Similarly, $\|G - g\|_\infty < r$. Thus, $f \in B(F, r)$ and $g \in B(G, r)$. It remains to show that $h = \wedge\{f, g\}$. Observe that

$$\begin{aligned}h(x) &= \alpha(x) + \wedge\{F, G\}(x) \\ &= \wedge\{F(x) + \alpha(x), G(x) + \alpha(x)\} \\ &= \wedge\{f, g\}(x).\end{aligned}$$

Thus, $h \in \wedge\{B(F, r), B(G, r)\}$. □

Multiplication in Spaces of Integrable Functions

In this section, we prove that $L^p(\Omega),\ 1 \leq p < \infty$ has what we shall define as the multiplication-almost uniform openness property. We also show that the multiplication on $L^\infty(\Omega)$ is uniformly open. A topic of our interest is to decide whether multiplication on L^p spaces ($1 \leq p < \infty$) is open. We observe that this operation is not defined for every pair of functions (i.e. the product of two p-integrable functions is not necessarily p-integrable). We explain this obstruction with an example:

Example 5.0.4. *Define* $P : L^1([0,1]) \times L^1([0,1]) \longrightarrow L^1([0,1])$*, where* $(f,g) \mapsto f \cdot g$*, and consider* $f(x) = g(x) = x^{-(1/2)} \in L^1([0,1])$*. Note that* $f(x) \cdot g(x) = x^{-1}$*. Thus,*

$$\|f \cdot g\|_1 = \left(\int_{[0,1]} x^{-1}\, dx \right) = \infty$$

but $x^{-1} \notin L^1([0,1])$*. Thus, the standard product is not necessarily defined on all of* $L^p([0,1]) \times L^p([0,1])$*. Similar examples can be constructed for* $L^p([0,1])$ *with* $1 \leq p < \infty$*.*

Certain classical spaces may have a multiplication that is not well defined for every pair of elements but is, however, defined on a dense subset of pairs. As explained before, the domain of the product $P(f,g) = f \cdot g$ on $L^p([0,1])$, with $1 \leq p < \infty$, is not $L^p([0,1]) \times L^p([0,1])$, but it clearly contains $L^p([0,1]) \times L^\infty([0,1]) \cup L^\infty([0,1]) \times L^p([0,1])$, where $L^\infty([0,1])$ denotes the set of all measurable and essentially bounded functions on $[0,1]$. This motivates the following definition:

Definition 5.0.5. *A topological space* X *is said to have the multiplication-almost (or uniform) openness property M-aOp (or M-aUOp, respectively) if and only if* $X \times X$ *contains a dense subset* A *such that multiplication, restricted to* A*, is well defined and open (or uniformly, respectively).*

We consider a product of p-integrable functions on the interval [0, 1]. We consider the dense-defined product $P(f,g) = f \cdot g$. The domain of P contains

$L^p([0,1]) \times L^\infty([0,1]) \cup L^\infty([0,1]) \times L^p([0,1])$, where $L^\infty([0,1])$ denotes the space of all essentially bounded functions. We stated the following result in [15] but did not provide the details of the proof: we do so here.

Theorem 5.0.6. *Let $1 \leq p < \infty$ and P be the multiplication on $L^p(\Omega, \mathcal{A}, \lambda)$ with domain $Dom(P)$. Then P has the M-aUOp.*

Proof. Without loss of generality we may assume that $\lambda(\Omega) = 1$. It is clear that for $t > 0$ the identity, $Id : L^p(\Omega, \mathcal{A}, \lambda) \to L^p(\Omega, \mathcal{A}, t\lambda)$ is a t-isometry, i.e. $\|f\|_p^{t\lambda} = t\|f\|_p^\lambda$.

We use a similar strategy to the one given in [5]. Specifically, we show that, for $\delta = \dfrac{\epsilon^2}{4^2}$, $B(f \cdot g, \delta) \subseteq B(f, \epsilon) \cdot B(g, \epsilon)$ for every $f, g \in L^p(\Omega)$. We assume, without loss of generality, that $\epsilon < 1$. We subdivide Ω into disjoint sets as follows:
$A_g = \{x : |g(x)| > \frac{\epsilon}{4}\}$, $A_f = \{x \notin A_g : |f(x)| > \frac{\epsilon}{4}\}$, and
$B = (A_f \cup A_g)^c = \{x : |f(x)| \text{ and } |g(x)| \leq \frac{\epsilon}{4}\}$.
Let $h \in B(f \cdot g, \delta)$. We set
$f_1 = \dfrac{h}{g} \cdot \chi_{A_g} + f \cdot \chi_{A_f} + \sqrt{|h|} \cdot \chi_B$ and $g_1 = g \cdot \chi_{A_g} + \dfrac{h}{f} \cdot \chi_{A_f} + \sqrt{|h|} \cdot e^{i\theta_h(x)} \cdot \chi_B$. We observe that many of the terms cancel, by definition of the characteristic function, so we have that

$$f_1 \cdot g_1 = h \cdot \chi_{A_g} + h \cdot \chi_{A_f} + h \cdot \chi_B = h.$$

We show that $g_1 \in B(g, \epsilon)$; the argument that $f_1 \in B(f, \epsilon)$ is similar. We have

$$\begin{aligned}
\|g - g_1\|_p^p &= \int |g - g_1|^p d\lambda \\
&= \int_{A_f} \left| \frac{h}{f} - g \right|^p d\lambda + \int_{A_g} |g - g|^p d\lambda + \int_B \left| \sqrt{|h|} \cdot e^{i\theta_h(x)} - g \right|^p d\lambda \\
&= \int_{A_f} \left| \frac{h}{f} - g \right|^p d\lambda + \int_B \left| \sqrt{|h|} \cdot e^{i\theta_h(x)} - g \right|^p d\lambda.
\end{aligned}$$

and

$$\|h - f \cdot g\|_p^p = \int_{A_f} |h - f \cdot g|^p d\lambda + \int_{A_g} |h - f \cdot g|^p d\lambda + \int_B |h - f \cdot g|^p d\lambda$$
$$\geq \int_{A_f} |h - f \cdot g|^p d\lambda + \int_B |h - f \cdot g|^p d\lambda.$$

Since $h \in B(f \cdot g, \delta)$, we have that $\|h - f \cdot g\|_p^p < \delta^p$. Thus, by the above inequality, we see that both $\int_{A_f} |h - f \cdot g|^p d\lambda < \delta^p$ and $\int_B |h - f \cdot g|^p d\lambda < \delta^p$. Thus,

$$\delta^p > \int_{A_f} |h - f \cdot g|^p d\lambda = \int_{A_f} |f|^p \left|\frac{h}{f} - g\right|^p d\lambda \geq \left(\frac{\epsilon}{4}\right)^p \int_{A_f} \left|\frac{h}{f} - g\right|^p d\lambda$$

and

$$\left(\frac{4\delta}{\epsilon}\right)^p > \int_{A_f} \left|\frac{h}{f} - g\right|^p d\lambda.$$

We also have that

$$\left(\int_B |h - f \cdot g|^p d\lambda\right)^{1/p} \geq \left(\int_B |h|^p d\lambda\right)^{1/p} - \left(\int_B |f \cdot g|^p d\lambda\right)^{1/p}$$

Now observe that

$$\left(\int_B |h|^p d\lambda\right)^{1/p} \leq \left(\int_B |h - f \cdot g|^p d\lambda\right)^{1/p} + \left(\int_B |f \cdot g|^p d\lambda\right)^{1/p} \leq \delta + \left(\frac{\epsilon}{4}\right)^2.$$

Using that $\left(\sqrt{|h|} \cdot e^{i\theta_h(x)}\right)^p \in L^2([0,1])$, an application of Holder's Inequality yields

$$\left(\int_B \left(\sqrt{|h|} \cdot e^{i\theta_h(x)}\right)^p d\lambda\right)^{1/p} \leq \left[\left(\int_B |h|^p d\lambda\right)^{1/2} \cdot \left(\int_B 1 d\lambda\right)^{1/2}\right]^{1/p} \leq \left(\int_B |h|^p d\lambda\right)^{1/2p}.$$

Hence,

$$\left(\int_B \left(\sqrt{|h|}\cdot e^{i\theta_h(x)}\right)^p d\lambda\right)^{1/p} \le \left(\int_B |h|^p d\lambda\right)^{1/(2p)} = \sqrt{\|h\cdot\chi_B\|_p} \le \sqrt{\delta + \left(\frac{\epsilon}{4}\right)^2}.$$

We have that

$$\begin{aligned}\|g-g_1\|_p^p &= \left(\left(\left(\int_B \left|\sqrt{|h|}\cdot e^{i\theta_h(x)} - g\right|^p d\lambda\right)^{1/p}\right)^p + \left(\int_{A_f} \left|\frac{h}{f} - g\right|^p d\lambda\right)\right.\\ &\le \left[\left(\int_B \left|\sqrt{|h|}\cdot e^{i\theta_h(x)}\right|^p d\lambda\right)^{1/p} + \left(\int_B |g|^p d\lambda\right)^{1/p}\right]^p + \left(\int_{A_f}\left|\frac{h}{f} - g\right|^p d\lambda\right)\\ &\le \left(\sqrt{\delta + \left(\frac{\epsilon}{4}\right)^2} + \frac{\epsilon}{4}\right)^p + \left(\frac{4\delta}{\epsilon}\right)^p.\end{aligned}$$

Therefore $\|g-g_1\|_p \le \frac{4\delta}{\epsilon} + \sqrt{\delta + \left(\frac{\epsilon}{4}\right)^2} + \frac{\epsilon}{4} < \epsilon$. This completes the proof. □

Corollary 5.0.7. $L^\infty(\Omega, \mathcal{A}, \lambda)$ *has the multiplication-uniform openness property (M-UOp)*

Proof. The proof follows the one given for Theorem 5.0.6. We just remark that $\|g-g_1\|_\infty = max\{\|g\cdot\chi_{A_f} - g_1\cdot\chi_{A_f}\|_\infty, \|g\cdot\chi_{A_g} - g_1\cdot\chi_{A_g}\|_\infty, \|g\cdot\chi_B - g_1\cdot\chi_B\|_\infty\}$. □

CHAPTER 6

CONNECTION BETWEEN OPENNESS AND DIMENSION

This chapter deals with the connection between openness of the multiplication on spaces of continuous functions and topological properties of the domain of those functions. More specifically, we begin by recalling the definition of the stable topological dimension of a space. Then we provide a brief overview of existing results on spaces of continuous functions defined on a topological space with zero topological dimension. Then, we include results that show an interconnection between openness of the multiplication on $C(\Omega, E)$, with Ω a compact Hausdorff space and E a unital algebra, and the topological structure of Ω. Many of these results are in [27]. The motivation for this topic comes from a paper by Draga and Kania (cf. [16]). We start with the definition of dimension of a topological space.

We follow Lebesgue's approach to dimension. Given two open coverings of a topological space X, $\mathcal{U}$ and $\mathcal{V}$, we say that $\mathcal{U}$ refines $\mathcal{V}$, denoted by $\mathcal{U} \prec \mathcal{V}$, if for every open set $U \in \mathcal{U}$ there exists at least one element $V \in \mathcal{V}$ such that $U \subset V$. The order of a finite open covering $\mathcal{U}$ of X is equal to the maximal number of elements in $\mathcal{U}$ that contain a point in X. More precisely, given $x \in X$, $ord_x\mathcal{U}$ is the number of elements in $\mathcal{U}$ that contain x. The order of $\mathcal{U}$ is equal to the maximum of $ord_x\mathcal{U}$, for every $x \in X$.

Definition 6.0.1. *(see [26]) If, for every finite open cover of X, there exists a finite open refinement of order less than or equal to $n+1$, then the dimension of X is less or equal to n, and we write $dim(X) \leq n$. If $dim(X) \leq n$ but $dim(X) \leq n-1$ does not hold, then $dim(X) = n$. If $dim(X) \leq n$ does not hold for every n, then $dim(X) = \infty$. By convention, $dim\emptyset = -1$. This number, $ordX$, is called the covering dimension or stable topological dimension of X.*

A similar definition to the one given above, but using cozero sets instead of coverings by open sets, yields to the strong covering dimension of X, and this is denoted by $Dim(X)$. We provide examples for some of the above terminology, and then we formulate some results about spaces of topological dimension zero.

Example 6.0.2. *Here are the dimensions of a few well-known sets:*

1. *The ternary Cantor set and a finite collection of points both have dimension zero;*

2. *The interval $I = [0, 1]$ has dimension one; and*

3. *The dimension of I^2 is two.*

Proposition 6.0.3. *(cf. [26], p.348-350 and p. 361)*
(1) A topological space X has zero topological dimension if and only if every finite open covering has a refinement that is a partition (i.e. a covering by pairwise disjoint clopen sets).
(2) In a normal topological space X, $dim(X) = 0$ if and only if $Dim(X) = 0$.
(3) If X is a compact space then $dim(X) = 0$ if and only if X is totally disconnected.
(4) A completely regular space X is strongly zero dimensional if and only if its Stone-Čech compactification (βX) is totally disconnected (e.g. $\beta\mathbb{N}$).

We shall invoke a proposition from [16] that we recall next for an easier reading.

Proposition 6.0.4. *(see Proposition 4.6 in [16]) Let X be a zero dimensional compact Hausdorff space. Then $C(X)$ (this refers to either the real or complex algebra) has the M-UOp.*

We now recall Proposition 3.0.10 (from Chapter 3), which concerns the openness property for spaces of algebra-valued continuous functions:

Proposition 6.0.5. *Let Ω be a compact Hausdorff topological space and E an algebra. The following holds:*

1. *If $C(\Omega, E)$ has the M-Op (or M-UOp) then E has the M-Op (or M-UOp, repectively), and*
2. *If $C(\Omega, E)$ has the M-Op (or M-UOp) and E is unital, then $C(\Omega)$ has the M-Op (or M-UOp, respectively).*

In [28], M. Rieffel introduced the left (right) topological stable rank of a unital Banach algebra A to be the least integer n such that the set of n-tuples of elements of A, which generate A as a left (right) ideal, is dense in the product A^n. If no such integer exists, then we say the respective rank is infinite. We shall consider commutative algebras, so we talk about topological stable rank. The next definition concerns the topological stable rank for $n = 1$.

Definition 6.0.6. *(cf. Definition 2.3 in [16]) A unital Banach algebra has topological stable rank 1 ($tsr A = 1$), when the group of all the invertible elements in A, $GL(A)$, is dense in A.*

It is easy to see that the algebra of all n square matrices with complex entries, $M_n(\mathbb{C})$, has topological rank 1. Further, if A is a unital C^* algebra, then the invertible elements of A are dense in A if and only if the invertible elements of $M_n(A)$ are dense in $M_n(A)$ (cf. Theorem 3.3 in [28]).

In a unital Banach algebra A, we say that $a \in A$ is a topological zero divisor if and only if

$$\inf\{\|x \cdot a\| + \|a \cdot x\| : x \in A, \|x\| = 1\} = 0.$$

This condition is equivalent to saying that there exists a sequence of norm 1 elements in A, $\{x_n\}$, such that $\lim_{n\to\infty} \|x_n \cdot a\| = 0$ and $\lim_{n\to\infty} \|a \cdot x_n\| = 0$. The next result can be found in [13], and we also include its proof for completeness of exposition.

Proposition 6.0.7. *Let A be a unital Banach algebra. Then the boundary of $GL(A)$ consists of topological zero divisors.*

Proof. We denote by $\partial GL(A)$ the boundary of $GL(A)$. Let $a \in \partial GL(A)$. Then $a \notin GL(A)$, since $GL(A)$ is open. Hence, there exists a sequence of invertible elements in A that converges to a, say $a_n \to a$. Then $\|a_n^{-1}\|$ is an unbounded sequence. Otherwise, if $M > 0$ is such that $\|a_n^{-1}\| \leq M$, for every $n \in \mathbb{N}$, we have

$$\|a_n^{-1} - a_m^{-1}\| \leq M^2 \|a_n - a_m\|,$$

for every n and $m \in \mathbb{N}$.

The sequence $\{a_n^{-1}\}_n$ satisfies the Cauchy condition, so it converges. If b denotes the limit of $\{a_n^{-1}\}_n$, we have that $a \cdot b = b \cdot a = 1$. This is impossible. Therefore $\|a_n^{-1}\|$ is unbounded, and we select a subsequence, $\{a_{n_k}\}$, such that, for every $k \in \mathbb{N}$, $\|a_{n_k}^{-1}\| \geq k$. We set $x_k = \|a_{n_k}^{-1}\|^{-1} \cdot a_{n_k}^{-1}$, and then we have

$$ax_k = (a - a_{n_k}) \cdot x_n + \|a_{n_k}^{-1}\|^{-1} \cdot e \to 0.$$

Similarly, we conclude that $\{x_k \cdot a\}_k$ converges to zero. This implies that a is a topological zero divisor. This completes the proof. □

The next proposition appears in [16] for unital Banach algebras. We formulate its statement and show a proof that gives a scheme to generalize the statement for a larger collection of spaces.

Proposition 6.0.8. *If Ω is a compact Hausdorff space with positive topological* *dimension, E is a unital commutative C^*-algebra, and $C(\Omega, E)$ has the M-Op, then the set of invertible elements of E is dense in E (i.e. the topological stable rank of E is equal to 1 ($tsr(E) = 1$)).*

Proof. Since $dim(\Omega) > 0$, there exists S connected with at least two points x_1 and x_2 (see Proposition 6.0.3 (3), and also [19]). We denote the unit in E by e, and, without loss of generality, we may assume that $\|e\| = 1$. We define f such that $f(x_1) = e$ and $f(x_2) \notin GL(E)$. We select $0 < \epsilon < 1$ so that we have $\mathcal{B}(f(x_2), \epsilon) \cap GL(E) = \emptyset$. We recall that the condition $tsr(E) \neq 1$ implies that the group of invertible elements in E is not dense. We apply a generalization of Tietze's extension theorem in [14] (see Dugundji's extension theorem for compact metric spaces in [18]) to ensure the existence of a continuous mapping F that extends f. Since $F(S)$ is connected, it intersects $\partial GL(E)$, which consists of topological zero divisors. Hence, there exists $s \in S$ such that $F(s)$ is a topological zero divisor of E. We notice that, if an element $a \in E$ is a topological zero divisor, then a^* is also a topological zero divisor. Moreover, the product of a topological zero divisor and an element in E is also a topological zero divisor. We consider the product $\mathcal{B}(F^*, \epsilon) \cdot \mathcal{B}(F, \epsilon)$. For every $g \in \mathcal{B}(F^*, \epsilon)$ and $h \in \mathcal{B}(F, \epsilon)$, we have that $g(x_1)$ and $h(x_1)$ are invertible elements in E. Moreover, $g(x_2)$ and $h(x_2)$ are not invertible, as $g(x_2)$ and $h(x_2) \in \mathcal{B}(f(x_2), \epsilon)$ and $\mathcal{B}(f(x_2), \epsilon) \cap GL(E) = \emptyset$. Then there exists s_0 such that $g(s_0)$ is a topological zero divisor and $g(s_0) \cdot h(s_0)$ is also a topological zero divisor. Since E is a $C^\star$-algebra, there exists a modulus 1 element in E, $\{z_n\}$, such that $g(s_0) \cdot z_n \to 0$. Then $\|g(s_0) \cdot h(s_0) \cdot z_n\| \leq \|g(s_0) \cdot z_n\| \cdot \|h(s_0)\| \to 0$, and $\|z_n \cdot g(s_0) \cdot h(s_0)\| \leq \|z_n \cdot g(s_0)\| \cdot \|h(s_0)\| \to 0$. Every function in $\mathcal{B}(F^*, \epsilon) \cdot \mathcal{B}(F, \epsilon)$ has a topological zero divisor in the range. Let $F^* \cdot F + \frac{1}{n} \cdot e$, for n large enough, be such that $\frac{1}{n} < \epsilon$. Then the spectrum of $F^* \cdot F + \frac{1}{n} \cdot e$, denoted $\sigma(F^* \cdot F + \frac{1}{n} \cdot e)$, is such that $\sigma(F^* \cdot F + \frac{1}{n} \cdot e) \subset \left[\frac{1}{n}, \infty\right)$. Since $0 \notin \left[\frac{1}{n}, \infty\right)$, we have that $F^* \cdot F + \frac{1}{n} \cdot e$

is invertible. Thus, the range does not contain any topological zero divisors. Therefore, $F^* \cdot F \notin int\,(\mathcal{B}(F^*, \epsilon) \cdot \mathcal{B}(F, \epsilon))$. Thus, $C(\Omega, E)$ does not have the M-Op. This completes the proof. □

We observe that for range spaces of $C(\Omega, E)$ with E a unital C^*-algebra, we have that $C(\Omega)$ has the M-Op, if $C(\Omega, E)$ has the M-Op. This implies that $tsr(E) = 1$. The converse is not true. Just consider $E = \mathbb{R}$. $\mathbb{R}$ has the M-UOp, but $C(\Omega, \mathbb{R})$ does not have the M-Op.

The proof given for Proposition 6.0.8 works for several other classes of spaces of vector-valued continuous functions. First we consider spaces of Lipschitz functions, $Lip(\Omega, E)$, with Ω a compact metric space and E a unital commutative C^* algebra. This space consists of all Lipschitz functions

$$Lip(\Omega, E) = \{f : \Omega \to E : \;\; f \;\text{ is Lipschitz}\}$$

endowed with any of the norms:

- $\|f\|_1 = \|f\|_\infty + L(f)$;
- $\|f\|_m = \max\{\|f\|_\infty, L(f)\}$.

These two norms are equivalent:

$$\|f\|_m \leq \|f\|_1 \leq 2\|f\|_m.$$

We recall that $L(f) = \sup_{x \neq y} \frac{\|f(x) - f(y)\|_E}{d(x,y)}$ with d denoting the metric on Ω.

We first notice that spaces of Lipschitz functions $(Lip(\Omega, E), \|\cdot\|_1)$ are Banach algebras with continuous $*$-operation. The existence of Lipschitz extensions is assured in the next theorem.

Theorem 6.0.9. *(cf. [34] p. 16) Let X be a metric space and X_0 be a subset of X.*

1. *For any $f_0 : X_0 \to \mathbb{R}$, there exists $f : X \to \mathbb{R}$ such that $f|_{X_0} = f_0$, $L(f) = L(f_0)$, and $\|f\|_\infty = \|f_0\|_\infty$.*

2. *For any $f_0 : X_0 \to \mathbb{C}$ there exists $f : X \to \mathbb{C}$ such that $f|_{X_0} = f_0$, $L(f) \leq \sqrt{2}L(f_0)$, and $\|f\|_\infty = \|f_0\|_\infty$.*

Sequence Spaces

We now consider multiplication in sequence spaces, particularly the spaces of convergent sequences (c) and ℓ^∞.

Corollary 6.0.10. *The Banach algebra of all convergent sequences c has the M-UOp, and so does the subalgebra c_0.*

Proof. We show that the one-point compactification of $\mathbb{N}$, $\mathbb{N} \cup \{p\}$, has topological dimension zero. Let $\mathcal{U}$ be a finite open covering of $\mathbb{N} \cup \{p\}$. We denote by U an open set in $\mathcal{U}$ that contains p. Then U^c is compact in $\mathbb{N}$, and, as compact subsets of $\mathbb{N}$ must be compact, we have that $\mathbb{N} \setminus U$ is finite (i.e. $\{n_1, \ldots, n_k\}$). Then we just consider the partition $\{\{n_1\}, \ldots, \{n_k\}, U\}$. This is a refinement of $\mathcal{U}$, consisting of clopen sets. Hence $dim(\mathbb{N} \cup \{p\}) = 0$ by (1) of Proposition 6.0.3. We also have that c is isometric to the space $C(\mathbb{N} \cup \{p\})$. By Proposition 6.0.4 in [16], we conclude that c has the M-UOp. Since c_0 is a closed subalgebra of c, we have that c_0 also has the M-UOp. □

We employ similar reasoning to prove that ℓ^∞ also has the M-UOp.

Corollary 6.0.11. *The Banach algebra of all bounded sequences* ℓ^∞ *has the M-UOp.*

Proof. We observe that ℓ^∞ is isometric to $C(\beta\mathbb{N})$, the space of all continuous functions defined on the Stone Čech compactification of $\mathbb{N}\,(\beta\mathbb{N})$. Proposition 3.9 in [32] formulates that $\beta\mathbb{N}$ is totally disconnected. Part (4) of Proposition 6.0.3 implies that $dim\beta\mathbb{N} = 0$. This completes the proof. □

Remark 6.0.12. *Applying Theorem 3.0.10 gives that* $c(C([0,1],\mathbb{R}))$ *does not have the M-Op. We recall that* $c(C([0,1],\mathbb{R}))$ *consists of all uniformly convergent sequences of functions in* $C([0,1],\mathbb{R})$.

CHAPTER 7

OPEN QUESTIONS

The topic of openness of binary maps brings forth several open problems. Initially, we wanted to see if we could extend w-openness to $C^{(n)}_{\langle D\rangle}[0,1]$, and we discovered that there were several results we could provide under this class of norms. However, there are still several open questions, some of which we now list:

1. One challenging problem seems to be a characterization of the compact and connected subsets of $[0,\, 1]^{n+1}$ that define a quasi-algebra norm on $C^{(n)}[0,\, 1]$.

2. Another interesting problem is how to prove the sufficiency in Theorem 4.0.24. This is almost certainly possible to show, but providing a rigorous proof is a tedious task. Part of the difficulty arises from the fact that the paths we take to avoid UCs must remain differentiable, and the strategy proposed by Behrends involves paths with sharp corners. Choosing a suitable bump function would "smooth out" the corners, but this construction seems to be an arduous undertaking.

In [10], Behrends lists some other open problems along these lines, some of which we include here:

3. It would be interesting to have a characterization of pairs of functions that are points of local openness for the multiplication in $C(\Omega, E)$, where E is an algebra.

4. Finally, Behrends mentions in [10] that it is difficult to provide conditions under which we can show that, if a pair of functions fails to be a point of local openness, the path γ has unacceptable crossings.

BIBLIOGRAPHY

[1] J.M. Almira, and U. Luther, *Inverse closedness of approximation algebras*, J. Math. Anal. Appl. 314 (2006), 30–44.

[2] M. Balcerzak, and A. Majchrzycki, *On multiplication in spaces of continuous functions*, Colloq. Math. 122 (2011), 247–253.

[3] M. Balcerzak, A. Majchrzycki, and P. Strobin, *On certain uniformly open multilinear mappings*, Banach J. Math. Anal. 10 (2016), 482–494.

[4] M. Balcerzak, A. Majchrzycki, and P. Strobin, *Uniform openness of multiplication in Banach spaces* L_p, Studia Mathematica 170 (2005), no. 2, 203-209.

[5] M. Balcerzak, A. Majchrzycki, and A. Wachowicz *Openness of multiplication in some functions spaces*, Taiwanese J. Math. 17 (20131), 1115–1126.

[6] M. Balcerzak, A. Wachowicz, and W. Wilczyński, *Multiplying balls in the space of continuous functions on* $[0,1]$, Studia Mathematica 170 (2005), no. 2, 203–209.

[7] M. Bartelt and J. Swetis, *An elementary extension of Tietze's Theorem*, Mathematics Magazine 66 (1993), no. 5, 330-332.

[8] A. Bátkai, M. Kamar Fijavž, and A. Rhandi, *Positive operator semigroups: From finite to infinite dimensions*, Birkhaüser series in Operator Theory: Advances and Applications, 257, 2017.

[9] E. Behrends, *Products of* n *open subsets in the space of continuous functions on* [0, 1], Studia Math. 204:1 (2011), 73-95.

[10] E. Behrends, *Walk the dog, or: Products of open balls in the space of continuous functions?*, Funct. Approx. 44:1 (2011), 153-164.

[11] E. Behrends, *Where is matrix multiplication locally open?*, Linear Algebra and its Applications 517 (2017), 167-176. https://doi.org/10.1016/j.laa.2016.12.014

[12] E. Behrends, *Where is pointwise multiplication in real CK-spaces locally open?*, Fund. Math. 236 (2017), 51-69.

[13] F.F.Bonsall and J.Duncan, *Complete normed algebras*, Springer Verlag, Berlin, 1973.

[14] F.Botelho and T.S.S.R.K.Rao, *A note on bi-contractive projections on spaces of vector valued continuous functions*, submitted (2018).

[15] F.Botelho and H.Renaud, *Topological properties of operations on spaces of differentiable functions*, Advances in Operator Theory 4 (2019), no. 1, 305–320.

[16] S. Draga and T. Kania, *When is multiplication in a Banach algebra open?*, Linear Algebra and its Applications 538 (2018), 149-165.

[17] J. Duchoň *A generalized Bernstein approximation theorem* Tatra Mt. Math. Publ. 49 (2011), 99–109. DOI: 10.2478/v10127-011-0029-x.

[18] J. Dugundji, *An extension of Tietze's theorem*, Pacific Journal of Mathematics 1 (1951), no. 3, 353-367.

[19] R. Engelking, *Theory of dimensions finite and infinite*, Sigma Series in Pure Math, 10. Heldermann, Lemgo, 1995.

[20] Y. Ge, J. Gu, S. Lin, and J. Zhu, *Almost-open maps, sequence-covering maps, and SN-networks*, Indian Journal of Pure Applied Math, 37 (2006), no. 2, 111–119.

[21] H. Glöckner, *Fundamentals of submersions and immersions between infinite-dimensional manifolds*, (2016) ArXiv:1502.05795v4. Helge GlÂÍockner

[22] H. Glöckner, *Implicit functions from topological vector spaces to Banach spaces*, Israel J. Math. 155 (2006), 205–252. Implicit functions from topological vector spaces to Banach spaces.

[23] K. Kawamura, H. Koshimizu, T. Miura, *Norms on* $C^1([0,1])$ *and their isometries*, Acta Sci. Math. 84 (2018), 239-261.

[24] A. Komisarski, *A connection between multiplication in* $C(X)$ *and the dimension of* X, Fundamenta Mathematicae 189 (2006), 149-154

[25] H. Elton Lacey, *The isometric theory of classical Banach spaces*, Die Grundlehren der mathematischen Wissenschaften, Band 208. Springer-Verlag, New York-Heidelberg, 1974.

[26] G. Naber, *Set theoretic topology, with emphasis on problems from the theory of coverings, zero dimensionality and cardinal invariants*, University Microfilms International, Ann Arbor, Michigan, 1977.

[27] H. Renaud, *Results on topological properties of operations on function spaces*, CONM (2019), in press.

[28] M.Rieffel, *Dimension and stable rank in the K-Theory of C*-algebras*, Proc. Lond. Math. Soc. 46:3 (1983), 301–333.

[29] W. Rudin *Functional Analysis*, McGraw-Hill series in Higher Mathematics, 1973.

[30] A. H. Shuchat, *Approximation of Vector-Valued Continuous Functions*, Proceedings of the American Mathematical Society 31:1 (1972), 97-103.

[31] A. Wachowicz, *Multiplying balls in* $C^{(N)}[0,1]$, Real Analysis Exchange 34 (2008/2009), no. 2, 445-450.

[32] R.C. Walker, *The stone-$\hat{C}$ech compactification*, Ergenbnisse der mathematik und ihrer Grenzgebiete, Band 83. Springer-Verlag, New York-Heidelberg, 1974.

[33] Z. Wang, and W. Zhang, *Isometries and isomorphisms in quasi-Banach algebras*, Acta Math. Univ. Comenianae LXXX (2011), no. 2, 277–285.

[34] N. Weaver, *Lipschitz algebras*, World Scientific Publishing Co, Singapore, 1999.

APPENDIX

Banach algebras

Remark 7.0.1. *Fremlin's example (c.f. [6]) and 3.0.6 both show that multiplication is not open in a real Banach algebra.*

Another important result on weak openness of multiplication comes from [16].

Corollary 7.0.2. *Let A be a unital Banach algebra, and let GLA be the set of all invertible elements in A. Suppose that GLA is dense in A. Then A has weakly open multiplication.*

Remark 7.0.3. *Since invertible elements are not dense in $C[0,1]$, the converse does not hold. To prove this, let $f : [0,1] \to \mathbb{R}$ be given as $f(x) = 1 - 2x$. Then for $\epsilon > 0$ (also < 1) and every $g \in C([0,\,1])$, ϵ-close to f, we have $g(0) > 0$ and $g(1) < 0$. This implies the existence of $x_0 \in (0,1)$, where g vanishes. Therefore, g is noninvertible.*

Now we cite some more results from [16]

Lemma 7.0.4. *Let A and B be Banach algebras, and let $J \subseteq A$ be a closed ideal.*

(i) If A has (uniformly, weakly) open multiplication, then so has A/J. Moreover, in both cases, the dependence of δ from ϵ in the quotient algebra is the same as in A.

(ii) If $h : A \longrightarrow B$ is a surjective homomorphism and A has (uniformly, weakly) open multiplication, then so has B.

Lemma 7.0.5. *Let* $(A_\gamma)_{\gamma\in\Gamma}$ *be a family of Banach algebras. Then multiplications in* A_γ *are equi-uniformly open if and only if* $A = (\bigoplus_{\gamma\in\Gamma} A_\gamma)_{l_\infty(\Gamma)}$ *has uniformly open multiplication. Moreover, in the latter case, multiplication in* A *is uniformly open with* δ *depending on* $\epsilon > 0$ *in the same way as in* A_γ*.*

Corollary 7.0.6. *Let* $(A_\gamma)_{\gamma\in\Gamma}$ *be a family of Banach algebras, and let* U *be an ultrafilter on the set* Γ*. If multiplications in* A_γ $(\gamma \in \Gamma)$ *are equi-uniformly open, then the ultraproduct* $\prod_{\gamma\in\Gamma}^{U} A_\gamma$ *has uniformly open multiplication.*

Proposition 7.0.7. *Suppose that* A *is a Banach algebra that contains a dense subalgebra* A_0 *such that multiplication restricted to this subalgebra is uniformly open. Then* A *has uniformly open multiplication.*

The contrapositive of the following result from [16] provides an easy way to check if multiplication in a Banach algebra is open.

Proposition 7.0.8. *Let* A *be a unital Banach algebra. Suppose that multiplication in* A *is an open mapping. Then the total stability rank of* A*, denoted tsr* A*, is* 1*.*