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ON DENSITY OF SMOOTH FUNCTIONS IN MUSIELAK ORLICZ SOBOLEVSPACES AND UNIFORM CONVEXITY OF THOSE SPACES

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by

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ABSTRACT

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We study density of smooth functions in Musialka Orlicz Sobolev $W^{k,\Phi}$ spaces as well as uniform convexity in those spaces.

We start with studying Musielak Orlicz spaces L^{Φ} and their basic properties. We improve some basic results by dropping the assumption of local integrability. Next, we characterize the density of compact supported smooth functions (C_C^{∞}) in a Musielak Orlicz spaces L^{Φ} generated by a functions Φ that are not necessarily locally integrable.

We proceed to study Musielak Orlicz Sobolev spaces $W^{k,\Phi}$. We start with characterizing those Musielak Orlicz spaces that embed into the space of locally integrable functions. We introduce the Musielak Orlicz Sobolev Space $W^{k,\Phi}$ and state some basic properties of those spaces. We connect the problem of density of C_C^{∞} in $W^{k,\Phi}$ with the problem of boundedness of the Hardy-Litllewood maximal operator $\mathcal{M}$ on L^{Φ}. We analyze the (A) conditions on the Musielak Orlicz function and compare them with other known conditions on Musielak Orlicz functions that are connected to the problem of density. We use the (A) condition to establish a fundamental convolution inequality in L^{Φ}. We use this result to prove results about approximation of compactly supported elements of $W^{k,\Phi}$ by elements of C_C^{∞}. We prove density of $C_C^{\infty}(\mathbb{R}^d)$ in $W^{k,\Phi}(\mathbb{R}^d)$. We then proceed to prove density in $W^{1,\Phi}(\Omega)$ of restrictions of compactly supported smooth functions on $\mathbb{R}^d$.

Finally we study uniform convexity of $W^{1,\Phi}$. We establish sufficient conditions for boundedness of integral operators, in particular Voltera operator on L^{Φ}. We discuss the Δ_2 condition for double phase Musielak Orlicz functions and investigate the boundedness of the Voltera operator on L^{Φ} spaces generated by those functions.

We study existence of isomorphic copies of ℓ^∞ and ℓ^1 in $W^{1,\Phi}$. Next, we provide the necessarily and sufficient conditions for $W^{1,\Phi}$ to be reflexive. We characterize uniform convexity of $W^{1,\Phi}$ for those spaces where the Voltera operator is bounded.

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CHAPTER 1

INTRODUCTION

In this thesis we are studying some density and geometrical properties of the Musielak Orlicz Sobolev (MOS) spaces. Musielak Orlicz spaces, at that point presented as an example of a modular space generated by an Orlicz function with a parameter, appeared first in the literature in 1951 in H. Nakano paper [46]. J. Musielak and W. Orlicz in 1959 in the paper [50] gave a more general definition often more suitable for applications. The name itself appeared first in the paper [52]. Musielak Orlicz spaces engendered some interest and were extensively studied during the seventies, eighties and nineties of the last century by various groups of mathematicians across the world. In particular the structural and geometrical properties of those spaces were well understood.

On the other hand, the Musielak Orlicz Sobolev spaces (MOS spaces) came to the light late in seventies. The first results about MOS spaces were established by H. Hudzik in series of paper between 1976-1979 (see [28],[29],[30], [31]). We remark that the author needed to assume some rather strange assumptions about the function Φ to establish his results, this is a theme common in the field of MOS spaces.

Parallel to the research on MO and MOS spaces the variable exponent spaces, also called the Nakano spaces, were of significant interest. In terms of MO spaces a variable exponent space $L^{p(\cdot)}$ is a space generated by a function Φ of the form $t^{p(x)}$ or $\frac{t^{p(x)}}{p(x)}$, where $p(x)$ is a measurable function such that $p(x) \geq 1$. Surprisingly, a serious investigation into Sobolev spaces based on $L^{p(\cdot)}$ began only in the nineties of the last century with the paper of O. Kovácik and J. Rákosník [43], where the authors proved some basic properties of variable exponent Sobolev spaces.

After the initial interest in the MOS spaces, the research in the area remained dormant for almost decade. Surprisingly, the reseach was reinvigorated by the interest of those spaces for their application in physics. In 1987 V.V. Zhikov [54] studied methods of variartional calculus employed in elasticity theory. There he noted that Lagrangians appearing in the theory of elasticity are, using the authors language, "generalized Orlicz functions". Energy functionals associated with them are expressed with the use of modulars associated with those Lagrangians. Moreover, the author states that the uniqueness of the Dirichlet problem associated with a particular Lagrangian is equivalent to showing the density of smooth, compactly supported functions in the "generalized Sobolev Orlicz space associated to the Lagrangian". In the language of the MOS spaces this is nothing else than density of C_C^∞ functions in $W^{k,\Phi}$.

Another application of MOS came from modeling electrorheological fluids - fluids whose viscosity changes in the presence of an electrical field. In 2000 M. Ruzicka [51] provided a model for mechanics of those fluids that employs the variable exponent Sobolev spaces. In 2002 L. Diening in his dissertation [14] expanded the theory of Ruzicka and provided, among other things, the sufficient condition on the regularity of the exponent $p(x)$ to guarantee the boundedness of Hardy-Litllewood maximal operator and thus establishing density of C_C^∞ in $W^{k,p(\cdot)}$. Those results rekindled the interest in MOS spaces and motivated other authors to study partial differential equations in the context of MOS spaces. In particular there was an effort to reproduce the results established by Diening in the context of more general MOS spaces which led to introduction of (A) conditions by P. Hästö in [23]. Those conditions play a central role in this dissertation.

This dissertation consists of four chapters. In the first chapter we introduce basic notation used throughout the dissertation. We provide some definitions considering the d-dimensional Euclidean space, its geometry and topology. We also introduce some notation for the Lebesgue measure and Lebesgue measurable functions as well as for some spaces of continuous functions.

We start the second chapter by introducing the notion of a Musielak Orlicz function Φ and a Musielak Orlicz space L^Φ. We recall basic facts about those spaces and provide their proofs for the sake of completeness. Additionally we enhance some results by dropping the assumption of local integrablility of the Musieak Orlicz functions (Theorem 2.1.8, Theorem 2.1.9).

In the section 2.2 we study the density of smooth compactly supported functions in $L^\Phi(\Omega)$, where Ω is an open subset of $\mathbb{R}^d$. We introduce the set $\operatorname{Sing}\Phi$ of singular points of Φ and use it to characterize those MO spaces for which density of smooth compactly supported functions holds true. The results of this section generalize the results know in the literature as we do not assume the local integrability of Φ (Theorem 2.2.7, Corollary 2.2.8).

In chapter 3 we introduce the Musielak Orlicz Sobolev spaces $W^{k,\Phi}(\Omega)$ and study density of compactly supported smooth functions ($C_C^\infty(\Omega)$) in it. In Section 3.1 we first study those MO spaces $L^\Phi(\Omega)$ that embed into the space of locally integrable functions $L^1_{loc}(\Omega)$. We introduce then the notion of the weak derivative of a locally integrable function f. We use the above to introduce the Musielak Orlicz Space $W^{k,\Phi}(\Omega)$. We provide some basics facts about the spaces $W^{k,\Phi}(\Omega)$ and some results that are used in further parts of the dissertation.

We start Section 3.2 by the discussion of the condition used in the literature to establish density of $C_C^\infty(\Omega)$ in $W^{k,\Phi}(\Omega)$. Here we note the connection of the problem of density to the boundedness of the Hardy-Littlewood maximal operator $\mathcal{M}$. We discuss the results in the literature that do not require the boundedness of $\mathcal{M}$ to establish density of $C_C^\infty(\Omega)$ in $W^{k,\Phi}(\Omega)$, namely the so called condition $\mathcal{M}_2$ and the condition (3.11). We proceed then to the discussion of the conditions (A0) and (A1) with the emphasis on the condition (A1). We establish the connection of condition (A1) to some good estimates on the norms of characteristic functions of small open balls $B \subset \mathbb{R}^d$ (Proposition 3.2.7). We end the section by showing that the condition (A1) is generally weaker then both the condition $\mathcal{M}_2$ and the condition (3.11)

(Corollary 3.2.13 , Proposition 3.2.14, Corollary 3.2.15).

In the section 3.3 we use the condition (A1) to establish a fundamental Lemma 3.3 and use it to prove results about approximation of compactly supported elements of $W^{k,\Phi}(\Omega)$ by elements of $C_C^\infty(\Omega)$ (Theorem 3.3.4, Corollary 3.3.5). We then proceed to establish density of smooth compactly supported functions in $W^{k,\Phi}(\mathbb{R}^d)$ (Theorem 3.3.6).

The section 3.4 is the most involved part of this dissertation. The whole section is a detailed proof of Theorem 3.4.11 where we prove the density of restrictions of elements of $C_C^\infty(\mathbb{R}^d)$ to Ω in $W^{1,\Phi}(\Omega)$. To establish this result we combine the results of section 3.3 with the method of local extension of the function Φ (Theorem 3.4.9). To our knowledge this is the first correct result of this kind in the literature.

Chapter 4 is categorically different from the previous chapters. First, it is a self contained work. Secondly, in this chapter we study some geometric properties of MOS spaces on an interval (α, β), which is qualitatively different then the work presented in previous chapters. The fact that we work in MOS space forces us to use different techniques then does used to establish similar results in other function spaces. In particular we need to guarantee the boundedness of the Voltera operator. This guaranties the existence of elements of $W^{1,\Phi}$ that enable us to determine if a particular MOS space is uniformly convex.

In section 4.2 we provide conditions sufficient for boundedness of integral operators, in particular the Voltera operator on L^Φ (Theorem 4.2.7). We also discuss when do the so called double phase MO functions Φ satisfy the Δ_2 condition and compute their conjugates (Proposition 4.2.11), as well as investigate the conditions for boundedness of the Voltera operator for MO spaces generated by those functions.

In section 4.3 we give conditions on Φ in order to $W^{1,\Phi}$ to contain an isomorphic copy of ℓ^∞. The section 4.4 contains similar results for copies of ℓ^1. In section 4.5 we provide necessary and sufficient conditions for $W^{1,\Phi}$ to be reflexive. In the section 4.6 we provide the main result of chapter 4, Theorem 4.6.5 where we provide the characterization of uniform convexity of $W^{1,\Phi}$ under the assumption of boundedness of the Voltera operator.

We end chapter 4 with section 4.7 where we the discuss superreflexivity and B-convexity in the context of $W^{1,\Phi}$. We obtain several equivalent conditions for spaces $W^{1,\Phi}$, $W^{1,\varphi}$, $W^{1,p(\cdot)}$.

1.1 Basic notation

We start by introducing basic notions and facts that will be used in setting up Musielak Orlicz and Musielak Orlicz Sobolev spaces.

The symbols $\mathbb{N}, \mathbb{R}, \mathbb{C}$ will stand for the set of natural, real and complex numbers respectively. We denote the d-dimensional euclidean space by $\mathbb{R}^d$ and its points will be denoted by Latin letters. As usual, for $(x_1, x_2, \ldots, x_d) = x \in \mathbb{R}^d$ and $(y_1, \ldots, y_d) = y \in \mathbb{R}^d$ we define the dot product $x \cdot y$ by the formula

$$x \cdot y = \sum_{n=1}^{d} x_n y_n,$$

and for $x \in \mathbb{R}^d$ we define its length $|x|$ as

$$|x| = (x \cdot x)^{1/2} = \left(\sum_{n=1}^{d} x_n^2 \right)^{1/2}.$$

For $n = 1, 2, \ldots, d$, let $-\infty < a_n < b_n < \infty$. By I_n we understand any interval of the forms (a_n, b_n), $[a_n, b_n)$, $(a_n, b_n]$, $[a_n, b_n]$. A cuboid is a set of the form

$$R = I_1 \times I_2 \times \cdots \times I_d.$$

For any point $(x_1, x_2, \ldots, x_d) = x \in \mathbb{R}^d$ and $l > 0$ we define the open cube of center

x and side-length l as

$$Q(x,l) = \left\{ (y_1, y_2, \dots, y_d) \in \mathbb{R}^d : \ |x_n - y_n| < \frac{l}{2}, \ \text{for } n = 1, 2, \dots, d \right\}.$$

Similarly, for any point $x \in \mathbb{R}^d$ and any $r > 0$ we define the open ball of center x and radius r as

$$B(x,r) = \{ y \in \mathbb{R}^d : \ |x - y| < r \}.$$

A general open ball or cube often will be given without specifying their centers and radii or side lengths. In those cases an open ball will be usually denoted by B and an open cube as Q.

We say that $\Omega \subset \mathbb{R}^d$ is open if for any $x \in \Omega$ there exists $r > 0$ such that $B(x,r) \subset \mathbb{R}^d$. On the other hand a set is said to be closed in $\mathbb{R}^d$ if it is a complement of an open set. For any subset $A \subset \mathbb{R}^d$ we define its interior A° as the largest open set U contained in A and its closure $\overline{A}$ as the smallest closed set containing A. For any set $A \subset \mathbb{R}^d$ we define its boundary as

$$\text{bd}(A) = \overline{A} \setminus A^\circ = \overline{A} \cap \overline{A^c}.$$

Clearly, any open cube and any open ball are open sets. Moreover for any $x \in \mathbb{R}^d$ and $l, r > 0$,

$$\overline{B(x,r)} = \{ y \in \mathbb{R}^d : \ |x - y| \leq r \},$$

$$\overline{Q(x,r)} = \{ (y_1, \dots, y_d) \in \mathbb{R}^d : \ |x_n - y_n| \leq \frac{l}{2}, \ \text{for } n = 1, 2, \dots, d \}.$$

In the sequel we will need the notion of the Lebesgue measure on $\mathbb{R}^d$, (see [19] for details). If $A \subset \mathbb{R}^d$ is a Lebesgue measurable set, then its Lebesgue measure will be denoted as $|A|$. Clearly, for any cuboid $R = I_1 \times \cdots \times I_d \subset \mathbb{R}^d$, where $I_1, \dots I_d$

are finite intervals, we have

$$|R| = |\overline{R}| = |I_1 \cdot \dots \cdot |I_d|,$$

where $|I_n|$ stands for the length of the interval I_n, $n = 1, \dots, d$. Similarly, for any $x \in \mathbb{R}^d$ and $l, r > 0$ we have

$$|Q(x,l)| = |\overline{Q(x,l)}| = l^d,$$

$$|B(x,r)| = |\overline{B(x,r)}| = \sigma_d r^d,$$

where

$$\sigma_d = \begin{cases} \frac{\pi^n}{n!} & d = 2n, \ n \in \mathbb{N} \\ \frac{2(n!)(4\pi)^n}{(2n+1)!} & d = 2n+1, \ n \in \mathbb{N}. \end{cases} \tag{1.1}$$

Let $A \subset \mathbb{R}^d$ be a Lebesgue measurable set. Recall that a function $f : A \to \mathbb{C}$ is called measurable if any preimage $f^{-1}[U]$ on an open set $U \subset \mathbb{C}$ is a Lebesgue measurable set. For a measurable function f its support is defined as the set

$$\text{supp}(f) = \{x \in \mathbb{R}^d : f(x) \neq 0\}.$$

The set of all Lebesgue measurable complex valued functions on A will be denoted as $L^0(A)$. A mensurable function $f : A \to \mathbb{C}$ is said to be simple if its support has a finite measure and the image $f[A]$ is a finite set. The set of all simple, complex valued functions on A is denoted as

$$S(A) = \{f \in L^0(A) : \ |\operatorname{supp} f| < \infty \text{ and } f \text{ has finitely many values}\}.$$

For any open $\Omega \subset \mathbb{R}^d$ and any $f \in L^0(\Omega)$ we define the essential support of f as

$$\operatorname{ess\,supp}(f) = \Omega \setminus \bigcup \{U \subset \Omega : U \text{ is open and } f = 0 \text{ a.e. on } U\}.$$

Notice that $\operatorname{ess\,supp}(f)$ is a closed subset of Ω.

For $f, g \in L^0(\mathbb{R}^d)$, we define the convolution $f * g$ by the formula

$$f * g(x) = \int_{\mathbb{R}^d} f(y)g(x-y)dy,$$

for $x \in \mathbb{R}^d$ for which the right hand side makes sense, which means that the integral exists.

In the sequel Ω will always stand for an open subset of $\mathbb{R}^d$.

A function f is said to be locally integrable in Ω if for any compact subset K of Ω

$$\int_K |f(x)|dx < \infty.$$

We denote the set of all locally integrable functions on Ω by $L^1_{loc}(\Omega)$.

By a restriction of a function f to Ω we understand the function $f_{|\Omega}$ defined by the formula

$$f_{|\Omega}(x) = \begin{cases} f(x) & x \in \Omega \\ 0 & x \notin \Omega. \end{cases}$$

In the sequel we will also need spaces of continuous and differentiable functions. For an open set $\Omega \subset \mathbb{R}^d$ the symbols $C(\Omega)$, $C^k(\Omega)$ stand for the set of all continuous, complex valued functions defined on Ω and the set of all k-times continuously differentiable functions defined on Ω, respectively. Recall that a function f is called smooth if it has all derivatives. The set of all smooth function defined on Ω will be denoted $C^\infty(\Omega)$. We remark that the elements of $C(\Omega)$, $C^k(\Omega)$,

$C^\infty(\Omega)$ are not necessarily bounded functions. Notice that, for $f \in C(\Omega)$,

$$\operatorname{ess\,supp}(f) = \overline{\operatorname{supp}(f)}.$$

By $C_C^\infty(\Omega)$ we denote the set of all smooth compactly supported function defined on Ω,

$$C_C^\infty(\Omega) = \{f \in C^\infty(\Omega) : \ \operatorname{ess\,supp}(f) \text{ is compact}\}.$$

As usual, a partial derivative of a function f at a point x with respect to the variable x_i will be denoted as $\partial_i f(x)$. To talk about partial derivatives of order higher then 1 we need the notion of multi-indices. A multi-index is an ordered d-tuple of nonnegative integers. If $\alpha = (\alpha_1, \ldots, \alpha_d)$ is an multi-index we set

(1) $|\alpha| = \alpha_1 + \cdots + \alpha_d$,

(2) $\alpha! = \alpha_1! \cdot \cdots \cdot \alpha_d!$,

(3) $\partial^\alpha = (\partial_1)^{\alpha_1} \ldots (\partial_d)^{\alpha_d}$.

Let $\alpha = (\alpha_1, \ldots, \alpha_d)$ and $\beta = (\beta_1, \ldots, \beta_d)$. We write $\alpha \leq \beta$ if for $i = 1, \ldots, d$, $\alpha_i \leq \beta_i$.

Now let us introduce the smooth cutoff of a function Let $s : \mathbb{R}^d \to [0,1]$, be a smooth compactly supported function such that $s(x) = 1$ for $|x| \leq 1$ ad $s(x) = 0$ for $|x| > 2$. For $R > 0$ define $s_R(x) = s\left(\frac{x}{R}\right)$. For any $f \in L^0(\mathbb{R}^d)$ and $R > 0$ we define the smooth cutoff of f as

$$f_R(x) = f(x)s_R(x).$$

Similarly, sometime we will need to smooth out a function, in order to do that we will use the standard mollifier. By the standard mollifier we understand the function $J : \mathbb{R}^d \to [0, \infty$ given by the formula

$$J(x) = Ce^{\frac{-1}{1-|x|^2}} \chi_{B(0,1)}(x),$$

where $C^{-1} = \int\limits_{\mathbb{R}^d} e^{\frac{-1}{1-|x|^2}} \chi_{B(0,1)}(x) dx$. For any $r > 0$, set $J_{(r)}(x) := \frac{1}{r^d} J\left(\frac{1}{r}x\right)$. For any locally integrable f and any $r > 0$ we define

$$f_r(x) = (f * J_{(r)})(x).$$

CHAPTER 2

MUSIELAK ORLICZ SPACES

2.1 Fundamental facts about Musielak Orlicz spaces

In this section we will define the Musielak Orlicz spaces and prove some basic facts about them. Most of the results of this section are well known, yet usually the Musielak Orlicz functions considered in literature were locally integrable. We drop this assumption and still arrive at the same conclusions, hence improve the known results.

We start by introducing some notation from the theory of function spaces.

Definition 2.1.1. *Let X be a complex linear space. A function $\rho : X \to [0, \infty]$ is called a modular if*

(1) $\rho(x) = 0$ *if and only if* $x = 0$,

(2) $\rho(zx) = \rho(x)$, *for every* $z \in \mathbb{C}$ *with* $|z| = 1$ *and any* $x \in X$

(3) $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ *for all* $x, y \in X$ *and* $\alpha, \beta \geq 0$ *such that* $\alpha + \beta = 1$.

Definition 2.1.2. *Let $A \subset \mathbb{R}^d$ be a Lebesgue measurable setand X be a linear subspace of $L^0(A)$ equipped with the norm $\|\cdot\|$. We say that it is a normed function space, if for every $f \in X$ and $g \in L^0(A)$ such that $0 \leq g \leq f$ a.e., we have $g \in X$ and $0 \leq \|g\| \leq \|f\|$. If a normed function space $(X, \|\cdot\|)$ is complete, we call it a Banach function space.*

We say that a normed function space $(X, \|\cdot\|)$ has the Fatou property provided that for every sequence $\{f_n\}_{n=1}^{\infty} \subset X$ such that $\sup_{n\in\mathbb{N}} \|f_n\| < \infty$, $\{f_n(x)\}_{n=1}^{\infty}$ is a non-decreasing sequence for a.a. $x \in A$ and $\lim_{n\to\infty} f_n = f$ a.e for some $f \in L^0(A)$ we have $f \in X$ and $\lim_{n\to\infty} \|f_n\| = \|f\|$. We abbreviate this fact as $f_n \uparrow f$.

Now we introduce the notion of the Musielak Orlicz function.

Definition 2.1.3. *Let $A \subset \mathbb{R}^d$, a function $\Phi : A \times [0, \infty) \to [0, \infty)$ is called a Musielak Orlicz function (MO function) if*

(a) *for every $x \in A$, $t \mapsto \Phi(x, t)$ is convex,*

(b) *for every $x \in A$, $\Phi(x, t) = 0$ if and only if $t = 0$,*

(c) *for every $t \in [0, \infty)$ $x \mapsto \Phi(x, t)$ is Lebesgue measurable on A.*

We say that Φ is locally integrable on A, if for every compact set $K \subset A$ and every $\lambda > 0$ $\int\limits_K \Phi(x, \lambda)dx < \infty$.

Few remarks are in place. At first, in the literature the set A is often considered to be a general measurable space and its elements are denoted by t, and at the same time, the elements of $[0, \infty)$ are denoted by x and then Φ is written as $\Phi : [0, \infty) \times A \to [0, \infty)$. Hence, in that convention the roles of x and t and their order are reversed compered to our convention. We have decided to adopt the convention of Definition (2.1.3) as this is the convention that dominates the literature concerning Musielak Orlicz Sobolev spaces. Moreover since the elements of Musielak Orlicz Sobolev spaces are functions defined on open subsets of the euclidean space $\mathbb{R}^d$ and it's points are usually denoted by x it reasonable to maintain this convention. Secondly, in the definition of general Musielak Orlicz function it is only required to for Φ to be non decreasing in the second variable. That assumption is to weak for obtaining a normed space. For this purpose we need convexity in the second variable of Φ.

Definition 2.1.4. *Let $A \subset \mathbb{R}^d$ and Φ be a MO function. We define the functional*

$$I_\Phi(f) = \int\limits_A \Phi(x, |f(x)|)dx.$$

It is not difficult to see that I_Φ is a modular on $L^0(A)$. We will use the modular to define three sets that will be of importance to us.

(1) $L_0^\Phi(A)$, the Musielak Orlicz class (MO class) associated with Φ,

$$L_0^\Phi(A) = \left\{ f \in L^0(A) : \ I_\Phi(f) < \infty \right\}.$$

(2) $L^\Phi(A)$, the Musielak Orlicz space (MO space) associated with Φ,

$$L^\Phi(A) = \left\{ f \in L^0(A) : \exists \lambda > 0 \ I_\Phi(\lambda f) < \infty \right\}.$$

(3) $E^\Phi(A)$, the set of finite elements of $L^\Phi(A)$,

$$E^\Phi(A) = \left\{ f \in L^0(A) : \forall \lambda > 0 \ I_\Phi(\lambda f) < \infty \right\}.$$

We also define the functional on $L^0(A)$,

$$\|f\|_\Phi = \inf \left\{ \lambda > 0 : I_\Phi\left(f/\lambda\right) \leq 1 \right\}.$$

We say that a sequence $\{f_n\}_{n=1}^\infty \subset L^0(A)$ converges in modular to the function $f \in L^0(A)$, if there exists $\lambda > 0$ such that $\lim\limits_{n \to \infty} I_\Phi(\lambda(f - f_n)) = 0$.

The next result is well known, we provide it with its proof for the sake of completeness.

Theorem 2.1.5. *Let A be a measurable subset of $\mathbb{R}^d$ and Φ a MO function on A. The following statements are true.*

(1) *$L^\Phi(A)$ and $E^\Phi(A)$ are linear spaces.*

(2) *$L_0^\Phi(A)$ is a convex subset of $L^0(A)$, $L^\Phi(A)$ is the smallest linear subspace of $L^0(A)$ containing $L_0^\Phi(A)$ and $E^\Phi(A)$ is the largest subspace of $L^0(A)$ contained in $L_0^\Phi(A)$.*

(3) *The space* $L^\Phi(A)$ *can be written as*

$$L^\Phi(A) = \{f \in L^0(A) : \|f\|_\Phi < \infty\}.$$

Moreover, the functional $\|\cdot\|_\Phi$ *is a norm on* $L^\Phi(A)$ *and is called the Luxemburg norm.* $L^\Phi(A)$ *equipped with the Luxemburg norm is a normed function space.*

(4) *A sequence* $\{f_n\}_{n=1}^\infty$ *is convergent in norm (Cauchy in norm) to* f *in* $L^\Phi(A)$, *if and only if for every* $\lambda > 0$ $\lim_{n\to\infty} I_\Phi(\lambda(f - f_n)) = 0$ *(for every* $\lambda, \varepsilon > 0$, *there exists* $N \in \mathbb{N}$, *such that for all* $n, m > N$ $I_\Phi(\lambda(f_m - f_n)) < \varepsilon$*).*

(5) *The Luxemburg norm* $\|\cdot\|_\Phi$ *has the Fatou property.*

(6) *The MO space* $L^\Phi(A)$ *equipped with the Luxemburg norm* $\|\cdot\|_\Phi$ *is a Banach function space and* $E^\Phi(A)$ *is a closed subspace of* $L^\Phi(A)$.

(7) *If* $\{f_n\}_{n=1}^\infty \subset L^\Phi(A)$ *converges to* 0 *in the norm or in the modular, then it converges to* 0 *in measure on sets of finite measure.*

Proof. To prove the statement (1) let us take $f, g \in L^\Phi(A)$ and $a \in \mathbb{C}$. Let λ_1 be such that $I_\Phi(\lambda_1 f) < \infty$. Then $I_\Phi\left(\frac{\lambda_1}{|a|} af\right) = I_\Phi(\lambda_1 f) < \infty$. We conclude that $af \in L^\Phi(A)$. Now let λ_2 be such that $I_\Phi(\lambda_2 g) < \infty$. By convexity of Φ we have

$$\begin{aligned} I_\Phi\left(\frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2}(f+g)\right) &= I_\Phi\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\lambda_1 f + \frac{\lambda_1}{\lambda_1+\lambda_2}\lambda_2 g\right) \\ &\leq \frac{\lambda_2}{\lambda_1+\lambda_2} I_\Phi(\lambda_1 f) + \frac{\lambda_1}{\lambda_1+\lambda_2} I_\Phi(\lambda_2 g) < \infty, \end{aligned}$$

so $f + g \in L^\Phi(A)$, which proves that $L^\Phi(A)$ is a linear space.

To prove that $E^\Phi(A)$ is a linear space take any $a, b \in \mathbb{C}$, $f, g \in E^\Phi(A)$ and

$\lambda > 0$. By convexity of Φ and definition of $E^\Phi(A)$ we have

$$I_\Phi(\lambda(af+bg)) \leq \frac{1}{2}\left(I_\Phi(2|a|\lambda f) + I_\Phi(2|b|\lambda g)\right) < \infty,$$

which proves (1).

For the proof of (2), we first show that $L_0^\Phi(A)$ is convex. Take any $f, g \in L_0^\Phi(A)$ and let $\alpha, \beta \geq 0$ be such that $\alpha + \beta = 1$. By definition of $L_0^\Phi(A)$ and convexity of Φ we have

$$I_\Phi(\alpha f + \beta g) \leq \alpha I_\Phi(f) + \beta I_\Phi(g) < \infty,$$

so $L_0^\Phi(A)$ is indeed convex.

To show that $L^\Phi(A)$ is the smallest linear subspace of $L^0(A)$ containing $L_0^\Phi(A)$ it suffices to show that any linear subspace X of $L^0(A)$ contains $L^\Phi(A)$. Let $X \subset L^0(A)$ be a linear subspace of $L_0^\Phi(A)$. For any $f \in L^\Phi(A)$, by definition of $L^\Phi(A)$ there exists $\lambda > 0$, such that $I_\Phi(\lambda f) < \infty$, and thus $\lambda f \in L_0^\Phi$. Since $L_0^\Phi \subset X$ and X is a linear space we have that $f = \frac{1}{\lambda}\lambda f \in X$, which shows that $L^\Phi(A) \subset X$.

Let us show now that $E^\Phi(A)$ is the largest linear subspace of $L^0(A)$ contained in $L_0^\Phi(A)$. Let X be a linear subspace of $L^0(A)$ such that $X \subset L_0^\Phi(A)$. Taking any $f \in X$ and $\lambda > 0$, we have $\lambda f \in X \subset L_0^\Phi(A)$, therefore $I_\Phi(\lambda f) < \infty$, so $f \in E^\Phi(A)$.

To prove (3) we start by showing that $L^\Phi(A) = \{f \in L^0(A) : \|f\|_\Phi < \infty\}$. Take any $f \in L^0(A)$, with $\|f\|_\Phi < \infty$. Then for any $\lambda > \|f\|_\Phi$ we have $I_\Phi\left(\frac{1}{\lambda}f\right) \leq 1$, and so $f \in L^\Phi(A)$. On the other hand, if $f \in L^\Phi(A)$ then there exists $\lambda > 0$ for which we have $I_\Phi(\lambda f) < \infty$. If $I_\Phi(\lambda f) \leq 1$, then $\|f\|_\phi \leq \frac{1}{\lambda} < \infty$. If $I_\Phi(\lambda f) > 1$, we argue as follows. Take any $0 \leq \lambda' \leq \lambda$ and let $\{\lambda_n\}_{n=1}^\infty$ be sequence such that for all $n \in \mathbb{N}$ $\lambda_n \leq \lambda$, and $\lim_{n\to\infty} \lambda_n = \lambda'$. By the Lebesgue Dominated Convergence Theorem

$$\lim_{n\to\infty} I_\Phi(\lambda_n f) = I_\Phi(\lambda' f),$$

and therefore $\lambda \mapsto I_\Phi(\lambda' f)$ is a continuous on $[0, \lambda]$ with respect to λ'. Since $I_\Phi(0f) = 0$ and $I_\Phi(\lambda) = 1$, by continuity of the above mapping, there exist $\lambda_0 > 0$, such that $I_\Phi(\lambda_0 f) = 1$, and so $\|f\|_\Phi \le \frac{1}{\lambda_0}$.

Clearly $\|0\|_\Phi = 0$. Assume now that $f \in L^\Phi(A)$ and $\|f\|_\Phi = 0$. Fix any $\lambda > 0$. Then for any $0 < \lambda' < 1$, by convexity of Φ

$$1 \ge \int\limits_A \Phi\left(x, \frac{\lambda}{\lambda'}|f(x)|\right) dx \ge \frac{1}{\lambda'} \int\limits_A \Phi(x, \lambda'|f(x)|) dx,$$

so for any $0 < \lambda' < 1$

$$\int\limits_A \Phi(x, \lambda|f(x)|) dx \le \lambda'.$$

Therefore $\int\limits_A \Phi(x, \lambda|f(x)|) dx = 0$ and $\Phi(x, \lambda|f(x)|) = 0$ a.e., therefore $f(x) = 0$ a.e. by the condition (b) of Definition 2.1.3. Now take $f \in L^\Phi(A)$ and any $a \in \mathbb{C}$ such that $a \neq 0$. For any $\lambda > \|f\|_\Phi$ we have

$$\int\limits_A \Phi\left(x, \frac{1}{\lambda}|f(x)|\right) dx \le 1,$$

so

$$\int\limits_A \Phi\left(x, \frac{1}{|a|\lambda}|af(x)|\right) dx \le 1.$$

Therefore $\|af\|_\Phi \le \lambda|a|$, and so $\|af\|_\Phi \le |a|\|f\|_\Phi$. On the other hand, for any $\lambda > \|af\|_\Phi$ we have

$$\int\limits_A \Phi\left(x, \frac{|a|}{\lambda}|f(x)|\right) dx \le 1.$$

Hence $\|f\|_\Phi \le \frac{\lambda}{|a|}$ and $|a|\|f\|_\Phi \le \|af\|_\Phi$.

As for the triangle inequality, take $f, g \in L^\Phi(A)$ and let $\lambda_1 > \|f\|_\Phi$ and

$\lambda_2 > \|g\|_\Phi$, we have

$$I_\Phi\left(\frac{1}{\lambda_1+\lambda_2}(f+g)\right) = I_\Phi\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{1}{\lambda_1}f + \frac{\lambda_2}{\lambda_1+\lambda_2}\frac{1}{\lambda_2}g\right) \leq$$

$$\frac{\lambda_2}{\lambda_1+\lambda_2}I_\Phi\left(\frac{1}{\lambda_1}f\right) + \frac{\lambda_1}{\lambda_1+\lambda_2}I_\Phi\left(\frac{1}{\lambda_2}g\right) \leq 1$$

and so $\|f+g\|_\Phi \leq \lambda_1+\lambda_2$, therefore $\|f+g\|_\Phi \leq \|f\|_\Phi + \|g\|_\Phi$.

Now take any $g \in L^0(A)$ and $f \in L^\Phi(A)$ such that $0 \leq g \leq f$. For any $\lambda > \|f\|_\Phi$ we have

$$I_\Phi\left(\frac{1}{\lambda}g\right) = \int\limits_A \Phi\left(x, \frac{1}{\lambda}|g(x)|\right)dx \leq \int\limits_A \Phi\left(x, \frac{1}{\lambda}|f(x)|\right)dx = I_\Phi\left(\frac{1}{\lambda}f\right) \leq 1,$$

therefore $\|f\|_\Phi \leq \lambda$ and so $\|f\|_\Phi \leq \|g\|_\Phi$, which proves that $(L^\Phi(A), \|\cdot\|_\Phi)$ is a normed function space.

To show (4) we will prove the first assertion, since the proof of the other one is analogous. Assume that $\lim\limits_{n\to\infty}\|f-f_n\|_\Phi = 0$. Since $\|\cdot\|_\Phi$ is a norm, for every $\lambda > 0$, $\lim\limits_{n\to\infty}\|\lambda(f-f_n)\|_\Phi = 0$. Take any $0 < \varepsilon < 1$, then there exist $N \in \mathbb{N}$ such that for all $n > N$,

$$\|\lambda(f-f_n)\|_\Phi < \varepsilon.$$

It implies that $I_\Phi\left(\frac{\lambda}{\varepsilon}(f-f_n)\right) \leq 1$ for $n \geq N$. Now for any $\varepsilon < \lambda' < 1$,

$$1 \geq I_\Phi\left(\frac{\lambda}{\lambda'}(f-f_n)\right) \geq \frac{1}{\lambda'}I_\Phi\left(\lambda(f-f_n)\right)$$

and so,

$$I_\Phi\left(\lambda(f-f_n)\right) \leq \lambda'.$$

Taking the infimum over $\varepsilon < \lambda' < 1$, we conclude

$$I_\Phi\left(\lambda(f - f_n)\right) \le \varepsilon,$$

therefore $\lim\limits_{n\to\infty} I_\Phi(\lambda(f - f_n)) = 0$. Conversely, if for every $\lambda > 0$, $\lim\limits_{n\to\infty} I_\Phi(\lambda(f - f_n)) = 0$, then for any $k \in \mathbb{N}$, there exist $N \in \mathbb{N}$, such that for all $n > N$,

$$I_\Phi(k(f - f_n)) \le 1.$$

Therefore $\|f - f_n\|_\Phi \le \frac{1}{k}$, and so $\lim\limits_{n\to\infty} \|f - f_n\|_\Phi = 0$.

To prove (5), take any $f \in L^0(A)$ and $\{f_n\}_{n=1}^\infty \subset L^\Phi(A)$ with $0 \le f_n \uparrow f$ and $\sup\limits_{n\in\mathbb{N}} \|f_n\|_\Phi < \infty$. Since $(L^\Phi(A), \|\cdot\|_\Phi)$ is a normed function space and $\{f_n\}_{n=1}^\infty$ is non decreasing, we have $\lim\limits_{n\to\infty} \|f_n\|_\Phi = \sup\limits_{n\in\mathbb{N}} \|f_n\|_\Phi$. Take any $\lambda > \sup\limits_{n\in\mathbb{N}} \|f_n\|_\Phi$. By Fatou's Lemma we have

$$\begin{aligned} I_\Phi\left(\frac{1}{\lambda} f\right) &= \int\limits_A \Phi\left(x, \frac{1}{\lambda} f(x)\right) dx = \int\limits_A \Phi\left(x, \frac{1}{\lambda} \lim_{n\to\infty} f_n(x)\right) dx \\ &= \int\limits_A \lim_{n\to\infty} \Phi\left(x, \frac{1}{\lambda} f_n(x)\right) dx \le \liminf_{n\to\infty} \int\limits_A \Phi\left(x, \frac{1}{\lambda} f_n(x)\right) dx \\ &= \liminf_{n\to\infty} I_\Phi\left(\frac{1}{\lambda} f_n\right) \le 1. \end{aligned}$$

Therefore $f \in L^\Phi(A)$ and $\|f\|_\Phi \le \sup\limits_{n\in\mathbb{N}} \|f_n\|_\Phi$. On the other hand, since $(L^\Phi(A), \|\cdot\|_\Phi)$ is a normed function space and for all $n \in \mathbb{N}$, we have $\|f_n\|_\Phi \le \|f\|_\Phi$. Therefore $\sup\limits_{n\in\mathbb{N}} \|f_n\|_\Phi \le \|f\|_\Phi$ which shows that $\|f\|_\Phi = \lim\limits_{n\to\infty} \|f_n\|_\Phi$.

To prove (6) at first we show that $(L^\Phi(A), \|\cdot\|_\Phi)$ is complete. We consider two cases. Let first $|A| < \infty$, and let $\{f_n\}_{n=1}^\infty$ be a Cauchy sequence in $(L^\Phi(A), \|\cdot\|_\Phi)$. For every $\varepsilon > 0$, we define the measure $\mu_\varepsilon(E) := \int\limits_E \Phi(x, \varepsilon) dx$, where $E \subset A$. Notice that, if $\mu_\varepsilon(E) = 0$, then $|E| = 0$, so the Lebesgue measure $|\cdot|$ is absolutely continuous with respect to μ_ε. Therefore, for any $\varepsilon > 0$ there exist $\delta > 0$ such that for any $E \subset A$ with $\mu_\varepsilon(E) < \delta$, we have $|A| < \varepsilon$. Take any $\lambda > 0$, $n, m \in \mathbb{N}$ and set

$$A_{n,m} = \{x \in A : \ \lambda|f_n(x) - f_m(x)| > \varepsilon\}.$$

Since $\{f_n\}_{n=1}^\infty$ is Cauchy there exists N such that for any $n, m > N$,

$$\int_A \Phi(x, \lambda|f_n(x) - f_m(x)|)dx < \delta.$$

Now

$$\mu_\varepsilon(A_{n,m}) = \int_{A_{n,m}} \Phi(x, \varepsilon)dx \leq \int_A \Phi(x, \lambda|f_n(x) - f_m(x)|)dx < \delta,$$

and so $|A_{n,m}| < \varepsilon$. Therefore $\{f_n\}_{n=1}^\infty$ is Cauchy in measure. Hence there exists $f \in L^0(A)$, such that $f_n \to f$ in measure. Then we find a subsequence $\{f_{n_k}\}_{k=1}^\infty$ converging to f almost everywhere on A. For any $\lambda > 0$, by continuity of $\Phi(x, \cdot)$ we get that $\Phi(x, \lambda|f_{n_k}(x) - f(x)|) \to 0$ for a.e. $x \in A$. Now for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n, m > N$,

$$\int_A \Phi(x, \lambda|f_n(x) - f_m(x)|)dx < \varepsilon.$$

Applying Fatou's Lemma, we get that for every $n > N$,

$$\int_A \Phi(x, \lambda|f_n(x) - f(x)|)dx \leq \liminf_{k\to\infty} \int_A \Phi(x, \lambda|f_{n_k}(x) - f_n(x)|)dx \leq \varepsilon.$$

We conclude that, for every $\lambda > 0$, $\lim\limits_{n\to\infty} \int\limits_A \Phi(x, \lambda|f_n(x) - f(x)|)dx = 0$ and so $\lim\limits_{n\to\infty} \|f_n - f\| = 0$, i.e. $L^\Phi(A)$ is complete.

Now assume that $|A| = \infty$, then there exists a sequence of sets of finite measure $\{A_i\}_{i=1}^\infty$, such that for every $i \in \mathbb{N}$ $A_i \subset A_{i+1}$ and $\bigcup\limits_{i=1}^\infty A_i = A$. For each $i \in \mathbb{N}$ consider the sequence $\{f_{n|A_i}\}_{n=1}^\infty$, where $f_{n|A_i}$ is the restriction of the function f_n to the set A_i. For each $i \in \mathbb{N}$ arguing as in the first case, we get that there exist a measurable function $f_i \in L^0(A_i)$, such that for every $i \in \mathbb{N}$ $\{f_{n|A_i}\}_{n=1}^{\infty}$ converges to f_i in measure on A_i. Notice now that for $i, j \in \mathbb{N}$ such that $j > i$ we have that $f_{j|A_i} = f_i$ a.e on A_i. Indeed, since $A_i \subset A_j$ we have that $f_{n|A_j}(x) = f_{n|A_i}(x) = f_n(x)$ for $x \in A_i$ and so, for any $\varepsilon > 0$,

$$\begin{aligned}\{x \in A_i : |f_i(x) - f_j(x)| > \varepsilon\} &\subset \{x \in A_i : |f_i(x) - f_n(x)| > \varepsilon/2\} \\ &\quad \cup \{x \in A_i : |f_n(x) - f_j(x)| > \varepsilon/2\} \\ &\subset \{x \in A_i : |f_i(x) - f_{n|A_i}(x)| > \varepsilon/2\} \\ &\quad \cup \{x \in A_j : |f_{n|A_j}(x) - f_j(x)| > \varepsilon/2\}.\end{aligned}$$

Since $f_{n|A_i}$ converges to f_i in measure on A_i and $f_{n|A_j}$ converges to f_j in measure on A_j we conclude that for any $\varepsilon > 0$, $|\{x \in A_i : |f_i(x) - f_j(x)| > \varepsilon\}| = 0$ and so $f_{j|A_i} = f_i$ a.e on A_i. For any $i < j \in \mathbb{N}$ let $E_{i,j} \subset A_i$ be such that $|E_{i,j}| = 0$ and $f_{j|A_i}(x) = f_i(x)$ for $x \in A_i \setminus E_{i,j}$ and let $E = \bigcup_{i,j \in \mathbb{N}, i<j} E_{i,j}$, clearly $|E| = 0$ as a countable union of sets of measure 0. For each $i \in \mathbb{N}$ the function f_i is defined on A_i so we extend it by 0 to the whole A, no we define a function $f : A \to \mathbb{C}$ by the formula

$$f(x) = \begin{cases} \lim_{i\to\infty} f_i(x) & x \in A \setminus E \\ 0 & x \in E. \end{cases}$$

The function f is well defined, since for every $x \in A \setminus E$ there exists $i \in \mathbb{N}$ such that $x \in A_i \setminus E$ and $f_j(x) = f_i(x)$ for $j > i$. By construction of function f we have that, for any $i \in \mathbb{N}$ the sequence $\{f_n\}_{n=1}^{\infty}$ converges to f in measure on A_i.

Fix $i \in N$, then there exist a subsequance $\{f_{n_j}\}$ that converges almost everywhere to f on A_i. By Fatou's lemma, for ever $n \in \mathbb{N}$ we have

$$\int\limits_{A_i} \Phi(x, \lambda|f_n(x) - f(x)|)dx = \int\limits_{A_i} \Phi(x, \lambda|f_n(x) - \liminf_{j\to\infty} f_{n_j}(x)|)dx =$$

$$\int\limits_{A_i} \liminf_{j\to\infty} \Phi(x, \lambda|f_n(x) - f_{n_j}(x)|)dx \le \liminf_{j\to\infty} \int\limits_{A_i} \Phi(x, \lambda|f_n(x) - f_{n_j}(x)|)dx \le$$

$$\liminf_{j\to\infty} \int\limits_{A} \Phi(x, \lambda|f_n(x) - f_{n_j}(x)|)dx.$$

Take any $\varepsilon > 0$, since $\{f_n\}$ is Cauchy, there exists $N \in \mathbb{N}$, such that for every $n, m > N$

$$\int\limits_{A} \Phi(x, \lambda|f_n(x) - f_m(x)|)dx < \varepsilon,$$

therefore, for $n > N$, we have

$$\int\limits_{A_i} \Phi(x, \lambda|f_n(x) - f(x)|)dx \le \liminf_{j\to\infty} \int\limits_{A} \Phi(x, \lambda|f_n(x) - f_{n_j}(x)|)dx \le \varepsilon$$

independent of i. Once again applying Fatou's lemma, we get that for $n > \mathbb{N}$

$$\int\limits_{A} \Phi(x, \lambda|f_n(x) - f(x)|)dx \le \liminf_{i\to\infty} \int\limits_{A_i} \Phi(x, \lambda|f_n(x) - f(x)|)dx \le \varepsilon,$$

since ε was chosen arbitrary, we conclude that $\lim\limits_{n\to\infty} \|f_n - f\|_\Phi = 0$.

The last thing to check is that $E^\Phi(A)$ is a closed subspace of $L^\Phi(A)$. Take a sequence $\{f_n\}_{n=1}^\infty \subset E^\Phi(A)$ and let $f \in L^\Phi(A)$ be such that $\lim\limits_{n\to\infty} \|f_n - f\|_\Phi = 0$. Taking any $\lambda > 0$, we arrive at

$$\int_A \Phi(x, \lambda|f(x)|)dx \le \frac{1}{2} \int_A \Phi(x, 2\lambda|f_n(x)|)dx + \frac{1}{2} \int_A \Phi(x, 2\lambda|f(x) - f_n(x)|)dx.$$

Since $f_n \in E^\Phi(A)$, the first term on the right hand side of the inequality is always finite, the second term tends to 0, therefore, there exists $n \in N$ such that it is finite. We conclude that $f \in E^{\Phi}(A)$.

As for the proof of (7), notice that in the course of the proof of (6) we showed that, if a sequence $\{f_n\}_{n=1}^{\infty}$ is Cauchy in $\|\cdot\|_{\Phi}$, then it is Cauchy in measure on sets of finite measure. Similar argument shows that, if $\{f_n\}_{n=1}^{\infty}$ converges in norm to 0, then it converges to 0 in measure on sets of finite measure. The argument for modular convergence is almost the same. □

From Theorem 2.1.5 (2) it follows that the Luxemburg norm $\|\cdot\|_{\Phi}$ is the Minkowski functional of the convex set $\{f \in L^0(A) : I_{\Phi}(f) \leq 1\}$ on the set $L^0(A)$ (see [42]). The name of the norm in the theory of Orlicz spaces and their generalizations comes from the doctoral thesis of W.J.A. Luxemburg [48].

In the sequel we will need the following technical lemma about the partion of the spatial domain of a MO function. We remark that this result is true for every σ-finite measure on any *sigma*-finite algebra of sets.

Lemma 2.1.6. *[39, p. 64] Let $A \subset \mathbb{R}^d$ be measurable and Φ be a MO function on A. There exists a sequence of pairwise disjoint set $\{A_n\}_{n=1}^{\infty}$ such that $|A \setminus \bigcup_{n=1}^{\infty} A_n| = 0$, for each $n \in \mathbb{N}$ $|A_n| < \infty$, and for any $n \in \mathbb{N}$ and any $t \geq 0$,*

$$\sup_{x \in A_n} \Phi(x,t) < \infty.$$

Proof. First we will construct a family of sets of finite measure $\{A_{n,j}\}_{n,j=1}^{\infty}$ such that for any $n, j \in \mathbb{N}$ and any $t > 0$

$$\sup_{x \in A_{n,j}} \Phi(x,t) < \infty,$$

and for any $J \in \mathbb{N}$, $\left| A \setminus \left(\bigcup_{j=1}^{J} \bigcup_{n=1}^{\infty} B_{j,n} \right) \right| < \dfrac{1}{J}$.

Let $\{B_n\}_{n=1}^{\infty}$ be a family of disjoint sets of finite measure such that $\bigcup\limits_{n=1}^{\infty} B_n = A$. For each $n, m, k \in \mathbb{N}$ set

$$B_{n,m}^k = \{x \in B_n : \ \Phi(x,k) \leq m\}.$$

Clearly $\bigcup\limits_{m=1}^{\infty} B_{n,m}^k = B_n$. Therefore, there exists a subsequence $\{m_k\}_{k=1}^{\infty}$ such that

$$\left|B_n \setminus B_{n,m_k}^k\right| < \frac{1}{2^n}\frac{1}{2^k}.$$

Define $A_{1,n} = \bigcap\limits_{k=1}^{\infty} B_{n,m_k}^k$, then we have

$$|B_n \setminus A_{1,n}| \leq \sum_{k=1}^{\infty} \left|A_n \setminus B_{n,m_k}^k\right| < \frac{1}{2^n}.$$

Take any $x \in A_{1,n}$, $t > 0$ and let $k \in \mathbb{N}$ be such that $t < k$, we have that $\Phi(x,t) \leq \Phi(x,k) \leq m_k < \infty$. Therefore we have $\sup\limits_{x \in \Omega_{1,n}} \Phi(x,t) < \infty$. Finally,

$$\left|A \setminus \bigcup_{n=1}^{\infty} A_{1,n}\right| = \left|\bigcup_{n=1}^{\infty} B_n \setminus A_{1,n}\right| < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Now assume that for some $J \in \mathbb{N}$ we have constructed a family of disjoint sets $\{A_{j,n}\}_{n=1}^{\infty}$ such that for any $1 \leq j \leq J$ and $n \in \mathbb{N}$ we have $|A_{j,n}| < \infty$, $\sup\limits_{x \in A_{n,j}} \Phi(x,t) < \infty$ and $\left|A \setminus \left(\bigcup\limits_{j=1}^{J} \bigcup\limits_{n=1}^{\infty} A_{j,n}\right)\right| < \frac{1}{J}$.
We will construct the family $\{A_{J+1,n}\}_{n=1}^{\infty}$. Let

$$A' = A \setminus \left(\bigcup_{j=1}^{J} \bigcup_{n=1}^{\infty} A_{j,n}\right), \ A'_n = A' \cap B_n \text{ and } B'^k_{n,m} = \{x \in B'_n : \ \Phi(x,k) \leq m\}.$$

Arguing as before, there exist a subsequence $\{m_k\}$ such that

$$\left|B'_n \setminus B'^k_{n,m_k}\right| < \frac{1}{J+1}\frac{1}{2^n}\frac{1}{2^k}.$$

Define $A_{J+1,n} = \bigcap_{k=1}^{\infty} B'^{k}_{n,m_k}$. We have

$$\sup_{x\in A_{J+1,n}} \Phi(x,t) < \infty,\ |B'_n \setminus A_{J+1,n}| \leq \frac{1}{J+1}\frac{1}{2^n}.$$

Therefore,

$$\left| A' \setminus \bigcup_{n=1}^{\infty} A_{J+1,n} \right| < \frac{1}{J+1}$$

and

$$\left| A \setminus \left(\bigcup_{j=1}^{J+1} \bigcup_{n=1}^{\infty} A_{j,n} \right) \right| = \left| A \setminus \left(\bigcup_{j=1}^{J} \bigcup_{n=1}^{\infty} A_{j,n} \cup \bigcup_{n=1}^{\infty} A_{J+1,n} \right) \right| = \left| A' \setminus \bigcup_{n=1}^{\infty} A_{J+1,n} \right| < \frac{1}{J+1}.$$

By induction, we have constructed the family $\{A_{n,j}\}_{n,j}^{\infty}$. Since for any $J \in \mathbb{N}$

$$\left| A \setminus \left(\bigcup_{j=1}^{\infty} \bigcup_{n=1}^{\infty} A_{j,n} \right) \right| \leq \left| A \setminus \left(\bigcup_{j=1}^{J} \bigcup_{n=1}^{\infty} A_{j,n} \right) \right| < \frac{1}{J},$$

we have that

$$\left| A \setminus \left(\bigcup_{j=1}^{\infty} \bigcup_{n=1}^{\infty} A_{j,n} \right) \right| = 0.$$

Since the family $\{A_{j,n}\}_{n,j=1}^{\infty}$ is countable, we can enumerate it as $\{A_n\}_{n=1}^{\infty}$. The sets in the family $\{A_n\}_{n=1}^{\infty}$ have finite measure, are disjoint and for any $n \in \mathbb{N}$ and any $t > 0$ satisfy

$$\sup_{x\in A_n} \Phi(x,t) < \infty.$$

□

The next Lemma was proved in [49] and we skip its proof.

Lemma 2.1.7. *[49, Lemma 8.3.] Let $A \subset \mathbb{R}^d$, $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive numbers, and let $\{f_n\}_{n=1}^{\infty} \subset L^0(A)$ be a sequence of finite, non-negative functions such that, for all $n \in \mathbb{N}$,*

$$\int_A f_n(x)dx \geq 2^n a_n.$$

Then there exists a sequence $\{A_k\}_{k=1}^{\infty}$ *of pairwise disjoint measurable sets, and a subsequence* $\{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$, *such that for all* $k \in \mathbb{N}$,

$$\int_{A_k} f_{n_k}(x)dx = a_{n_k}.$$

Next we establish the relation between pairs of MO functions and their corresponding MO spaces. Let Φ and Ψ be two MO functions on A. We say that Ψ is weaker than Φ if there exist positive constant C and a positive, integrable function h, such that for all $t \geq 0$,

$$\Psi(x,t) \leq \Phi(x,Ct) + h(x) \text{ a.e. in } A. \tag{2.1}$$

It means that there exists $B \subset A$, $|B| = 0$, such that for every $x \in A \setminus B$ we have

$$\Psi(x,t) \leq \Phi(x,Ct) + h(x).$$

for all $t \geq 0$. If Ψ is weaker than Φ, we write $\Psi \prec \Phi$.

A MO function Φ is said to satisfy the Δ_2 condition if there exist a constant $C > 0$ and an integrable, positive function h such that for $t \geq 0$,

$$\Phi(x,2t) \leq C\Phi(x,t) + h(x) \text{ a.e in } A.$$

It means that there exists $B \subset A$, $|B| = 0$, such that for every $x \in A \setminus B$ we have

$$\forall t > 0 \ \Phi(x,2t) \leq C\Phi(x,t) + h(x).$$

The results (1), (2), (3) in the following theorem are well known and can be found in [49]. On the other hand (4) is known to be true under the assumption of local integrability of Φ. Here we prove this result and drop the assumption of local integrability.

Theorem 2.1.8. *Let $A \subset \mathbb{R}^n$ and Φ, Ψ be MO functions on A. Then*

(1) *$L_0^{\Phi}(A) \subset L_0^{\Psi}(A)$ if and only if there exist a non negative $h \in L^1(A)$ and $C > 0$ such that for any $t \geq 0$ and a.e $x \in A$,*

$$\Psi(x,t) \leq C\Phi(x,t) + h(x). \tag{2.2}$$

(2) *$L^{\Phi}(A) \subset L^{\Psi}(A)$ if and only if $\Psi \prec \Phi$.*

(3) *If $\Psi \prec \Phi$, then $E^{\Phi}(A) \subset E^{\Psi}(A)$.*

(4) *If $\Psi \prec \Phi$, then modular (resp. norm) convergence in $L^{\Phi}(A)$ is stronger then modular (resp. norm) convergence in $L^{\Psi}(A)$.*

Proof. Since the proofs of (1), (2), (3) are very similar we will provide only the proofs of (1) and (4).

The proof of sufficiency in (1) is straightforward. Take any $f \in L_0^{\Psi}(A)$, then we have

$$\int_A \Psi(x,|f(x)|)dx \leq C \int_A \Phi(x,|f(x)|) + h(x)dx < \infty,$$

so $f \in L_0^{\Psi}(A)$.

To show necessity of the condition in (1) first let us write $A = \bigcup_{i=1}^{\infty} A_i$, where A_i's are pairwise disjoint and of finite measure. Let $f_{r,i} = r\chi_{A_i}$, where i is a natural number and $r \geq 0$ is rational. For each $n \in \mathbb{N}$ let us write

$$h_n(x) = \sup_{t \geq 0} \left(\Psi(x,t) - 2^n\Phi(x,t)\right) \tag{2.3}$$

First we will show that $h_n(x) = \sup_{(r,i)\in(\mathbb{Q}\cap[0,\infty))\times\mathbb{N}} (\Psi(x, f_{r,i}(x)) - 2^n\Phi(x, f_{r,i}(x)))$. Fix $x \in A$, and let $x \in A_i$. Clearly

$$\sup_{(r,i)\in(\mathbb{Q}\cap[0,\infty))\times\mathbb{N}} (\Psi(x, f_{r,i}(x)) - 2^n\Phi(x, f_{r,i}(x))) \leq \sup_{t\geq 0} (\Psi(x,t) - 2^n\Phi(x,t)) = h_n(x).$$

To establish the other inequality we notice that there exists a sequence $\{t_m\}_{m=1}^{\infty} \subset [0,\infty)$ such that

$$h_n(x) = \lim_{m\to\infty} \Psi(x, t_m) - 2^n\Phi(x, t_m).$$

Since $\Psi(x,\cdot) - 2^n\Phi(x,\cdot)$ is a continuous function, by density of rational numbers in $\mathbb{R}$, for each $m \in \mathbb{N}$ there exist a rational number r_m such that

$$|\Psi(x, t_m) - 2^n\Phi(x, t_m) - (\Psi(x, r_m) - 2^n\Phi(x, r_m))| < \frac{1}{m}.$$

Therefore

$$h_n(x) = \lim_{m\to\infty} \Psi(x, t_m) - 2^n\Phi(x, t_m) = \lim_{m\to\infty} \Psi(x, r_m) - 2^n\Phi(x, r_m) \tag{2.4}$$
$$\leq \sup_{r,i\in(\mathbb{Q}\cap[0,\infty))\times\mathbb{N}} (\Psi(x, f_{r,i}(x)) - 2^n\Phi(x, f_{r,i}(x))). \tag{2.5}$$

Since the set $(\mathbb{Q}\cap[0,\infty))\times\mathbb{N}$ is countable, $\{f_{r,i}\}_{(r,i)\in(\mathbb{Q}\cap[0,\infty))\times\mathbb{N}}$ can be rewritten as $\{f_k\}_{k=1}^{\infty}$, where $f_1 = 0$. Therefore $h_n(x) = \sup_{k\in\mathbb{N}} (\Psi(x, f_k(x)) - 2^n\Phi(x, f_k(x)))$ and so, for each $n \in \mathbb{N}$ the function h_n is measurable and the equality 2.3 holds.

Observe that the inequality 2.2 holds if and only if there exists $n \in \mathbb{N}$ such that $h_n \in L^1(A)$. To show $L^\Phi(A) \subset L^\Psi(A)$ implies $\Psi \prec \Phi$, we argue by contradiction. In view of the above reasoning it suffices to show that, if for any $n \in \mathbb{N}$, $h_n \notin L^1(A)$, then $L^\Phi(A) \not\subset L^\Psi(A)$. Suppose that for all $n \in \mathbb{N}$ we have $\int_A h_n(x)dx = \infty$. Let us write

$$b_{n,m}(x) = \max_{1 \le k \le m} \left(\Psi(x, f_k(x)) - 2^n \Phi(x, f_k(x)) \right).$$

Notice that, for each $m \in \mathbb{N}$, $b_{n,m}(x) \ge 0$, $b_{n,m}(x)$ is a non decreasing sequence and $\lim\limits_{m \to \infty} b_{n,m}(x) = h_n(x)$. Therefore

$$\lim_{m \to \infty} \int\limits_A b_{n,m}(x) = \int\limits_A h_n(x) dx = \infty.$$

Hence, for each $n \in \mathbb{N}$ there exists $m_n \in \mathbb{N}$ such that

$$\int\limits_A b_{n,m_n}(x) dx \ge 2^n.$$

Writing $b_n = b_{n,m_n}$, we get that $\int\limits_A b_n(x) \ge 2^n$. Now let us denote

$$B_{n,k} = \{x \in A : \Psi(x, f_k(x)) - 2^n \Phi(x, f_k(x) = b_n(x)\},$$

Notice that $\bigcup\limits_{k=1}^{m_n} B_{n,k} = A$. Define now $\widetilde{f_n}(x) = \sum\limits_{k=1}^{m_n} f_k(x) \chi_{B_{n,k}}(x)$. Thus for all $n \in \mathbb{N}$, a.e. $x \in A$

$$b_n(x) = \Psi(x, \widetilde{f_n}(x)) - 2^n \Phi(x, \widetilde{f_n}(x)). \tag{2.6}$$

Hence we arrive at

$$\int\limits_A \Psi(x, \widetilde{f_n}(x)) dx \ge \int\limits_A \left(\Psi(x, \widetilde{f_n}(x)) - 2^n \Phi(x, \widetilde{f_n}(x)) \right) dx = \int\limits_A b_n(x) dx \ge 2^n.$$

For all $n \in \mathbb{N}$ by Lemma 2.1.7 there exist a family $\{B_k\}_{k=1}^{\infty}$ of pairwise disjoint subsets of A and a subsequence $\{n_k\}_{k=1}^{\infty}$, such that for every $k \in N$,

$$\int_{B_k} \Psi(x, \widetilde{f}_{n_k}(x))dx = 1.$$

Finally, defining $f(x) = \sum_{k=1}^{\infty} \widetilde{f}_{n_k}(x)\chi_{B_k}(x)$, we have

$$I_{\Psi}(f) = \sum_{k=1}^{\infty} \int_{B_k} \Psi(x, \widetilde{f}_{n_k}(x))dx = \infty.$$

Therefore $f \notin L_0^{\Psi}(A)$. On the other hand by (2.6),

$$\begin{aligned} I_{\Phi}(f) &= \int_A \Phi\left(x, \sum_{k=1}^{\infty} \widetilde{f_{n_k}}(x)\chi_{B_k}(x)\right) dx = \sum_{k=1}^{\infty} \int_{B_k} \frac{1}{2^{n_k}} \left(\Psi(x, \widetilde{f}_{n_k}(x)) - b_{n_k}(x)\right) dx \\ &\leq \sum_{k=1}^{\infty} \frac{1}{2^{n_k}} \int_{B_k} \Psi(x, \widetilde{f}_{n_k}(x))dx = 1, \end{aligned}$$

therefore $f \in L_0^{\Phi}(A)$.

To show (4) it suffices to show that for any sequence $\{f_n\}_{n=1}^{\infty} \subset L^{\Phi}(A)$, such that $\lim_{n\to\infty} I_{\Phi}(\lambda f_n) = 0$ for some $\lambda > 0$, there exists another $\lambda_1 > 0$, such that $\lim_{n\to\infty} I_{\Psi}(\lambda_1 f_n) = 0$.

Take any $\varepsilon > 0$. For the function Ψ consider the partition $\{A_n\}_{n=1}^{\infty}$ of A, from Lemma 2.1.6. By integrability of h, there exists $N \in \mathbb{N}$, such that for $A' = \bigcup_{n=1}^{N} A_n$ we have

$$\int_{A\setminus A'} h(x)dx < \frac{\varepsilon}{4},$$

where h is the function from (2.1). Since, for every $n \in N$ and every $\lambda > 0$, $\sup_{x\in A_n} \Psi(x, \lambda) < \infty$, we get that

$$I_\Psi(\lambda\chi_{A'}) = \sum_{n=1}^{N} \int_{A_n} \Psi(x,\lambda)dx < \infty.$$

Therefore $\chi_{A'} \in E^\Psi(A)$. Let C be the constant from the inequality (2.2). Take now $a > 0$ such that

$$I_\Psi\left(\frac{a}{C+1}\lambda\chi_{A'}\right) < \varepsilon.$$

Since $\lim_{n\to\infty} I_\Phi(\lambda f_n) = 0$ and $|A'| < \infty$, by Theorem 2.1.5 (7), the sequence $\{f_n\}_{n=1}^\infty$ converges in measure to 0 on the set A'. Let us write $B_n = \{x \in A' : |f_n(x)| \leq a\}$ and $D_n = A' \setminus B_n$. Notice that $A' = B_n \cup D_n$ and so $D_n = \{x \in A' : |f_n(x)| > a\}$. Since $\{f_n\}_{n=1}^\infty$ converges to 0 in measure on A' we have that $\lim_{n\to\infty} |D_n| = 0$. Therefore there exists $N_1 \in \mathbb{N}$ such that for any $n > N_1$ we have

$$\int\limits_{D_n} h(x)dx < \frac{\varepsilon}{4}.$$

Moreover we can choose N_1 is such a way that for all $n > N_1$,

$$I_\Phi(f_n) < \frac{\varepsilon}{4}.$$

Hence, by the fact that $\frac{1}{C+1} < 1$, (2.1) and convexity of Φ we have

$$\begin{aligned}
I_\Psi\left(\frac{\lambda}{C+1}f_n\right) &= \int_A \Psi\left(x, \frac{\lambda}{C+1}|f_n(x)|\right)dx \\
&= \int_{A'} \Psi\left(x, \frac{\lambda}{C+1}|f_n(x)|\right)dx + \int_{A\setminus A'} \Psi\left(x, \frac{\lambda}{C+1}|f_n(x)|\right)dx \\
&= \int_{B_n} \Psi\left(x, \frac{\lambda}{C+1}|f_n(x)|\right)dx + \int_{D_n} \Psi\left(x, \frac{\lambda}{C+1}|f_n(x)|\right)dx \\
&+ \int_{A\setminus A'} \Psi\left(x, \frac{\lambda}{C+1}|f_n(x)|\right)dx \le \int_{B_n} \Psi\left(x, \frac{a\lambda}{C+1}\right)dx \\
&+ \int_{D_n} \frac{C}{C+1}\Phi\left(x, \lambda|f_n(x)|\right)dx + \int_{D_n} h(x)dx \\
&+ \frac{C}{C+1}\int_{A\setminus A'} \Phi\left(x, \lambda|f_n(x)|\right)dx + \int_{A\setminus A'} h(x)dx \\
&\le \int_{A'} \Psi\left(x, \frac{a\lambda}{C}\right)dx + \frac{\varepsilon}{4} + I_\Phi(x, \lambda f_n) + \frac{\varepsilon}{4} < \varepsilon.
\end{aligned}$$

Since the above inequality holds for all $n > N_1$ and ε was arbitrarily chosen we conclude that $\lim\limits_{n\to\infty} I_\Psi\left(\frac{\lambda}{C+1}f_n\right) = 0$ which shows (4). $\square$

Now we characterize the equality of spaces $L^\Phi(A) = E^\Phi(A)$. We remark that this theorem was known to be true under the assumption of local integrability of Φ, hence the Theorem below lets us drop this assumption.

Theorem 2.1.9. *Let $A \subset \mathbb{R}^n$ and Φ be a MO function on A. The following are equivalent:*

(1) $L^\Phi(A) = E^\Phi(A)$,

(2) *modular convergence is equivalent to norm convergence,*

(3) Φ *satisfies* Δ_2 *condition.*

Proof. First we show that (1) implies (3) Notice that, since always $E^{\Phi}(A) \subset L^{\Phi}(A)$, to show that $L^{\Phi}(A) = E^{\Phi}(A)$ it suffices to show that $L^{\Phi}(A) \subset E^{\Phi}(A)$, which is equivalent to the inclusion $L_0^{\Phi}(A) \subset L_0^{\Psi}(A)$, where $\Psi(x,t) = \Phi(x,2t)$. By Theorem 2.1.8 (1), this is equivalent to the inequality

$$\Phi(x,2t) \leq C\Phi(x,t) + h(x),$$

for all $t \geq 0$, a.e. $x \in A$ where $C > 0$ and H is a non-negative function from $L^1(A)$. This exactly the Δ_2 condition.

To show that (3) implies (2), notice that Δ_2 implies that for any $t \geq 0$ and a.e. $x \in A$

$$\Psi(x,t) \leq C\Phi(x,t) + h(x)$$

where $\Psi(x,t) = \Phi(x,2t)$, in other words $\Psi \prec \Phi$. By Theorem 2.1.8 (4) modular convergence in $L^{\Phi}(A)$ implies modular convergence in $L^{\Psi}(A)$. That is, if for some n $\lambda > 0$ and some sequence $\{f_n\}_{n=1}^{\infty} \subset L^{\Psi}(A)$ we have $\lim\limits_{n\to\infty} \int\limits_A \Phi\left(x, \lambda|f_n(x)|\right) dx = 0$, then $\int\limits_A \Phi\left(x, 2\lambda|f_n(x)|\right) dx = 0$. It follows that for every $\lambda' > 0$, $\lim\limits_{n\to\infty} I_{\Phi}(\lambda' f_n) = 0$, so $\lim\limits_{n\to\infty} \|f_n\|_{\Phi} = 0$.

To prove that (2) implies (3), we argue by contraposition. Assume that Δ_2 does not hold for Φ. Denoting $\Psi(x,t) = \Phi(x,2t)$ and repeating the argument from Theorem 2.1.8 (1), we find a sequence $\{B_k\}_{k=1}^{\infty}$ of disjoint subsets of A and a sequence of functions $\{f_k\}_{k=1}^{\infty}$, such that

$$I_{\Psi}\left(\sum_{k=1}^{\infty} f_k \chi_{B_k}\right) = \sum_{k=1}^{\infty} I_{\Psi}\left(f_k \chi_{B_k}\right) = \infty$$

and

$$I_{\Phi}\left(\sum_{k=1}^{\infty} f_k \chi_{B_k}\right) = \sum_{k=1}^{\infty} I_{\Phi}\left(f_k \chi_{B_k}\right) = 1.$$

Now if we define $g_n = \sum_{k=n}^{\infty} f_k \chi_{B_k}$, $n \in \mathbb{N}$, then $\lim_{n\to\infty} I_\Phi(g_n) = 0$ and $\lim_{n\to\infty} I_\Phi(2g_n) = \infty$. Therefore $\{g_n\}_{n=1}^{\infty}$ converges in the modular I_Φ, but it does not converge in norm in $L^\Phi(A)$. □

Since, in the end we want to characterize when smooth, compactly supported functions are dense in L^Φ we first need to understand the relation between simple functions and elements of E^Φ. We have the following theorem.

Theorem 2.1.10. *Let $A \subset \mathbb{R}^n$ and Φ be a MO function on A. Let $S(A)$ be the set of all simple, complex valued functions defined on A. Then $S(A) \cap E^\Phi(A)$ is dense in $(E^\Phi(A), \|\cdot\|_\Phi)$.*

Proof. We start with showing that any non-negative function $f \in E^\Phi(A)$ can be approximated by a sequence of simple functions from $E^\Phi(A)$. Take any non-negative $f \in E^\Phi(A)$. There exists a sequence $\{s_n\}_{n=1}^{\infty}$ of non-negative simple functions, such that, $s_n \uparrow f$, a.e. in A. Since for every $n \in \mathbb{N}$, $s_n \leq f$ a.e., we have that $\{s_n\}_{n=1}^{\infty} \subset E^\Phi(A)$. For any $\lambda > 0$, $\Phi(x, \lambda(f(x) - s_n(x))) \leq \Phi(x, \lambda f(x))$ a.e. in A, and $I_\Phi(\lambda f) < \infty$. Hence, by the Lebesgue Dominated Convergence Theorem $\lim_{n\to\infty} I_\Phi(\lambda(f - s_n)) = 0$, and therefore $\lim_{n\to\infty} \|f - s_n\|_\Phi = 0$.

Now take an arbitrary $f \in E^\Phi(A)$ and write it as $f = f_1 - f_2 + i(f_3 - f_4)$, where $f_1, f_2, f_3, f_4 \geq 0$. For each of the functions f_j, $j = 1, \ldots 4$ we can repeat the previous argument and since a linear combination of simple functions is simple, we can construct a sequence $\{g_n\}_{n=1}^{\infty} \subset E^\Phi(A)$ of complex valued simple functions such that $\lim_{n\to\infty} \|f - g_n\|_\Phi = 0$. □

Definition 2.1.11. *Let $A \subset \mathbb{R}^d$ be a Lebesgue measurable set and $\Phi : A \times [0, \infty) \to [0, \infty)$ be a MO function. For every $x \in A$ and $t > 0$, we define*

$$\Phi^*(x, t) = \sup_{s \geq 0}(st - \Phi(x, s)).$$

We call Φ^* the complementary function of Φ (often in the literature Φ^* referred as the Legandre transform of Φ).

The function Φ^* is not necessarily finite valued. For example, let $\Phi : \mathbb{R} \times [0, \infty) \to [0, \infty)$ be given by $\Phi(x, t) = t$. The function Φ^* is then given by

$$\Phi^*(x,t) = \begin{cases} 0 & 0 \leq t \leq 1, \ x \in \mathbb{R} \\ \infty & t > 1, \ x \in \mathbb{R}. \end{cases}$$

Still the function $t \mapsto \Phi^*(x, t)$ is convex for any $x \in \mathbb{R}$, in the sense that the inequality

$$\Phi(x, \alpha t_1 + \beta t_2) \leq \alpha \Phi(x, t_1) + \beta \Phi(x, t_2),$$

holds for all $t_1, t_2 \geq 0$ and $\alpha, \beta > 0$ such that $\alpha + \beta = 1$. It turns out that Φ^* is an example of an *extended Musielak Orlicz function.*

Definition 2.1.12. *Let $A \subset \mathbb{R}^d$ be a measurable set. A function $\Phi : A \times [0, \infty) \to [0, \infty]$ is an extended Musielak Orlicz function if*

(1) *$t \mapsto \Phi(x, t)$ is convex for $t \geq 0$, for a.e. $x \in A$,*

(2) *$x \mapsto \Phi(x, t)$ is Lebesgue measurable on A for every $t \geq 0$,*

(3) *for a.e. $x \in A$, $\lim_{t \to 0^+} \Phi(x, 0) = 0$ and $\lim_{t \to \infty} \Phi(x, t) = \infty$.*

It is easy to see that if Φ is an extended MO function on a Lebesgue measurable set $A \subset \mathbb{R}^d$, then for $a.e.x \in A$ the function $t \mapsto \Phi(x, t)$ is a non-decreasing function. We will need to be able to take inverses of such functions.

Definition 2.1.13. *Let $A \subset \mathbb{R}^d$ be a Lebesgue measurable set and $\Phi : A \times [0, \infty) \to [0, \infty]$ be an extended MO function. A generalized inverse of Φ is a function defined by the formula*

$$\Phi^{-1}(x, t) = \inf\{s \geq 0 : \Phi(x, s) \geq t\}.$$

Notice that if Φ is just a MO function the generalized inverse coincides with regular inverse of Φ with respect to the variable t. We also have the following proposition.

Proposition 2.1.14. *[26, Lemma.2.3.3, p. 24] Let $A \subset \mathbb{R}^d$ be a Lebesgue measurable set and $\Phi : A \times [0, \infty) \to [0, \infty]$ be an extended MO function. For $x \in A$ let $a_x = \sup\{t \geq 0 : \Phi(x, t) = 0\}$, $b_x = \inf\{t \geq 0 : \Phi(x, t) = \infty\}$. We have $\Phi^{-1}(x, 0) = 0$ and $0 < \Phi^{-1}(x, t) < \infty$ for $t > 0$ and*

$$\Phi^{-1}(x, \Phi(x, t)) = \begin{cases} 0 & 0 \leq t \leq a_x \\ t & a_x < t \leq b_x \\ b_x & t > b_x. \end{cases}$$

On the other hand

$$\Phi(x, \Phi^{-1}(x, t)) = \min\{t, \Phi(x, b_x)\}.$$

The next proposition shows the relation between Φ and Φ^*

Proposition 2.1.15. *Let $A \subset \mathbb{R}^d$ be a Lebesgue measurable setand Φ be MO function defined on A. The following statements are true.*

(1) *Φ^* is an extended Musielak-Orlicz function defined on A.*

(2) *For every $x \in A$ and $s, t > 0$*

$$st \leq \Phi(x, s) + \Phi^*(x, t).$$

We call the inequality above Young's inequality.

(3) *For every $x \in A$ and $t > 0$ we have*

$$t \leq \Phi^{-1}(x, t)(\Phi^*)^{-1}(x, t) \leq 2t.$$

Proof. (1) Let Φ be as in the assumption. Let $x \in A$ be such that $t \mapsto \Phi(x,t)$ is convex. To show convexity of $\Phi^*(x,t)$ take any $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$ and $t_1, t_2 \geq 0$. By convexity of Φ we have

$$\begin{aligned}
\Phi^*(x, \alpha t_1 + \beta t_2) &= \sup_{s \geq 0}(s(\alpha t_1 + \beta t_2) - \Phi(x,s)) \\
&= \sup_{s \geq 0}(\alpha(st_1 - \Phi(x,s)) + \beta(st_2 - \Phi(x,s))) \\
&\leq \alpha \sup_{s \geq 0}(st_1 - \Phi(x,s)) + \beta \sup_{s \geq 0}(st_2 - \Phi(x,s)) \\
&= \alpha \Phi^*(x,t_1) + \beta \Phi^*(x,t_2).
\end{aligned}$$

On the other hand for any $s > 0$ we have $\Phi(x,s) \geq 0$ and since $\Phi(x,0) = 0$ we conclude that $\Phi^*(x,0) = 0$. Now we will show that $\lim_{t \to 0^+} \Phi^*(x,t) = 0$. Since $\Phi(x,1) \neq 0$ by convexity of $s \mapsto \Phi(x,s)$ we conclude that for $s \geq 1$ we have $\Phi(x,s) \geq s\Phi(x,1)$. Hence, for any $t \geq 0$,

$$\Phi^*(x,t) \leq \max\{t, \sup_{s > 1}(st - \Phi(x,1)s)\}$$

and since $\lim_{t \to 0^+} t = 0$ we have that $\lim_{t \to 0^+} \Phi^*(x,t) = 0$. Similarly, since

$$\Phi^*(x,t) \geq t - \Phi(x,1)$$

we get that $\lim_{t \to \infty} \Phi^*(x,t) = \infty$.

The proof of (2) follows immediately from the definition of Φ^*. As for proof of (3), the second inequality follows from Young's inequality and the first one is true since for every $x \in A$ and $t > 0$, $\Phi^*\left(x, \frac{\Phi(x,t)}{t}\right) < \Phi(x,t)$ (see [26, Theorem 2.4.10, p.33]). $\square$

Theorem 2.1.16. *Let $A \subset \mathbb{R}^d$ be Lebesgue measurable and let Φ be a*

Musielak-Orlicz function on A, for every $f \in L^\Phi$ we define

$$\|f\|_\Phi^0 = \sup_{\|g\|_{\Phi^*} \le 1} \left| \int_A f(x)g(x)dx \right| = \sup_{I_{\Phi^*}(g) \le 1} \left| \int_A f(x)g(x)dx \right|,$$

where $\|\cdot\|_\Phi^0$ is a function norm on $L^\Phi(A)$, called the Orlicz norm. Moreover, for any $f \in L^\Phi(A)$,

$$\|f\|_\Phi \le \|f\|_\Phi^0 \le 2\|f\|_\Phi.$$

Proof. It is easy to show that $\|\cdot\|_\Phi^0$ is a function norm. As for the inequality it follows form [49, page 9]. □

Theorem 2.1.17. *Let $A \subset \mathbb{R}^d$ be a Lebesgue measurable setand let Φ be a MO function on A. For every $f \in L^\Phi$ and $g \in L^{\Phi^*}$ we have*

$$\left| \int_A f(x)g(x)dx \right| \le \|f\|_\Phi \|g\|_{\Phi^*}^0,$$

$$\left| \int_A f(x)g(x)dx \right| \le \|f\|_\Phi^0 \|g\|_{\Phi^*}.$$

We call those inequalities Hölder inequalities.

Proof. The proof follows directly form Theorems (A3), (A4) in [41]. □

For a MO function $\Phi : E \times [0,\infty) \to [0,\infty)$ and a measurable set $E \subset \mathbb{R}^d$ we define

$$\Phi_{E^+}(t) = \operatorname*{ess\,sup}_{x \in E} \Phi(x,t),$$

$$\Phi_{E^-}(t) = \operatorname*{ess\,inf}_{x \in E} \Phi(x,t).$$

Clearly, $\Phi_{E^+}(t)$ is an Orlicz function. That is a convex, non-negative function on $[0,\infty)$ such that $\Phi_{E^+}(t) = 0$ if and only if $t = 0$. On the other hand, the function Φ_{E^-} is not an Orlicz function. Still, the following lemma holds true.

Lemma 2.1.18. *Let $E \subset \mathbb{R}^d$ be measurable and Φ be a MO function defined on E. There exists an Orlicz function $\Psi : [0,\infty) \to [0,\infty]$, such that for all $t \geq 0$,*

$$\Psi(t) \leq \Phi_{E^-}(t) \leq \Psi(2t).$$

Proof. Let us show first that $\frac{\Phi_{E^-}(t)}{t}$ is a non-decreasing function. Let $0 < t_1 < t_2$, then for any $x \in E$, by convexity,

$$\Phi(x, t_1) = \Phi\left(x, \frac{t_1}{t_2} t_2\right) \leq \frac{t_1}{t_2}\Phi(x, t_2).$$

Hence

$$\frac{\Phi_{E^-}(t_1)}{t_1} = \operatorname*{ess\,inf}_{x\in E} \frac{\Phi(x,t_1)}{t_1} \leq \operatorname*{ess\,inf}_{x\in E} \frac{\Phi(x,t_2)}{t_2} = \frac{\Phi_{E^-}(t_2)}{t_2}.$$

Now for any $s, t > 0$ define $f_t(s) = \left(\frac{s}{t} - 1\right)\Phi_{E^-}(t)$. It follows that for any $t \geq 0$ the function $s \mapsto f_t(s)$ is convex and for any $s \geq 0$ we have

$$f_t(s) \leq \Phi_{E^-}(s).$$

Indeed, if $s \leq t$, then

$$f_t(s) = \left(\frac{s}{t} - 1\right)\Phi_{E^-}(t) \leq 0 \leq \Phi_{E^-}(s).$$

On the other hand, if $s \geq t$, by the fact that $\frac{\Phi_{E^-}(s)}{s}$ is a non-decreasing,

$$f_t(s) = \left(\frac{s}{t} - 1\right)\Phi_{E^-}(t) \leq \frac{s}{t}\Phi_{E^-}(t) \leq s\frac{\Phi_{E^-}(s)}{s} = \Phi_{E^-}(s).$$

Defining $\Psi(s) = \max\{0, \sup_{t>0} f_t(s)\}$, we see that Ψ is convex and $\Psi(0) = 0$. Moreover for any $s > 0$, $\Psi(s) \leq \Phi_{E^-}(s)$. On the other hand, for any $t > 0$,

$$\Phi_{E^-}(t) = \left(\frac{2t}{t} - 1\right) \Phi_{E^-}(t) = f_t(2t) \leq \Psi(2t).$$

Hence, for any $t \geq 0$,

$$\Psi(t) \leq \Phi_{E^-}(t) \leq \Psi(2t).$$

□

2.2 Density of $C_C^\infty(\Omega)$ in $L^\Phi(\Omega)$

In this section we want to answer the question of density of smooth functions in the space $L^\Phi(\Omega)$, where Ω is an open subset of $\mathbb{R}^d$. The main point here is that we consider MO functions that are not necessarily locally integrable. A simple example of such function is a function $\Phi_1 : \mathbb{R} \times [0, \infty) \to [0, \infty)$ given by the formula

$$\Phi_1(x, t) = \begin{cases} \frac{t}{|x|} & t \geq 0,\ x \neq 0 \\ 0 & t \geq 0,\ x = 0. \end{cases}$$

A slightly more involved function that is not locally integrable can be constructed as follows. Let $\{r_n\}_{n=1}^\infty$ be an enumeration of rational numbers from the interval $(0, 1)$. For any natural number n and any $x \in (0, 1)$ we set

$$g_n(x) = \begin{cases} \frac{1}{x - r_n} & x > r_n \\ 0 & x \leq r_n. \end{cases}$$

Notice that for any $n \in \mathbb{N}$ the function g_n is not integrable on any interval containing r_n, but the function $\sqrt{g_n}$ is an element of $L^1(0, 1)$ and $\|\sqrt{g_n}\|_1 \leq \frac{2}{3}$. Hence $g(x) = \sum\limits_{n=1}^{\infty} \frac{\sqrt{g_n(x)}}{2^n}$ is an element of $L^1(0, 1)$. Therefore there exists a subset A of $(0, 1)$ of measure 0 such that for any $x \in (0, 1) \setminus A$, $g(x)$ is finite. Notice now that for any $x \in (0,1) \setminus A$ we have that

$$w(x) = \sum_{n=1}^{\infty} \frac{g_n(x)}{4^n} \leq g(x)^2 < \infty.$$

We define now the function $\Phi_2 : \mathbb{R} \times [0,\infty) \to [0,\infty)$ by the formula

$$\Phi_2(x,t) = \begin{cases} tw(x) & x \in (0,1) \setminus A,\ t \geq 0 \\ 0 & x \in A,\ t \geq 0 \\ t & x \in \mathbb{R} \setminus (0,1),\ t \geq 0. \end{cases},$$

By construction, the function Φ_2 is also a non locally integrable MO function. In fact w is not integrable on any open subinterval of $(0,1)$. It is clear that both Φ_1 and Φ_2 satisfy the Δ_2 condition. Hence by Theorem 2.1.10 simple functions that belong to $L^{\Phi_i}(\mathbb{R})$ are dense $L^{\Phi_i}(\mathbb{R})$, for $i = 1, 2$. But what about density of compactly supported smooth functions? It turns out that they are dense in $L^{\Phi_1}(\mathbb{R})$ but not in $L^{\Phi_2}(\mathbb{R})$. The discussion in this subsection explains why that is the case and characterizes those MO functions, satisfying Δ_2 condition, for which elements of $C_C^{\infty}(\Omega)$ are dense in $L^{\Phi}(\Omega)$.

We start with the following definition.

Definition 2.2.1. *Let $\Omega \subset \mathbb{R}^d$ and Φ be a MO function defined on Ω. We define the set* $\operatorname{Sing}\Phi$*, the set of singular points of Φ, as*

$$\operatorname{Sing}\Phi = \{x \in \mathbb{R}^d :\ \forall r > 0\ \exists t_r > 0 \int_{B(x,r)\cap\Omega} \Phi(y,t_r)dy = \infty\}.$$

It is easy to see that for Φ_1 and Φ_2 defined above we have

$$\operatorname{Sing}\Phi_1 = \{0\},$$

$$\text{Sing}\,\Phi_2 = [0,1].$$

Notice that both of those sets are closed. This is not a coincidence. The following Proposition holds.

Proposition 2.2.2. *For any open set* Ω *and MO function* Φ *defined on* Ω *the set* $\text{Sing}\,\Phi$ *is a closed subset of* $\mathbb{R}^d$*.*

Proof. Let $x \in \mathbb{R}^d$, $\{x_n\}_{n=1}^{\infty} \subset \text{Sing}\ \Phi$ and $\lim\limits_{n\to\infty} x_n = x$. We will show that $x \in \text{Sing}\ \Phi$. For any $r > 0$, then there exists $N \in \mathbb{N}$ such that for all $n > N$, $|x - x_n| < \frac{r}{2}$. Notice that, for each $n > N$ we have $B(x_n, \frac{r}{2}) \subset B(x,r)$. For any $n > N$, by definition of Sing Φ there exists $t_{\frac{r}{2}} > 0$ such that

$$\int\limits_{B(x_n,\frac{r}{2})\cap\Omega} \Phi(y, t_{\frac{r}{2}})dy = \infty,$$

therefore

$$\int\limits_{B(x,r)\cap\Omega} \Phi(y, t_{\frac{r}{2}})dy \geq \int\limits_{B(x_n,\frac{r}{2})\cap\Omega} \Phi(y, t_{\frac{r}{2}})dy = \infty.$$

Since r was arbitrary chosen, we conclude that $x \in \text{Sing}\ \Phi$. $\square$

Now we show that for density of $C_C^{\infty}(\Omega) \cap E^{\Phi}(\Omega)$ in $E^{\Phi}(\Omega)$ the set $\text{Sing}\,\Phi$ has to be of zero measure.

Theorem 2.2.3. *Let* $\Omega \subset \mathbb{R}^d$ *and* Φ *be a MO function on* Ω*. If* $|\,\text{Sing}\,\Phi| > 0$*, then* $C_C^{\infty}(\Omega) \cap E^{\Phi}(\Omega)$ *is not dense in* $E^{\Phi}(\Omega)$*.*

Proof. We argue by contradiction. Assume that $C_C^{\infty}(\Omega) \cap E^{\Phi}(\Omega)$ is dense in $E^{\Phi}(\Omega)$ and $|\text{Sing}\ \Phi| > 0$. Let $\{\Omega_n\}_{n=1}^{\infty}$ be a sequence of sets such that $A_n = \Omega_n$, where $\{A_n\}_{n=1}^{\infty}$ is the sequence of sets from the conclusion of Lemma 2.1.6. Since $|\Omega \setminus \bigcup\limits_{n=1}^{\infty} \Omega_n| = 0$, there exists n_0 such that $|\,\text{Sing}\,\Phi \cap \Omega_{n_0}| > 0$. By inner regularity of the Lebesgue measure there exists a compact set $K \subset \text{Sing}\,\Phi \cap \Omega_{n_0}$ such that

$0 < |K| < |\operatorname{Sing}\Phi \cap \Omega_{n_0}|$. For any $t > 0$,

$$\int_K \Phi(x,t)dx \leq \int_{\Omega_{n_0}} \Phi(x,t)dx < \infty,$$

therefore $\chi_K \in E^\Phi(\Omega)$. By density of $C_C^\infty(\Omega) \cap E^\Phi(\Omega)$ in $E^\Phi(\Omega)$ there exists a sequence of functions $f_n \in C_C^\infty(\Omega) \cap E^\Phi(\Omega)$ such that,

$$\lim_{n\to\infty} \|\chi_K - f_n\|_\Phi = 0.$$

Therefore, by Theorem 2.1.5 (7), $\{\chi_K - f_n\}_{n=1}^\infty$ converges to 0 in measure for any set $A \subset \Omega$ with $|A| < \infty$. It follows that there exists a subsequence $\{n_k\}_{k=1}^\infty \subset \mathbb{N}$ such that $\lim_{k\to\infty} \chi_K(x) - f_{n_k}(x) = 0$ for almost all $x \in \Omega$. For convenience let us rewrite f_{n_k} as f_k. Since $|K| > 0$ there exists $x_0 \in K$ such that $\lim_{k\to\infty} f_k(x_0) = 1$. Take now k_0 such that

$$\frac{1}{2} < f_{k_0}(x_0) < \frac{3}{2}.$$

By continuity of f_{k_0}, there exists a ball $B(x_0, r)$, such that for any $y \in B(x_0, r)$,

$$-\frac{1}{4} < f_{k_0}(y) - f_{k_0}(x_0) < \frac{1}{4}.$$

Combining both of the above inequalities, we get that for any $y \in B(x_0, r)$,

$$\frac{1}{4} < |f_{k_0}(y)| < \frac{7}{4},$$

therefore $\frac{1}{4}\chi_{B(x_0,r)} < |f_{k_0}|$, so $\chi_{B(x_0,r)} \in E^\Phi(\Omega)$. Hence, for each $\lambda > 0$ we have

$$\int_{B(x_0,r)} \Phi(y,\lambda)dy < \infty.$$

On the other hand, $x_0 \in K \subset \operatorname{Sing}\Omega$, and therefore there exists a t_r such that

$$\int\limits_{B(x_0,r)} \Phi(y, t_r) = \infty,$$

and so $\chi_{B(x_0,r)} \notin E^{\Phi}(\Omega)$, a contradiction. $\square$

To prove that the condition $|\operatorname{Sing}\Phi| = 0$ is sufficient for density of C_C^{∞} functions in E_{Φ} we need the following fact about compact sets.

Lemma 2.2.4. *Let K be a compact subset of $\mathbb{R}^d$. There exists a sequence of open, bounded sets $\{U_n\}_{n=1}^{\infty}$ such that, for each $n \in \mathbb{N}$,*

(1) $K \subset U_n$,

(2) $|\overline{U_n}| = |U_n| < |K| + \frac{1}{n}$,

(3) $\overline{U_{n+1}} \subset U_n$, $|U_{n+1}| < |U_n|$.

Proof. Let $K \subset \mathbb{R}^d$ be compact. We will construct the sequence $\{U_n\}_{n=1}^{\infty}$ by induction.

Let $\{Q_{1,n}\}_{n=1}^{\infty}$ be a sequence of open cubes such that $\left|\bigcup\limits_{n=1}^{\infty} Q_{1,n}\right| < |K| + 1$ and $K \subset \bigcup\limits_{n=1}^{\infty} Q_{1,n}$. By compactness of K, there exists M_1 cubes, $\{Q_{1,n}\}_{n=1}^{M_1}$, such that $K \subset \bigcup\limits_{n=1}^{M_1} Q_{1,n}$. Notice that $\bigcup\limits_{n=1}^{M_1} Q_{1,n}$ can be written as a finite union of disjoint cuboids $\{R_j\}_{j=1}^{J_1}$,

$$\bigcup_{n=1}^{M_1} Q_{1,n} = \bigcup_{j=1}^{J_1} R_j.$$

Moreover for each $1 \le j \le J_1$, $|\overline{R_j}| = |R_j|$. Therefore

$$\left|\overline{\bigcup_{n=1}^{M_1} Q_{1,n}}\right| = \left|\overline{\bigcup_{j=1}^{J_1} R_j}\right| = \left|\bigcup_{j=1}^{J_1} \overline{R_j}\right| \le \sum_{j=1}^{J_1} |\overline{R_j}| = \sum_{j=1}^{J_1} |R_j| = \left|\bigcup_{j=1}^{J_1} R_j\right| = \left|\bigcup_{n=1}^{M_1} Q_{1,n}\right|,$$

so

$$\left|\overline{\bigcup_{n=1}^{M_1} Q_{1,n}}\right| = \left|\bigcup_{n=1}^{M_1} Q_{1,n}\right|.$$

Defining $U_1 = \bigcup_{n=1}^{M_1} Q_{1,n}$ we have $K \subset U_1$, $|\overline{U_1}| = |U_1|$ and $|U_1| < |K| + 1$.

Assume that the set U_n is constructed. Now we will construct the set U_{n+1}. First notice that $\overline{U_n}$ is compact, therefore the boundary of U_n, $\operatorname{bd}(U_n)$ is also compact.

Let $f(x) = \operatorname{dist}(x, \operatorname{bd}(U_n))$, the distance of x from $\operatorname{bd}(U_n)$, the function f is continuous on $\mathbb{R}^d$ and for any $x \in K$, $f(x) > 0$. By compactness of K, there exists $x_0 \in K$ such that $\min_{x \in K} f(x) = f(x_0) = \alpha > 0$, so for any $x \in K$,

$$\operatorname{dist}(x, \operatorname{bd}(U_n)) \geq \alpha.$$

Denote

$$\mathcal{Q}_\alpha(K) = \left\{ Q(x,l) : \ x \in K, \ l \leq \frac{\alpha}{\sqrt{d}} \right\}.$$

Notice that, if $Q \in \mathcal{Q}_\alpha(K)$, then $\overline{Q} \subset U_n$. Indeed, $Q = Q(x,l)$ for some $(x_1, \ldots, x_d) = x \in K$ and $l \leq \frac{\alpha}{\sqrt{d}}$. If $(y_1, \ldots, y_d) = y \in \overline{Q}$, then for any $i = 1, \ldots, d$ we have

$$|x_i - y_i| \leq \frac{l}{2}$$

and so

$$|x - y| \leq \frac{l\sqrt{d}}{2} \leq \frac{\alpha}{2}.$$

For any $z \in \mathbb{R}^d \setminus U_n$ we have

$$|z - x| \geq \operatorname{dist}(x, \operatorname{bd}(U_n)) \geq \alpha$$

and so

$$|z-y| \geq ||z-x|-|x-y|| \geq |z-x|-|x-y| \geq \alpha - \frac{\alpha}{2} = \frac{\alpha}{2}.$$

In particular $y \notin \mathbb{R}^d \setminus U_n$ and so $y \in U_n$, i.e. $\overline{Q} \subset U_n$. Since $\overline{Q}$ is closed and U_n is open we have that $\overline{Q} \cap \mathrm{bd}(U_n) = \emptyset$. For each $x \in K$ denote

$$\mathcal{Q}_\alpha(x) = \left\{ Q(x,l) : \; l \leq \frac{\alpha}{\sqrt{d}} \right\}.$$

Clearly $\mathcal{Q}_\alpha(x) \subset \mathcal{Q}_\alpha(K)$ and $\mathcal{Q}_\alpha(x)$ forms a topological basis for neighborhoods of x in the Euclidean topology of $\mathbb{R}^d$. Take now any open set U, such that $K \subset U$. For each $x \in K$, there exists $l_x > 0$ with $Q(x,l_x) \in \mathcal{Q}_\alpha(x)$, such that $Q(x,l_x) \subset U$.

Let U be an open set, such that $K \subset U$ and $|U| < |K| + \frac{1}{n+1}$. Then there exists a family $\{Q(x,l_x)\}_{x\in K} \subset \mathcal{Q}_\alpha(K)$ such that $\bigcup\limits_{x \in K} Q(x,l_x) \subset U$. By compactness of K, there exists $M_{n+1} \in \mathbb{N}$ and a finite subfamily $\{Q(x_m, l_{x_m})\}_{m=1}^{M_{n+1}}$ such that $K \subset \bigcup\limits_{m=1}^{M_{n+1}} Q(x_m, l_{x_m})$. Let us write $Q_{m,n+1} = Q(x_m, l_{x_m})$, then $K \subset \bigcup\limits_{m=1}^{M_{n+1}} Q_{m,n+1}$ and $\left| \bigcup\limits_{m=1}^{M_{n+1}} Q_{m,n+1} \right| < |U| < |K| + \frac{1}{n+1}$. Since $Q_{m,n+1} \in \mathcal{Q}_\alpha(K)$, we have

$$\overline{Q_{m,n+1}} \cap \mathrm{bd}(U_n) = \emptyset.$$

Define $U_{n+1} = \bigcup\limits_{m=1}^{M_{n+1}} Q_{m,n+1}$. Arguing as in case $n=1$, we get that

$$\left| \overline{U_{n+1}} \right| = |U_{n+1}| \, .$$

For every $m = 1, 2, \ldots, M_{n+1}$, we have, $\overline{Q_{m,n+1}} \subset U_n$, therefore

$$\overline{U_{n+1}} = \bigcup_{m=1}^{M_{n+1}} \overline{Q_{m,n+1}} \subset U_n.$$

Recall that for any open set $U \subset \mathbb{R}^d$ with $|U| < \infty$ and any compact set $F \subset U$ we have that $|F| < |U|$. Hence,

$$\left|\overline{U_{n+1}}\right| < |U_n| .$$

By induction, we have constructed a sequence of open, bounded sets $\{U_n\}_{n=1}^{\infty}$ with desired properties. □

We will need the following result.

Theorem 2.2.5. *[10, Theorem 2.6.1] Let $\Omega \subset \mathbb{R}^d$ be open. For every open $U \subset \Omega$ and compact $K \subset U$ there exists $f \in C_C^\infty(\Omega)$, $f : \Omega^d \to [0,1]$ such that $f|_K \equiv 1$ and* $\operatorname{supp} f \subset U$.

Now we can prove that if $|\operatorname{Sing}\Phi| = 0$ and for some compact set K the function $\chi_K \in L^\Phi(\Omega)$ then χ_K can be approximated by elements of $C_C^\infty(\Omega)$.

Theorem 2.2.6. *Let Φ be a MO function defined on an open $\Omega \subset \mathbb{R}^d$. If $|\operatorname{Sing}\Phi| = 0$, then for any compact $K \subset \Omega$ such that $\chi_K \in E^\Phi(\Omega)$, there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset C_C^\infty(\Omega)$, such that $\lim_{n\to\infty} \|f_n - \chi_K\|_\Phi = 0$.*

Proof. Let $K \subset \Omega$ be a compact set such that $\chi_K \in E^\Phi(\Omega)$. Then there exists an open and bounded set U such that $K \subset U$. Define

$$(\operatorname{Sing}\Phi)_U = \overline{\operatorname{Sing}\Phi \cap U}.$$

By Proposition 2.2.2, $\operatorname{Sing}\Phi$ is closed. Hence,

$$(\operatorname{Sing}\Phi)_U = \overline{\operatorname{Sing}\Phi \cap U} = \operatorname{Sing}\Phi \cap \overline{U},$$

and $(\operatorname{Sing}\Phi)_U$ is compact. By Lemma 2.2.4, there exists a sequence of open sets $\{U_n\}_{n=1}^{\infty}$, such that for every $n \in \mathbb{N}$,

(1) $(\operatorname{Sing}\Phi)_U \subset U_n$,

(2) $|\overline{U_n}| = |U_n| < \frac{1}{n}$,

(3) $\overline{U_{n+1}} \subset U_n$ and $|U_{n+1}| < |U_n|$.

Take any $n \in \mathbb{N}$ and define $K_n = K \setminus U_n$. Clearly K_n is compact and $K = (K \cap U_n) \cup K_n$. Setting $V_n = U \setminus \overline{U_{n+1}}$ we have that $K_n \subset V_n$ and V_n is open. First we will show that $\chi_{V_n} \in E^\Phi(\Omega)$. Notice that

$$V_n \subset \overline{U} \setminus U_{n+2} \subset \overline{U} \setminus (\overline{U} \cap \operatorname{Sing}\Phi) \subset \overline{U} \setminus \operatorname{Sing}\Phi.$$

Hence, $V_n \cap \operatorname{Sing}\Phi \subset (\overline{U} \setminus U_{n+2}) \cap \operatorname{Sing}\Phi = \emptyset$. By definition of $\operatorname{Sing}\Phi$, for each $x \in \overline{U} \setminus U_{n+2}$ there exists $r_x > 0$ such that for all $\lambda > 0$,

$$\int_{B(x,r_x)\cap\Omega} \Phi(y,\lambda)dy < \infty.$$

On the other hand, since $V_n \subset \overline{U} \setminus U_{n+2}$ and $\overline{U} \setminus U_{n+2}$ is closed, so $\overline{V_n} \subset \overline{U} \setminus U_{n+2}$. Hence, for every $x \in \overline{V_n}$ here exists $r_x > 0$ such that for all $\lambda > 0$,

$$\int_{B(x,r_x)\cap\Omega} \Phi(y,\lambda)dy < \infty.$$

The family $\{B(x,r_x)\}_{x\in\overline{V_n}}$ is an open cover of $\overline{V_n}$, so by compactness of $\overline{V_n}$ there exists a finite subcover $\{B(x_i,r_{x_i})\}_{i=1}^k$ of $\overline{V_n}$. Then for any $\lambda > 0$,

$$\int_\Omega, \Phi(x,\lambda\chi_{V_n}(x))dx \le \frac{1}{k}\sum_{i=1}^k \int_{B(x_i,r_{x_i})\cap\Omega} \Phi(y,\lambda k)dy < \infty,$$

and therefore $\chi_{V_n} \in E^\Phi(\Omega)$. Since $\chi_{K_n} \le \chi_{V_n}$, we have $\chi_{K_n} \in E^\Phi(\Omega)$. Recall that $K_n = K \setminus U_n$ and $|U_n| < \frac{1}{n}$, hence $\lim_{n\to\infty} \chi_K - \chi_{K_n} = 0$ in measure. We can find a subsequence $\{n_k\}_{k=1}^\infty \subset \mathbb{N}$, such that $\lim_{k\to\infty} \chi_{K_{n_k}} = \chi_K$ a.e.. Without loss of generality, assume that $\{\chi_{K_{n_k}}\}_{k=1}^{\infty} = \{\chi_{K_n}\}_{n=1}^{\infty}$. For any $\lambda > 0$ we have $I_\Phi(\lambda\chi_K) < \infty$ and

$$\Phi(x, \lambda(\chi_K(x) - \chi_{K_n}(x)) \leq \Phi(x, \lambda_K(x)) \text{ for a.e. } x \in \mathbb{R}^d.$$

Hence by the Lebesgue Dominated Convergence Theorem $\lim_{n\to\infty} I_\Phi(\lambda(\chi_K - \chi_{K_n})) = 0$ so

$$\lim_{n\to\infty} \|\chi_K - \chi_{K_n}\|_\Phi = 0. \tag{2.7}$$

For a fixed $n \in \mathbb{N}$, by Lemma 2.2.4, there exists a sequence of open sets $\{W'_m\}_{m=1}^{\infty}$ such that for every $m \in \mathbb{N}$,

(1′) $K_n \subset W'_m$,

(2′) $|\overline{W'_m}| = |W'_m| < |K_n| + \frac{1}{m}$,

(3′) $\overline{W'_{m+1}} \subset W'_m$.

Define now $W_m = W'_m \cap V_n$. For each $m \in \mathbb{N}$ we have $K_n \subset W_m$ and W_m is open. Hence in view of Theorem 2.2.5 there exists a function $f_{m,n}$, such that $f_{m,n} \in C_C^\infty(\Omega)$, $f_{m,n} : \Omega^d \to [0,1]$ and $f_{m,n}|_{K_n} \equiv 1$ and $\operatorname{supp} f_{m,n} \subset W_m$. Notice that $f_{m,n} \leq \chi_{W_m}$ a.e., and hence $\chi_{W_m} - f_{m,n} \leq \chi_{W_m \setminus K_n}$ a.e. For a fixed $\lambda > 0$ and any $m \in \mathbb{N}$ by (2') we have

$$\begin{aligned}
|\{x : \chi_{W_m}(x) - f_{m,n}(x) > \lambda\}| &\leq |\{x : \chi_{W_m \setminus K_n}(x) > \lambda\}| \\
&\leq |W_m \setminus K_n| = |W_m| - |K_n| \\
&\leq |W'_m| - |K_n| < \frac{1}{m}.
\end{aligned}$$

Therefore $\lim_{m\to\infty} f_{m,n} = \chi_{K_n}$ in measure for every $n \in \mathbb{N}$. We can find a subsequence $\{m_k\}_{k=1}^{\infty} \subset \mathbb{N}$, such that $\lim_{k\to\infty} f_{m_k,n} = \chi_{K_n}$ a.e. Without loss of generality, assume that $\{f_{m_k,n}\}_{k=1}^{\infty} = \{f_{m,n}\}_{m=1}^{\infty}$. Since $f_{m,n} \leq \chi_{W_m} \leq \chi_{V_n}$ a.e. and $\chi_{V_n} \in E^\Phi(\Omega)$, so

$f_{m,n} \in E^\Phi(\Omega)$. For any $\lambda > 0$ we have $I_\Phi(\lambda\chi_{V_n}) < \infty$ and

$$\Phi(x, \lambda(f_{m,n}(x) - \chi_{K_n}(x))) \leq \Phi(x, \lambda\chi_{V_n}(x)) \text{ for a.e. } x \in \mathbb{R}^d.$$

By the Lebesgue Dominated Convergence Theorem $\lim_{m\to\infty} I_\Phi(\lambda(f_{m,n} - \chi_{K_n})) = 0$, so, for every $n \in \mathbb{N}$

$$\lim_{m\to\infty} \|f_{m,n} - \chi_{K_n}\|_\Phi = 0.$$

Now, for every $n \in \mathbb{N}$ define $f_n = f_{m_n,n}$, where m_n is the smallest integer such that

$$\|f_{m_n,n} - \chi_{K_n}\|_\Phi < \frac{1}{n}. \tag{2.8}$$

Finally, by (2.7) and (2.8),

$$\lim_{n\to\infty} \|f_n - \chi_K\|_\Phi \leq \lim_{n\to\infty} \|f_n - \chi_{K_n}\|_\Phi + \lim_{n\to\infty} \|\chi_{K_n} - \chi_K\|_\Phi = 0.$$

□

Finally we can show that if $|\operatorname{Sing}\Phi| = 0$, then functions from $C_C^\infty(\Omega) \cap E^\Phi(\Omega)$ are dense in $E^\Phi(\Omega)$.

Theorem 2.2.7. *Let $\Omega \subset \mathbb{R}^d$ be open and Φ be a MO function defined on Ω. If $|\operatorname{Sing}\Phi| = 0$, then $C_C^\infty(\Omega) \cap E^\Phi(\Omega)$ is dense in $E^\Phi(\Omega)$. In other words $\overline{C_C^\infty(\Omega) \cap E^\Phi(\Omega)} = E^\Phi(\Omega)$.*

Proof. By Theorem 2.1.10, $E^\Phi(\Omega) \cap S(\Omega)$ is dense in $E^\Phi(\Omega)$. Every simple function is a finite linear combination of characteristic functions of measurable sets with finite measure. Hence, by linearity and in the view of Theorem 2.2.6 it suffices to show that any characteristic function χ_A of a set of finite measure A such that $\chi_A \in E^\Phi(\Omega)$ can be approximated by a characteristic function of a compact set.

For any such A, by inner regularity of the Lebesgue measure there exist a sequence of compact sets $K_n \subset A$, such that $|A \setminus K_n| < \frac{1}{n}$ for every $n \in \mathbb{N}$. For $\lambda > 0$ we have

$$I_\Phi(\lambda(\chi_A - \chi_{K_n})) = I_\Phi(\lambda(\chi_{A\setminus K_n})).$$

Clearly $\Phi(x, \lambda\chi_{A\setminus K_n}(x)) \leq \Phi(x, \lambda\chi_A(x))$ for a.e. $x \in \Omega$. Since $I_\Phi(\lambda\chi_A) < \infty$, so by the absolute continuity of the Lebesgue integral, we deduce that

$$\lim_{n\to\infty} I_\Phi(\lambda(\chi_A - \chi_{K_n})) = 0,$$

and so $\lim_{n\to\infty} \|\chi_K - \chi_{K_n}\|_\Phi = 0.$ □

Since $E^\Phi(\Omega) = L^\Phi(\Omega)$ if and only if Φ satisfies Δ_2 (Theorem 2.1.9), the next result is an immediate consequence of Theorems 2.1.9, 2.2.7, 2.2.3.

Corollary 2.2.8. *Let $\Omega \subset \mathbb{R}^d$ be open and Φ be a MO function defined on Ω satisfying Δ_2 condition. Then $C_C^\infty(\Omega) \cap L^\Phi(\Omega)$ is dense in $L^\Phi(\Omega)$ if and only if $|\operatorname{Sing}\Phi| = 0$.*

We end this section by remarking that this is an almost full description of the problem of density of $C_C^\infty(\Omega)$ functions in $L^\Phi(\Omega)$. It remains to answer the question: is the Δ_2 condition is is necessary for $\overline{C_C^\infty(\Omega) \cap L^\Phi(\Omega)} = L^\Phi(\Omega)$? At the moment of writing this work the author is not able to provide an answer to this question.

CHAPTER 3

MUSIELAK ORLICZ SOBOLEV SPACES

3.1 Basic facts about Musielak Orlicz Sobolev spaces

In this chapter we will introduce and study the Musielak Orlicz Sobolev spaces. To this end we need to introduce the notion of weakly differientable functions. We start by studying the structure of locally integrable functions defined on an open set Ω. Recall that for any set $A \subset \mathbb{R}^d$ the symbol A° stands for the interior of A, that is the largest open subset of A. First we notice that for any open subset Ω of $\mathbb{R}^d$ there exists a family $\{K_n\}_{n=1}^{\infty}$ of compact subsets of Ω such that

$(K1)$ For every $n \in \mathbb{N}$ we have $\overline{K_n^\circ} = K_n \subset K_{n+1}^\circ$,

$(K2)$ $\bigcup\limits_{n=1}^{\infty} K_n = \Omega$.

Conditions (K1) and (K2) imply that for any compact $K \subset \Omega$ there exists $n \in \mathbb{N}$ such that $K \subset K_n$. Indeed, by (K1) and (K2),

$$\Omega = \bigcup_{n=1}^{\infty} K_n \subset \bigcup_{n=1}^{\infty} K_{n+1}^\circ.$$

For each $n \in N$ the set K_{n+1}° is open, therefore for any compact $K \subset \Omega$ there exists $N \in \mathbb{N}$ such that $K \subset \bigcup\limits_{n=1}^{N} K_{n+1}^\circ$. By (K1) we have that

$$K \subset \bigcup_{n=1}^{N} K_{n+1}^\circ = K_{N+1}^\circ \subset \overline{K_{N+1}^\circ} = K_{N+1}.$$

For each $n \in \mathbb{N}$ we define a seminorm $\|\cdot\|_{K_n}$ on $L^1_{loc}(\Omega)$ via the formula

$$\|f\|_{K_n} = \|f\chi_{K_n}\|_1,$$

where f is an element of $L^1_{loc}(\Omega)$. We define a function $d : L^1_{loc}(\Omega) \times L^1_{loc}(\Omega) \to [0,1]$ by the formula

$$d(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f-g\|_{K_n}}{1+\|f-g\|_{K_n}}. \tag{3.1}$$

The following proposition shows that $L^1_{loc}(\Omega)$ equipped with the metric d is a Fréchet space. It is a complete linear metric space with translation invariant metric given by a countable family of seminorms.

Proposition 3.1.1. *Let d be a function defined by* (3.1). *The following statements are true.*

(1) *The function d is a metric on $L^1_{loc}(\Omega)$ and $(L^1_{loc}(\Omega), d)$ is a Fréchet space.*

(2) *For any sequence $\{f_k\}_{k=1}^{\infty} \subset L^1_{loc}(\Omega)$ and any $f \in L^1_{loc}(\Omega)$, $\lim_{k\to\infty} d(f_k, f) = 0$ if and only if for all compact sets $K \subset \Omega$ we have*

$$\lim_{k\to\infty} \int_K |f(x) - f_k(x)| dx = 0.$$

Proof. Let $\{K_n\}_{n=1}^{\infty}$ be a family of compact subsets of Ω satisfying conditions (K1) and (K2) and be d the function defined by the formula (3.1).

(1) First we show that d is translation invariant, that is, for every $f, g, h \in L^1_{loc}(\Omega)$ we have

$$d(f+h, g+h) = d(f,g).$$

Indeed, for every $f, g, h \in L^1_{loc}(\Omega)$ we have

$$\begin{aligned} d(f+h, g+h) &= \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f+h-(g+h)\|_{K_n}}{1+\|f+h-(g+h)\|_{K_n}} = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f-g\|_{K_n}}{1+\|f+h-g\|_{K_n}} \\ &= d(f,g). \end{aligned}$$

Let us show now that d is a metric on $L^1_{loc}(\Omega)$. For any $f \in L^1_{loc}(\Omega)$, we have

$$d(f,f) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f-f\|_{K_n}}{1+\|f-f\|_{K_n}} = 0.$$

On the other hand if for some $f, g \in L^1_{loc}(\Omega)$ we have $d(f,g) = 0$, then for each $n \in \mathbb{N}$,

$$\|f-g\|_{K_n} = \|(f-g)\chi_{K_n}\|_1 = 0$$

and so $f(x) = g(x)$ for a.a. $x \in K_n$. Hence $f = g$ a.e. in Ω. Therefore $d(f,g) = 0$ if and only if $f = g$ a.e..

Take any $f, g \in L^1_{loc}(\Omega)$, clearly we have

$$d(f,g) = d(g,f).$$

Notice that the function $F : [0,\infty) \to [0,1)$, defined by the formula

$$F(t) = \frac{t}{t+1},$$

for $t \geq 0$, is increasing. Indeed, for any $t \geq 0$,

$$F'(t) = \frac{1}{(t+1)^2} > 0.$$

Moreover, F is subadditive on the interval $[0,\infty)$. In fact, for every $t_1, t_2 \geq 0$ we have

$$F(t_1+t_2) = \frac{t_1+t_2}{1+t_1+t_2} = \frac{t_1}{1+t_1+t_2} + \frac{t_2}{1+t_1+t_2} \leq \frac{t_1}{1+t_1} + \frac{t_2}{1+t_2} = F(t_1)+F(t_2).$$

Hence, for any $f, g, h \in L^1_{loc}(\Omega)$ it follows that

$$\begin{aligned} d(f,g) &= \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f-g\|_{K_n}}{1+\|f-g\|_{K_n}} = \sum_{n=1}^{\infty} \frac{1}{2^n} F(\|f-g\|_{K_n}) \\ &\leq \sum_{n=1}^{\infty} \frac{1}{2^n} F(\|f-h\|_{K_n} + \|h-g\|_{K_n}) \\ &\leq \sum_{n=1}^{\infty} \frac{1}{2^n} F(\|f-h\|_{K_n}) + \sum_{n=1}^{\infty} \frac{1}{2^n} F(\|h-g\|_{K_n}) = d(f,h) + d(h,g). \end{aligned}$$

Hence the triangle inequality is satisfied.

Finally let us show that $(L^1_{loc}(\Omega), d)$ is complete. Let $\{f_k\}_{k=1}^{\infty} \subset L^1_{loc}(\Omega)$ be a Cauchy sequence in $(L^1_{loc}(\Omega), d)$. Then for each $n \in \mathbb{N}$, $\{\|f_k\|_{K_n}\}_{k=1}^{\infty}$ is a Cauchy sequence. Since for every $n, k \in \mathbb{N}$ we have that $\|f_k\|_{K_n} = \|f_k \chi_{K_n}\|_1$ we conclude that, for every $n \in \mathbb{N}$ the sequence $\{f_k \chi_{K_n}\}_{k=1}^{\infty}$ is Cauchy in $L^1(\Omega)$. Hence, for each $n \in \mathbb{N}$ there exists $g_n \in L^1(\Omega)$ such that

$$\lim_{k \to \infty} \|g_n - f_k \chi_{K_n}\|_1 = 0. \tag{3.2}$$

Moreover, since for each $k \in \mathbb{N}$ we have $\operatorname{supp} f_k \chi_{K_n} \subset K_n$ we get that $\operatorname{supp} g_n \subset K_n$. We also notice that, if $n_1 \leq n_2$, then $g_{n_1} = g_{n_2}$ a.e. in K_{n_1}. Let us define a function f by the formula

$$f(x) = \lim_{n \to \infty} g_n(x),$$

for a.e. $x \in \Omega$. Then for any $n \in \mathbb{N}$, $g(x) = f_n(x)$ a.e. on K_n. First let us show that f is in $L^1_{loc}(\Omega)$. Take any compact set $K \subset \Omega$, since $\Omega = \bigcup_{n=1}^{\infty} K_n = \bigcup_{n=1}^{\infty} K_n^{\circ}$, there exists $n \in \mathbb{N}$ such that $K \subset K_n^{\circ} \subset K_n$. Hence

$$\int_K |f(x)| dx \leq \int_{K_n} |f(x)| dx = \int_{K_n} |g_n(x)| dx < \infty.$$

Therefore we conclude that f is an element of $L^1_{loc}(\Omega)$. The only thing left to show is that $\lim_{k\to\infty} d(f, f_k) = 0$. Take any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\sum\limits_{N+1}^{\infty} \frac{1}{2^n} < \frac{\varepsilon}{2}$. By (3.2), there exists $M \in \mathbb{N}$ such that for any $n = 1, \ldots, N$ and any $k > M$ we have $\|g_n - f_k\chi_{K_n}\|_1 < \frac{\varepsilon}{2}$. Therefore for any $k > M$,

$$\begin{aligned} d(f, f_k) &= \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|(f - f_k)\chi_{K_n}\|_1}{1 + \|(f - f_k)\chi_{K_n}\|_1} \\ &< \sum_{n=1}^{N} \frac{1}{2^n} \frac{\|g_n - f_k\chi_{K_n}\|_1}{1 + \|g_n - f_k\chi_{K_n}\|_1} + \frac{\varepsilon}{2} < \sum_{n=1}^{N} \frac{1}{2^n} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

Hence $\lim_{k\to\infty} d(f, f_k) = 0$.

(2) First assume that for all compact set $K \subset \Omega$ we have

$$\lim_{k\to\infty} \int_K |f(x) - f_k(x)| dx = 0.$$

Thus for any $n \in \mathbb{N}$ we have

$$\lim_{k\to\infty} \int_{K_n} |f(x) - f_k(x)| dx = \lim_{k\to\infty} \|f - f_k\|_{K_n} = 0. \tag{3.3}$$

Take any $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that $\sum\limits_{n=N+1}^{\infty} \frac{1}{2^n} < \frac{\varepsilon}{2}$. By the equation (3.3), there exists $M \in \mathbb{N}$ such that for $n = 1, \ldots, N$ and all $k > M$ we have $\|f - f_k\|_{K_n} < \frac{\varepsilon}{2}$. Hence, for any $k > M$,

$$d(f, f_k) = \sum_{n=1}^{N} \frac{1}{2^n} \frac{\|f - f_k\|_{K_n}}{1 + \|f - f_k\|_{K_n}} + \sum_{n=N+1}^{\infty} \frac{1}{2^n} \frac{\|f - f_k\|_{K_n}}{1 + \|f - f_k\|_{K_n}} < \varepsilon.$$

We conclude that $\lim_{k\to\infty} d(f, f_k) = 0$.

Assume now that $\lim_{k\to\infty} d(f, f_k) = 0$. For each $n \in \mathbb{N}$, we have that $\frac{1}{2^n} \frac{\|f - f_k\|_{K_n}}{1 + \|f - f_k\|_{K_n}} \leq d(f, f_k)$ and so $\lim_{k\to\infty} \frac{1}{2^n} \frac{\|f - f_k\|_{K_n}}{1 + \|f - f_k\|_{K_n}} = 0$. Since the sequence $\left\{ \frac{\|f - f_k\|_{K_n}}{1 + \|f - f_k\|_{K_n}} \right\}_{k=0}^{\infty}$ converges to 0 only if the enumerator converges to 0 we conclude that for any $n \in \mathbb{N}$ we have $\lim_{k\to\infty} \int_{K_n} |f(x) - f_k(x)|dx = 0$. Take any compact subset $K \subset \Omega$. There exists $n \in \mathbb{N}$ such that $K \subset K_n$ and so, for any $k \in \mathbb{N}$ we have

$$\int_K |f(x) - f_k(x)|dx \leq \int_{K_n} |f(x) - f_k(x)|dx.$$

Therefore we conclude that $\lim_{k\to\infty} \int_K |f(x) - f_k(x)|dx = 0$, which completes the proof. □

It follows from (2) of Theorem 3.1.1 that the topology generated by the metric d is independent of the choice of the family $\{K_n\}_{n=1}^{\infty}$ satisfying conditions (K1) and (K2).

Now we characterize those MO functions Φ for which the inclusion map $L^\Phi(\Omega) \subset L^1_{loc}(\Omega)$ is continuous.

Theorem 3.1.2. *Let Ω be an open subset of $\mathbb{R}^d$ and Φ be a MO function defined on Ω. $L^\Phi(\Omega) \subset L^1_{loc}(\Omega)$ if and only if for any compact subset $K \subset \Omega$ there exists a constant $C_K > 0$ and a nonnegative function $h_K \in L^1(K)$ such that for a.a. $x \in K$ and any $t \geq 0$ we have*

$$t \leq C_k\Phi(x,t) + h_k(x). \tag{3.4}$$

Moreover, if $L^\Phi(\Omega) \subset L^1_{loc}(\Omega)$, then the inclusion map is continuous.

Proof. Notice that if $L^\Phi(\Omega) \subset L^1_{loc}(\Omega)$, then for any compact $K \subset \Omega$ and any $f \in L^\Phi$ we have $f\chi_K \in L^1(K)$, in other words

$$L^\Phi(K) \subset L^1(K).$$

On the other hand if for any compact $K \subset \Omega$ and any $f \in L^\Phi$ we have $f\chi_K \in L^1(K)$ we get that f is locally integrable on Ω. Hence, $L^\Phi(\Omega) \subset L^1_{loc}(\Omega)$ if and only if $L^\Phi(K) \subset L^1(K)$ for every compact $K \subset \Omega$.

Since $L^1(K)$ is MO space generated by the function $t\chi_K(x)$, by Theorem 2.1.8 (2), $L^\Phi(K) \subset L^1(K)$, if and only if, there exist a constant $C_K > 0$ and a non-negative function $h_K \in L^1(K)$ such that for a.a. $x \in K$ and any $t \geq 0$ we have

$$t \leq C_K \Phi(x,t) + h_K(x).$$

Which proves the first part of the theorem.

Let $L^\Phi(\Omega) \subset L^1_{loc}(\Omega)$. Hence the inequality (3.4) is satisfied. Take any compact set $K \subset \Omega$ and any function $f \in L^\Phi(\Omega)$ such that $\|f\|_\Phi > 0$. We have

$$\begin{aligned} \int\limits_K \frac{|f(x)|}{\|f\|_\Phi} dx &\leq C_K \int\limits_K \Phi\left(x, \frac{|f(x)|}{\|f\|_\Phi}\right) dx + \|h_K\|_1 \leq C_K I_\Phi\left(\frac{f}{\|f\|_\Phi}\right) + \|h_K\|_1 \\ &\leq C_K + \|h_K\|_1. \end{aligned}$$

Hence,

$$\int\limits_K |f(x)| dx \leq (C_K + \|h_K\|_1)\|f\|_\Phi.$$

Therefore for any sequence $\{f_k\}_{k=1}^\infty \subset L^\Phi(\Omega)$ converging in norm to some $f \in L^\Phi(\Omega)$ we have

$$\lim_{k\to\infty} \int\limits_K |f(x) - f_k(x)| dx = 0.$$

By Proposition 3.1.1 (2), it follows that

$$\lim_{k\to\infty} d(f_k, f) = 0,$$

which shows the continuity of the inclusion map.

□

We recall now the notion of the weak derivative.

Definition 3.1.3. (*Weak derivative*)*[2, p. 22] We say that a function $f \in L^1_{loc}(\Omega)$ is weakly k-differentiable if for every multi-index α with $|\alpha| \le k$ there exists a function $f_\alpha \in L^1_{loc}(\Omega)$ such that for every $u \in C_C^\infty(\Omega)$ we have*

$$\int_\Omega f(x)\partial^\alpha u(x)dx = (-1)^{|\alpha|} \int_\Omega f_\alpha(x)u(x)dx.$$

If the above identity holds, we denote $\partial^\alpha f = f_\alpha$ and say that f_α is a weak α-derivative of f.

Weak differentiability has the following characterization.

Lemma 3.1.4. *Let Ω be an open subset of $\mathbb{R}^d$, f, g be locally integrable functions on Ω and α be an multi-index. The function f is the α-weak derivative of g if and only if for every $u \in C_C^k(\Omega)$, where $k = |\alpha|$, we have*

$$\int_\Omega g(x)\partial^\alpha u(x)dx = (-1)^{|\alpha|} \int_\Omega f(x)u(x)dx.$$

Proof. Let Ω be an open subset of $\mathbb{R}^d$, f, g be locally integrable functions defined on Ω and α be an multi-index. Clearly if for every $u \in C_C^k(\Omega)$, where $k = |\alpha|$, we have

$$\int_\Omega g(x)\partial^\alpha u(x)dx = (-1)^{|\alpha|} \int_\Omega f(x)u(x)dx,$$

then f is α-weak derivative of g, as $C_C^\infty(\Omega) \subset C_C^k(\Omega)$.

On the other hand, assume that f is the α-weak derivative of g and take any $u \in C_C^k(\Omega)$. Recall that standard mollifier is the function $J : \mathbb{R}^d \to [0, \infty)$ given by the formula

$$J(x) = Ce^{\frac{-1}{1-|x|^2}}\chi_{B(0,1)}(x),$$

and that, for $r > 0$ we have $J_{(r)}(x) := \frac{1}{r^d} J\left(\frac{1}{r}x\right)$. We define the function u_r by

$$u_r(x) = (u * J_{(r)})(x),$$

where $x \in \mathbb{R}^d$. Notice now that since $u \in C^k_C(\Omega)$, then for small enough r, say for $r < R$, we have $\operatorname{ess\,supp} u_r \subset \Omega$. Clearly u_r is a smooth function, since $J_{(r)}$ is smooth, hence for $r < R$ we have $u_r \in C^\infty_C(\Omega)$. Since f is the α-weak derivative of g we get that

$$\int\limits_\Omega g(x)\partial^\alpha u_r(x)dx = (-1)^{|\alpha|} \int\limits_\Omega f(x)u_r(x)dx.$$

Moreover, by the fact that we have $\partial^\alpha u_r = J_{(r)} * \partial^\alpha u$ and that both u and $\partial^\alpha u$ are continuous functions, we have that $\partial^\alpha u_r$ and u_r converge uniformly to $\partial^\alpha u$ and u respectively, as r goes to 0 (see [19, Theorem 8.14, p. 242]). Notice now that, for $r < R/2$ the supports of all functions u_r are contained in one compact set $\operatorname{ess\,supp} u_{R/2} = K \subset \Omega$. Since both f and g are locally integrable functions $f\chi_K$ and $g\chi_K$ generate continuous functionals on $L^\infty(K)$ of the form $u \mapsto \int\limits_K g(x)u(x)dx$ and $u \mapsto \int\limits_K f(x)u(x)dx$, where $u \in L^\infty(K)$. Using the continuity of those functionals and the fact that both $\partial^\alpha u_r$ and u_r converge uniformly on Ω respectively to $\partial^\alpha u$ and u as r goes to 0, we conclude that

$$\begin{aligned}\int\limits_\Omega g(x)\partial^\alpha u(x)dx &= \lim_{r\to 0} \int\limits_K g(x)\partial^\alpha u_r(x)dx = \lim_{r\to 0}(-1)^{|\alpha|} \int\limits_K f(x)u_r(x)dx \\ &= (-1)^{|\alpha|} \int\limits_\Omega f(x)u(x)dx.\end{aligned}$$

□

Finally we can define the notion of a *Musielak Orlicz Sobolev* space.

Definition 3.1.5. *Let $\Omega \subset \mathbb{R}^d$ be open and let Φ be a MO function on Ω such that $L^\Phi(\Omega) \subset L^1_{loc}(\Omega)$. For any $k \in \mathbb{N}$ we define the Musielak Orlicz Sobolev spaces, in* *short MOS spaces as*

$$W^{k,\Phi}(\Omega) = \left\{ f \in L^0(\Omega) : \partial^\alpha f \in L^\Phi(\Omega),\ |\alpha| \le k \right\},$$

$$E^{\Phi,k}(\Omega) = \left\{ f \in L^0(\Omega) : \partial^\alpha f \in E^\Phi(\Omega),\ |\alpha| \le k \right\}.$$

Clearly $E^{\Phi,k}(\Omega) \subset W^{k,\Phi}(\Omega)$. *We endow* $W^{k,\Phi}(\Omega)$ *with the norm*

$$\|f\|_{W^{k,\Phi}} = \sum_{|\alpha| \le k} \|\partial^\alpha f\|_\Phi.$$

Let $\{f_n\}_{n=1}^\infty \subset W^{k,\Phi}(\Omega)$ *and* $f \in W^{k,\Phi}(\Omega)$. *We say that* f_n *converges to* f *in modular in* $W^{k,\Phi}(\Omega)$ *if there exists a* $\lambda > 0$ *such that for any multi-index* α *with* $|\alpha| \le k$,

$$\lim_{n\to\infty} \int_\Omega \Phi\left(x, \lambda \left|\partial^\alpha f_n(x) - \partial^\alpha f(x)\right|\right) dx = 0.$$

Sometimes it will be convenient to talk about MOS spaces by using the language of modulars. We have the following fact relating MOS with equivalent modular definition.

Proposition 3.1.6. *Let* $\rho : W^{k,\Phi}(\Omega) \to [0,\infty]$ *be a modular defined by the formula*

$$\rho(f) = \sum_{|\alpha| \le k} I_\Phi(\partial^\alpha f).$$

For every $f \in W^{k,\Phi}(\Omega)$ *let*

$$\|f\| = \inf\left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) \le 1\right\}.$$

The functional $\|\cdot\|$ *is a norm on* $W^{k,\Phi}(\Omega)$. *Moreover* $\|\cdot\|$ *is equivalent to* $\|\cdot\|_{W^{k,\Phi}}$ *on* $W^{k,\Phi}(\Omega)$.

Proof. Notice that, by convexity of Φ, the functional ρ is a modular and $\|\cdot\|$ is a norm on $W^{k,\Phi}$. Let $C(k,d) = \sum\limits_{|\alpha|\le k} 1$, then for $f \in W^{k,\Phi}(\Omega)$ such that $f \neq 0$ we have

$$\rho\left(\frac{f}{C(k,d)\|f\|_{W^{k,\Phi}}}\right) = \sum_{|\alpha|\le k} I_\Phi\left(\frac{\partial^k f}{C(k,d)\sum\limits_{|\alpha|\le k}\|\partial^\alpha f\|_\Phi}\right) \le \sum_{|\alpha|\le k}\frac{1}{C(k,d)} = 1,$$

therefore $\|f\| \le C(k,d)\|f\|_{W^{k,\Phi}}$. On the other hand, for any multi-index α with $|\alpha| \le k$ and $\lambda > 0$ such that $\rho\left(\frac{f}{\lambda}\right) = \sum\limits_{|\alpha|\le k} I_\Phi\left(\frac{\partial^\alpha f}{\lambda}\right) \le 1$ we have

$$I_\Phi\left(\frac{\partial^\alpha f}{\lambda}\right) \le \rho\left(\frac{f}{\lambda}\right) \le 1.$$

Hence, for each multi-index λ with $|\lambda| \le k$ we have $\|\partial^\alpha f\|_\Phi \le \lambda$. Therefore

$$\|\partial^\alpha f\|_\Phi \le \inf\left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) \le 1\right\} = \|f\|,$$

and so $\|f\|_{W^{k,\Phi}} = \sum\limits_{|\alpha|\le k}\|\partial^\alpha f\|_\Phi \le C(k,d)\|f\|$.

□

The following theorem establishes the basic properties of MOS spaces.

Theorem 3.1.7. *Let $\Omega \subset \mathbb{R}^d$ be open, Φ be a MO function on Ω such that $L^\Phi(\Omega) \subset L^1_{loc}(\Omega)$ and $k \in \mathbb{N}$. The following statements holds true.*

(1) *$E^{\Phi,k}(\Omega)$ is a closed subspace of $W^{k,\Phi}(\Omega)$.*

(2) *$W^{k,\Phi}(\Omega)$ is a Banach space.*

(3) *If Φ satisfies Δ_2 condition, then $E^{\Phi,k}(\Omega) = W^{k,\Phi}(\Omega)$.*

Proof. (1) First we will show that $E^{\Phi,k}(\Omega)$ is a closed subspace of $W^{k,\Phi}(\Omega)$. Let $\{f_n\}_{n=1}^\infty \subset E^{\Phi,k}(\Omega)$ be such that $\lim\limits_{n\to\infty}\|f_n - f\|_{W^{k,\Phi}} = 0$ for some $f \in W^{k,\Phi}(\Omega)$. We will prove that $f \in E^{\Phi,k}(\Omega)$. Taking any multi-index α with $|\alpha| \le k$, since for every

$n \in \mathbb{N}$, $\|\partial^\alpha f_n - \partial^\alpha f\|_\Phi \leq \|f_n - f\|_{W^{k,\Phi}}$ we conclude that

$$\lim_{n\to\infty} \|\partial^\alpha f_n - \partial^\alpha f\|_\Phi = 0.$$

Hence, for all $\lambda > 0$, $\lim\limits_{n\to\infty} I_\Phi(\lambda(\partial^\alpha f_n - \partial^\alpha f)) = 0$. For any $\lambda > 0$ take $n \in \mathbb{N}$ such that $I_\Phi(2\lambda(\partial^\alpha f_n - \partial^\alpha f)) < 1$. Then since $\partial^\alpha f_n \in E^{\Phi,k}(\Omega)$,

$$I_\Phi(\lambda\partial^\alpha f) \leq \frac{1}{2} I_\Phi(2\lambda(\partial^\alpha f - \partial^\alpha f_n)) + \frac{1}{2} I_\Phi(2\lambda\partial^\alpha f_n) < \infty.$$

Hence, for each multi-index α with $|\alpha| \leq k$ we have $\partial^\alpha f \in E^\Phi(\Omega)$ and so $f \in E^{\Phi,k}(\Omega)$.

(2) Let now $\{f_n\}_{n=1}^\infty$ be a Cauchy sequence in $W^{k,\Phi}(\Omega)$. Since for every $n, m \in \mathbb{N}$, $\|f_n - f_m\|_{W^{k,\Phi}} = \sum\limits_{|\alpha|\leq k} \|\partial^\alpha f_n - \partial^\alpha f_m\|_\Phi$ we get that for each multi-index α with $|\alpha| \leq k$, $\{\partial^\alpha f_n\}_{n=1}^\infty$ is a Cauchy sequence in $L^\Phi(\Omega)$. Hence for each multi-index α with $|\alpha| \leq k$ there exists $f_\alpha \in L^\Phi(\Omega)$ such that

$$\lim_{n\to\infty} \|f_\alpha - \partial^\alpha f_n\|_\Phi = 0. \tag{3.5}$$

In particular for $\alpha = 0$ there exists $f \in L^\Phi(\Omega)$ such that

$$\lim_{n\to\infty} \|f - f_n\|_\Phi = 0.$$

Now it suffices to show that f is α-weakly differentiable and that $\partial^\alpha f = f_\alpha$ for any multi-index α with $|\alpha| \leq k$. Take any $u \in C_C^\infty(\Omega)$ and any multi-index α with

$|\alpha| \leq k$. For any $n \in \mathbb{N}$ we have

$$\int_{\Omega} f(x)\partial^{\alpha}u(x)dx = \int_{\Omega} \left(f(x) - f_n(x) + f_n(x)\right)\partial^{\alpha}u(x)dx$$
$$= \int_{\Omega} \left(f(x) - f_n(x)\right)\partial^{\alpha}u(x)dx - \int_{\Omega} \partial^{\alpha}f_n(x)u(x)dx = I(n) - II(n)$$

It follows from the continuous embedding of $L^{\Phi}(\Omega)$ into $L^1_{loc}(\Omega)$ that for any compact set $K \subset \Omega$ we have

$$\lim_{n\to\infty} \|(f_{\alpha} - \partial^{\alpha}f_n)\chi_K\|_1 = 0, \tag{3.6}$$

in particular,

$$\lim_{n\to\infty} \|(f - f_n)\chi_K\|_1 = 0. \tag{3.7}$$

Letting now $K = \operatorname{ess\,supp} u$, (3.6) and (3.7) show that $\{f_{n|K}\}_{n=1}^{\infty}$ and $\{\partial^{\alpha}f_{n|K}\}_{n=1}^{\infty}$ converge in $L^1(K)$ to $f_{|K}$ and $(\partial^{\alpha}f)_{|K}$ respectively. Since u and $\partial^{\alpha}u$ are both bounded functions they define continuous linear functionals on $L^1(\Omega)$ by the formulas $g \mapsto \int_K g(x)u(x)dx$ and $g \mapsto \int_K g(x)\partial^{\alpha}u(x)dx$ respectively, where $g \in L^1(K)$. Therefore, by (3.7) and (3.6) we have

$$\lim_{n\to\infty} I(n) = \lim_{n\to\infty} \int_{\Omega} \left(f(x) - f_n(x)\right)\partial^{\alpha}u(x)dx = 0,$$

$$\lim_{n\to\infty} II(n) = \lim_{n\to\infty} \int_{\Omega} \partial^{\alpha}f_n(x)u(x)dx = \int_{\Omega} \partial^{\alpha}f_{\alpha}(x)u(x)dx.$$

Combining the above we arrive at

$$\int_{\Omega} f(x)\partial^{\alpha}u(x)dx = \lim_{n\to\infty} I(n) - II(n) = -\int_{\Omega} \partial^{\alpha}f_{\alpha}(x)u(x)dx.$$

Therefore f is α-weakly differentiable for any multi-index alpha with $|\alpha| \leq k$ and

$\partial^\alpha f = f_\alpha$.

(3) Assume that Φ satisfies Δ_2 and $f \in W^{k,\Phi}(\Omega)$. For every multi-index α with $|\alpha| \le k$ we have that $\partial^\alpha f \in L^\Phi(\Omega)$. Since Φ satisfies Δ_2, we conclude that $\partial^\alpha f \in E^\Phi(\Omega)$ and so $f \in E^{\Phi,k}(\Omega)$.

□

Notice that, in general for $f \in W^{k,\Phi}(\Omega)$ and compact set $K \subset \Omega$ we do not have that $f\chi_K \in W^{k,\Phi}(\Omega)$. Fortunately the following fact holds.

Theorem 3.1.8. *[1, Lemma 3] Let Φ be a Musielak Orlicz function on an open $\Omega \subset \mathbb{R}^d$ and $k \in \mathbb{N}$. For any $f \in W^{k,\Phi}(\Omega)$, $f_R = f s_R$ converges in modular in $W^{k,\Phi}(\Omega)$ to f as $R \to \infty$.*

Proof. Take any $f \in W^{k,\Phi}(\Omega)$. Let α, γ, δ be multi-indexes. Recall that for $R > 0$, s_R is the smooth cutoff function, that is $s_R : \mathbb{R}^d \to [0,1]$ and is given by the formula $s_R\left(\frac{x}{R}\right)$, where $x \in \mathbb{R}^d$ and $s : \mathbb{R}^d \to [0,1]$ is a smooth function such that $s(x) = 1$ for $x \in B(0,1)$ and $\operatorname{ess\,supp} s \subset \overline{B(0,2)}$. Define

$$K_k = \max_{|\gamma| \le k} \|\partial^\gamma s\|_\infty, \quad C_\alpha = \max_{\delta+\gamma=\alpha} \frac{\alpha!}{\delta!\gamma!}, \quad C = 1 + \sum_{|\alpha| \le k} C_\alpha K_k.$$

Taking multi-index α with $|\alpha| \le k$, by the Leibnitz formula [27, p. 13], for every $x \in \mathbb{R}^d$ and $R > 1$,

$$|\partial^\alpha f_R(x)| = |\partial^\alpha (f s_R)(x)| = \left| \sum_{\delta+\gamma=\alpha} \frac{\alpha!}{\delta!\gamma!} \frac{1}{R^{|\gamma|}} \partial^\delta f(x) \partial^\gamma s\left(\frac{x}{R}\right) \right| \le C_\alpha K_k \sum_{\delta \le \alpha} |\partial^\delta f(x)|.$$

Hence

$$\begin{aligned} \|f_R\|_{W^{k,\Phi}} &= \sum_{|\alpha| \le k} \|\partial^\alpha f_R\|_\Phi \le \sum_{|\alpha| \le k} C_\alpha K_k \sum_{\delta \le \alpha} \|\partial^\delta f\|_\Phi \\ &\le \sum_{|\alpha| \le k} C_\alpha K_k \|f\|_{W^{k,\Phi}} \le C\|f\|_{W^{k,\Phi}}. \end{aligned}$$

Therefore for $R > 1$ and $\lambda = \frac{1}{2C\|f\|_{W^{k,\Phi}}}$,

$$\int_\Omega \Phi\left(x, \lambda\left|\partial^\alpha f_R(x) - \partial^\alpha f(x)\right|\right) dx \le \frac{1}{2}(I_\Phi(2\lambda\partial^\alpha f_R) + I_\Phi(2\lambda\partial^\alpha f)) \le 1. \tag{3.8}$$

Notice that for a.e. $x \in \Omega$ we have $|\partial^\beta f(x)| < \infty$ for any multi-index β with $|\beta| \le k$. For $R > 1$, we have

$$|\partial^\alpha(f_R(x) - f(x))| = |\partial^\alpha(fs_R)(x) - \partial^\alpha f(x)| = \left|\sum_{\substack{\delta+\gamma=\alpha\\ \delta\neq\alpha}} \frac{\alpha!}{\delta!\gamma!}\frac{1}{R^{|\gamma|}}\partial^\delta f(x)\partial^\gamma s\left(\frac{x}{R}\right)\right|$$
$$\le \frac{K_k}{R}\sum_{\substack{\delta+\gamma=\alpha\\ \delta\neq\alpha}} \frac{\alpha!}{\delta!\gamma!}\left|\partial^\delta f(x)\right|.$$

Hence for a.e. $x \in \Omega$ we have

$$\lim_{R\to\infty} |\partial^\alpha(f_R(x) - f(x))| = 0.$$

Finally (3.8) shows that the function defined for $x \in \Omega$ by the formula

$$\Phi(x, 2|\lambda\partial^\alpha f_R(x)|) + \Phi(x, |2\lambda\partial^\alpha f(x)|)$$

is an integrable majorant of $\Phi(x, \lambda|\partial^\alpha f_R(x) - \partial^\alpha f(x)|)$. By the Lebesgue Dominated Convergence Theorem we have

$$\lim_{R\to\infty} \int_\Omega \Phi\left(x, \lambda\left|\partial^\alpha f_R(x) - \partial^\alpha f(x)\right|\right) dx = 0.$$

We conclude that f_R converges in modular in $W^{k,\Phi}(\Omega)$ to f as $R \to \infty$. □

The next result is an immediate corollary of Theorem 3.1.8.

Corollary 3.1.9. *Let Φ be a MO function on $\Omega \subset \mathbb{R}^d$ satisfying the Δ_2 condition* *and* $k \in \mathbb{N}$. *For any* $f \in W^{k,\Phi}(\Omega)$ *we have*

$$\lim_{R\to\infty} \|f - f_R\|_{W^{k,\Phi}} = 0.$$

3.2 The (A) conditions

The problem of density of smooth functions in a Musielak Orlicz Sobolev spaces turns out to be more subtle then that of Musielak Orlicz spaces. In the context of MOS we do not only need to be able to approximate functions in $L^\Phi(\Omega)$ but also we need to have control over the size of the derivatives of the approximants. In particular, the method that worked for approximating a function in $L^\Phi(\Omega)$ will not work for functions in $W^{k,\Phi}(\Omega)$. To see this, notice that the proof of the main result from Chapter 2 was based on approximating characteristic functions of compact sets which are not weakly differentiable, hence cannot belong to $W^{k,\Phi}(\Omega)$.

The way to circumvent this problem is to construct the approximants by the means of convolution with a mollifier. Recall that the standard mollifier is the function $J : \mathbb{R}^d \to [0, \infty)$ given by the formula

$$J(x) = Ce^{\frac{-1}{1-|x|^2}} \chi_{B(0,1)}(x),$$

where C is such that $\|J\|_1 = 1$. For any $r > 0$, we set $J_{(r)}(x) := \frac{1}{r^d} J\left(\frac{1}{r}x\right)$. Now, for any $f \in W^{k,\Phi}(\Omega)$ if we extend it by 0 for $x \in \mathbb{R}^d \setminus \Omega$. Then for $x \in \mathbb{R}^d$ we set

$$f_r(x) = (f * J_{(r)})(x)$$

and since f is a locally integrable function we get that f_r is of class $C^\infty(\mathbb{R}^d)$. Now if f would be compactly supported then f_r had been the natural candidate for the approximant of f as for α-weakly differentiable function f one has
$\partial^\alpha f_r = (\partial^\alpha f) * J_{(r)}$. Therefore two natural questions arise: what are the conditions

sufficient for f_r to be an element of $L^\Phi(\Omega)$ and for f_r to converge to f as r goes to 0?

The following inequality can give us the right idea where to search for such conditions. Let $f \in L^1_{loc}(\mathbb{R}^d)$, $r > 0$ and $x \in \mathbb{R}^d$. It is easy to see that

$$|f_r(x)| = \left| \frac{1}{r^d} \int\limits_{\mathbb{R}^d} J\left(\frac{x-y}{r}\right) f(y)dy \right| \leq \frac{Ce^{-1}}{r^d} \int\limits_{B(x,r)} |f(y)|dy \leq \frac{\sigma_d Ce^{-1}}{|B(x,r)|} \int\limits_{B(x,r)} |f(y)|dy$$

$$\leq \sigma_d Ce^{-1} \sup_{r>0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y)|dy = \mathcal{M}f(x), \tag{3.9}$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal operator. Hence the boundedness of that maximal operator on $L^\Phi(\mathbb{R}^d)$ is sufficient at least for f_r to be an element of $L^\Phi(\mathbb{R}^d)$. It turns out that it also suffices for convergence of f_r to f, when $r \to 0$ in the norm of $W^{k,\Phi}(\mathbb{R}^d)$, see for example [24, Theorem 6.5].

Unfortunately, there is no one universal approach to the problem of boundedness of maximal operator on $L^\Phi(\mathbb{R}^d)$, as different forms of the function Φ require different methods. Roughly speaking, those methods lie on a spectrum between two extreme cases: the weighted L^p case, that this $\Phi(x,t) = t^p w(x)$, where $p \geq 1$ and w is a locally integrable weight $w : \mathbb{R}^d \to [0,\infty)$ and the variable exponent case $\Phi(x,t) = t^{p(x)}$ where $p : \mathbb{R}^d \to [1,\infty)$ is a measurable function.

The weighed L^p case is well understood. If $\Phi(x,t) = t^p w(x)$, then the maximal operator $\mathcal{M}$ is bounded on $L^\Phi(\mathbb{R}^d)$ if and only if $p > 1$ and w is of Muckpenhaupt class A_p that is

$$\sup_{B \subset \mathbb{R}^d} \left(\frac{1}{|B|} \int\limits_B w(x)dx \right) \left(\int\limits_B w(x)^{-\frac{q}{p}} \right)^{\frac{p}{q}} dx < \infty,$$

where the supremum above is taken over all open balls $B \subset \mathbb{R}^d$ and $\frac{1}{p} + \frac{1}{q}$. There is a plethora of results based on the ideas of Muckenhaupt. For example A.

Gogatishvili and V. Kokilashvili had shown in [21] a characterisation of boudedness of $\mathcal{M}$ on $L^{\Phi}(\mathbb{R}^d)$, where Φ is of the form $\Phi(x,t) = \phi(t)w(x)$, where w is a locally integrable weight and ϕ is an Orlicz function. Another example is a sufficient condition for boudedness of $\mathcal{M}$ for Φ such that for each $t > 0$, $\Phi(\cdot, t) = w_t(x)$ is an element of some class A_p (L.D Ky [45]).

The case of variable exponent $\Phi(x,t) = t^{p(x)}$ turns out to be more diverse than the weighted case. First of all it was shown by Diening in [15] that if one assumes that the exponent $p(\cdot)$ is constant outside of a compact set, strictly bigger than 1 and satisfies the log-Hölder condition, namely

$$\left|\frac{1}{p(x)} - \frac{1}{p(y)}\right| \leq \frac{C}{\log\left(e + 1/|x-y|\right)}, \tag{3.10}$$

for $x, y \in \mathbb{R}^d$ and some constant $C > 0$, then $\mathcal{M}$ is bounded on $L^{\Phi}(\mathbb{R}^d) = L^{p(\cdot)}(\mathbb{R}^d)$. Later Cruz-Uribe, Fiorenza, Martell and Perez [13] were able to replace the assumption of $p(\cdot)$ being constant outside a compact set by the following log-Hölder decay condition

$$\left|\frac{1}{p(x)} - \frac{1}{p_\infty}\right| \leq \frac{c_2}{\log(e + |x|)},$$

where $x \in \mathbb{R}^d$ and some constants $c_2 > 0$, $p_\infty > 1$ and still recover the boundedness of $\mathcal{M}$. Further developments have shown that even if log-Hölder is the weakest modulus of continuity that guarantees boundedness of $\mathcal{M}$, there exist exponents that are not log-Hölder continuous yet the maximal operator is still bounded.

The full description of boudedness of $\mathcal{M}$ in the variable exponent case was given by Dienieng in [16, Theorem 8.1] and involves the so called generalized Muckenhaupt condition $\mathcal{A}$ which, roughly speaking, says that a certain family of averaging operators is uniformly bounded on $L^{\Phi}(\mathbb{R}^d)$. Even if the boundedness of $\mathcal{M}$ is described by condition $\mathcal{A}$ the condition itself is not necessarily easy to check. Hence there is a natural need for conditions that would guaranty boundedness of $\mathcal{M}$

both on variable exponent spaces and spaces somehow similar to them, i.e. spaces given by MO functions that are essentially non-weighted. An example of such conditions was given by Hästö in [23] and they are the conditions (A0), (A1), (A2) two of which, namely (A0) and (A1) are discussed in this section.

On the other hand, the maximal operator approach to problem of density is restrictive in some aspects. The arguments based on maximal operator require that the function Φ is either defined on $\mathbb{R}^d \times [0,\infty)$ or, if Φ is defined on $\Omega \times [0,\infty)$, $\Omega \subset \mathbb{R}^d$ then it needs be extended to a function $\Psi : \mathbb{R}^d \times [0,\infty) \to [0,\infty)$ in such a fashion that if $f \in L^{\Psi}(\mathbb{R}^d)$ then $f_{|\Omega} \in L^{\Phi}(\Omega)$. Moreover, boundedness of the maximal operator is, in general, not required for density of compactly supported smooth functions. For example $C_C^{\infty}(\mathbb{R}^d)$ is dense in the Sobolev space $W^{1,1}(\mathbb{R}^d)$ but $\mathcal{M}$ is not bounded in $L^1(\mathbb{R}^d)$.

The observation above suggest that there exists conditions weaker than required for boundedness of $\mathcal{M}$ which are sufficient for density of smooth functions. Surprisingly there was little research done in this direction. To the author's knowledge there is only two papers that study the density of $C_C^{\infty}(\Omega)$ in $W^{k,\Phi}(\Omega)$ without an implicit or explicit assumption of boundedness of $\mathcal{M}$. The first one is [6] by Benkirane, Douieb and Sidi El Vally. Here the authors assume that the function Φ is locally integrable and satisfies the following condition.

$$\frac{\Phi(x,t)}{\Phi(y,t)} \leq t^{\frac{A}{\log\left(\frac{1}{|x-y|}\right)}}, \tag{3.11}$$

where $A > 0$, $x, y \in \Omega$ and $|x-y| < \frac{1}{2}$ and $t \geq 1$. Moreover they require additional assumption about uniform continuity of some family of convolution operators. We will show that condition (3.11) implies condition (A1) discussed below. Hence, the results established in the section 3.3 will generalize results of (3.11). The second paper dealing with the problem of density is [1] by Y. Ahmida, I. Chlebicka, P.

Gwiazda, A. Youssfi. Once again the authors assume local integrability of Φ but now also the assume the following condition.

Definition 3.2.1. *Let $\Omega \subset \mathbb{R}^d$ be an open set. Let Φ be a MO function defined on Ω. We say that Φ satisfies $\mathcal{M}_2$ if there exists a function*
$\omega : [0, 1/2] \times [1, \infty) \to [1, \infty)$ *such that,*

(1) *$\omega(s, \cdot)$ and $\omega(\cdot, t)$ are non-decreasing for every $s \in [0, 1/2]$, $t \geq 0$,*

(2) *$\limsup_{t \to 0} \omega\left(t, Ct^{-d}\right) < \infty$ for any $C > 0$,*

(3) *for every $x, y \in \Omega$ with $|x - y| \leq \frac{1}{2}$ and $t \geq 0$,*

$$\Phi(x, t) \leq \omega(|x - y|, t)\Phi(y, t).$$

Under this condition the authors establish the density of $C_C^\infty(\mathbb{R}^d)$ in $W^{k,\Phi}(\mathbb{R}^d)$. Moreover, the authors claim to have shown that, under a mild condition on the regularity of the boundary of Ω, namely the segment condition (see [2, Definition 3.21, p. 68]), they have established the density of restrictions of $C_C^\infty(\mathbb{R}^d)$ functions in $W^{k,\Phi}(\Omega)$. Unfortunately their claim is not true, as in the course of their proof they have assumed existence of an extension $\overline{\Phi}$ of the function Φ from Ω to whole $\mathbb{R}^d$ such that the extension still satisfies the $\mathcal{M}_2$ condition, existence of such an extension was not established in the paper. Moreover, which is more problematic, in the course of the proof of their result the author have used strong continuity of group of translation operators $\{\tau_x\}_{x \in \mathbb{R}^d}$ on $L^{\overline{\Phi}}(\mathbb{R}^d)$, which is true only in very specific cases (see [37, Theorem 2.1]). In section 3.4 we establish a version of the mentioned theorem while avoiding using the strong continuity of translations and existence of global extensions of Φ. To our knowledge this is the first correct result of this kind for MOS spaces.

It turns out that if Φ satisfies the $\mathcal{M}_2$ it also satisfies a condition very close to condition (A1) (see Proposition 3.2.11 below). Moreover if we also assume that Φ satisfies (A0) it turn out that Φ satisfies (A1) (see Corollary 3.2.13 below). Hence our results generalize those of [1], as long as one assumes condition (A0). Moreover, we also establish that in the case of $\Phi(x,t) = t^{p(x)}$ the condition $\mathcal{M}_2$ collapses to the case of a constant exponent. Hence, $\mathcal{M}_2$ is in general very restrictive.

Now we will introduce and study the conditions (A0) and (A1).

Definition 3.2.2. *Let $\Omega \subset \mathbb{R}^d$ be open and Φ be a MO function defined on Ω. We say that Φ satisfies (A0) if there exists $\beta \in (0,1]$ such that for every $x \in \Omega$,*

$$\beta \leq \Phi^{-1}(x,1) \leq \frac{1}{\beta}.$$

This condition states that the function Φ is essentially non weighted. The following theorem characterizes the relation between the condition (A0) and the inverse and conjugate function.

Theorem 3.2.3. *Let Φ be a* MO *function defined on an open set $\Omega \subset \mathbb{R}^d$. The following statements are true.*

(1) *The function Φ satisfies (A0) if and only if there exists $\beta \in (0,1]$ such that for all $x \in \Omega$,*

$$\Phi(x,\beta) \leq 1 \leq \Phi\left(x, \frac{1}{\beta}\right).$$

(2) *If Φ satisfies (A0), then Φ^* also satisfies (A0).*

Proof. Clearly the statement (1) is true.

(2) By Proposition 2.1.15 (3) we have

$$1 \leq \Phi^{-1}(x,1)(\Phi^*)^{-1}(x,1) \leq 2.$$

Hence, by the fact that Φ satisfies (A0) for $x \in \Omega$ we have

$$\frac{\beta}{2} \leq \beta \leq \frac{1}{\Phi^{-1}(x,1)} \leq (\Phi^*)^{-1}(x,1) \leq \frac{2}{\Phi^{-1}(x,1)} \leq \frac{2}{\beta}.$$

Therefore Φ^* satisfies (A0) with the constant $\frac{\beta}{2}$.

□

The next condition is the analogue of log-Hölder continuity and will be crucial in further discussion.

Definition 3.2.4. *Let $\Omega \subset \mathbb{R}^d$ be an open set. Let Φ be a extended MO function defined on Ω. We say that Φ satisfies* (A1) *if there exist constants $\beta, \delta \in (0,1)$ such that for all open balls B with $|B| < \delta$ and almost all $x, y \in B \cap \Omega$ we have for all $t \in \left[\Phi^{-1}(y,1), \Phi^{-1}\left(y, \frac{1}{|B|}\right)\right]$,*

$$\Phi(x, \beta t) \leq \Phi(y,t).$$

The conditions (A0), (A1) and their variants appear in literature in many forms (see [23], [24], [25], [26]). The one we adopted correspond to the ones in [26]. Our definition of (A1) is a variant of the condition (A1') in [26]. Instead of requiring the volumes of the balls B to be less then one, we only require them to be less then some constant δ smaller then 1. This difference is essentially cosmetic, in the sense that all of the theorems bellow stay true if we replace δ with 1. We adopt this convention to show the connection between the condition (A1) and the condition $\mathcal{M}_2$ which we introduce later.

Notice condition (A1) is symmetric, that is if we change the role of x and y we get the same condition.

The proof of this lemma follows from [26, 4.1.3, 4.1.5]. Since the authors of [26] work under slightly weaker assumptions on convexity of the function Φ we decide to provide its proof here for the sake of completeness.

Lemma 3.2.5. *Let $\Omega \subset \mathbb{R}^d$ be an open set. Let Φ be a MO function defined on Ω. The following statements are equivalent.*

(1) *Φ satisfies* (A1) *with the constants β and δ.*

(2) *For all open balls B with $|B| < \delta$ and almost all $x, y \in B \cap \Omega$ we have*

$$\beta \Phi^{-1}(x,t) \leq \Phi^{-1}(y,t),$$

for $t \in \left[1, \frac{1}{|B|}\right]$ where β is the constant from the condition (A1) *for Φ.*

(3) *Φ^* satisfies* (A1) *in the following form: for all open balls B with $|B| < \delta$ and almost all $x, y \in B \cap \Omega$ we have for all $t \in \left[(\Phi^*)^{-1}(y,1), (\Phi^*)^{-1}\left(y, \frac{1}{|B|}\right)\right]$,*

$$\Phi^*\left(x, \frac{\beta}{2}t\right) \leq \Phi^*(y,t),$$

where β, δ are constants from condition (A1) *for function Φ, and $(\Phi^*)^{-1}$ is the generalized inverse in the sense of Definition 2.1.13.*

Proof. Throughout the proof B is an open ball with $|B| < \delta$ and $x, y \in B \cap \Omega$. Assume that (1) holds, for any $t \in \left[1, \frac{1}{|B|}\right]$ there exists $u \in \left[\Phi^{-1}(x,1), \Phi^{-1}\left(x, \frac{1}{|B|}\right)\right]$, such that $u = \Phi^{-1}(x,t)$. By (A1) we have

$$\Phi(y, \beta u) \leq \Phi(x,u).$$

Notice now that $\Phi(x,u) = t$. Hence,

$$\beta \Phi^{-1}(x,t) = \beta u = \Phi^{-1}(y, \Phi(y, \beta u)) \leq \Phi^{-1}(y, \Phi(x,u)) = \Phi^{-1}(y,t).$$

Which proves (1) implies (2).

Assume that (2) holds and let $\Phi^{-1}(y,1) \le t \le \Phi^{-1}\left(y, \frac{1}{|B|}\right)$. Since $\Phi(y,\cdot)$ is invertible we have $\Phi(y,t) \in \left[1, \frac{1}{|B|}\right]$. Denoting $u = \Phi(y,t)$, by symmetry of condition (2) we have

$$\Phi(x, \beta t) = \Phi(x, \beta\Phi^{-1}(y,u)) \le \Phi(x, \Phi^{-1}(x,u)) = u = \Phi(y,t).$$

Which shows that (2) implies (1).

Take any $1 \le t \le \frac{1}{|B|}$. By Proposition 2.1.15 (3), for any $t > 0$, $x \in \Omega$,

$$\frac{\beta}{2}(\Phi^*)^{-1}(x,t) \le \frac{\beta t}{\Phi^{-1}(x,t)}.$$

Applying (2) and again Proposition 2.1.15 (3) we arrive at

$$\frac{\beta t}{\Phi^{-1}(x,t)} \le \frac{\beta t}{\beta\Phi^{-1}(y,t)} \le (\Phi^*)^{-1}(y,t).$$

Hence, by the above inequalities we have

$$\frac{\beta}{2}(\Phi^*)^{-1}(x,t) \le (\Phi^*)^{-1}(y,t).$$

We conclude that $(\Phi^*)^{-1}$ satisfies (2) with the constant $\frac{\beta}{2}$. Since (2) implies (1), it follows that Φ^* satisfies (A1) with the constant $\frac{\beta}{2}$, so (2) implies (3).

Since $(\Phi^*)^* = \Phi$ we can repeat the previous argument to conclude that (3) implies (1). Indeed, assuming that Φ^* satisfies (A1), we deduce (2) with Φ^{-1} replaced by $(\Phi^*)^{-1}$. Since (2) implies (3) we deduce that $(\Phi^*)^* = \Phi$ satisfies (A1). □

Lemma 3.2.6. *Let $\Omega \subset \mathbb{R}^d$ be an open set and let Φ be a MO function defined on Ω. If Φ satisfies* (A1)*, then for any open ball B with $|B| < \delta$, and any*

$[\Phi^{-1}]_{B\cap\Omega^+}(1) \le t \le [\Phi^{-1}]_{B\cap\Omega^-}(\frac{1}{|B|})$,

$$\Phi_{B\cap\Omega^+}(\beta t) \le \Phi_{B\cap\Omega^-}(t),$$

where β is the constant from condition (A1)*.*

Proof. Let Φ be as in the assumption, take any open ball B with $|B| < \delta$. For any $y \in B \cap \Omega$ and $[\Phi^{-1}]_{B\cap\Omega^+}(1) \le t \le [\Phi^{-1}]_{B\cap\Omega^-}(\frac{1}{|B|})$, we have

$$\operatorname*{ess\,sup}_{x\in B\cap\Omega} \Phi(x, \beta t) \le \Phi(y, t).$$

Therefore

$$\Phi_{B\cap\Omega^+}(\beta t) = \operatorname*{ess\,sup}_{x\in B\cap\Omega} \Phi(x, \beta t) \le \operatorname*{ess\,inf}_{y\in B\cap\Omega} \Phi(y, t) = \Phi_{B\cap\Omega^-}(t).$$

□

The usefulness of condition (A1) comes from the fact that it lets us control norms of characteristic functions of balls that are in $L^\Phi(\Omega)$.

Proposition 3.2.7. *Let $\Omega \subset \mathbb{R}^d$ be open and Φ be a MO function defined on Ω. Assume that Φ satisfies* (A1) *and $\beta, \delta \in (0, 1)$ are the constants from this condition. For any open ball B with $|B| < \delta$ and almost all $x \in B \cap \Omega$, the following statements hold true.*

(1)

$$\beta \frac{1}{|B\cap\Omega|} \int_{B\cap\Omega} \frac{1}{\Phi^{-1}\left(y, \frac{1}{|B|}\right)} dy \le \frac{1}{\Phi^{-1}\left(x, \frac{1}{|B|}\right)} \le \frac{1}{\beta} \frac{1}{|B\cap\Omega|} \int_{B\cap\Omega} \frac{1}{\Phi^{-1}\left(y, \frac{1}{|B|}\right)} dy.$$

(2)

$$\frac{\beta}{2}\frac{1}{|B|}\int\limits_{B\cap\Omega}\frac{1}{\Phi^{-1}\left(y,\frac{1}{|B\cap\Omega|}\right)}dy \le \|\chi_{B\cap\Omega}\|_{\Phi} \le \frac{1}{\beta}\frac{1}{|B\cap\Omega|}\int\limits_{B\cap\Omega}\frac{1}{\Phi^{-1}\left(y,\frac{1}{|B|}\right)}dy.$$

(3)

$$\begin{aligned}\frac{1}{|B\cap\Omega|}\int\limits_{B\cap\Omega}\frac{1}{\Phi^{-1}\left(y,\frac{1}{|B\cap\Omega|}\right)}dy &\le \int\limits_{B\cap\Omega}(\Phi^*)^{-1}\left(y,\frac{1}{|B\cap\Omega|}\right)dy\\ &\le \frac{2}{|B\cap\Omega|}\int\limits_{B\cap\Omega}\frac{1}{\Phi^{-1}\left(y,\frac{1}{|B\cap\Omega|}\right)}dy.\end{aligned}$$

(4)

$$\begin{aligned}\beta\frac{1}{|B\cap\Omega|}\int\limits_{B\cap\Omega}(\Phi^*)^{-1}\left(y,\frac{1}{|B|}\right)dy &\le (\Phi^*)^{-1}\left(x,\frac{1}{|B|}\right)\\ &\le \frac{1}{\beta}\frac{1}{|B\cap\Omega|}\int\limits_{B\cap\Omega}(\Phi^*)^{-1}\left(y,\frac{1}{|B|}\right)dy.\end{aligned}$$

Proof. Assume that Φ satisfies (A1). Let B be an open ball with $|B| < \delta$ such that $B \cap \Omega \neq \emptyset$. By symmetry of the condition (A1) and Lemma 3.2.5 (2) for a.a. $x, y \in B \cap \Omega$ we have

$$\beta\Phi^{-1}\left(y,\frac{1}{|B|}\right) \le \Phi^{-1}\left(x,\frac{1}{|B|}\right) \le \frac{1}{\beta}\Phi^{-1}\left(y,\frac{1}{|B|}\right).$$

Taking the reciprocals, integrating over $B \cap \Omega$ with respect to y and dividing by $|B \cap \Omega|$ we get

$$\beta\frac{1}{|B\cap\Omega|}\int\limits_{B\cap\Omega}\frac{1}{\Phi^{-1}\left(y,\frac{1}{|B|}\right)}dy \le \frac{1}{\Phi^{-1}\left(x,\frac{1}{|B|}\right)} \le \frac{1}{\beta}\frac{1}{|B\cap\Omega|}\int\limits_{B\cap\Omega}\frac{1}{\Phi^{-1}\left(y,\frac{1}{|B|}\right)}dy,$$

which proves (1). To prove (2) notice that by (1) we have

$$\begin{aligned}
&\int\limits_{\Omega} \Phi\left(x, \beta|B\cap\Omega| \frac{1}{\int\limits_{B\cap\Omega} \frac{1}{\Phi^{-1}\left(y, \frac{1}{|B|}\right)} dy} \chi_{B\cap\Omega}(x)\right) dx \\
&\leq \int\limits_{\Omega} \Phi\left(x, \Phi^{-1}\left(x, \frac{1}{|B|}\chi_{B\cap\Omega}(x)\right)\right) dx \\
&= \frac{|B\cap\Omega|}{|B|} \leq 1.
\end{aligned}$$

Therefore we have

$$\|\chi_{B\cap\Omega}\|_{\Phi} \leq \frac{1}{\beta}\frac{1}{|B\cap\Omega|}\int\limits_{B} \frac{1}{\Phi^{-1}\left(y, \frac{1}{|B|}\right)} dy.$$

For the other inequality notice that

$$\int\limits_{\Omega} \Phi^{*}\left(y, (\Phi^{*})^{-1}\left(y, \frac{1}{|B\cap\Omega|}\chi_{B\cap\Omega}(y)\right)\right) dy = \frac{|B\cap\Omega|}{|B\cap\Omega|} = 1,$$

therefore we have

$$\left\|(\Phi^{*})^{-1}\left(\cdot, \frac{1}{|B\cap\Omega|}\chi_{B\cap\Omega}(\cdot)\right)\right\|_{\Phi^{*}} \leq 1.$$

Hence, by Proposition 2.1.15 (3),

$$\begin{aligned}
\|\chi_{B\cap\Omega}\|_{\Phi} \geq \|\chi_{B\cap\Omega}\|_{\Phi}^{0} = \sup_{\|f\|_{\Phi^{*}}\leq 1} \left|\int\limits_{\Omega} \chi_{B\cap\Omega}(y) f(y) dy\right| \geq \int\limits_{B\cap\Omega} (\Phi^{*})^{-1}\left(y, \frac{1}{|B\cap\Omega|}\right) dy \\
\geq \frac{1}{2}\frac{1}{|B\cap\Omega|}\int\limits_{B\cap\Omega} \frac{1}{\Phi^{-1}\left(y, \frac{1}{|B\cap\Omega|}\right)} dy \geq \frac{\beta}{2}\frac{1}{|B|}\int\limits_{B\cap\Omega} \frac{1}{\Phi^{-1}\left(y, \frac{1}{|B\cap\Omega|}\right)} dy,
\end{aligned}$$

which shows (2).

To show (3) notice that by Lemma 2.1.15 (2), we have

$$\frac{1}{|B\cap\Omega|}\frac{1}{\Phi^{-1}\left(y,\frac{1}{|B\cap\Omega|}\right)}\le(\Phi^*)^{-1}\left(y,\frac{1}{|B\cap\Omega|}\right)\le\frac{2}{|B\cap\Omega|}\frac{1}{\Phi^{-1}\left(y,\frac{1}{|B\cap\Omega|}\right)}.$$

Integrating with respect to y,

$$\begin{aligned}\frac{1}{|B\cap\Omega|}\int\limits_{B\cap\Omega}\frac{1}{\Phi^{-1}\left(y,\frac{1}{|B\cap\Omega|}\right)}dy&\le\int\limits_{B\cap\Omega}(\Phi^*)^{-1}\left(y,\frac{1}{|B\cap\Omega|}\right)dy\\&\le\frac{2}{|B\cap\Omega|}\int\limits_{B\cap\Omega}\frac{1}{\Phi^{-1}\left(y,\frac{1}{|B\cap\Omega|}\right)}dy.\end{aligned}$$

Finally, since Φ satisfies (A1),for $y\in B\cap\Omega$ by Lemma 3.2.5 (2) and (3) we have

$$\beta(\Phi^*)^{-1}\left(y,\frac{1}{|B|}\right)\le(\Phi^*)^{-1}\left(x,\frac{1}{|B|}\right)\le\frac{1}{\beta}(\Phi^*)^{-1}\left(y,\frac{1}{|B|}\right).$$

Hence,

$$\begin{aligned}\beta\frac{1}{|B\cap\Omega|}\int\limits_{B\cap\Omega}(\Phi^*)^{-1}\left(y,\frac{1}{|B|}\right)dy&\le(\Phi^*)^{-1}\left(x,\frac{1}{|B|}\right)\\&\le\frac{1}{\beta}\frac{1}{|B\cap\Omega|}\int\limits_{B\cap\Omega}(\Phi^*)^{-1}\left(y,\frac{1}{|B|}\right)dy.\end{aligned}$$

□

Notice that the statement (3) in Proposition 3.2.7 holds true without the assumption (A1). The next two lemmas show that if Φ satisfies either (A0) or (A1) then the elements of $L^\Phi(\Omega)$ are locally integrable functions.

Lemma 3.2.8. *Let $\Omega\subset\mathbb{R}^d$ be open and Φ be a MO function defined on Ω. Assume that Φ satisfies* (A0) *or* (A1)*. Then for any compact set $K\subset\Omega$ we have $\chi_K\in L^\Phi(\Omega)$, $\chi_K\in L^{\Phi^*}(\Omega)$.*

Proof. Assume that Φ is satisfying (A1). Let $K\subset\Omega$ be compact. First we will show that $\chi_K \in L^{\Phi^*}(\Omega)$. Let $R > 0$ be such that $|B(0, R)| < \delta$ and for every $r < R$ and $x \in K$, $B(x, r) \subset \Omega$. By compactness of K there exist $N \in \mathbb{N}$, $\{x_i\}_{i=1}^N \subset K$ and $\{r_i\}_{i=1}^N \subset (0, R)$ such that $K \subset \bigcup_{i=1}^N B(x_i, r_i)$. By Lemma 2.1.15 (3), Φ^* satisfies (A1) and so by Proposition 3.2.7 (2), (3) and (4),

$$\begin{aligned}\|\chi_K\|_{\Phi^*} &\leq \sum_{i=1}^N \|\chi_{B(x_i,r_i)}\|_{\Phi^*} \leq \frac{1}{\beta} \sum_{i=1}^N \int\limits_{B(x_i,r_i)} \frac{1}{|B(x_i, r_i)|} \frac{1}{(\Phi^*)^{-1}\left(x, \frac{1}{|B(x_i,r_i)|}\right)} dx \\ &\leq \frac{1}{\beta} \sum_{i=1}^N \int\limits_{B(x_i,r_i)} \Phi^{-1}\left(x, \frac{1}{|B(x_i, r_i)|}\right) dx \\ &\leq \frac{1}{\beta^2} \sum_{i=1}^N |B(x_i, r_i)| \Phi^{-1}\left(x_i, \frac{1}{|B(x_i, r_i)|}\right) < \infty.\end{aligned}$$

Hence $\chi_K \in L^{\Phi^*}(\Omega)$. Since $(\Phi^*)^* = \Phi$ the same argument shows that $\chi_K \in L^{\Phi}(\Omega)$.

Assume now that Φ satisfies (A0). For any compact $K \subset \Omega$, we have

$$\int\limits_\Omega \Phi(x, \beta\chi_K(x))dx \leq \int\limits_K \Phi(x, \Phi^{-1}(x, 1))dx = 1 < \infty.$$

Hence $\chi_K \in L^\Phi(\Omega)$. Notice that if Φ satisfies (A0), then Φ^* also does, therefore $\chi_K \in L^{\Phi^*}(\Omega)$. □

Lemma 3.2.9. *Let $\Omega \subset \mathbb{R}^d$ be open and Φ be a MO function defined on Ω. Assume that Φ satisfies* (A0) *or* (A1)*. If $f \in L^\Phi(\Omega)$ is such that* $\operatorname{ess\,supp} f \subset \Omega$ *is compact, then $f \in L^1(\Omega)$.*

Proof. Let $f \in L^\Phi(\Omega)$ be such that $\operatorname{ess\,supp} f \subset \Omega$ is compact. By Hölder's inequality and Lemma 3.2.8,

$$\int\limits_\Omega |f(x)|dx \leq \|f\|_\Phi \|\chi_{\operatorname{ess\,supp} f}\|_{\Phi^*} < \infty.$$

□

Now we will study the connection between the condition (A1) and the conditions from [1] and [6]. We start with condition $\mathcal{M}_2$ from [1].

Definition 3.2.10. *Let $\Omega \subset \mathbb{R}^d$ be an open set. Let Φ be a MO function defined on Ω. We say that Φ satisfies $\mathcal{M}_2$ if there exists a function $\omega : [0, 1/2] \times [1, \infty) \to [1, \infty)$ such that,*

(1) *$\omega(s, \cdot)$ and $\omega(\cdot, t)$ are non-decreasing for every $s \in [0, 1/2]$, $t \geq 0$,*

(2) *$\limsup_{t \to 0} \omega\left(t, Ct^{-d}\right) < \infty$ for any $C > 0$,*

(3) *for every $x, y \in \Omega$ with $|x - y| \leq \frac{1}{2}$ and $t \geq 0$,*

$$\Phi(x, t) \leq \omega(|x - y|, t)\Phi(y, t).$$

Notice that if Φ satisfied $\mathcal{M}_2$ then the function ω from condition $\mathcal{M}_2$ can be replaced by a function ω_1 defined by the formula

$$\omega_1(s, t) = \sup_{0 \leq t' \leq t} \sup_{\substack{x', y' \in \Omega, \\ |x' - y'| \leq s}} \frac{\Phi(x', t')}{\Phi(y', t')},$$

where $t \geq 0$ and $s \in [0, 1/2]$. Indeed, take any $t \geq 0$ and $s \in [0, 1/2]$. Let $0 \leq t' \leq t$ and $x', y' \in \Omega$ be such that $|x' - y'| \leq s$. Since Φ satisfies $\mathcal{M}_2$ we have

$$\frac{\Phi(x', t')}{\Phi(y', t')} \leq \omega(|x' - y'|, t').$$

Hence, in view of Definition 3.2.10 (1),

$$\omega_1(s, t) = \sup_{0 \leq t' \leq t} \sup_{\substack{x', y' \in \Omega, \\ |x - y| \leq s}} \frac{\Phi(x', t')}{\Phi(y', t')} \leq \sup_{0 \leq t' \leq t} \sup_{\substack{x', y' \in \Omega, \\ |x' - y'| \leq s}} \omega(|x' - y'|, t') = \omega(s, t),$$

which shows that ω_1 is well defined. Clearly ω_1 is non-decreasing in both variables and for every $x, y \in \Omega$ with $|x-y| \leq \frac{1}{2}$ and $t \geq 0$,

$$\Phi(x,t) \leq \omega_1(|x-y|,t)\Phi(y,t).$$

On the other hand, since $\omega_1 \leq \omega$ we have

$$\limsup_{s\to 0} \omega_1\left(s, Cs^{-d}\right) \leq \limsup_{s\to 0} \omega\left(s, Cs^{-d}\right) < \infty.$$

Hence, ω_1 satisfies conditions (1), (2), (3) from Definition 3.2.10.

Proposition 3.2.11. *Let $\Omega \subset \mathbb{R}^d$ be an open set. Let Φ be a MO function defined on Ω satisfying $\mathcal{M}_2$. Then for any constant $C > 0$ there exist constants $C_1 \geq 1$, $R \in \left(0, \frac{1}{2}\right]$ such that for every $x, y \in \Omega$ with $|x-y| \leq R$ and $0 \leq t \leq \frac{C}{|x-y|^d}$,*

$$\Phi(x,t) \leq 2C_1\Phi(y,t).$$

Proof. Since Φ satisfies $\mathcal{M}_2$, for any $C > 0$ there exists a constant $C_1 \geq 1$ such that

$$\limsup_{t\to 0} \omega\left(t, \frac{C}{t^d}\right) = C_1.$$

By definition of $\limsup$, there exists $R \in (0, 1/2]$ such that for $0 \leq t \leq R$, $\omega\left(t, \frac{C}{t^d}\right) \leq 2C_1$. Hence for any $x, y \in \Omega$, with $0 \leq |x-y| \leq R$, $0 \leq t \leq \frac{C}{|x-y|^d}$, we get

$$\Phi(x,t) \leq \omega(|x-y|,t)\Phi(y,t) \leq \omega\left(|x-y|, \frac{C}{|x-y|^d}\right)\Phi(y,t) \leq 2C_1\Phi(y,t).$$

□

Corollary 3.2.12. *Assume that a MO function Φ defined on an open set $\Omega \subset \mathbb{R}^d$ satisfies $\mathcal{M}_2$ condition. For any $C > 0$ there exist constants $\beta, \delta \in (0,1)$ such that* *for every ball* $B \subset \Omega$*, with* $|B| \le \delta$ *and any* $x, y \in B$ *and* $t \in \left[0, \frac{C}{|B|}\right]$ *we have*

$$\Phi(x, \beta t) \le \Phi(y, t).$$

Proof. Take any $C > 0$ and recall that $\sigma_d = |B(0,1)|$. By Proposition 3.2.11, there exist constant $C_1 \ge 1$ and $R \in \left(0, \frac{1}{2}\right]$ such that for every $x, y \in \Omega$ with $|x - y| \le R$ and $0 \le t \le \frac{2^d C}{\sigma_d |x-y|^d}$,

$$\Phi(x,t) \le 2C_1 \Phi(y,t).$$

For any ball B with $|B| \le \frac{\sigma_d R^d}{2^d} = |B(0, R/2)|$ and any $x, y \in B$ we have $|x - y| \le R$. Hence, for every $x, y \in B$ we have

$$\frac{\sigma_d |x-y|^d}{2^d} \le |B|.$$

By convexity of Φ, for any $0 \le t \le \frac{C}{|B|} \le \frac{2^d C}{\sigma_d |x-y|^d}$ we have

$$\Phi\left(x, \frac{1}{2C_1} t\right) \le \frac{1}{2C_1} \Phi(x,t) \le \Phi(y,t).$$

Notice now, that since $R \le \frac{1}{2}$, we have that for all $d \in \mathbb{N}$,

$$\frac{\sigma_d R^d}{2^d} < 1.$$

Setting $\beta = \frac{1}{2C_1}$ and $\delta = \frac{\sigma_d R^d}{2^d}$ we get the desired conclusion.

□

Corollary 3.2.13. *Assume that a MO function* Φ *is defined on an open set* $\Omega \subset \mathbb{R}^d$*. If* Φ *satisfies* $\mathcal{M}_2$ *and* (A0)*, then it satisfies* (A1)*.*

Proof. Since Φ satisfies (A0) here exists $C \in (0, 1]$ such that for every $x \in \Omega$,

$$C \le \Phi^{-1}(x, 1) \le \frac{1}{C}.$$

By Corollary 3.2.12 there exist $\beta, \delta \in (0,1)$ such that for every ball $B \subset \Omega$, with $|B| \leq \delta$ and any $x, y \in B$ and $t \in \left[0, \frac{1}{C|B|}\right]$ we have

$$\Phi(x, \beta t) \leq \Phi(y, t). \tag{3.12}$$

Take any ball $B \subset \Omega$, with $|B| \leq \delta$ and any $y \in B$, by concavity of $\Phi^{-1}(y, \cdot)$ and the fact that $1/|B| \geq 1/\delta > 1$ we have

$$\Phi^{-1}\left(y, \frac{1}{|B|}\right) \leq \frac{1}{|B|}\Phi^{-1}(y,1) \leq \frac{1}{C|B|}.$$

Hence for every ball $B \subset \Omega$, with $|B| \leq \delta$ and any $x, y \in B$ and any $t \in \left[\Phi^{-1}(y,1), \Phi^{-1}\left(y, \frac{1}{|B|}\right)\right]$ by (3.12) we have

$$\Phi(x, \beta t) \leq \Phi(y, t).$$

In other words, Φ satisfies (A1).

□

Proposition 3.2.14. *Assume $\Omega \subset \mathbb{R}^d$ is an open set. Let $\{\Omega_n\}_{n=1}^{\infty}$ be the decomposition of Ω into open and connected disjoint components. Define the function*

$$\Phi(x,t) = t^{p(x)}\chi_\Omega(x) \quad \textit{for every} \quad x \in \Omega, \quad t \geq 0,$$

where $p(x) \geq 1$. Assume that Φ satisfies $\mathcal{M}_2$. Let $R > 0$ be the constant from Proposition 3.2.11. If for every two distinct numbers $n, m \in \mathbb{N}$ such that $\Omega_n \neq \emptyset$ and $\Omega_m \neq \emptyset$ there exists a finite sequence of natural numbers $s = (k_1, k_2, \ldots, k_N)$ such that $k_1 = n$ and $k_N = m$ and for every $i = 1, \ldots, N-1$ there exist points $x_i \in \Omega_{k_i}$ and $x_{i+1} \in \Omega_{k_{i+1}}$ with $|x_i - x_{i+1}| \leq R$, then $p(x) = C$ for some constant C.

Proof. By Proposition 3.2.11, there exist constants $C_1 \geq 1$ and $R \in (0, \frac{1}{2}]$ such that for every $x, y \in \Omega$ with $|x - y| \leq R$ and $0 \leq t \leq \frac{1}{|x-y|^d}$,

$$\Phi(x,t) \leq 2C_1\Phi(y,t).$$

Take any $n \in \mathbb{N}$ for which we have $\Omega_n \neq \emptyset$. For any $x, y \in \Omega_n$ with $|x - y| \leq R$ and $0 < t < \min\left\{\frac{1}{|x-y|^d}, 1\right\}$ we have

$$t^{p(x)} \leq 2C_1 t^{p(y)} \quad \text{and} \quad t^{p(y)} \leq 2C_1 t^{p(x)}.$$

Therefore

$$(p(x) - p(y)) \log t \leq \log(2C_1) \quad \text{and} \quad (p(y) - p(x)) \log t \leq \log(2C_1).$$

If $p(x) \neq p(y)$, then without loss of generality, we can assume that $(p(x) - p(y)) < 0$. Since $t < 1$, for small enough t, $(p(x) - p(y)) \log t > \log(2C_1)$, which contradicts the above inequalities. Hence for any $x, y \in \Omega_n$ with $|x - y| \leq R$, we have $p(x) = p(y)$. Since Ω_n is connected there exists an constant C_n such that for every $x \in \Omega_n$ we have $p(x) = C_n$.

Now take any $n, m \in \mathbb{N}$ such that $n \neq m$ and $\Omega_n \neq \emptyset$, $\Omega_m \neq \emptyset$. By our assumption on Ω there exists a finite sequence of natural numbers $s = (k_1, k_2, \ldots, k_N)$ such that $k_1 = n$ and $k_N = m$ and for every $i = 1\ldots, N-1$ there exist points $x_i \in \Omega_{k_i}$ and $x_{i+1} \in \Omega_{k_{i+1}}$ with $|x_i - x_{i+1}| \leq R$. By the argument above, for each $i = 1, \ldots, N$ there exists a constant C_i such that for all $x \in \Omega_{k_i}$ we have $p(x) = C_i$. Fix any $i = 1, \ldots, N-1$. Since $|x_i - x_{i+1}| \leq R$ we repeat the argument from the first part of the proof replacing x, y by x_i, x_{i+1} respectively and deduce that $p(x_i) = p(x_{i+1})$. Hence for any $n, m \in \mathbb{N}$ such that $n \neq m$ and $\Omega_n \neq \emptyset$, $\Omega_m \neq \emptyset$ we have $C_n = C_m$, which finishes the proof.

□

The next result follows immediately form Proposition 3.2.14.

Corollary 3.2.15. *Assume $\Omega \subset \mathbb{R}^d$ is an open and connected set. Define the function*

$$\Phi(x,t) = t^{p(x)}\chi_\Omega(x) \quad \text{for every} \quad x \in \Omega, \quad t \geq 0,$$

where $p(x) \geq 1$. If Φ satisfies $\mathcal{M}_2$ then $p(x) = C$ for some constant $C > 0$.

Now we compere the condition from [6] with (A1).

Proposition 3.2.16. *Let $\Omega \subset \mathbb{R}^d$ be open and connected. Let Φ be a MO function on Ω. Assume that*

$$\frac{\Phi(x,t)}{\Phi(y,t)} \leq t^{\frac{A}{\log\left(\frac{1}{|x-y|}\right)}}, \tag{3.13}$$

where $A > 0$, $x, y \in \Omega$ and $|x-y| < \frac{1}{2}$ and $t \geq 1$. Then Φ satisfies (A1). In particular if $\Phi(x,t) = t^{p(x)}$, where $1 \leq p(x) < \infty$ a.e. on Ω and $p(\cdot)$ is log-*Hölder continuous (3.10), then Φ satisfies (A1).*

Proof. Let $\Omega \subset \mathbb{R}^d$ be open and Φ be a MO function as in the assumption. Let $\delta = \frac{\sigma_d}{4^d}$, then by (1.1), $\delta < 1$. For any open ball B with $|B| < \delta$ and any $x, y \in B$ we have $|x-y| < 2\left(\frac{|B|}{\sigma_d}\right)^{1/d} < \frac{1}{2}$. Taking any $x, y \in \Omega$ such that $|x-y| < \frac{1}{2}$ by (3.13) we have

$$\frac{\Phi(x,1)}{\Phi(y,1)} \leq 1, \quad \frac{\Phi(y,1)}{\Phi(x,1)} \leq 1.$$

Therefore $\Phi(x,1) = \Phi(y,1)$ and since Ω is connected we conclude that for every $y \in \Omega$ we have

$$\Phi(x,1) = 1$$

and so

$$\Phi^{-1}(y,1) = 1.$$

For any open ball B with $|B| < \delta$ and any $x, y \in B$ we have $\frac{\sigma_d |x-y|^d}{2^d} < |B|$,

$$\frac{1}{|B|} \leq \frac{2^d}{\sigma_d |x-y|^d}. \tag{3.14}$$

Denote $C_d = \left(\frac{2^d}{\sigma_d}\right)^{\frac{A}{\log(2)}} + 1$. Take any $x, y \in \Omega \cap B$, where $|B| < \delta$ and let $t \in \left[\Phi^{-1}(y,1), \Phi^{-1}\left(y, \frac{1}{|B|}\right)\right] = \left[1, \Phi^{-1}\left(y, \frac{1}{|B|}\right)\right]$. It follows that $1 \leq t \leq \Phi^{-1}\left(y, \frac{1}{|B|}\right) \leq \frac{1}{|B|}\Phi^{-1}(y,1) = \frac{1}{|B|}$. By (3.14) and in view of $\frac{\sigma_d}{2^d} > 1$ for all $d \in \mathbb{N}$ we have

$$\begin{aligned}\frac{\Phi(x,t)}{\Phi(y,t)} &\leq t^{\frac{A}{\log\left(\frac{1}{|x-y|}\right)}} \leq \left(\frac{1}{|B|}\right)^{\frac{A}{\log\left(\frac{1}{|x-y|}\right)}} \leq \left(\frac{2^d}{\sigma_d |x-y|^d}\right)^{\frac{A}{\log\left(\frac{1}{|x-y|}\right)}} \\ &\leq \left(\frac{2^d}{\sigma_d}\right)^{\frac{A}{\log(2)}} \left(\frac{1}{|x-y|}\right)^{\frac{Ad}{\log\left(\frac{1}{|x-y|}\right)}} \leq C_d e^{Ad\frac{\log\left(\frac{1}{|x-y|}\right)}{\log\left(\frac{1}{|x-y|}\right)}} = C_d e^{Ad}.\end{aligned}$$

Since $A > 0$ and $C_d > 1$ we have that $C_d e^{Ad} > 1$. Defining $\beta = \frac{1}{C_d e^{Ad}}$ we get $0 < \beta < 1$ and so

$$\frac{\Phi(x,t)}{\Phi(y,t)} \leq \frac{1}{\beta}.$$

Hence, by convexity of Φ,

$$\Phi(x, \beta t) \leq \beta \Phi(x,t) \leq \Phi(y,t),$$

in other words Φ satisfies (A1).

□

3.3 The condition (A1) and approximation of compactly supported functions in $W^{k,\Phi}(\Omega)$

In this section we prove the main tool of this chapter, namely the Lemma 3.3.2 and then use it to establish the density of $C_C^\infty(\mathbb{R}^d)$ in the MOS space $W^{k,\Phi}(\Omega)$. The crucial thing here is that we don't use the boundedness of the maximal operator, neither do we need to have some global regularity of function Φ like the log-Hölder decay condition. Even though we use the (A1) condition we don't require that our function is essentially non weighted. Hence, our results generalize all results about density of smooth functions that were established using the (A) conditions as we have dropped the condition $(A0)$ and $(A2)$.

We start with the following observation

Lemma 3.3.1. *Let $\Omega \subset \mathbb{R}^d$ be open and Φ be a MO function defined on Ω. Assume that Φ satisfies $(A1)$. If $f \in L^\Phi(\Omega)$ then there exist a constant $C' > 1$ such that for any $x \in \Omega$ and $r > 0$ with $|B(x,r)| < \delta$, and for almost all $z \in B(x,r) \cap \Omega$, we have*

$$|f_r(x)| \leq C'\Phi^{-1}\left(z, \frac{1}{|B(x,r)|}\right)\|f\|_\Phi.$$

Proof. Let $x \in \Omega$ and $r > 0$ be as in the assumption. For any $f \in L^\Phi(\Omega)$, by Hölder's inequality,

$$\begin{aligned}|f_r(x)| &= |(f * J_{(r)})(x)| = \left|\int_{\mathbb{R}^d} J_{(r)}(x-y)f(y)dy\right| \\ &= \left|C\int_{\mathbb{R}^d} \frac{1}{r^d} e^{\frac{-1}{1-\frac{|x-y|^2}{r^2}}} \chi_{B(0,1)}\left(\frac{x-y}{r}\right) f(y)dy\right| \\ &\leq \frac{Ce^{-1}}{r^d}\int\limits_{\mathbb{R}^d} \chi_{B(x,r)\cap\Omega}(y)|f(y)|dy \leq \frac{2Ce^{-1}}{r^d}\|\chi_{B(x,r)\cap\Omega}\|_{\Phi^*}\|f\|_\Phi.\end{aligned}$$

Since Φ satisfies (A1), by Lemma 3.2.5 (3), Φ^* satisfies (A1) with the constant $\frac{\beta}{2}$ instead of β. By Propositions 3.2.7 (1), (2), Proposition 2.1.15 (3), condition (A1) applied to $y, z \in B(x,r)\cap\Omega$, for a.a $z \in B(x,r)\cap\Omega$ we have that

$$\begin{aligned}\|\chi_{B(x,r)\cap\Omega}\|_{\Phi^*} &\leq \frac{2}{\beta}\frac{1}{|B(x,r)\cap\Omega|}\int\limits_{B(x,r)\cap\Omega}\frac{1}{(\Phi^*)^{-1}\left(y, \frac{1}{|B(x,r)|}\right)}dy\\ &\leq \frac{4}{\beta^2}\frac{1}{(\Phi^*)^{-1}\left(z, \frac{1}{|B(x,r)|}\right)} \leq \frac{4|B(x,r)|}{\beta^2}\Phi^{-1}\left(z, \frac{1}{|B(x,r)|}\right).\end{aligned}$$

Since $|B(x,r)| = \sigma_d r^d$, where $\sigma_d = |B(0,1)|$, we conclude

$$\begin{aligned}|f_r(x)| &\leq \frac{2Ce^{-1}}{r^d}\frac{4|B(x,r)|}{\beta^2}\Phi^{-1}\left(z, \frac{1}{|B(x,r)|}\right)\|f\|_\Phi\\ &\leq \frac{8Ce^{-1}\sigma_d}{\beta^2}\Phi^{-1}\left(z, \frac{1}{|B(x,r)|}\right)\|f\|_\Phi.\end{aligned}$$

Setting $C' = \frac{8Ce^{-1}\sigma_d}{\beta^2} + 1$ we get the desired inequality. □

Now we can proceed to the main tool of this section.

Lemma 3.3.2. *Let $\Omega \subset \mathbb{R}^d$ be open and Φ be a MO function defined on Ω. Assume that Φ satisfies* (A1). *Let $C' > 1$ be the constant from* Lemma 3.3.1 *and β be the constant from the condition* (A1). *For any $N \in \mathbb{N}$, let $f_0, \ldots, f_N \in L^\Phi(\Omega)$ be such that for $i = 0, \ldots, N$, $\operatorname{ess\,supp} f_i \subset \Omega$ is compact. There exist sequence $\{r_n\}_{n=1}^\infty \subset (0,\infty)$, and sequences $\{R(f_i, r_n)\}_{n=1}^\infty \subset (0,\infty)$ for $i = 1,\ldots N$, with $\lim\limits_{n\to\infty} r_n = 0$ and $\lim\limits_{n\to\infty} R(f_i, r_n) = 0$ for $i = 0,\ldots,N$, such that for each $i = 0,\ldots,N$,*

$$\lim_{n\to\infty}(f_i)_{r_n} = f \text{ a.e. in } \Omega,$$

$$\int\limits_\Omega \Phi\left(x, \frac{\beta}{2C''}\frac{|(f_i)_{r_n}(x)|}{\|f_i\|_\Phi}\right)dx \leq 2\int\limits_\Omega \Phi\left(x, \frac{|f_i(x)|}{C'\|f_i\|_\Phi}\right)dx + R(f_i, r_n),$$

where $(f_i)_{r_n}(x) = (f_i * J_{(r_n)})(x)$ and $C'' = \frac{4C'}{\beta}$.

Proof. Let C' be the constant from Lemma 3.3.1 and β, δ be the constants from condition (A1). By defining $C'' = \frac{4C'}{\beta}$ we notice that

$$C' < C''. \tag{3.15}$$

Take any $N \in \mathbb{N}$ and let $\{f_0, \dots, f_N\} \subset L^{\Phi}(\Omega)$ be such that for any $i = 0, \dots, N$, $\operatorname{ess\,supp} f_i \subset \Omega$ is compact. There exists $R > 0$ with $|B(0,R)| < \delta$ such that for every $r < R$ and $x \in \bigcup_{i=0}^{N} \operatorname{ess\,supp} f_i$ we have $B(x,r) \subset \Omega$. For any $i = 0, \dots, N$ and $0 < r < R$ define the sets

$$E_{1,r}(i) = \left\{ x \in \operatorname{ess\,supp}(f_i)_r : [\Phi^{-1}]_{B(x,r)^+}(1) \leq \frac{|(f_i)_r(x)|}{2C'\|f_i\|_{\Phi}} \leq [\Phi^{-1}]_{B(x,r)^-}\left(\frac{1}{|B(x,r)|}\right) \right\}$$

$$E_{2,r}(i) = \left\{ x \in \operatorname{ess\,supp}(f_i)_r : \frac{|(f_i)_r(x)|}{2C'\|f\|_{\Phi}} < [\Phi^{-1}]_{B(x,r)^+}(1) \right\},$$

$$E'_{2,r}(i) = \left\{ x \in E_{2,r}(i) : |(f_i)_r(x)| \leq |f_i(x)| \right\},$$

$$E''_{2,r}(i) = \left\{ x \in E_{2,r}(i) : |(f_i)_r(x)| > |f_i(x)| \right\}.$$

By Lemma 3.3.1, for every $0 \leq i \leq N$, every $x \in \Omega$ and a.a. $z \in B(x,r)$ we have

$$\frac{|(f_i)_r(x)|}{2C'\|f_i\|_{\Phi}} \leq \frac{|(f_i)_r(x)|}{C'\|f_i\|_{\Phi}} \leq \Phi^{-1}\left(z, \frac{1}{|B(x,r)|}\right).$$

Therefore,

$$\frac{|(f_i)_r(x)|}{2C'\|f_i\|_{\Phi}} \leq [\Phi^{-1}]_{B(x,r)^-}\left(\frac{1}{|B(x,r)|}\right).$$

Hence $\operatorname{ess\,supp}(f_i)_r = E_{1,r}(i) \cup E_{2,r}(i)$ and $E_{2,r}(i) = E'_{2,r}(i) \cup E''_{2,r}(i)$. Let $r < R$ be such that $|B(0,r)| < \delta$. Set

$$I_{1,r}(i) = \int_{E_{1,r}(i)} \Phi\left(x, \beta \frac{|(f_i)_r(x)|}{2C''\|f_i\|_{\Phi}}\right) dx,$$

$$I_{2,r}(i) = \int\limits_{E_{2,r}(i)} \Phi\left(x, \beta \frac{|(f_i)_r(x)|}{2C''\|f_i\|_\Phi}\right) dx.$$

For a fixed $0 \le i \le N$,

$$\begin{aligned} I_{1,r}(i) + I_{2,r}(i) &= \int\limits_{E_{1,r}(i)} \Phi\left(x, \beta \frac{|(f_i)_r(x)|}{2C''\|f_i\|_\Phi}\right) dx + \int\limits_{E_{2,r}(i)} \Phi\left(x, \beta \frac{|(f_i)_r(x)|}{2C''\|f_i\|_\Phi}\right) dx \\ &= \int\limits_{\operatorname{ess\,supp}(f_i)_r} \Phi\left(x, \beta \frac{|(f_i)_r(x)|}{2C''\|f_i\|_\Phi}\right) dx = \int\limits_{\Omega} \Phi\left(x, \beta \frac{|(f_i)_r(x)|}{2C''\|f_i\|_\Phi}\right) dx. \end{aligned}$$

Take any $x \in E_{1,r}(i)$ and let Ψ be the Orlicz function from Lemma 2.1.18. By the inequality (3.15), the definition of $E_{1,r}(i)$, Lemma 3.2.6, Lemma 2.1.18 and Jensen's inequality applied to Ψ we get that

$$\begin{aligned} \Phi\left(x, \beta \frac{|(f_i)_r(x)|}{2C''\|f_i\|_\Phi}\right) &\le \Phi\left(x, \beta \frac{|(f_i)_r(x)|}{2C'\|f_i\|_\Phi}\right) \le \Phi_{B(x,r)^+}\left(\beta \frac{|(f_i)_r(x)|}{2C'\|f_i\|_\Phi}\right) \\ &\le \Phi_{B(x,r)^-}\left(\frac{|(f_i)_r(x)|}{2C'\|f_i\|_\Phi}\right) \\ &\le \Phi_{B(x,r)^-}\left(\frac{1}{2C'\|f_i\|_\Phi} \int_{\mathbb{R}^d} J_{(r)}(x-y)|f_i(y)|dy\right) \\ &\le \Psi\left(\frac{1}{C'\|f_i\|_\Phi} \int_{\mathbb{R}^d} J_{(r)}(x-y)|f_i(y)|dy\right) \\ &\le \int_{\mathbb{R}^d} J_{(r)}(x-y)\Psi\left(\frac{|f_i(y)|}{C'\|f\|_\Phi}\right) dy \\ &\le \int\limits_{\mathbb{R}^d} J_{(r)}(x-y)\Phi_{B(x,r)^-}\left(\frac{|f_i(y)|}{C'\|f_i\|_\Phi}\right) dy \\ &\le \int\limits_{\mathbb{R}^d} J_{(r)}(x-y)\Phi\left(y, \frac{|f_i(y)|}{C'\|f_i\|_\Phi}\right) dy. \end{aligned}$$

Hence by Fubini's Theorem we conclude that

$$I_{1,r}(i) = \int\limits_{E_{1,r}(i)} \Phi\left(x, \beta\frac{|(f_i)_r(x)|}{2C''\|f_i\|_\Phi}\right) dx \leq \int\limits_{E_{1,r}(i)} \int\limits_{\mathbb{R}^d} J_{(r)}(x-y)\Phi\left(y, \frac{|f_i(y)|}{C'\|f_i\|_\Phi}\right) dydx$$
$$= \int\limits_{\mathbb{R}^d} \int\limits_{E_{1,r}(i)} J_{(r)}(x-y)\Phi\left(y, \frac{|f_i(y)|}{C'\|f_i\|_\Phi}\right) dxdy \leq \int\limits_{\mathbb{R}^d} \Phi\left(x, \frac{|f_i(x)|}{C'\|f_i\|_\Phi}\right) dx$$
$$= \int\limits_{\Omega} \Phi\left(x, \frac{|f_i(x)|}{C'\|f_i\|_\Phi}\right) dx,$$

and so

$$I_{1,r}(i) \leq \int\limits_{\Omega} \Phi\left(x, \frac{|f_i(x)|}{C'\|f_i\|_\Phi}\right) dx. \tag{3.16}$$

On the other hand, by definition of $E'_{2,r}(i)$, the inequality (3.15) and since $\beta \in (0,1)$ we have

$$I_{2,r}(i) = \int\limits_{E_{2,r}(i)} \Phi\left(x, \beta\frac{|(f_i)_r(x)|}{2C''\|f_i\|_\Phi}\right) dx$$
$$= \int\limits_{E'_{2,r}(i)} \Phi\left(x, \beta\frac{|(f_i)_r(x)|}{2C''\|f_i\|_\Phi}\right) dx + \int\limits_{E''_{2,r}(i)} \Phi\left(x, \beta\frac{|(f_i)_r(x)|}{2C''\|f_i\|_\Phi}\right) dx$$
$$\leq \int\limits_{E'_{2,r}(i)} \Phi\left(x, \frac{|(f_i)_r(x)|}{2C''\|f_i\|_\Phi}\right) dx + \int\limits_{E''_{2,r}(i)} \Phi\left(x, \frac{|(f_i)_r(x)|}{2C''\|f_i\|_\Phi}\right) dx$$
$$\leq \int\limits_{E'_{2,r}(i)} \Phi\left(x, \frac{|f_i(x)|}{C'\|f_i\|_\Phi}\right) dx + \frac{1}{2}\int\limits_{E''_{2,r}(i)} \Phi\left(x, \frac{|f_i(x)|}{C'\|f_i\|_\Phi}\right) dx$$
$$+ \frac{1}{2}\int\limits_{E''_{2,r}(i)} \Phi\left(x, \frac{|(f_i)_r(x) - f_i(x)|}{C''\|f_i\|_\Phi}\right) dx.$$

Defining the quantity

$$R(f_i, r) = \int\limits_{E''_{2,r}(i)} \Phi\left(x, \frac{|(f_i)_r(x) - f_i(x)|}{C''\|f_i\|_\Phi}\right) dx,$$

we have

$$I_{2,r}(i) \leq \int\limits_{E'_{2,r}(i)} \Phi\left(x, \frac{|f_i(x)|}{C'\|f_i\|_\Phi}\right) dx + \frac{1}{2} \int\limits_{E''_{2,r}(i)} \Phi\left(x, \frac{|f_i(x)|}{C'\|f_i\|_\Phi}\right) dx + \frac{1}{2} R(f_i, r). \quad (3.17)$$

Now we will find a sequence $\{r_n\}_{n=1}^{\infty} \subset (0, \infty)$ such that for every $i = 0, \ldots, N$ we have

$$\lim_{n\to\infty} R(f_i, r_n) = 0.$$

First notice that by Lemma 3.2.9 for each $i = 0, \ldots, N$, f_i is an element of $L^1(\Omega)$. Hence, by [19, 8.14 Theorem], $(f_i)_r \to f_i$ in $L^1(\Omega)$ as $r \to 0$. Therefore there exists $\{r_n\}_{n=1}^{\infty} \subset (0, R)$ such that $\lim_{n\to\infty} r_n = 0$ and $\lim_{n\to\infty} (f_i)_{r_n} = f$ a.e. in Ω, for every $i = 0, \ldots, N$. Letting $r_0 = \sup_{n\in\mathbb{N}} r_n$, clearly $r_0 < R$. For any $n \in \mathbb{N}$ we have

$$E''_{2,r_n}(i) \subset E_{2,r_n}(i) \subset \operatorname{ess\,supp}(f_i)_{r_n} \subset \operatorname{ess\,supp} f_i + B(0, r_n) \subset \operatorname{ess\,supp} f_i + B(0, r_0).$$

By definition of $E''_{2,r_n}(i)$, for every $x \in E''_{2,r_n}(i)$ it follows that

$$\frac{|f_i(x)|}{2C'\|f_i\|_\Phi} < \frac{|(f_i)_{r_n}(x)|}{2C'\|f_i\|_\Phi} \leq [\Phi^{-1}]_{B(x,r_n)^+}(1).$$

Hence, by the fact that $C'' = \frac{4C'}{\beta}$, for every $n \in \mathbb{N}$ we have

$$\begin{aligned}\frac{|(f_i)_{r_n}(x) - f_i(x)|}{C''\|f_i\|_\Phi} \chi_{E''_{2,r_n}(i)}(x) &\leq \beta \frac{|(f_i)_{r_n}(x)| + |f_i(x)|}{4C'\|f_i\|_\Phi} \chi_{E''_{2,r_n}(i)}(x) \\ &\leq \beta [\Phi^{-1}]_{B(x,r_n)^+}(1) \chi_{E''_{2,r_n}(i)}(x).\end{aligned}$$

Notice that, by the fact that $r_0 < R$ we have that $|B(x, r_0)| < \delta < 1$ and so by (A1), for a.e. $z \in B(x, r_0)$,

$$\beta[\Phi^{-1}]_{B(x,r_0)^+}(1) = \beta \operatorname*{ess\,sup}_{y \in B(x,r_0)} \Phi^{-1}(y,1) \leq \operatorname*{ess\,sup}_{y \in B(x,r_0)} \Phi^{-1}(z,1)$$
$$= \Phi^{-1}(z,1) < \Phi^{-1}\left(z, \frac{1}{|B(x,r_0)|}\right).$$

Taking the essential infimum over $z \in B(x, r_0)$ on the both sides of the above inequality we arrive at $\beta[\Phi^{-1}]_{B(x,r_0)^+}(1) \leq [\Phi^{-1}]_{B(x,r_0)^-}\left(\frac{1}{|B(x,r_0)|}\right)$. By concavity of $[\Phi^{-1}]_{B(x,r_0)^-}$ and the fact that $B(x, r_n) \subset B(x, r_0)$, for every $i = 0, \ldots, N$ we conclude that

$$\begin{aligned}
\Phi\left(x, \frac{(|(f_i)_{r_n}(x) - f(x)|}{C''\|f_i\|_\Phi}\chi_{E''_{2,r_n}(i)}(x)\right) &\leq \Phi\left(x, \beta[\Phi^{-1}]_{B(x,r_n)^+}(1)\chi_{E''_{2,r_n}(i)}(x)\right) \\
&\leq \Phi\left(x, \beta[\Phi^{-1}]_{B(x,r_0)^+}(1)\chi_{E''_{2,r_n}(i)}(x)\right) \\
&\leq \Phi\left(x, [\Phi^{-1}]_{B(x,r_0)^-}\left(\frac{1}{|B(x,r_0)|}\right)\chi_{E''_{2,r_n}(i)}(x)\right) \\
&\leq \Phi\left(x, \Phi^{-1}\left(x, \frac{1}{|B(x,r_0)|}\chi_{E''_{2,r_n}(i)}(x)\right)\right) \\
&= \frac{1}{|B(x,r_0)|}\chi_{E''_{2,r_n}(i)}(x) \\
&\leq \frac{1}{|B(x,r_0)|}\chi_{\operatorname{ess\,supp} f_i + B(0,r_0)}(x) \\
&= \frac{1}{|B(0,r_0)|}\chi_{\operatorname{ess\,supp} f_i + B(0,r_0)}(x).
\end{aligned}$$

Then the Lebesgue Dominated Convergence Theorem gives us

$$\lim_{n\to\infty} R(f_i, r_n) = \lim_{n\to\infty} \int_{E''_{2,r_n}(i)} \Phi\left(x, \frac{|(f_i)_{r_n}(x) - f_i(x)|}{C'\|f_i\|_\Phi}\right) dx = 0.$$

Finally, by (3.16) and (3.17), we conclude that for every $i = 0, \dots, N$,

$$\int_\Omega \Phi\left(x, \beta\frac{|(f_i)_{r_n}(x)|}{2C''\|f_i\|_\Phi}\right) dx = I_{1,r_n}(i) + I_{2,r_n(i)} \leq \int_\Omega \Phi\left(x, \frac{|f_i(x)|}{C'\|f_i\|_\Phi}\right) dx$$
$$+ \int_{E'_{2,r_n}(i)} \Phi\left(x, \frac{|f_i(x)|}{C'\|f_i\|_\Phi}\right) dx$$
$$+ \frac{1}{2}\int_{E''_{2,r_n}(i)} \Phi\left(x, \frac{|f_i(x)|}{C'\|f_i\|_\Phi}\right) dx$$
$$+ \frac{1}{2}\int_{E''_{2,r_n}(i)} \Phi\left(x, \frac{|(f_i)_{r_n}(x) - f_i(x)|}{C'\|f_i\|_\Phi}\right) dx$$
$$\leq 2\int_\Omega \Phi\left(x, \frac{|f_i(x)|}{C'\|f_i\|_\Phi}\right) dx + R(f_i, r_n).$$

This completes the proof. □

Theorem 3.3.3. *Let $\Omega \subset \mathbb{R}^d$ be open and Φ be a MO function defined on Ω satisfying* (A1) *condition. For any $N \in \mathbb{N}$ let $f_0, \dots, f_N \in L^\Phi(\Omega)$ be such that* ess supp $f_i \subset \Omega$ *is compact for $i = 0, \dots, N$. Let β be the constant form the condition* (A1) *and C'' be the constant from Lemma 3.3.2. There exists a sequence $\{r_n\}_{n=1}^\infty \subset (0, \infty)$ such that for every for $i = 0, \dots, N$,*

$$\lim_{n\to\infty} \int_\Omega \Phi\left(x, \frac{\beta}{4C''}\frac{|f_i(x) - (f_i)_{r_n}(x)|}{\|f_i\|_\Phi}\right) dx = 0.$$

In particular $\{(f_i)_{r_n}\}_{n=1}^\infty$ converges to f_i in modular for all $i = 0, \dots, N$.

Proof. Let C'' be the constant from Lemma 3.3.2, recall that $C'' = \frac{4C'}{\beta}$ where C' is the constant from 3.3.1 and β is the constant from condition (A1). Take any $N \in \mathbb{N}$ and $\{f_0, \dots f_N\} \subset L^\Phi(\Omega)$ such that for any $i = 0, \dots, N$, ess supp $f_i \subset \Omega$ is compact. Let $\{r_n\} \subset (0, \infty)$ be the sequence from Lemma 3.3.2. For each $i = 0, \dots, N$, define

$$F_i(x) = \sup_{n\in\mathbb{N}} \Phi\left(x, \frac{\beta}{2C''}\frac{|(f_i)_{r_n}(x)|}{\|f_i\|_\Phi}\right).$$

By Lemma 3.3.2 and the fact that $C'' > C' > 1$,

$$\int_\Omega F_i(x)dx = \int_\Omega \sup_{n\in\mathbb{N}} \Phi\left(x, \frac{\beta}{2C''}\frac{|(f_i)_{r_n}(x)|}{\|f_i\|_\Phi}\right) dx \le \sup_{n\in\mathbb{N}} \int_\Omega \Phi\left(x, \frac{\beta}{2C'}\frac{|(f_i)_{r_n}(x)|}{\|f_i\|_\Phi}\right) dx$$
$$\le 2\int_\Omega \Phi\left(x, \frac{|f_i(x)|}{C'\|f_i\|_\Phi}\right) dx + \sup_{n\in\mathbb{N}} R(f_i, r_n) < \infty.$$

For every $x \in \Omega$, we have

$$\Phi\left(x, \frac{\beta}{4C''}\frac{|f_i(x) - (f_i)_{r_n}(x)|}{\|f_i\|_\Phi}\right) \le \frac{1}{2}\Phi\left(x, \frac{\beta}{2C''}\frac{|f_i(x)|}{\|f_i\|_\Phi}\right) + \frac{1}{2}\Phi\left(x, \frac{\beta}{2C''}\frac{|(f_i)_{r_n}(x)|}{\|f_i\|_\Phi}\right)$$
$$\le \frac{1}{2}\Phi\left(x, \frac{\beta}{2C''}\frac{|f_i(x)|}{\|f_i\|_\Phi}\right) + \frac{1}{2}F_i(x).$$

Hence $\Phi\left(\cdot, \frac{\beta}{4C''}\frac{|f_i(\cdot) - (f_i)_{r_n}(\cdot)|}{\|f_i\|_\Phi}\right)$ is dominated by an integrable function. By Lemma 3.3.2, $\lim_{n\to\infty} (f_i)_{r_n} = f_i$ a.e. in Ω, therefore, by The Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty} \int_\Omega \Phi\left(x, \frac{\beta}{4C''}\frac{|f_i(x) - (f_i)_{r_n}(x)|}{\|f_i\|_\Phi}\right) dx = 0.$$

□

Now we have the tools to prove the concrete results. We start with the fact that condition (A1) is enough to be able to approximate a compactly supported function f from $W^{k,\Phi}(\Omega)$ with functions from $C_C^\infty(\Omega)$, provided that the support of f is far away from boundary of Ω. Using the theorem above, and the fact that we have smooth cutoff functions, we can show that (A1) is enough to guarantee the modular density of $C_C^\infty(\mathbb{R}^d)$ in $W^{k,\Phi}(\mathbb{R}^d)$.

Theorem 3.3.4. *Let $k \in \mathbb{N}$, $\Omega \subset \mathbb{R}^d$ be open and Φ be a MO function defined on Ω satisfying both the* (A1) *condition. Let β and C'' be the constants from condition* (A1) *and Lemma 3.3.2. Let $f \in E^{\Phi,k}(\Omega)$ be such that* $\operatorname{ess\,supp} f \subset \Omega$ *is compact.*

There exists a sequence $\{r_m\}_{m=1}^{\infty} \subset (0,\infty)$ *such that, for every multi-index* α *with* $|\alpha| \le k$,

$$\lim_{m\to\infty} \int_{\Omega} \Phi\left(x, \frac{\beta|\partial^\alpha f_{r_m}(x) - \partial^\alpha f(x)|}{12C''\|f\|_{W^{k,\Phi}}}\right) dx = 0.$$

Proof. Take any $k \in \mathbb{N}$, and throughout the proof we assume that α is a multi-index such that $|\alpha| \le k$. Let Φ be as in the assumption. For every multi-index α denote $f_\alpha = \partial^\alpha f$ with $f_0 = f$. Since Φ is locally integrable $\operatorname{Sing}\Phi = \emptyset$, so by Theorem 2.2.7 for each multi-index α, there exists a sequence $\{u_{\alpha,j}\}_{j=1}^{\infty} \subset C_C^\infty(\Omega)$ such that,

$$\lim_{k\to\infty} \|f_\alpha - u_{\alpha,j}\|_\Phi = 0. \tag{3.18}$$

Recall that $\|f\|_{W^{k,\Phi}} = \sum_{|\alpha|\le k} \|\partial^\alpha f\|_\Phi$. Let $R > 0$ be such that for each $x \in \operatorname{ess\,supp} f$ and $r < R$ we have $B(x,r) \subset \Omega$ and $B(x,r) < \delta$ where δ is the constant form condition (A1). For each multi-index α define

$$I(r,\alpha) = \int_{\Omega} \Phi\left(x, \frac{\beta|f_\alpha(x) - \partial^\alpha f_r(x)|}{12C''\|f\|_{W^{k,\Phi}}}\right) dx.$$

Recall that for each $x \in \mathbb{R}^d$ and each multi-index α,

$$\partial^\alpha f_r(x) = \partial^\alpha (J_{(r)} * f)(x) = (J_{(r)} * \partial^\alpha f)(x).$$

Taking any $0 < r < R$, $j \in \mathbb{N}$ and any multi-index α, we have

$$\begin{aligned}
I(r,\alpha) &= \int\limits_{\Omega} \Phi\left(x, \frac{\beta|\partial^\alpha f_r(x) - f_\alpha(x)|}{12C''\|f\|_{W^{k,\Phi}}}\right) dx \\
&\leq \frac{1}{3}\int\limits_{\Omega} \Phi\left(x, \frac{\beta|(J_{(r)} * f_\alpha)(x) - (J_{(r)} * u_{\alpha,j})(x)|}{4C''\|f\|_{W^{k,\Phi}}}\right) dx \\
&+ \frac{1}{3}\int\limits_{\Omega} \Phi\left(x, \frac{\beta|(J_{(r)} * u_{\alpha,k})(x) - u_{\alpha,j}(x)|}{4C''\|f\|_{W^{k,\Phi}}}\right) dx \\
&+ \frac{1}{3}\int\limits_{\Omega} \Phi\left(x, \frac{\beta|u_{\alpha,j}(x) - f_\alpha(x)|}{4C''\|f\|_{W^{k,\Phi}}}\right) dx = I_1(r,\alpha,j) + I_2(r,\alpha,j) + I_3(\alpha,j).
\end{aligned}$$

First we estimate the term $I_3(\alpha, j)$. Notice that for each multi-index α, since $\beta \in (0,1)$, by convexity of Φ,

$$I_3(\alpha,j) = \frac{1}{3}\int\limits_{\Omega} \Phi\left(x, \frac{\beta|u_{\alpha,j}(x) - f_\alpha(x)|}{4C''\|f\|_{W^{k,\Phi}}}\right) dx \leq \frac{1}{3}\int\limits_{\Omega} \Phi\left(x, \frac{|u_{\alpha,j}(x) - f_\alpha(x)|}{2C''\|f_\alpha\|_{\Phi}}\right) dx.$$

Fix any $\varepsilon > 0$, in view of (3.18), there exists $K(\varepsilon) \in \mathbb{N}$ such that for all multi-index α and all $j > K(\varepsilon)$,

$$\frac{1}{3}\int\limits_{\Omega} \Phi\left(x, \frac{|u_{\alpha,j}(x) - f_\alpha(x)|}{2C''\|f_\alpha\|_{\Phi}}\right) dx < \varepsilon, \tag{3.19}$$

so for all $j > K(\varepsilon)$

$$I_3(\alpha, j) < \varepsilon. \tag{3.20}$$

For any $j > K(\varepsilon)$, by Lemma 3.3.2, there exists a sequence $\{r_n(j)\}_{n=1}^\infty \subset \left(0, \frac{R}{2}\right)$ such that for every multi-index α,

$$I_1(r_n(j),\alpha,k) = \frac{1}{3}\int\limits_\Omega \Phi\left(x, \frac{\beta|(J_{(r_n(j))} * f_\alpha)(x) - (J_{(r_n(j))} * u_{\alpha,j})(x)|}{4C''\|f\|_{W^{k,\Phi}}}\right) dx \tag{3.21}$$
$$= \frac{1}{3}\int\limits_\Omega \Phi\left(x, \frac{\beta|(J_{(r_n(j))} * (f_\alpha - u_{\alpha,j}))(x)|}{4C'\|f\|_{W^{k,\Phi}}}\right) dx$$
$$\le \frac{2}{3}\int\limits_\Omega \Phi\left(x, \frac{|u_{\alpha,j}(x) - f_\alpha(x)|}{2C'\|f_\alpha\|_\Phi}\right) dx + R(r_n(j), |f_\alpha - u_{\alpha,j}|)$$
$$< 2\varepsilon + R(r_n(j), |f_\alpha - u_{\alpha,j}|).$$

To estimate $I_2(r,\alpha,j)$ notice that since $u_{\alpha,j} \in C_C^\infty(\Omega)$, by [19, Theorem 8.14 c] $J_{(r_n(j))} * u_{\alpha,j}$ converges uniformly on Ω to $u_{\alpha,j}$ as $n \to \infty$, that is for all $j \in \mathbb{N}$,

$$\lim_{n\to\infty} \sup_{x\in\Omega} \left|(J_{(r_n(j))} * u_{\alpha,j})(x) - u_{\alpha,j}(x)\right| = 0.$$

On the other hand, by Hölder inequality, the fact that $\overline{\operatorname{ess\,supp} u_{\alpha,j} + B\left(0,\frac{R}{2}\right)} \subset \Omega$ is compact for any multi-index α and by local integrability of Φ for every

$$\Phi\left(x, \frac{\beta|(J_{(r_n(j))} * u_{\alpha,j})(x) - u_{\alpha,j}(x)|}{4C''\|f\|_{W^{k,\Phi}}}\right)$$
$$\le \Phi\left(x, \frac{\beta|(J_{(r_n(j))} * u_{\alpha,j})(x)| + \beta|u_{\alpha,j}(x)|}{4C''\|f\|_{W^{k,\Phi}}}\right)$$
$$\le \Phi\left(x, \frac{|(J_{(r_n(j))} * u_{\alpha,j})(x)| + |u_{\alpha,j}(x)|}{4C''\|f\|_{W^{k,\Phi}}}\right)$$
$$\le \Phi\left(x, \frac{\|J_{(r_n(j))}\|_1\|u_{\alpha,j}\|_\infty + \|u_{\alpha,j}\|_\infty}{4C''\|f\|_{W^{k,\Phi}}}\chi_{\operatorname{ess\,supp} u_{\alpha,j} + B(0,r_n(j))}(x)\right)$$
$$= \Phi\left(x, \frac{\|u_{\alpha,j}\|_\infty}{2C''\|f\|_{W^{k,\Phi}}}\chi_{\operatorname{ess\,supp} u_{\alpha,j} + B(0,r_n(j))}(x)\right)$$
$$\le \Phi\left(x, \frac{\|u_{\alpha,j}\|_\infty}{2C''\|f\|_{W^{k,\Phi}}}\chi_{\operatorname{ess\,supp} u_{\alpha,j} + B\left(0,\frac{R}{2}\right)}(x)\right) \in L^1(\Omega).$$

Hence, by the Lebesgue Dominated Convergence Theorem,

$$\lim_{n\to\infty} I_2(r_n(j),\alpha,j) = \lim_{n\to\infty} \frac{1}{3}\int\limits_\Omega \Phi\left(x, \frac{\beta|(J_{(r_n(j))} * u_{\alpha,j})(x) - u_{\alpha,j}(x)|}{4C''\|f\|_{W^{k,\Phi}}}\right) dx = 0. \quad (3.22)$$

Let now $N_1(j) \in \mathbb{N}$ be such that for all $n > N_1(j)$,

$$I_2(r_n(j),\alpha,j) < \varepsilon.$$

Finally we estimate $I_1(r,\alpha,j)$ By Lemma 3.3.2 there exists $N_2(j) \in \mathbb{N}$ such that for all $n > N_2(j)$ and all multi-indices α with $|\alpha| < k$,

$$R(r_n(j), |f_\alpha - u_{\alpha,j}|) < \varepsilon.$$

For $n > \max\{N_1(j), N_2(j)\}$ we have by, (3.21)

$$\begin{aligned} I(r_n(j),\alpha) &= \int\limits_\Omega \Phi\left(x, \frac{\beta|f_\alpha(x) - \partial^\alpha f_{r_n(j)}(x)|}{12C''\|f\|_{W^{k,\Phi}}}\right) dx \\ &\leq I_1(r_n(j),\alpha,j) + I_2(r_n(j),\alpha,j) + I_3(\alpha,j) < 5\varepsilon. \end{aligned}$$

Take any $m \in \mathbb{N}$ and set $\varepsilon = \frac{1}{m}$. By (3.19) for $j_m = K\left(\frac{1}{m}\right) + 1$ we have

$$I_3(\alpha, j_m) < \frac{1}{m}$$

Set

$$r_m = r_{\max\{N_1(j_m),N_2(j_m)\}+1}(j_m).$$

Then for every $m \in \mathbb{N}$ and multi-index α,

$$I(r_m,\alpha) = I(r_{\max\{N_1(j_m),N_2(j_m)\}+1}(j_m),\alpha) < \frac{5}{m}.$$

Hence by (3.22)

$$I(r_m, \alpha) = \int\limits_{\Omega} \Phi\left(x, \frac{\beta|\partial^\alpha f_{r_m}(x) - f_\alpha(x)|}{12C'\|f\|_{W^{k,\Phi}}}\right) dx < \frac{5}{m}.$$

Therefore, $\{r_m\}_{m=1}^{\infty}$ is the desired sequence.

□

The Theorem 3.3.4 is the main result of this section as it allows us to approximate compactly supported functions $f \in E^{k,\Phi}(\Phi)$ with smooth, compactly supported functions as long as the support of f is a subset of Ω. The next result (Corollary 3.3.5) is just an application of the above result to MOS spaces $W^{k,\Phi}(\Omega)$ fro which the function Φ satisfies Δ_2 condition.

Now we have an immediate corollary.

Corollary 3.3.5. *Let $k \in \mathbb{N}$, $\Omega \subset \mathbb{R}^d$ be open and Φ be a MO function defined on Ω satisfying both* (A1) *and Δ_2 condition. For every compactly supported $f \in W^{k,\Phi}(\Omega)$ such that* $\operatorname{ess\,supp} f \subset \Omega$ *there exists a sequence of smooth functions* $\{u_n\}_{n=1}^{\infty} \subset C_C^{\infty}(\Omega)$ *such that*

$$\lim_{n\to\infty} \|f - u_n\|_{W^{k,\Phi}} = 0.$$

Proof. By the fact that Φ satisifes the Δ_2 condition and the Theorem 3.1.7 (3) we have that $E^{k,\Phi}(\Omega) = W^{k,\Phi}(\Omega)$. For any $f \in W^{k,\Phi}(\Omega)$, by Theorem 3.3.4, there exists a sequence of positive numbers $\{r_n\}_{n=1}^{\infty}$ such that for any multi-index with $|\alpha| < k$ we have

$$\lim_{n\to\infty} \int\limits_{\Omega} \Phi\left(x, \frac{\beta|\partial^\alpha f_{r_n}(x) - \partial^\alpha f(x)|}{12C''\|f\|_{W^{k,\Phi}}}\right) dx = 0.$$

In other word the sequence $\{\partial^\alpha f_{r_n}\}_{n=1^\infty}$ converges in the modular I_Φ to $\partial^\alpha f$. Since

Φ satisfies Δ_2 condition, by Theorem 2.1.9 we conclude that, for any multi-index α with $|\alpha| \leq k$ we have

$$\lim_{n\to\infty} \|\partial^\alpha (f - f_{r_n})\|_\Phi = 0.$$

Since for every $n \in \mathbb{N}$, $\|f - f_{r_n}\|_{W^{k,\Phi}} = \sum_{|\alpha|\leq k} \|\partial^\alpha (f - f_{r_n})\|_\Phi$ we conclude that

$$\lim_{n\to\infty} \|f - f_{r_n}\|_{W^{k,\Phi}} = 0.$$

If we write $u_n = f_{r_n}$, then

$$\lim_{n\to\infty} \|f - u_n\|_{W^{k,\Phi}} = 0.$$

□

In the case of $\Omega = \mathbb{R}^d$ we can use the Theorem 3.3.4 and the smooth cutoff functions to establish the modular density of compactly supported smooth functions in $E^{\Phi,k}(\mathbb{R}^d)$. In particular, if Φ satisfies Δ_2 condition we establish density of $C_C^\infty(\mathbb{R}^d)$ in $W^{k,\Phi}(\mathbb{R}^d)$.

Theorem 3.3.6. *Let Φ be a MO function defined on $\mathbb{R}^d$ satisfying* (A1)*. Then for every $f \in E^{\Phi,k}(\mathbb{R}^d)$ there exists a sequence $\{f_n\}_{n=1}^\infty \subset C_C^\infty(\mathbb{R}^d)$ and a constant $\lambda' > 0$ such that for any multi-index α with $|\alpha| \leq k$ we have*

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^d} \Phi(x, \lambda' |\partial^\alpha (f_n(x) - f(x))|) dx = 0. \tag{3.23}$$

In particular if Φ satisfies Δ_2 condition then $C_C^\infty(\mathbb{R}^d)$ is dense in $W^{k,\Phi}(\mathbb{R}^d)$.

Proof. Let $f \in E^{\Phi,k}(\mathbb{R}^d)$. Take any $R > 0$ and let that $f_R = f s_R$. Then $\operatorname{ess\,supp} f_R \subset \overline{B(0, 2R)}$ is compact. In view of Theorem 3.3.4 there exists a sequence $\{r_{m(R)}\}_{m=1}^\infty \subset (0, \infty)$ such that

$$\lim_{m\to\infty} \int\limits_{\mathbb{R}^d} \Phi\left(x, \frac{\beta |\partial^\alpha (f_R)_{r_{m(R)}}(x) - \partial^\alpha f_R(x)|}{12C'' \|f_R\|_{W^{k,\Phi}}}\right) dx = 0.$$

Let α, γ, δ be multi-indexes. Define $K_k = \max\limits_{|\gamma| \le k} \|\partial^\gamma s\|_\infty$, $C_\alpha = \max\limits_{\delta+\gamma=\alpha} \frac{\alpha!}{\delta!\gamma!}$, $C = 1 + \sum\limits_{|\alpha| \le k} C_\alpha K_k$. Taking multi-index α with $|\alpha| \le k$, by Leibnitz formula, for every $x \in \mathbb{R}^d$ and $R > 1$,

$$|\partial^\alpha f_R(x)| = |\partial^\alpha (f s_R)(x)| = \left| \sum_{\delta+\gamma=\alpha} \frac{\alpha!}{\delta!\gamma!} \frac{1}{R^{|\gamma|}} \partial^\delta f(x) \partial^\gamma s\left(\frac{x}{R}\right) \right| \le C_\alpha K_k \sum_{\delta \le \alpha} |\partial^\delta f(x)|.$$

In view of the above, for every $\lambda > 0$ we have

$$\int\limits_\Omega \Phi\left(x, \lambda |\partial^\alpha f_R(x)|\right) dx \le \int\limits_\Omega \Phi\left(x, \lambda C_\alpha K_k \sum_{\delta \le \alpha} |\partial^\delta f(x)|\right) dx < \infty,$$

and so, for every $R > 1$ we have $f_R \in E^{\Phi,k}(\mathbb{R}^d)$. We also observe that

$$\begin{aligned}\|f_R\|_{W^{k,\Phi}} &= \sum_{|\alpha| \le k} \|\partial^\alpha f_R\|_\Phi \le \sum_{|\alpha| \le k} C_\alpha K_k \sum_{\delta \le \alpha} \|\partial^\delta f\|_\Phi \le \sum_{|\alpha| \le k} C_\alpha K_k \|f\|_{W^{k,\Phi}} \\ &\le C\|f\|_{W^{k,\Phi}}.\end{aligned}$$

For any $R > 1$ and $x \in \mathbb{R}^d$, we have

$$\Phi\left(x, \frac{\beta|\partial^\alpha (f_R)_{r_m(R)}(x) - \partial^\alpha f_R(x)|}{12C''C\|f\|_{W^{k,\Phi}}}\right) \le \Phi\left(x, \frac{\beta|\partial^\alpha (f_R)_{r_m(R)}(x) - \partial^\alpha f_R(x)|}{12C''\|f_R\|_{W^{k,\Phi}}}\right),$$

therefore

$$\lim_{m\to\infty} \int\limits_{\mathbb{R}^d} \Phi\left(x, \frac{\beta|\partial^\alpha (f_R)_{r_m(R)}(x) - \partial^\alpha f_R(x)|}{12C''C\|f\|_{W^{k,\Phi}}}\right) dx = 0.$$

By Theorem 3.1.8 there exists $\lambda > 0$ such that for every multi-index α such that $|\alpha| \le k$,

$$\lim_{R\to\infty} \int\limits_{\mathbb{R}^d} \Phi\left(x, \lambda \left|\partial^\alpha f_R(x) - \partial^\alpha f(x)\right|\right) dx = 0.$$

Therefore, for each $n \in \mathbb{N}$ there exists $R_n > 1$ such that

$$\int_{\mathbb{R}^d} \Phi\left(x, \lambda \left|\partial^\alpha f_{R_n}(x) - \partial^\alpha f(x)\right|\right) dx < \frac{1}{n}.$$

Similarly, let $m(R_n)$ be such that

$$\int_{\mathbb{R}^d} \Phi\left(x, \frac{\beta|\partial^\alpha (f_{R_n})_{r_{m(R_n)}}(x) - \partial^\alpha f_{R_n}(x)|}{12C'C\|f\|_{W^{k,\Phi}}}\right) dx < \frac{1}{n}.$$

Define

$$f_n = (f_{R_n})_{r_{m(R_n)}}.$$

For a fixed $n \in \mathbb{N}$ and λ' such that $2\lambda' \leq \max\left\{\lambda, \frac{\beta}{12C'C\|f\|_{W^{k,\Phi}}}\right\}$, we have

$$\int_{\mathbb{R}^d} \Phi\left(x, \lambda' \left|\partial^\alpha f_n(x) - \partial^\alpha f(x)\right|\right) dx \leq$$
$$\frac{1}{2}\int_{\mathbb{R}^d} \Phi\left(x, 2\lambda' \left|\partial^\alpha f_n(x) - \partial^\alpha f_{R_n}(x)\right|\right) dx + \frac{1}{2}\int_{\mathbb{R}^d} \Phi\left(x, 2\lambda' \left|\partial^\alpha f_{R_n}(x) - \partial^\alpha f(x)\right|\right) dx \leq$$
$$\frac{1}{2}\int_{\mathbb{R}^d} \Phi\left(x, \lambda \left|\partial^\alpha f_n(x) - \partial^\alpha f_{R_n}(x)\right|\right) dx + \frac{1}{2}\int_{\mathbb{R}^d} \Phi\left(x, \frac{\beta\left|\partial^\alpha f_{R_n}(x) - \partial^\alpha f(x)\right|}{12C'C\|f\|_{W^{k,\Phi}}}\right) dx < \frac{1}{n}.$$

We conclude that (3.23) holds.

If Φ satisfies Δ_2 condition, then for every multi-index α by (3.23) and Theorem (2.1.9) (3) we have

$$\lim_{n\to\infty} \|f - f_n\|_\Phi = 0.$$

Hence $C_c^\infty(\mathbb{R}^d)$ is dense in $W^{k,\Phi}(\mathbb{R}^d)$.

□

3.4 Density of compactly supported smooth functions in $W^{1,\Phi}(\Omega)$

This section is the most involved in this chapter. In essence the whole section is a step by step proof of Theorem 3.4.11. Our goal here is to establish the density of restrictions of $C_C^\infty(\mathbb{R}^d)$ to Ω in the space $W^{1,\Phi}(\Omega)$.

First let us remark that analogous theorem holds in classical Sobolev spaces $W^{k,p}(\Omega)$, for any $1 \leq p < \infty$, any $k \in \mathbb{N}$ and domain Ω whose boundary is locally a graph of a continuous function, (see for example [2]). Unfortunately, in the case of MOS spaces the method used to prove the mentioned result cannot work. First of, it uses the fact that for any $x \in \mathbb{R}^d$ the translation operators $\tau_x : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$, given by the formula $(\tau_x f)(y) = f(y - x)$, $f \in L^p(\mathbb{R}^d)$, $y \in \mathbb{R}^d$ are all isometries, which is not true in a general MO space as they may be even ill-defined. Secondly the group $\{\tau_x\}_{x\in\mathbb{R}^d}$ is strongly continuous on $L^p(\mathbb{R}^d)$ in the sense that for $f \in L^p$ we have $\lim_{x\to 0} \|\tau_x f - f\|_p = 0$, once again this not true for $L^\Phi(\Omega)$ (see [37, Theorem 2.1]). The third problem is that in the case of $L^p(\Omega)$ one can think of it as a MO space on Ω which is given by the function $\Phi(x,t) = t^p$, where $x \in \Omega$ and $t \geq 0$. Such function can be extended to a MO function on the whole $\mathbb{R}^d$ in an obvious fashion, with the extension having all nice properties of the original function. In the case of a general MO function the existence of a good extension is not guaranteed.

The reasons above suggest that a different method is required to achieve our goal. We approached the problem via the means of local extensions of both the function Φ and the element $f \in W^{1,\Phi}(\Omega)$. Since both extension are constructed by reflection of parts of the domain Ω about its boundary, it only works if we expect our functions f to by once weakly differentiable. Moreover the method also requires higher regularity of the boundary of Ω.

We start our exposition by introducing some notation. We base our approach on that of [36]. For any $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ we define $x' \in \mathbb{R}^{d-1}$ to be

$$x' = (x_1, .., x_{d-1}).$$

Moreover for $x \in \mathbb{R}^d$ such that $x = 0$, we have that $x' = 0$ in $\mathbb{R}^{d-1}$. With a slight abuse of notation we write

$$x = (x', x_d).$$

Recall that for $y \in \mathbb{R}^d$, $d \in \mathbb{N}$ and $r > 0$, $Q(y, r)$ is a cube with center y and side length $2r$. For a fixed $d \in \mathbb{N}$ we will need both cubes in $\mathbb{R}^d$ and in $\mathbb{R}^{d-1}$. For a fixed $(y', y_d) = y \in \mathbb{R}^d$ we denote the cube in $\mathbb{R}^{d-1}$, centered at y' and of side length $2r$ as $Q_{d-1}(y', r)$.

We will also need the notion of *rigid motion* on $\mathbb{R}^d$. A rigid motion on $\mathbb{R}^d$ is the map $T : \mathbb{R}^d \to \mathbb{R}^d$ of the form

$$T = R + c,$$

where R is a rotation in $\mathbb{R}^d$ and $c \in \mathbb{R}^d$. For a fixed rigid motion T and a point $x \in \mathbb{R}^d$ we define the *local coordinates of* x as the vector y, where

$$y := T(x).$$

Now we recall the notion of differentiable transformations between open subsets of $\mathbb{R}^d$.

Definition 3.4.1. *Let U, V be open subsets of $\mathbb{R}^d$ and $\mathcal{R} : U \to V$ be a map. For a fixed $m \in \mathbb{N}$ we say that $\mathcal{R}$ belongs to $C^m(U, V)$, if for $i = 1, \dots, d$ there exist functions $\varrho_i : U \to \mathbb{R}$ such that for $x \in U$,*

$$\mathcal{R}(x) = (\varrho_1(x), \dots, \varrho_d(x))$$

and ϱ_i is in $C^m(U)$.

For $\mathcal{R} \in C^m(U,V)$ and $x \in \mathbb{R}^d$ we define the *Jacobian matrix* of $\mathcal{R}$ at x as

$$J_{\mathcal{R}}(x) = (\partial_j \varrho_i(x))_{i,j=1}^d.$$

We call $|\det J_{\mathcal{R}}(x)|$ the *Jacobian* of $\mathcal{R}$ at x. The following substitution formula will be needed in the sequel. The proof of this result can be found in [36, 11.53, p.341].

Theorem 3.4.2. *(Change of variables) Let U, V be open sets in $\mathbb{R}^d$ and $\mathcal{R} : U \to V$ be an invertable map of class $C^1(U,V)$, such that its inverse $\mathcal{R}^{-1}$ is of class $C^1(V,U)$. For any $f \in W^{1,1}(V)$ the function $f \circ \mathcal{R}$ is and element of $W^{1,1}(U)$, moreover, for any $i = 1, \dots, d$ the following formula holds for a.a $x \in U$,*

$$\partial_i (f \circ \mathcal{R})(x) = \sum_{j=1}^d \partial_j f(\mathcal{R}(x)) \partial_i \varrho_j(x).$$

Now we can make a precise definition of a set with regular boundary.

Definition 3.4.3. *Given an open set $\Omega \subset \mathbb{R}^d$ we say that $\mathrm{bd}(\Omega)$ is of class C^1 if for every $x_0 \in \mathrm{bd}(\Omega)$ there exist an open neighborhood U of x_0, a rigid motion T, a number $r > 0$ and $\mathfrak{f} \in C^1(Q_{d-1}(0,r))$ such that*

$$T(x_0) = 0,$$

$$T(\Omega \cap U) = \{y \in Q(0,r) : y_d > \mathfrak{f}(y')\}.$$

In the sequel the symbol $\mathfrak{f}$ will be restricted only to the function $\mathfrak{f} \in C^1(Q_{d-1}(0,r))$ from the above definition.

The next theorem will be crucial in further constructions as it provides us with both a family of very regular open neighborhoods of points in $\mathrm{bd}(\Omega)$ and reflection maps on those neighborhoods.

Theorem 3.4.4. *Let $\Omega \subset \mathbb{R}^d$ be open and its boundary $\mathrm{bd}(\Omega)$ be of class C^1. Then for each $x_0 \in \mathrm{bd}(\Omega)$ there exist two triples $(V, \{V_{(t_0,t_1)}\}_{(t_0,t_1)\subset(-1,1)}, \mathcal{R})$ and $(W, \{W_{(t_0,t_1)}\}_{(t_0,t_1)\subset(-1,1)}, \mathcal{R}')$ where V is an open neighborhood of x_0, W is an open neighborhood of 0, $\{V_{(t_0,t_1)}\}_{(t_0,t_1)\subset(-1,1)}$ and $\{W_{(t_0,t_1)}\}_{(t_0,t_1)\subset(-1,1)}$ are families of open sets and $\mathcal{R}, \mathcal{R}'$ are respectively $C^1(V,V)$ and $C^1(W,W)$ maps. The triple $(W, \{W_{(t_0,t_1)}\}_{(t_0,t_1)\subset(-1,1)}, \mathcal{R}')$ is of the form*

$$W = \bigcup_{t\in(-1,1)} S + Ate_d,$$

and for $-1 \le t_0 < t_1 \le 1$,

$$W_{(t_0,t_1)} = \bigcup_{t\in(t_0,t_1)} S + Ate_d,$$

where $A > 0$ is some number and

$$S = \{(x, \mathfrak{f}(x)) \in \mathbb{R}^d : x \in Q_{d-1}(0, r')\},$$

where $\mathfrak{f}$ is the $C^1(Q_{d-1}(0,r))$ function associated to x_0 by Definition 3.4.3 and $0 < r' < r$ is such that, for each $i = 1, \ldots, d-1$ we have

$$\sup_{x\in Q_{d-1}(0,r')} |\partial_i \mathfrak{f}(x)| < \infty.$$

Furthermore the map $\mathcal{R}' : W \to W$ is given by the formula

$$\mathcal{R}'(y', \mathfrak{f}(y') + At) = (y', \mathfrak{f}(y') - At),$$

where $(y', \mathfrak{f}(y') + At) = y \in W$.

The triple $(V, \{V_{(t_0,t_1)}\}_{(t_0,t_1)\subset(-1,1)}, \mathcal{R})$ is connected to $(W, \{W_{(t_0,t_1)}\}_{(t_0,t_1)\subset(-1,1)}, \mathcal{R}')$ by the formulas $V = T^{-1}(W)$, $V_{(t_0,t_1)} = T^{-1}(W_{(t_0,t_1)})$

and $\mathcal{R} = T^{-1} \circ \mathcal{R}' \circ T$*, where* T *is the rigid motion associated with* x_0 *via the definition 3.4.3. Moreover the triple* $(V, \{V_{(t_0,t_1)}\}_{(t_0,t_1)\subset(-1,1)}, \mathcal{R})$ *satisfies the following conditions*

(1) $V_\emptyset = \emptyset$, $V_{(-1,1)} = V$ *and for each* $-1 \leq t_0 < t_1 \leq 1$ *and* $-1 \leq t_0' < t_1' \leq 1$,

$$V_{(t_0,t_1)} \cap V_{(t_0',t_1')} = V_{(t_0,t_1)\cap(t_0',t_1')}.$$

(2)

$$V_{(-1,0)} = V \cap \overline{\Omega}^C \text{ and } V_{(0,1)} = V \cap \Omega.$$

(3) *If* $-1 \leq t_0 < 0 < t_1 \leq 1$*, then*

$$\mathrm{bd}(\Omega) \cap V \subset V_{(t_0,t_1)}$$

(4) *There exists a constant* $C > 0$ *such that for any* $(t_0, t_1) \subset (-1, 1)$,

$$|V_{(t_0,t_1)}| = C|t_1 - t_0|$$

(5) $\mathcal{R} \circ \mathcal{R} = Id_V$,

(6) $\mathcal{R}_{|V\cap\mathrm{bd}(\Omega)} = Id_{|V\cap\mathrm{bd}(\Omega)}$,

(7) *for any* $(t_0, t_1) \subset (-1, 1)$,

$$\mathcal{R}(V_{(t_0,t_1)}) = V_{(-t_1,-t_0)}$$

(8) *If* $\mathcal{R}(x) = (\varrho_i(x), \ldots, \varrho_d(x))$*, then there exists a constant* $C_\mathcal{R} > 0$ *such that, for*

each $i = 1, \dots, d$ *and each multi-index* α *with* $|\alpha| = 1$*,*

$$\sup_{x \in V} |\partial^\alpha \varrho_i(x)| < C_{\mathcal{R}}.$$

.

Proof. Let Ω be as in the assumption. Take any $x_0 \in \mathrm{bd}(\Omega)$ and let $r > 0, U, T, \mathfrak{f}$ be as in the Definition 3.4.3. We have that

$$T(\Omega \cap U) = \{y \in Q(0, r) : y_d > \mathfrak{f}(y')\}.$$

Notice that

$$T(\mathrm{bd}(\Omega) \cap U) = \{y \in Q(0, r) : y_d = \mathfrak{f}(y')\} \text{ and } \mathfrak{f}(0) = 0.$$

Indeed, take any $x \in \mathrm{bd}(\Omega) \cap U$. Then there exists a sequence $\{x_n\}_{n=1}^\infty \subset \Omega \cap U$ such that $\lim_{n\to\infty} x_n = x$. Denote $y_n = T(x_n)$, where $y_n = (y_n', y_{n,d})$ and $T(x) = y$, then $\{y_n\}_{n=1}^\infty \subset T(\Omega \cap U)$ and $\lim_{n\to\infty} y_n = y$. Hence for any $n \in \mathbb{N}$ we have $y_{n,d} > \mathfrak{f}(y_n')$ and $\lim_{n\to\infty} y_{n,d} = y_d$. Continuity of $\mathfrak{f}$ implies that $y_d \geq \mathfrak{f}(y')$. On the other hand, suppose that $y_d > \mathfrak{f}(y')$, then $T(x) \in T(\Omega \cap U)$. Hence x would be an element of $U \cap \Omega$, which cannot be true as $x \in \mathrm{bd}(\Omega)$. Therefore $y_d = \mathfrak{f}(y')$. In particular, since $T(x_0) = 0$, we conclude that $\mathfrak{f}(0) = 0$.

By continuity of $\mathfrak{f}$ and $\mathfrak{f}(0) = 0$ there exists $0 < r' < \frac{r}{2}$ such that, for $y \in Q(0, r') \cap T(\mathrm{bd}(\Omega) \cap U)$ we have $|\mathfrak{f}(y')| < \frac{r}{2}$. Taking any $z \in \mathrm{bd}(Q(0, r))$ and

$y \in Q(0,r') \cap T(\mathrm{bd}(\Omega) \cap U)$ we have $z = (z', z_d)$ and $y = (y', \mathfrak{f}(y'))$ and

$$\begin{aligned} |z-y| &= |(z',z_d) - (y',\mathfrak{f}(y'))| = |(z'-y', z_d - \mathfrak{f}(y'))| \\ &\geq \max\{\max_{i=1,\dots,d-1} |z_i - y_i|, |z_d - \mathfrak{f}(y')|\} \\ &\geq \max\{\max_{i=1,\dots,d-1} |\,|z_i| - |y_i|\,|, |\,|z_d| - |\mathfrak{f}(y')|\,|\}. \end{aligned}$$

Since $z \in \mathrm{bd}(Q(0,r))$ there exists $1 \leq i \leq d$ for which $|z_i| = r$. From the fact that $y \in Q_{d-1}(0,r')$ and $|y_d| = |\mathfrak{f}(y')| < \frac{r}{2}$ and $r' < \frac{r}{2}$ we have

$$2dr \geq |(z',z_d) - (y',\mathfrak{f}(y'))| \geq \frac{r}{2}. \tag{3.24}$$

Defining now the set $S = \{y \in Q(0,r') : y_d = \mathfrak{f}(y')\}$, we have

$$S \subset \{y \in Q(0,r) : y_d = \mathfrak{f}(y')\} = T(\mathrm{bd}(\Omega) \cap U).$$

Moreover, by inequality (3.24) we get

$$\frac{r}{2} \leq \mathrm{dist}(S, \mathrm{bd}(Q(0,r))) \leq 2dr.$$

Let $A := \mathrm{dist}(S, \mathrm{bd}(Q(0,r)))/(8d)$, then $A \leq \frac{r}{4}$. Define now

$$W := \bigcup_{-1<t<1} tAe_d + S$$

$$W_{(t_0,t_1)} := \bigcup_{t_0<t<t_1} tAe_d + S$$

where $(t_0,t_1) \subset (-1,1)$ and e_d is the unit vector $(0,\dots,0,1)$ in $\mathbb{R}^d$. We will show that the sets

$$V := T^{-1}(W),$$

$$V_{(t_0,t_1)} := T^{-1}(W_{(t_0,t_1)})$$

satisfy the conditions (1)-(4). Notice that, by definition of $W_{(t_0,t_1)}$, we have $W_\emptyset = \emptyset$, $W_{(-1,1)} = W$ and for each $-1 \leq t_0 < t_1 \leq 1$ and $-1 \leq t_0' < t_1' \leq 1$,

$$W_{(t_0,t_1)} \cap W_{(t_0',t_1')} = W_{(t_0,t_1)\cap(t_0',t_1')}.$$

So, since T a bijection, condition (1) is satisfied.

Next we show that $W \subset Q(0,r)$ and $W_{(0,1)} \subset T(\Omega \cap U)$. Take any $z \in W$,

$$z = (z', z_d) = (y', \mathfrak{f}(y') + At)$$

for some $t \in (-1,1)$ and $y \in S$. If $t = 0$, then

$$z = (z', z_d) = (y', \mathfrak{f}(y')) = y \in S \subset T(\mathrm{bd}(\Omega) \cap U) \subset Q(0,r).$$

If $t \neq 0$, then

$$z = (y', \mathfrak{f}(y') + At).$$

Since $y \in Q(0,r')$ we have that y' is an element of $Q_{d-1}(0,r')$. Hence, for $i = 1, \ldots, d-1$,

$$|z_i| = |y_i| \leq r' < r/2.$$

We also have

$$|z_d| = |\mathfrak{f}(y') + tA| \leq |\mathfrak{f}(y')| + |t|A \leq \frac{r}{2} + A \leq \frac{r}{2} + \frac{r}{4} < r,$$

which implies that for all $i = 1, \ldots, d$ we have $|z_i| < r$ and so $z \in Q(0,r)$. Therefore $W \subset Q(0,r)$. Moreover, since elements of $W_{(0,1)}$ are of the form $(y', \mathfrak{f}(y') + tA)$, where $t > 0$, we have that $W_{(0,1)} \subset T(\Omega \cap U)$.

Notice that, since $W_{(0,1)} \subset W$ and $W_{(0,1)} \subset T(\Omega \cap U)$ we have

$$V_{(0,1)} = V_{(0,1)} \cap V = T^{-1}(W_{(0,1)}) \cap V \subset (\Omega \cap U) \cap V = \Omega \cap V.$$

On the other hand, if $x \in \Omega \cap V$ then there exist $(y', y_d) = y \in W$ such that $x = T^{-1}(y)$. By the definition of W there exists $(z', \mathfrak{f}(z')) = z \in S$ such that $(y', y_d) = (z', \mathfrak{f}(z') + At)$ for some $t \in (-1, 1)$. Since $x \in \Omega \cap V \subset \Omega \cap U$ we conclude that $\mathfrak{f}(z') + At > \mathfrak{f}(z')$ and so $t > 0$. In particular $y = (y', y_d) = (z', \mathfrak{f}(z') + At) \in W_{(0,1)}$ and so $x = T^{-1}(y) \in T^{-1}(W_{(0,1)}) = V_{(0,1)}$. Therefore

$$\Omega \cap V \subset V_{(0,1)},$$

and so

$$V_{(0,1)} = V \cap \Omega.$$

Similarly, by the fact that T is an isometry on $\mathbb{R}^d$, we have

$$W_{(-1,0)} \subset [T(\overline{\Omega} \cap U)]^C = T((\overline{\Omega} \cap U)^C)$$

Hence by the fact that $V_{(-1,0)} \subset V$,

$$V_{(-1,0)} = T^{-1}(W_{(-1,0)}) \subset (\overline{\Omega}^C \cup U^C) \cap V = \overline{\Omega}^C \cap V.$$

Similar argument as in the proof of inclusion $V \cap \Omega \subset V_{(0,1)}$ shows that any $x \in \overline{\Omega}^C \cap V$ is of the form $x = T^{-1}(y)$ where $y \in W$. Moreover $y = (y', \mathfrak{f}(y') + At)$ for some $t < 0$, hence $x \in V_{(-1,0)}$ and so

$$\overline{\Omega}^C \cap V \subset V_{(-1,0)}.$$

Hence

$$\overline{\Omega}^C \cap V = V_{(-1,0)}.$$

We have shown that the condition (2) is satisfied.

Since $S \subset T(U \cap \mathrm{bd}(\Omega))$, then $T^{-1}(S) \subset U \cap \mathrm{bd}(\Omega)$. Moreover, $T^{-1}(S) \subset V \subset U$, Hence

$$T^{-1}(S) = V \cap \mathrm{bd}(\Omega)$$

If $1 < t_0 < 0 < t_1 < 1$, then $S \subset W_{(t_0,t_1)}$ and so $V \cap \mathrm{bd}(\Omega) \subset V_{(t_0,t_1)}$, hence condition (3) is satisfied.

To show that (4) holds notice that

$$|W_{(t_0,t_1)}| = \int\limits_{Q_{d-1}(0,r')} [\mathfrak{f}(y') + At_1 - (\mathfrak{f}(y') - At_0)]dy' = A(2r')^{d-1}|t_1 - t_0|,$$

and since T is an isometry we conclude that

$$|V_{(t_0,t_1)}| = |W_{(t_0,t_1)}| = A(2r'^{d-1})|t_1 - t_0|,$$

which shows (4).

Next we show for any $(t_0, t_1) \subset (-1, 1)$, $W_{(t_0,t_1)}$ is open. Take any $z_0 \in W_{(t_0,t_1)}$, then there exists $y_0 \in Q(0, r')$ and $t_0 < t < t_1$ such that

$$z_0 = (y_0', \mathfrak{f}(y_0') + tA).$$

Take any $\varepsilon > 0$ such that $At + 2\varepsilon < At_1$ and $At_0 < At - 2\varepsilon$. Since $\mathfrak{f}$ is continuous there exits $s > 0$ such that $Q_{d-1}(y_0', s) \subset Q_{d-1}(0, r')$ and for $y' \in Q_{d-1}(y_0', s)$, we have

$$\mathfrak{f}(y_0') - \varepsilon < \mathfrak{f}(y') < \mathfrak{f}(y_0') + \varepsilon.$$

Now we define

$$U_{z_0} = Q_{d-1}(y_0', s) \times (At - \varepsilon + \mathfrak{f}(y_0'), At + \mathfrak{f}(y_0') + \varepsilon).$$

Then U_{z_0} is an open cube in $\mathbb{R}^d$ and for any $z \in U_{z_0}$ we have that

$$z = (y', z_d)$$

for some $y' \in Q_{d-1}(y_0', s)$ and $z_d \in (At - \varepsilon + \mathfrak{f}(y_0'), At + \mathfrak{f}(y_0') + \varepsilon)$. Notice that

$$\begin{aligned} At_0 < At - 2\varepsilon < At - \varepsilon + \mathfrak{f}(y_0') - \mathfrak{f}(y') < z_d - \mathfrak{f}(y') \\ < At + \mathfrak{f}(y_0) - \mathfrak{f}(y') + \varepsilon < At + 2\varepsilon < At_1. \end{aligned}$$

Let $t_{y'} := \frac{z_d - \mathfrak{f}(y')}{A}$, then $t_{y'} \in (t_0, t_1)$ and $z_d = \mathfrak{f}(y') + t_{y'}A$. Therefore $z = (y', \mathfrak{f}(y') + t_{y'}A) \in W_{(t_0,t_1)}$ and so $U_{z_0} \subset W_{(t_0,t_1)}$, i.e. $W_{(t_0,t_1)}$ is open.

Let us now define the map $\mathcal{R}' : W \to W$. For $(y', \mathfrak{f}(y') + tA) = (y', y_d) = y \in W$, we define

$$\mathcal{R}'(y) = \mathcal{R}'(y', y_d) = \mathcal{R}'(y', \mathfrak{f}(y') + tA) = (y', \mathfrak{f}(y') - tA) = (y', 2\mathfrak{f}(y') - y_d). \quad (3.25)$$

Notice that for $y \in W$,

$$(\mathcal{R}' \circ \mathcal{R}')(y) = \mathcal{R}'((y', \mathfrak{f}(y') - tA)) = (y', \mathfrak{f}(y') + tA) = y.$$

Hence $\mathcal{R}' \circ \mathcal{R}' = Id_W$, and for $(t_0, t_1) \subset (-1, 1)$ we have

$$\mathcal{R}'(W_{(t_0,t_1)}) = W_{(-t_1,-t_0)}.$$

For $y \in W$, if we write

$$\mathcal{R}'(y) = (\varrho_1'(y), \ldots, \varrho_d'(y)),$$

then we have

$$\varrho_i'(y) = y_i,$$

for $i = 1, \ldots, d-1$ and for $i = d$,

$$\varrho_d'(y) = 2\mathfrak{f}(y') - y_d.$$

Since $\mathfrak{f}$ is of class $C^1(Q_{d-1}(0,r))$, its restriction to $Q_{d-1}(0,r')$ is also of class $C^1(Q_{d-1}(0,r'))$ and for each $i = 1, \ldots, d$ we have

$$\sup_{y' \in Q_{d-1}(0,r')} |\partial_i \mathfrak{f}(y)| < C_{r'} \tag{3.26}$$

for some $C_{r'} > 0$. Moreover,

$$\partial_j \varrho_i' = \delta_{ij} \tag{3.27}$$

for $i = 1, \ldots, d-1$, $j = 1, \ldots, d-1$,

$$\partial_d \varrho_i' = 0$$

for $i = 1, \ldots, d-1$,

$$\partial_j \varrho_d' = 2\partial_j \mathfrak{f}$$

for $j = 1, \ldots, d-1$, finally

$$\partial_d \varrho_d = -1.$$

Hence, $\mathcal{R}'$ is of class $C^1(W, W)$, and for each $y \in W$,

$$|\det J_{\mathcal{R}'(y)}| = 1. \tag{3.28}$$

Finally we define $\mathcal{R} : V \to V$ as

$$\mathcal{R} = T^{-1} \circ \mathcal{R}' \circ T.$$

Notice that $\mathcal{R}$ together with the set V and the family $V_{(t_1,t_2)}$ satisfy conditions (1)-(7) and since T is a composition of a translation and a rotation, the condition (8) is also satisfied. □

Lemma 3.4.5. *Let* $(W, \{W_{(t_0,t_1)}\}_{(t_0,t_1)\subset(-1,1)}, \mathcal{R}')$ *be the triple from Theorem 3.4.4. Let* g *be a function in* $W^{1,1}(W_{(0,1)})$*. Then the function* $\widetilde{g}$ *defined by the formula*

$$\widetilde{g}(y) = \begin{cases} g(y) & y \in W_{(0,1)} \\ 0 & y \in W \setminus (W_{(-1,0)} \cup W_{(0,1)}) \\ g(\mathcal{R}'(y)) & y \in W_{(-1,0)} \end{cases}$$

is an element of $W^{1,1}(W)$ *and for a.a* $y = (y', y_d) \in W$ *we have*

$$\partial_i \widetilde{g}(y) = \begin{cases} \partial_i g(y) & y \in W_{(0,1)},\ i = 1, \dots, d \\ (\partial_i g)(\mathcal{R}'(y)) + 2(\partial_d g)(\mathcal{R}'(y))\partial_i \mathfrak{f}(y') & y \in W_{(-1,0)}\ i = 1, \dots, d-1 \\ -(\partial_d g)(\mathcal{R}'(y)) & y \in W_{(-1,0)},\ i = d. \end{cases}$$

Proof. By assumption, the triple $(W, \{W_{(t_0,t_1)}\}_{(t_0,t_1)\subset(-1,1)}, \mathcal{R}')$ satisfies the conclusions of the Theorem 3.4.4, in particular the sets $W_{(t_0,t_1)}$ are open for each

$(t_0, t_1) \subset (-1, 1)$ and of the form

$$W_{(t_0,t_1)} = \bigcup_{t\in(t_0,t_1)} S + Ate_d,$$

where $A > 0$ is some number and

$$S = \{(x, \mathfrak{f}(x)) \in \mathbb{R}^d : x \in Q_{d-1}(0, r')\},$$

where $\mathfrak{f}$ is of class $C^1(Q_{d-1}(0, r))$ such that, for some $0 < r' < r$ and any $i = 1, \ldots, d-1,$

$$\sup_{x\in Q_{d-1}0,r')} |\partial_i \mathfrak{f}(x)| < \infty.$$

Take any $0 < \varepsilon < 1/3$, and for $(y', y_d) = y \in W$ define

$$F_{\varepsilon,1}(y) = \int_{-\infty}^{y_d} \frac{15}{16A^5\varepsilon^5}(\mathfrak{f}(y') + A\varepsilon - t)^2(\mathfrak{f}(y') + 3A\varepsilon - t)^2\chi_{[\mathfrak{f}(y')+A\varepsilon,\mathfrak{f}(y')+3A\varepsilon]}(t)dt,$$

$$F_{-1,-\varepsilon}(y) = \int_{y_d}^{\infty} \frac{15}{16A^5\varepsilon^5}(\mathfrak{f}(y') - A\varepsilon - t)^2(\mathfrak{f}(y') - 3A\varepsilon - t)^2\chi_{[\mathfrak{f}(y')-3A\varepsilon,\mathfrak{f}(y')-A\varepsilon]}(t)dt,$$

$$F_{-\varepsilon,\ \varepsilon}(y) = \chi_W(y) - F_{\varepsilon,1}(y) - F_{-1,-\varepsilon}(y).$$

Clearly, $F_{\varepsilon,1}$ and $F_{-1,-\varepsilon}$ are non-negative functions. Moreover, by the Fundamental Theorem of Calculus, for any $(y', y_d) = y \in W$ the function $t \mapsto F_{\varepsilon,1}(y', t)$ is non-decreasing and $t \mapsto F_{-1,-\varepsilon}(y', t)$ is non-increasing function, where $t \in (-A, A)$. Moreover we have

$$F_{\varepsilon,1} : W \to [0, 1]. \tag{3.29}$$

Indeed, take any $(y', y_d) = y \in W_{(\varepsilon,1)}$. By definition of $W_{(\varepsilon,1)}$ we have that $y_d < \mathfrak{f}(y') + A\varepsilon$, and so

$$F_{\varepsilon,1}(y) = \int_{-\infty}^{y_d} \frac{15}{16A^5\varepsilon^5}(\mathfrak{f}(y') + A\varepsilon - t)^2(\mathfrak{f}(y') + 3A\varepsilon - t)^2\chi_{[\mathfrak{f}(y')+A\varepsilon,\mathfrak{f}(y')+3A\varepsilon]}(t)dt = 0.$$

For $(y', y_d) = y \in W$ such that $\mathfrak{f}(y') + A\varepsilon \leq y_d \leq \mathfrak{f}(y') + 3A\varepsilon$, i.e. $y \in W \setminus \left(W_{(-1,\varepsilon)} \cup W_{(3\varepsilon,1)}\right)$, we have that

$$\begin{aligned}F_{\varepsilon,1}(y) &= \int_{-\infty}^{y_d} \frac{15}{16A^5\varepsilon^5}(\mathfrak{f}(y') + A\varepsilon - t)^2(\mathfrak{f}(y') + 3A\varepsilon - t)^2\chi_{[\mathfrak{f}(y')+A\varepsilon,\mathfrak{f}(y')+3A\varepsilon]}(t)dt \\ &= \int_{\mathfrak{f}(y')+A\varepsilon}^{y_d} \frac{15}{16A^5\varepsilon^5}(\mathfrak{f}(y') + A\varepsilon - t)^2(\mathfrak{f}(y') + 3A\varepsilon - t)^2dt\end{aligned}$$

By substitution $u = \mathfrak{f}(y') + A\varepsilon - t$,

$$\begin{aligned}F_{\varepsilon,1}(t) &= -\frac{15}{16A^5\varepsilon^5}\int_{0}^{\mathfrak{f}(y')+A\varepsilon-y_d} t^2(t+2A\varepsilon)^2dt = \frac{15}{16A^5\varepsilon^5}\int_{\mathfrak{f}(y')+A\varepsilon-y_d}^{0} t^2(t+2A\varepsilon)^2dt \\ &= \frac{15}{16A^5\varepsilon^5}\left(\frac{(y_d - \mathfrak{f}(y') - A\varepsilon)^5}{5} - \frac{4A\varepsilon(\mathfrak{f}(y') + A\varepsilon - y_d)^4}{4} - \frac{4A^2\varepsilon^2(\mathfrak{f}(y') + A\varepsilon - y_d)^3}{3}\right) \\ &= \frac{-\left(\mathfrak{f}(y') + A\varepsilon - y_d\right)^3\left(3(\mathfrak{f}(y') + A\varepsilon - y_d)^2 + 15A\varepsilon(\mathfrak{f}(y') + A\varepsilon - y_d) + 20A^2\varepsilon^2\right)}{16A^5\varepsilon^5}.\end{aligned}$$

Notice that also, for $y = (y', \mathfrak{f}(y + 3A\varepsilon) \in W$ we have

$$F_{\varepsilon,1}(y) = \frac{-1}{16A^5\varepsilon^5}(-2A\varepsilon)^3(3(-2A\varepsilon)^2 + 15A\varepsilon(-2A\varepsilon) + 20A^2\varepsilon^2) = 1.$$

Now, if $(y', y_d) \in W$ and $y_d > \mathfrak{f}(y') + 3A\varepsilon$ we have that

$$F_{\varepsilon,1}(y) = \int_{-\infty}^{y_d} \frac{15}{16A^5\varepsilon^5}(\mathfrak{f}(y') + A\varepsilon - t)^2(\mathfrak{f}(y') + 3A\varepsilon - t)^2 \chi_{[\mathfrak{f}(y')+A\varepsilon, \mathfrak{f}(y')+3A\varepsilon]}(t)dt =$$

$$= \int_{\mathfrak{f}(y')+A\varepsilon}^{\mathfrak{f}(y')+3A\varepsilon} \frac{15}{16A^5\varepsilon^5}(\mathfrak{f}(y') + A\varepsilon - t)^2(\mathfrak{f}(y') + 3A\varepsilon - t)^2 dt = 1.$$

And so the function $F_{\varepsilon,1}$ is of the form

$$F_{\varepsilon,1}(y) = \begin{cases} 0 & y \in W_{(-1,\varepsilon)} \\ \frac{(y_d - \mathfrak{f}(y') - A\varepsilon)^3\left(3(\mathfrak{f}(y')+A\varepsilon - y_d)^2 + 15A\varepsilon(\mathfrak{f}(y')+A\varepsilon - y_d) + 20A^2\varepsilon^2\right)}{16A^5\varepsilon^5} & y \in W \cap \overline{W_{(\varepsilon,3\varepsilon)}} \\ 1 & y \in W_{(3\varepsilon,1)}. \end{cases} \tag{3.30}$$

Now we can also see that

$$\operatorname{ess\,supp} F_{\varepsilon,1} = \overline{W_{(\varepsilon,1)}}. \tag{3.31}$$

Indeed, if $y \in \overline{W_{(\varepsilon,1)}}$, then there exists a sequence $\{y_k\}_{k=1}^{\infty} \subset W_{(\varepsilon,1)}$ such that $\lim_{k\to\infty} y_k = y$. By the formula (3.30) we have that, for each $k \in \mathbb{N}$, $F_{\varepsilon,1}(y_k) \neq 0$, hence $y_k \in \operatorname{ess\,supp} F_{\varepsilon,1}$. But $\operatorname{ess\,supp} F_{1,\varepsilon}$ is a closed set and so $\overline{W_{(\varepsilon,1)}} \subset \operatorname{ess\,supp} F_{1,\varepsilon}$. Similarly, if $y \in \operatorname{ess\,sup} F_{\varepsilon,1}$, there exists a sequence of points $\{y_k\}_{k=1}^{\infty}$, such that, for each $k \in \mathbb{N}$ we have $F_{\varepsilon,1}(y_k) > 0$ and $\lim_{k\to\infty} y_k = y$. Inspection of the formula (3.30) shows that $F_{\varepsilon,1}(z) \neq 0$ only for $z \in W_{(\varepsilon,1)}$. Therefore, we conclude that $\{y_k\}_{k=1}^{\infty} \subset W_{(1,\varepsilon)}$ and so we conclude that $y_k \in \overline{W_{(0,1)}}$.

Similar reasoning to the one above shows that for $y \in W$ we have

$$F_{-1,-\varepsilon}(y) = \begin{cases} 1 & y \in W_{(-1,-3\varepsilon)} \\ \frac{(\mathfrak{f}(y')-A\varepsilon-y_d)^3\left(2(\mathfrak{f}(y')+A\varepsilon+y_d)^2+15A\varepsilon(y_d-\mathfrak{f}(y')+A\varepsilon)+20A^2\varepsilon^2\right)}{16A^5\varepsilon^5} & y \in W \cap \overline{W_{(-3\varepsilon,-\varepsilon)}} \\ 0 & y \in W_{(-\varepsilon,1)}, \end{cases} \tag{3.32}$$

and

$$F_{-1,-\varepsilon} : W \to [0,1], \tag{3.33}$$

$$\operatorname{ess\,supp} F_{-1,-\varepsilon} = \overline{W_{(-1,-\varepsilon)}}. \tag{3.34}$$

Since $F_{-\varepsilon,\varepsilon} = \chi_W - F_{\varepsilon,1} - F_{-1,-\varepsilon}$, by the fact that supports of $F_{\varepsilon,1}$ and $F_{-1,-\varepsilon}$ are disjoint (see (3.31) and (3.34)) and the fact that ranges of both $F_{\varepsilon,1}$ and $F_{-1,-\varepsilon}$ are equal to $[0,1]$ (see (3.29) and (3.33)) we have that

$$F_{-\varepsilon,\varepsilon} : W \to [0,1]. \tag{3.35}$$

Examining the formulas (3.30) and (3.32) we see that $F_{-\varepsilon,\varepsilon}(z) \neq 0$ only for $z \in W_{(-3\varepsilon,3\varepsilon)}$. Hence, reasoning as in the case of $F_{\varepsilon,1}$ we deduce that

$$\operatorname{ess\,supp} F_{-\varepsilon,\varepsilon} = \overline{W_{(-3\varepsilon,3\varepsilon)}}. \tag{3.36}$$

Moreover we also see that, for $y \in W_{(-\varepsilon,\varepsilon)}$, we have

$$F_{-\varepsilon,\varepsilon}(y) = 1. \tag{3.37}$$

Notice that, by the Fundamental Theorem of Calculus and definitions of $F_{\varepsilon,1}$, $F_{-1,-\varepsilon}$, for $(y, y_d) = y \in W$ we have

$$\partial_d F_{\varepsilon,1}(y) = \frac{15}{16A^5\varepsilon^5}(\mathfrak{f}(y') + A\varepsilon - y_d)^2(\mathfrak{f}(y') + 3A\varepsilon - y_d)^2\chi_{[\mathfrak{f}(y')+A\varepsilon,\mathfrak{f}(y')+3A\varepsilon]}(y_d), \tag{3.38}$$

and

$$\partial_d F_{-1,-\varepsilon}(y) = \frac{-15}{16A^5\varepsilon^5}(\mathfrak{f}(y') - A\varepsilon - y_d)^2(\mathfrak{f}(y') - 3A\varepsilon - y_d)^2\chi_{[\mathfrak{f}(y')-3A\varepsilon,\mathfrak{f}(y')-A\varepsilon]}(y_d). \tag{3.39}$$

Take any $y \in W$ and if we set $a = \mathfrak{f}(y')$, $b = A\varepsilon$ and $x = y_d$, then

$$\partial_d F_{\varepsilon,1}(y) = \frac{15}{16b^5}(a + b - x)^2(a + 3b - x)^2\chi_{[a+b,a+3b]}(x).$$

Hence, for a fixed a and b the supremum over x of the right side of the equation above is attained at $x = a + 2b$ and is equal to $\frac{15}{b}$. And so, for a fixed $y' \in Q_{d-1}(0, r')$, we have that

$$\sup_{\mathfrak{f}(y')-A<y_d<\mathfrak{f}(y')+A} \partial_d F_{\varepsilon,1}(y', y_d) = \frac{15}{A\varepsilon}.$$

Consequently,

$$\sup_{y\in W} \partial_d F_{\varepsilon,1}(y) = \sup_{y'\in Q_{d-1}(0,r')} \left(\sup_{\mathfrak{f}(y')-A<y_d<\mathfrak{f}(y')+A} \partial_d F_{\varepsilon,1}(y', y_d) \right) = \frac{15}{A\varepsilon}. \tag{3.40}$$

Now for any $(y', t) \in Q_{d-1}(0, r') \times \mathbb{R}$ define

$$K_{\varepsilon,1}(y', t) = \frac{15}{16A^5\varepsilon^5}(\mathfrak{f}(y') + A\varepsilon - t)^2(\mathfrak{f}(y') + 3A\varepsilon - t)^2\chi_{[\mathfrak{f}(y')+A\varepsilon,\mathfrak{f}(y')+3A\varepsilon]}(t),$$

and notice that for $t \in \mathbb{R}$ and $y' \in Q_{d-1}(0, r')$, we have

$$\chi_{[\mathfrak{f}(y')+A\varepsilon, \mathfrak{f}(y')+3A\varepsilon]}(t) = \chi_{[t-3A\varepsilon, t-A\varepsilon]}(\mathfrak{f}(y')).$$

Fix any $t \in \mathbb{R}$ and take any $y' \in Q_{d-1}(0, r')$. First assume that $\mathfrak{f}(y') \in (-\infty, t - 3A\varepsilon) \cup (t - A\varepsilon, \infty)$, then for $i = 1, \ldots, d-1$ we have

$$\partial_i K_{\varepsilon,1}(y', t) = 0.$$

If now $\mathfrak{f}(y') \in (t - 3A\varepsilon, t - A\varepsilon)$, then

$$\begin{aligned}\partial_i K_{\varepsilon,1}(y', t) = \frac{15}{4A^5\varepsilon^5} \partial_i \mathfrak{f}(y')(\mathfrak{f}(y') + A\varepsilon - t)(\mathfrak{f}(y') + 3A\varepsilon - t)(\mathfrak{f}(y') + 2A\varepsilon - t) \\ \cdot \chi_{[\mathfrak{f}(y')+A\varepsilon, \mathfrak{f}(y')+3A\varepsilon]}(t),\end{aligned}$$

Now consider the case $\mathfrak{f}(y') = t - 3A\varepsilon$. Let $i = 1, \ldots, d-1$ and take any $h \in \mathbb{R}$ such that $y' + he_i \in Q_{d-1}(0, r')$ we have

$$\begin{aligned}\frac{K_{\varepsilon,1}(y' + he_i, t) - K_{\varepsilon,1}(y', t)}{h} &= \frac{K_{\varepsilon,1}(y' + he_i, \mathfrak{f}(y') + 3A\varepsilon) - K_{\varepsilon,1}(y', \mathfrak{f}(y') + 3A\varepsilon)}{h} \\ &= \frac{K_{\varepsilon,1}(y' + he_i, \mathfrak{f}(y') + 3A\varepsilon)}{h} \\ &= \frac{15}{16A^5\varepsilon^5 h}(\mathfrak{f}(y' + he_i) - \mathfrak{f}(y') - 2A\varepsilon)^2 \\ &\cdot (\mathfrak{f}(y' + he_i) - \mathfrak{f}(y'))^2 \chi_{[\mathfrak{f}(y'), \mathfrak{f}(y')+2A\varepsilon]}(\mathfrak{f}(y' + he_i)).\end{aligned}$$

By the Mean Value Theorem, for some $\theta \in [0, 1]$ we have

$$\begin{aligned}\frac{K_{\varepsilon,1}(y' + he_i, t) - K_{\varepsilon,1}(y', t)}{h} &= \frac{15}{16A^5\varepsilon^5 h}(\partial_i \mathfrak{f}(y' + \theta he_i)h - 2A\varepsilon)^2(\partial_i \mathfrak{f}(y' + \theta he_i)h)^2 \\ &\cdot \chi_{[\mathfrak{f}(y'), \mathfrak{f}(y')+2A\varepsilon]}(\mathfrak{f}(y') + \partial_i \mathfrak{f}(y' + \theta he_i)h).\end{aligned}$$

Thus, for small enough h, if $\partial_i\mathfrak{f}(y'+\theta he_i)h>0$ we have that

$$\frac{K_{\varepsilon,1}(y'+he_i,t)-K_{\varepsilon,1}(y',t)}{h}=\frac{15}{16A^5\varepsilon^5}(\partial_i\mathfrak{f}(y'+\theta he_i)h-2A\varepsilon)^2(\partial_i\mathfrak{f}(y'+\theta he_i)h)^2$$

and if $\partial_i\mathfrak{f}(y'+\theta he_i)h\leq 0$ we have

$$\frac{K_{\varepsilon,1}(y'+he_i,t)-K_{\varepsilon,1}(y',t)}{h}=0.$$

Finally, for small enough h, we have

$$\left|\frac{K_{\varepsilon,1}(y'+he_i,t)-K_{\varepsilon,1}(y',t)}{h}\right|\leq\left|\frac{15}{16A^5\varepsilon^5}(\partial_i\mathfrak{f}(y'+\theta he_i)h-2A\varepsilon)^2(\partial_i\mathfrak{f}(y'+\theta he_i)h)^2\right|.$$

By the assumption $\sup_{x\in Q_{d-1}(0,r')}|\partial_i\mathfrak{f}(x)|<\infty$, we have for $i=1,\ldots,d-1$,

$$\partial_i K_{\varepsilon,1}(y',t)=\lim_{h\to 0}\frac{K_{\varepsilon,1}(y'+he_i,t)-K_{\varepsilon,1}(y',t)}{h}=0.$$

Similarly, if $\mathfrak{f}(y')=t-A\varepsilon$ we have

$$\partial_i K_{\varepsilon,1}(y',t)=0.$$

From this we conclude that the function $K_{\varepsilon,1}$ has continuous partial derivatives, for $i=1,\ldots,d-1$, on $Q_{d-1}(0,r')\times\mathbb{R}$, given by the formula

$$\partial_i K_{\varepsilon,1}(y',t)=\frac{15}{4A^5\varepsilon^5}\partial_i\mathfrak{f}(y')(\mathfrak{f}(y')+A\varepsilon-t)(\mathfrak{f}(y')+3A\varepsilon-t)(\mathfrak{f}(y')+2A\varepsilon-t)$$
$$\cdot\chi_{[\mathfrak{f}(y')+A\varepsilon,\mathfrak{f}(y')+3A\varepsilon]}(t).$$

Hence, by the Leibniz Integral Rule [19, Theorem 2.27, p.56] we have that

$$\partial_i F_{\varepsilon,1}(y) = \int_{-\infty}^{y_d} \partial_i K_{\varepsilon,1}(y',t)dt = \int_{-\infty}^{y_d} \frac{15}{4A^5\varepsilon^5}\partial_i\mathfrak{f}(y')(\mathfrak{f}(y')+A\varepsilon-t)(\mathfrak{f}(y')+3A\varepsilon-t)$$
$$\cdot(\mathfrak{f}(y')+2A\varepsilon-t)\chi_{[\mathfrak{f}(y')+A\varepsilon,\mathfrak{f}(y')+3A\varepsilon]}(t)dt$$
$$= \frac{-15}{16A^5\varepsilon^5}\partial_i\mathfrak{f}(y')(\mathfrak{f}(y')+A\varepsilon-y_d)^2(\mathfrak{f}(y')+3A\varepsilon-y_d)^2\chi_{[\mathfrak{f}(y')+A\varepsilon,\mathfrak{f}(y')+3A\varepsilon]}(y_d)$$
$$= -\partial_i\mathfrak{f}(y')K_{\varepsilon,1}(y),$$

and so, by (3.38), for $i = 1, \dots, d-1$,

$$\partial_i F_{\varepsilon,1}(y) = -\partial_i\mathfrak{f}(y')\partial_d F_{\varepsilon,1}(y). \tag{3.41}$$

Arguing similarly, we can show that

$$\partial_i F_{-1,-\varepsilon}(y) = -\partial_i\mathfrak{f}(y')\partial_d F_{-1,-\varepsilon}(y). \tag{3.42}$$

Since any $y \in W$ can be written as $y = (y', y_d) = (y', \mathfrak{f}(y') + At)$ for some $t \in (-1, 1)$, and since

$$\mathcal{R}'(y) = (y', \mathfrak{f}(y') - At) = (y', 2\mathfrak{f}(y') - y_d),$$

we have that

$$\partial_d F_{\varepsilon,1}(\mathcal{R}'(y)) = \frac{15}{16A^5\varepsilon^5}(-\mathfrak{f}(y')+A\varepsilon+y_d)^2(-\mathfrak{f}(y')+3A\varepsilon+y_d)^2$$
$$\cdot\chi_{[\mathfrak{f}(y')+A\varepsilon,\mathfrak{f}(y')+3A\varepsilon]}(2\mathfrak{f}(y')-y_d))$$
$$= \frac{15}{16A^5\varepsilon^5}(\mathfrak{f}(y')-A\varepsilon-y_d)^2(\mathfrak{f}(y')-3A\varepsilon-y_d)^2\chi_{[\mathfrak{f}(y')-3A\varepsilon,\mathfrak{f}(y')-A\varepsilon]}(y_d)$$
$$= -\partial_d F_{-1,-\varepsilon}(y).$$

We conclude that for any $y \in W$,

$$\partial_d F_{\varepsilon,1}(\mathcal{R}'(y)) = -\partial_d F_{-1,-\varepsilon}(y), \tag{3.43}$$

and for $i = 1, \dots d-1$, by (3.41), (3.42), (3.43) we have

$$\partial_i F_{\varepsilon,1}(\mathcal{R}'(y)) = -\partial_i \mathfrak{f}(y')\partial_d F_{\varepsilon,1}(\mathcal{R}'(y)) = \partial_i \mathfrak{f}(y')\partial_d F_{-1,-\varepsilon}(y) = -\partial_i F_{-1,-\varepsilon}(y). \tag{3.44}$$

From the above equalities, the fact that $\mathcal{R}'(W_{(\varepsilon,1)}) = W_{(-1,-\varepsilon)}$, and by (3.40) we conclude that

$$\sup_{y \in W} |\partial_d F_{\varepsilon,1}(y)| = \sup_{y \in W} |\partial_d F_{-1,-\varepsilon}(y)| = \frac{15}{A\varepsilon}. \tag{3.45}$$

and for $i = 1, \dots, d-1$, by the assumption $\sup_{x \in Q_{d-1}(0,r')} |\partial_i \mathfrak{f}(x)| < \infty$ and (3.41), the fact that any $y \in W$ can be written as $y = \mathcal{R}'(\mathcal{R}'(y))$ and (3.44) we have

$$\sup_{y \in W} |\partial_i F_{-1,-\varepsilon}(y)| = \sup_{y \in W} |\partial_i F_{\varepsilon,1}(y)| = \sup_{y \in W} |\partial_i \mathfrak{f}(y')||\partial_d F_{\varepsilon,1}(y)| \tag{3.46}$$

$$\leq \frac{15}{A\varepsilon} \sup_{x \in Q_{d-1}(0,r')} |\partial_i \mathfrak{f}(x)| < \infty. \tag{3.47}$$

We use the function $F_{\varepsilon,1}$, $F_{-\varepsilon,\varepsilon}$ and $F_{-1,-\varepsilon}$ to partition any function form $C^1_C(W)$ into $C^1_C(W)$ functions with supports that will be placed conveniently placed in W. Take any $u \in C^1_C(W)$ and $i = 1, \dots, d$ and $0 < \varepsilon < \frac{1}{3}$ and define the functions

$$u_{1,\varepsilon} = F_{\varepsilon,1} u, \tag{3.48}$$

$$u_{2,\varepsilon} = F_{-\varepsilon,\varepsilon} u, \tag{3.49}$$

$$u_{3,\varepsilon} = F_{-1,-\varepsilon} u. \tag{3.50}$$

By definitions of functions $F_{\varepsilon,1}$, $F_{-\varepsilon,\varepsilon}$, $F_{-1,-\varepsilon}$, we conclude that

$$u_{1,\varepsilon} + u_{2,\varepsilon} + u_{3,\varepsilon} = u.$$

Notice now, that $u_{1,\varepsilon}(y) \neq 0$ only if both $u(y) \neq 0$ and $F_{\varepsilon,1}(y) \neq 0$, therefore if $u_{1,\varepsilon}(y) \neq 0$, by formula (3.30), we have that $y \in W_{(1,\varepsilon)}$. Hence $\operatorname{ess\,supp} u_{1,\varepsilon} \subset \overline{W_{(\varepsilon,1)}} \cap \operatorname{ess\,supp} u$. And since $\operatorname{ess\,supp} u$ is a compact subset of W, we have that $\operatorname{ess\,supp} u \cap \operatorname{bd}(W)$ is empty and so

$$\operatorname{ess\,supp} u_{1,\varepsilon} \subset \overline{W_{(\varepsilon,1)}} \setminus (\operatorname{bd}(W) \cap \operatorname{ess\,supp} u). \tag{3.51}$$

Recall that

$$W_{(\varepsilon,1)} = \bigcup_{t \in (\varepsilon,1)} S + Ate_d,$$

where S is the graph of $\mathfrak{f}$,

$$S = \{(y', y_d) \in \mathbb{R}^d : y' \in Q_{d-1}(0, r'), y_d = \mathfrak{f}(y')\}.$$

Since $\mathfrak{f}$ is a bounded continuous function on $Q_{d-1}(0, r')$ it can be uniquely extended to a continuous function $\bar{\mathfrak{f}}$ on $\overline{Q_{d-1}(0, r')}$, such that $\bar{\mathfrak{f}}(x) = \mathfrak{f}(x)$ for $x \in Q_{d-1}(0, r')$. If $\{y_k\}_{k=1}^{\infty} = \{(y_k', \mathfrak{f}(y_k'))\}_{k=1}^{\infty}$ is a sequence of points from S converging to some point (y', y_d), then $y' \in \overline{Q_{d-1}(0, r')}$ and $y_d = \lim_{k\to\infty} \mathfrak{f}(y_k') = \lim_{k\to\infty} \bar{\mathfrak{f}}(y_k') = \bar{\mathfrak{f}}(y')$. Therefore

$$\overline{S} \subset \{(y', y_d) \in \mathbb{R}^d : \ y' \in \overline{Q_{d-1}(0, r')}, \ y_d = \bar{\mathfrak{f}}(y')\}$$

Similarly, for any point $(y', y_d) = y \in \{(y', y_d) \in \mathbb{R}^d : \ y' \in \overline{Q_{d-1}(0, r')}, \ y_d = \bar{\mathfrak{f}}(y')\}$ we have $y_d = \bar{\mathfrak{f}}(y')$. Now we show that $(y', y_d) \in \overline{S}$. Since y' is an element of $\overline{Q_{d-1}(0, r')}$

there exists a sequence $\{x_k\}_{k=1}^{\infty}$ in $Q_{d-1}(0, r')$ convergent to y' in $\mathbb{R}^{d-1}$. By continuity of $\mathfrak{f}$ the sequence $\{(x_k, \mathfrak{f}(x_k))\}_{k=1}^{\infty} \subset S$ convergent to $(y', \bar{\mathfrak{f}}(y')) = (y', y_d)$ in $\mathbb{R}^d$. Hence

$$\overline{S} = \{(y', y_d) \in \mathbb{R}^d : \ y' \in \overline{Q_{d-1}(0, r')}, \ y_d = \bar{\mathfrak{f}}(y')\}.$$

Now we will show that, for any $(a, b) \subset (-1, 1)$ we have that

$$\overline{W_{(a,b)}} = \bigcup_{t \in [a,b]} \overline{S} + Ate_d. \tag{3.52}$$

If $(y', y_d) = y$ is an element of $\overline{W_{(a,b)}}$, then there exists a sequence $\{y_k\}_{k=1}^{\infty} = \{(y'_k, y_{k,d})\}_{k=1}^{\infty}$ of points in $W_{(a,b)}$ such that $\lim_{k\to\infty} y_k = y$. For each $k \in \mathbb{N}$, y_k is an element of $W_{(a,b)}$ and so it can be written as $(y'_k, \mathfrak{f}(y'_k) + At_k)$, where $t_k \in (a, b)$ and $y'_k \in Q_{d-1}(0, r')$. Since the sequence $\{y_k\}_{k=1}^{\infty}$ is convergent so are the sequences $\{y'_k\}_{k=1}^{\infty}$ and $\{\mathfrak{f}(y'_k) + At_k\}$ are convergent to y' in $\mathbb{R}^{d-1}$ and to y_d in $\mathbb{R}$, respectively. Since each y'_k is an element of $Q_{d-1}(0, r')$ we have that $y' \in \overline{Q_{d-1}(0, r')}$. Moreover, since for $x \in Q_{d-1}(0, r')$ we have $\bar{\mathfrak{f}}(x) = \mathfrak{f}(x)$, we can write

$$y_d = \lim_{k\to\infty} (\mathfrak{f}(y'_k) + At_k) = \lim_{k\to\infty} (\bar{\mathfrak{f}}(y'_k) + At_k).$$

By continuity of $\bar{\mathfrak{f}}$ we conclude that $\lim_{k\to\infty} \bar{\mathfrak{f}}(y'_k) = \bar{\mathfrak{f}}(y')$ and so the sequence $\{t_k\}_{k=1}^{\infty}$ is also convergent to some $t \in [a, b]$. Hence

$$y \in \{(y', y_d) \in \mathbb{R}^d : y' \in \overline{Q_{d-1}(0, r')}, \ y_d = \bar{\mathfrak{f}}(y') + At, \ t \in [a, b]\}.$$

Therefore

$$\overline{W_{(a,b)}} \subset \bigcup_{t \in [a,b]} \left\{(y', y_d) \in \mathbb{R}^d : y' \in \overline{Q_{d-1}(0, r')}, \ y_d = \bar{\mathfrak{f}}(y')\right\} + Ate_d = \bigcup_{t \in [a,b]} \overline{S} + Ate_d.$$

If now (y', y_d) is an element of $\bigcup_{t\in[a,b]} \overline{S} + Ate_d$. it can be written as $(y', \bar{\mathfrak{f}}(y') + At)$, where $y' \in \overline{Q_{d-1}(0,r')}$ and $t \in [a,b]$. Then there exists a sequence of points $\{x_k\}_{k=1}^{\infty} \in Q_{d-1}(0,r')$ such that $\lim_{k\to\infty} x_k = y'$ in $\mathbb{R}^{d-1}$ and a sequence of numbers $\{t_k\} \subset (a,b)$ convergent to t. Therefore the sequence $\{(x_k, \mathfrak{f}(x_k) + At_k)\}_{k=1}^{\infty} \subset W_{(a,b)}$ converges to (y', y_d) in $\mathbb{R}^d$ and so $\bigcup_{t\in[a,b]} \overline{S} + Ate_d \subset \overline{W_{(a,b)}}$. In particular, we have (3.52).

For $i = 1, \ldots, d-1$ and $(a,b) \subset (-1,1)$, we define

$$\mathrm{Face}_i(W_{(a,b)}) = \bigcup_{t\in[a,b]} \left\{(y', y_d) = (y_1, \ldots, y_i, \ldots, \mathfrak{f}(y') + At) \in \overline{Q_{d-1}(0,r')} :\ |y_i| = r'\right\}.$$

Then we can write

$$\begin{aligned}\mathrm{bd}(W_{(a,b)}) = \overline{W_{(a,b)}} \setminus W_{(a,b)} &= \left(\bigcup_{t\in[a,b]} \overline{S} + Ate_d\right) \setminus \left(\bigcup_{t\in(a,b)} S + Ate_d\right)\\ &= (\overline{S} + ae_d) \cup (\overline{S} + be_d) \cup \bigcup_{i=1}^{d-1} \mathrm{Face}_i(W_{(a,b)}).\end{aligned}$$

Hence, we get that for any $0 < \varepsilon < 1$, we have that

$$\overline{W_{(\varepsilon,1)}} \setminus \mathrm{bd}(W) = W_{(\varepsilon,1)} \cup (S + A\varepsilon e_d) = \bigcup_{t\in[\varepsilon,1)} S + Ate_d.$$

and so, by (3.51) we have that

$$\begin{aligned}\operatorname{ess\,supp} u_{1,\varepsilon} \subset (\overline{W_{(1,\varepsilon)}} \setminus \mathrm{bd}(W)) \cap \operatorname{ess\,supp} u &\subset \bigcup_{t\in[\varepsilon,1)} S + Ate_d \qquad (3.53)\\ &\subset \bigcup_{t\in(\varepsilon/2,1)} S + Ate_d = W_{(\varepsilon/2,1)}.\end{aligned}$$

Hence

$$u_{1,\varepsilon} \in C^1_C(W_{(\varepsilon/2,1)}). \tag{3.54}$$

Similarly, one can show that

$$u_{3,\varepsilon} \in C^1_C(W_{(-1,-\varepsilon/2)}), \quad \operatorname{ess\,supp} u_{2,\varepsilon} \subset \overline{W_{(-3\varepsilon,3\varepsilon)}}. \tag{3.55}$$

Take now any $y \in W_{(0,1)}$. By the definition of $W_{(0,1)}$, there exists $t_y \in (0,1)$ such that $y \in S + At_y e_d$, and so if $3\varepsilon < t_y$ we have that $y \in W_{(3\varepsilon,1)}$. Hence for $0 < \varepsilon < t_y/3$, by formula (3.30),

$$F_{1,\varepsilon}(y) = 1.$$

It follows in view of (3.53),

$$\lim_{\varepsilon \to 0^+} u_{1,\varepsilon} = u\chi_{W_{(0,1)}}, \tag{3.56}$$

where the above limit is taken pointwise. Similar reasoning shows that

$$\lim_{\varepsilon \to 0^+} u_{3,\varepsilon} = u\chi_{W_{(-1,0)}}. \tag{3.57}$$

Take now a function g, such that $g \in W^{1,1}(W_{(0,1)})$. We will show, that the function $\widetilde{g}$ defined by the formula

$$\widetilde{g}(y) = \begin{cases} g(y) & y \in W_{(0,1)} \\ 0 & y \in W \setminus (W_{(-1,0)} \cup W_{(0,1)}) \\ g(\mathcal{R}'(y)) & y \in W_{(-1,0)} \end{cases}$$

is an element of $W^{1,1}(W)$ and for a.a $(y', y_d) = y \in W$ we have

$$\partial_i \widetilde{g}(y) = \begin{cases} \partial_i g(y) & y \in W_{(0,1)},\ i = 1, \ldots, d \\ (\partial_i g)(\mathcal{R}'(y)) + 2(\partial_d g)(\mathcal{R}'(y))\partial_i \mathfrak{f}(y') & (y', y_d) \in W_{(-1,0)}\ i = 1, \ldots, d-1 \\ -(\partial_d g)(\mathcal{R}'(y)) & (y', y_d) \in W_{(-1,0)},\ i = d. \end{cases}$$

It is clear that $\widetilde{g}_{|W_{(0,1)}}$ is an element of $W^{1,1}(W_{(0,1)})$. Next we show that $\widetilde{g}_{|W_{(-1,0)}}$ is an element of $W^{1,1}(W_{(-1,0)})$. Notice that $\widetilde{g}_{|W_{(-1,0)}}$ is given by the formula

$$\widetilde{g}_{|W_{(-1,0)}}(y) = g(\mathcal{R}'(y)),$$

where $y \in W_{(-1,0)}$. By assumption, the map $\mathcal{R}'$ is of class $C^1(W, W)$ and $\mathcal{R}'(W_{(0,1)}) = W_{(-1,0)}$. Moreover, $\mathcal{R}'$ is its own inverse, therefore $\mathcal{R}'_{|W_{(0,1)}} : W_{(0,1)} \to W_{(-1,0)}$ is an invertable map of class $C^1(W_{(0,1)}, W_{(-1,0)})$. Now by the Theorem 3.4.2, we conclude that $\widetilde{g}_{|W_{(-1,0)}}$ is an element of $W^{1,1}(W_{(-1,0)})$ and

$$\partial_i \widetilde{g}_{|W_{(-1,0)}}(y) = \sum_{j=1}^{d} (\partial_j g)(\mathcal{R}'(y))\partial_i \varrho'_j(y),$$

for $i = 1, \ldots, d$ and a.a $y \in W_{(-1,0)}$. Since we know the explicit formulas for $\partial_i \varrho'_j$ (see (3.27) and below) we rewrite the above formula as

$$\partial_i \widetilde{g}_{|W_{(-1,0)}}(y) = \begin{cases} (\partial_i g)(\mathcal{R}'(y)) + 2(\partial_d g)(\mathcal{R}'(y))\partial_i \mathfrak{f}(y') & (y', y_d) \in W_{(-1,0)}, 1 \le i < d \\ -(\partial_d g)(\mathcal{R}'(y)) & (y', y_d) \in W_{(-1,0)},\ i = d. \end{cases}$$

Now we show that $\widetilde{g}$ is indeed an element of $W^{1,1}(W)$. First we will show that $\widetilde{g}$ is weakly differentiable on W. By Lemma 3.1.4 it suffices to show that for each

$u \in C^1_C(W)$ and $i = 1, \dots, d$ we have

$$\int_W \widetilde{g}(x)\partial_i u(x)dx = -\int_W \partial_i \widetilde{g}(x)u(x)dx.$$

In view of (3.48), (3.49), (3.50) and (3.54), (3.55), for any $i = 1, \dots, d$ we have

$$\begin{aligned}\int_W \widetilde{g}(y)\partial_i u(y)dy &= \int_{W_{(\varepsilon/2,1)}} \widetilde{g}(y)\partial_i u_{1,\varepsilon}(y)dy + \int_{W_{(-3\varepsilon,3\varepsilon)}} \widetilde{g}(y)\partial_i u_{2,\varepsilon}(y)dy \\ &+ \int_{W_{(-1,-\varepsilon/2)}} \widetilde{g}(y)\partial_i u_{3,\varepsilon}(y)dy = I_1(\varepsilon) + I_2(\varepsilon) + I_3(\varepsilon).\end{aligned} \tag{3.58}$$

We deal with each of $I_j(\varepsilon)$, for $j = 1, 2, 3$ separately. Notice that, since $g \in W^{1,1}(W_{(0,1)})$ and $u_{1,\varepsilon} \in C^1_C(W_{(\varepsilon/2,1)}) \subset C^1_C(W_{(0,1)})$,

$$I_1(\varepsilon) = \int_{W_{(\varepsilon/2,1)}} \widetilde{g}(y)\partial_i u_{1,\varepsilon}(y)dy = \int_{W_{(0,1)}} g(y)\partial_i u_{1,\varepsilon}(y)dy = -\int_{W_{(0,1)}} \partial_i g(y)u_{1,\varepsilon}(y)dy.$$

Similarly, since $\widetilde{g}_{|W_{(-1,0)}} \in W^{1,1}(W_{(-1,0)})$ and $u_{-1,-\varepsilon} \in C^1_C(W_{(-1,-\varepsilon/2)}) \subset C^1_C(W_{(0,1)})$ we have

$$\begin{aligned}I_3(\varepsilon) &= \int_{W_{(-1,-\varepsilon/2)}} \widetilde{g}(y)\partial_i u_{3,\varepsilon}(y)dy = \int_{W_{(-1,0)}} \widetilde{g}_{|W_{(-1,0)}}(y)\partial_i u_{3,\varepsilon}(y)dy \\ &= -\int_{W_{(-1,0)}} \partial_i \widetilde{g}_{|W_{(-1,0)}}(y)u_{3,\varepsilon}(y).\end{aligned}$$

The case of I_2 is slightly more involved. First we rewrite it as a sum of two integrals

$$I_2(\varepsilon) = \int\limits_{W_{(-3\varepsilon,3\varepsilon)}} \widetilde{g}(y)\partial_i u_{2,\varepsilon}(y)dy = \int\limits_{W_{(-3\varepsilon,3\varepsilon)}} \widetilde{g}(y)\partial_i u(y)F_{-\varepsilon,\varepsilon}(y)dy$$
$$+ \int\limits_{W_{(-3\varepsilon,3\varepsilon)}} \widetilde{g}(y)u(y)\partial_i F_{-\varepsilon,\varepsilon}(y)dy = I_{2,1}(\varepsilon) + I_{2,2}(\varepsilon).$$

If we denote by $C_u = \sup\limits_{y\in W} |\partial_i u(y)|$, then we get that for every $0 < \varepsilon < \frac{1}{3}$ and $y \in W$,

$$|\widetilde{g}(y)\partial_i u(y)F_{-\varepsilon,\varepsilon}(y)| \leq C_u|\widetilde{g}(y)|.$$

In view of $\widetilde{g}_{|W_{(0,1)}} = g \in W^{1,1}(W_{(0,1)})$ and $\widetilde{g}_{|W_{(-1,0)}} = (g \circ \mathcal{R}')_{W_{(-1,0)}} \in W^{1,1}(W_{(-1,0)})$ we have $\widetilde{g}_{|W_{(-1,0)}} \in L^1(W_{(-1,0)})$ and $\widetilde{g}_{|W_{(0,1)}} \in L^1(W_{(0,1)})$. By the fact that $|W \setminus (W_{(-1,0)} \cup W_{(0,1)})| = 0$ we get that $\widetilde{g} \in L^1(W)$ and since $W_{(-3\varepsilon,3\varepsilon)} \subset W$ we conclude $\widetilde{g} \in L^1(W_{(-3\varepsilon,3\varepsilon)})$. Hence, by the Lebesgue Dominated Convergence Theorem, we have that

$$\lim_{\varepsilon\to 0^+} I_{2,1}(\varepsilon) = 0.$$

Now we want to deal with the term $I_{2,2}(\varepsilon)$. By (3.28), (3.43), (3.44), the fact that

$\mathcal{R}' \circ \mathcal{R}' = Id_W$ and the integration by substitution formula, we have

$$
\begin{aligned}
I_{2,2}(\varepsilon) &= \int\limits_{W_{(-3\varepsilon,3\varepsilon)}} \widetilde{g}(y)u(y)\partial_i F_{-\varepsilon,\varepsilon}(y)dy \\
&= -\int\limits_{W_{(-3\varepsilon,3\varepsilon)}} \widetilde{g}(y)u(y)(\partial_i F_{-1,-\varepsilon}(y) + \partial_i F_{\varepsilon,1}(y))dy \\
&= -\int\limits_{W_{(-3\varepsilon,0)}} \widetilde{g}(y)u(y)\partial_i F_{-1,-\varepsilon}(y)dy - \int\limits_{W_{(0,3\varepsilon)}} \widetilde{g}(y)u(y)\partial_i F_{\varepsilon,1}(y)dy \\
&= -\int\limits_{W_{(-3\varepsilon,0)}} g(\mathcal{R}'(y))u(y)\partial_i F_{-1,-\varepsilon}(y)dy - \int\limits_{W_{(0,3\varepsilon)}} g(y)u(y)\partial_i F_{\varepsilon,1}(y)dy \\
&\overset{\mathcal{R}'\circ\mathcal{R}'=Id_W}{=} -\int_{W_{(0,3\varepsilon)}} g(y)u(\mathcal{R}'(y))\partial_i F_{-1,-\varepsilon}(\mathcal{R}'(y))|\det J_{\mathcal{R}'}(y)|dy \\
&- \int\limits_{W_{(0,3\varepsilon)}} g(y)u(y)\partial_i F_{\varepsilon,1}(y)dy \\
&\overset{(3.28),\ (3.43),\ (3.44)}{=} \int_{W_{(0,3\varepsilon)}} g(y)u(\mathcal{R}'(y))\partial_i F_{\varepsilon,1}(y)dy \\
&- \int\limits_{W_{(0,3\varepsilon)}} g(y)u(y)\partial_i F_{\varepsilon,1}(y)dy = \int\limits_{W_{(0,3\varepsilon)}} g(y)\partial_i F_{\varepsilon,1}(y)(u(\mathcal{R}'(y)) - u(y))dy. \quad (3.59)
\end{aligned}
$$

Recall that for any $y \in W_{(0,3\varepsilon)}$, $y = (y', y_d)$ and is of the form

$$(y', y_d) = (y', \mathfrak{f}(y') + tA)$$

for some $t \in (0, 3\varepsilon)$, and also we have

$$\mathcal{R}'(y) = (y', \mathfrak{f}(y') - tA).$$

By the Mean Value Theorem, applied to the d-th variable on the interval $(\mathfrak{f}(y') - tA, \mathfrak{f}(y') + tA)$ for some $\theta \in (-1, 1)$ we have

$$u(y) - u(\mathcal{R}'(y)) = u(y', \mathfrak{f}(y') + tA) - u(y', \mathfrak{f}(y') - tA) = 2A\partial_d u(y', \mathfrak{f}(y') + \theta tA)t.$$

Denote now $C'_u = \sup_{y \in W} |\partial_d u(y)|$. Then for any $y \in W_{(0,3\varepsilon)}$ we have

$$|u(y) - u(\mathcal{R}'(y))| \le 6AC'_u\varepsilon.$$

And so by (3.46), for $y \in W_{(0,3\varepsilon)}$ and $i = 1, \ldots, d-1$ we have

$$|\partial_i F_{\varepsilon,1}(y)(u(\mathcal{R}'(y)) - u(y))| \le 6AC'_u\varepsilon \sup_{x \in Q_{d-1}(0,r')} |\partial_i \mathfrak{f}(x)| \frac{15}{A\varepsilon} = 90C'_u \sup_{x \in Q_{d-1}(0,r')} |\partial_i \mathfrak{f}(x)|,$$

and also by (3.45) we have

$$|\partial_d F_{\varepsilon,1}(y)(u(\mathcal{R}'(y)) - u(y))| \le 6AC'_u\varepsilon \frac{15}{A\varepsilon} \le 90C'_u.$$

And so, by (3.59) and the Lebesgue Dominated Convergence Theorem we conclude that

$$\lim_{\varepsilon \to 0^+} I_{2,2}(\varepsilon) = 0.$$

Therefore,

$$\lim_{\varepsilon \to 0^+} I_2(\varepsilon) = \lim_{\varepsilon \to 0^+} I_{2,1}(\varepsilon) + \lim_{\varepsilon \to 0^+} I_{2,2}(\varepsilon) = 0 \tag{3.60}$$

Recall that $C_u = \sup_{y \in W} |\partial_i u(y)| < \infty$ and so $|\widetilde{g}(y)\partial_i u(y)| \le C_u \widetilde{g}(y)$ for all $y \in W$. Hence, by the fact that $\widetilde{g} \in L^1(W)$, (3.58), (3.60), and the Lebesgue Dominated

Convergence Theorem,

$$\begin{aligned}\int\limits_W \widetilde{g}(y)\partial_i u(y)dy &= \lim_{\varepsilon\to 0^+}(I_1(\varepsilon)+I_2(\varepsilon)+I_3(\varepsilon)) = \lim_{\varepsilon\to 0^+} I_1(\varepsilon) + \lim_{\varepsilon\to 0^+} I_2(\varepsilon) + \lim_{\varepsilon\to 0^+} I_3(\varepsilon) \\ &= -\lim_{\varepsilon\to 0^+}\int\limits_{W_{(0,1)}} \partial_i g(y) u_{1,\varepsilon}(y)dy - \lim_{\varepsilon\to 0^+}\int\limits_{W_{(-1,0)}} \partial_i \widetilde{g}_{|W_{(-1,0)}}(y) u_{3,\varepsilon}(y)dy \\ &= -\int\limits_W \left(\partial_i g(y)u(y)\chi_{W_{(0,1)}}(y) + \partial_i\widetilde{g}_{|W_{(-1,0)}}(y)u(y)\chi_{W_{(-1,0)}}(y)\right)dy \\ &= -\int\limits_W \partial_i\widetilde{g}(y)u(y)dy,\end{aligned}$$

and so $\widetilde{g}$ is weakly differentiable on W. Since both $\widetilde{g}$ and its weak derivatives are integrable on W, we conclude that $\widetilde{g} \in W^{1,1}(W)$.

□

For any path $\gamma : [0,1] \to \mathbb{R}^d$ we denote its length by $l(\gamma)$.

Lemma 3.4.6. *Let $(W, \{W_{(t_0,t_1)}\}_{(t_0,t_1)\subset(-1,1)}, \mathcal{R}')$ be the triple from Theorem 3.4.4. For any $x, y \in W$ we have*

$$|\mathcal{R}'(x) - \mathcal{R}'(y)| \le C_1|x-y|,$$

for some constant $C_1 > 0$ independent of points x, y. Moreover, there exist a family of paths $\gamma_{x,y} : [0,1] \to W$ indexed by $x, y \in W$, and a constant $C_2 > 0$ independent of x, y, such that

(1) $\gamma_{x,y}(0) = x$, $\gamma_{x,y}(1) = y$,

(2) $\mathcal{R}'(\gamma_{x,y}(s)) = \gamma_{\mathcal{R}'(x),\mathcal{R}'(y)}(s)$, *for* $s \in [0,1]$,

(3) *if we write* $x = (x', \mathfrak{f}(x') + t_x)$, $y = (y', \mathfrak{f}(y') + t_y)$ *for some* $t_x, t_y \in (-1,1)$ *such that* $t_x \le t_y$, *then for any* $s \in [0,1]$ *and any interval* (a,b) *such that*

$[t_x, t_y] \subset (a, b) \subset (-1, 1)$ *we have*

$$\gamma_{x,y}(s) \in W_{(a,b)}$$

and

$$\mathcal{R}'(\gamma_{x,y}(s)) \in W_{(-b,-a)},$$

(4) $l(\gamma_{x,y}) \leq C_2|x - y|$,

(5) $l(\mathcal{R}'(\gamma_{x,y})) \leq C_1C_2|x - y|$.

Proof. Take any $x = (x', x_d) \in W$ and $y = (y', y_d) \in W$, by the Mean Value Theorem we have

$$|\mathfrak{f}(x') - \mathfrak{f}(y')| = |\nabla\mathfrak{f}(x' + t_0(y' - x')) \cdot (x' - y')|,$$

for some $t_0 \in [0, 1]$. Let

$$C_f = \sup_{x \in Q_{d-1}(0,r')} |\nabla\mathfrak{f}(x)|, \tag{3.61}$$

by assumption on $\mathfrak{f}$, $C_f < \infty$. Hence

$$|\mathfrak{f}(x') - \mathfrak{f}(y')| \leq C_f|x' - y'| \tag{3.62}$$

for any $x', y' \in Q_{d-1}(0, r')$ and so for any $x, y \in W$,

$$
\begin{aligned}
|\mathcal{R}'(x) - \mathcal{R}'(y)| &= \sqrt{|x' - y'|^2 + |2(\mathfrak{f}(x') - \mathfrak{f}(y')) + (y_d - x_d)|^2} \\
&\leq \sqrt{8|x' - y'|^2 + 8|\mathfrak{f}(x') - \mathfrak{f}(y')|^2 + 8|x_d - y_d|^2} \\
&\leq \sqrt{8|x' - y'|^2 + 8C_f^2|x' - y'|^2 + 8|x_d - y_d|^2} \\
&\leq \sqrt{8|x' - y'|^2 + 8C_f^2|x' - y'|^2 + 8C_f^2|x_d - y_d|^2 + 8|x_d - y_d|^2} \\
&= \sqrt{8|x - y|^2 + 8C_f^2|x - y|^2} = \sqrt{8 + 8C_f^2}|x - y|.
\end{aligned}
$$

We rewrite it as

$$
|\mathcal{R}'(x) - \mathcal{R}'(y)| \leq C_1|x - y|, \tag{3.63}
$$

where $C_1 = \sqrt{8 + 8C_f^2}$, and $x, y \in W$.

Letting $x, y \in W$, $x = (x', x_d) = (x', \mathfrak{f}(x') + At_x)$ and $y = (y', y_d) = (y', \mathfrak{f}(y') + At_y)$ for some $t_x, t_y \in (-1, 1)$. Define now a path $\gamma_{x,y} : [0, 1] \to W$ by the formula

$$
\gamma_{x,y}(s) = ((1 - s)x' + sy', \mathfrak{f}((1 - s)x' + sy') + (1 - s)At_x + sAt_y), \tag{3.64}
$$

for $s \in [0, 1]$. It is clear that $\gamma_{x,y}(0) = (x', \mathfrak{f}(x') + At_x) = x$ and $\gamma_{x,y}(1) = y$ which is (1). We also have for $s \in [0, 1]$,

$$
\begin{aligned}
\mathcal{R}'(\gamma_{x,y}(s)) &= ((1 - s)x' + sy', \mathfrak{f}((1 - s)x' + sy') - (1 - s)At_x - sAt_y) \\
&= ((1 - s)x' + sy', \mathfrak{f}((1 - s)x' + sy') + (1 - s)(-At_x) + s(-At_y)) \\
&= \gamma_{\mathcal{R}'(x), \mathcal{R}'(y)}(s), \qquad (3.65)
\end{aligned}
$$

which is (2).

Condition (3) is quite obvious. In fact for each $s \in [0,1]$ we have $\gamma_{x,y}(s) \in S + A((t_y - t_x)s + t_x)e_d$, so for any interval $(a,b) \subset (-1,1)$ such that $[t_x, t_y] \subset (a,b)$, provided that $t_x \le t_y$, we have

$$\gamma_{x,y}(s) \in W_{(a,b)}, \tag{3.66}$$

and

$$\mathcal{R}'(\gamma_{x,y}(s)) \in W_{(-b,-a)}. \tag{3.67}$$

We will show now condition (4) that is the length of $\gamma_{x,y}$ is bounded by $C_2|x-y|$, where the constant C_2 is independent of the choice of x, y. By the formula for the length of a parametric curve and the Mean Value Theorem, there exists $s_1 \in (0,1)$ such that

$$\begin{aligned} l(\gamma_{x,y}) &= \int_0^1 \sqrt{|x'-y'|^2 + |(\nabla f)(x' + (y'-x')s) \cdot (y'-x') + A(t_y - t_x)|^2} ds \\ &= \sqrt{|x'-y'|^2 + |(\nabla f)(x' + (y'-x')s_1) \cdot (y'-x') + A(t_y - t_x)|^2}. \end{aligned}$$

Hence by the inequality (3.62), Cauchy Schwartz Inequality and convexity of the function $u(t)=t^2$ we have

$$
\begin{aligned}
l(\gamma_{x,y}) &= \int_0^1 \sqrt{|x'-y'|^2+|(\nabla f)(x'+(y'-x')s)\cdot(y'-x')+A(t_y-t_x)|^2}ds \\
&\overset{MVT}{=} \sqrt{|x'-y'|^2+|(\nabla f)(x'+(y'-x')s_1)\cdot(y'-x')+A(t_y-t_x)|^2} \\
&\overset{CSI}{\leq} \sqrt{|x'-y'|^2+2|(\nabla f)(x'+(y'-x')s_1)|^2|y'-x'|^2+2|A(t_y-t_x)|^2} \\
&\overset{(3.61)}{\leq} \sqrt{(2C_f^2+1)|y'-x'|^2+2|A(t_y-t_x)+\mathfrak{f}(y')-\mathfrak{f}(x')-(\mathfrak{f}(y')-\mathfrak{f}(x'))|^2} \\
&\leq \sqrt{(2C_f^2+2)}\sqrt{\begin{aligned}&|y'-x'|^2+|\mathfrak{f}(y')-\mathfrak{f}(x')+A(t_y-t_x)|^2\\ &+2|\mathfrak{f}(y')-\mathfrak{f}(x')+A(t_y-t_x)||\mathfrak{f}(y')-\mathfrak{f}(x')|+|\mathfrak{f}(y')-\mathfrak{f}(x')|^2\end{aligned}} \\
&\overset{(3.65)}{\leq} \sqrt{(2C_f^2+2)}\sqrt{2|x-y|^2+2C_f|x-y||x'-y'|+C_f^2|x'-y'|^2} \\
&\leq \sqrt{(2C_f^2+2)(C_f^2+2C_f+2)}|x-y| = C_2|x-y|.
\end{aligned}
$$

Hence we get condition (4) that is for $x, y \in W$,

$$l(\gamma_{x,y}) \leq C_2|x-y|,$$

where $C_2=\sqrt{(2C_f^2+2)(C_f^2+2C_f+2)}$. Now by (3.65), we conclude that

$$l(\mathcal{R}'(\gamma_{x,y})) = l(\gamma_{\mathcal{R}'(x),\mathcal{R}'(y)}) \leq C_2|\mathcal{R}'(x)-\mathcal{R}'(y)| \leq C_2C_1|x-y|,$$

which shows (5) and finishes the proof.

□

Lemma 3.4.7. *Let $(W, \{W_{(t_0,t_1)}\}_{(t_0,t_1)\subset(-1,1)}, \mathcal{R}')$ be the triple from Theorem 3.4.4. For a MO function Φ on $W_{(0,1)}$, define the function $\Phi^W : W\times[0,\infty)\to[0,\infty)$ by* *the formula*

$$\Phi^W(x,t) = \begin{cases} \Phi(x,t) & x \in W_{(0,1)} \\ 0 & x \in S \\ \Phi(\mathcal{R}'(x),t) & x \in W_{(-1,0)}. \end{cases}$$

Then Φ^W is a MO function. Moreover

(1) *if Φ satisfies Δ_2 condition, so does Φ^W.*

(2) *Let $N = \lceil 2C_1C_2 \rceil$ and $N_1 = \lceil 2C_1 \rceil$, where constants $C_1,\ C_2$ are from the Lemma (3.4.6). If Φ satisfies condition* (A1) *with constant β, then Φ^W satisfies* (A1) *with the constant β^{N+N_1}.*

Proof. Let S, W, Φ and $\mathcal{R}'$ be as in the assumptions. Since $\mathcal{R}'$ is of class $C^1(W,W)$, the function $\Phi(\cdot,t) \circ \mathcal{R}'$ is measurable on $W_{(-1,0)}$, for each $t \geq 0$. Moreover, since

$$W \setminus (W_{(-1,0)} \cup W_{(0,1)}) = S$$

and $|S| = 0$ we conclude that $\Phi^W(\cdot,t)$ is measurable for each $t \geq 0$. On the other hand, if $x \in W_{(-1,0)}$, then $\mathcal{R}'(x) \in W_{(0,1)}$ and so $\Phi(\mathcal{R}'(x),\cdot)$ is a non-negative, convex function such that $\Phi(\mathcal{R}'(x),t) = 0$ if and only if $t = 0$. Hence Φ^W is a MO function on W.

(1) Assume now that $\Phi \in \Delta_2$, then there exist $C > 0$ and a non-negative $h \in L^1(W_{(0,1)})$ such that for any $t \geq 0$ and a.e. $x \in W_{(0,1)}$,

$$\Phi(x,2t) \leq C\Phi(x,t) + h(x).$$

Define now $h_W : W \to [0, \infty)$ by the formula

$$h_W(x) = \begin{cases} h(x) & x \in W_{(0,1)} \\ 0 & x \in S \\ h(\mathcal{R}'(x)) & x \in W_{(-1,0)}. \end{cases}$$

Since $\mathcal{R}'$ is a differentiable involution, we have that $|\det J_{\mathcal{R}'}(x)| = 1$ for all $x \in W$ and so

$$\begin{aligned} \int\limits_W h_W(x)dx &= \int\limits_{W_{(1,0)}} h_W(x)dx + \int\limits_{W_{(1,0)}} h_W(\mathcal{R}'(x))dx \\ &\leq \|h\|_1 + \int\limits_{W_{(1,0)}} h_W(x)dx|\det J_{\mathcal{R}'}(x)|dx \leq 2\|h\|_1. \end{aligned}$$

Hence h_W is a non-negative element of $L^1(W)$. Taking now any $x \in W_{(0,1)}$ and $t \geq 0$,

$$\Phi^W(x, 2t) = \Phi\left(x, 2t\right) \leq C\Phi(x,t) + h(x) = C\Phi^W(x,t) + h_W(x).$$

Similarly, if $x \in W_{(-1,0)}$, then

$$\Phi^W(x, 2t) = \Phi\left(\mathcal{R}'(x), 2t\right) \leq C\Phi\left(\mathcal{R}'(x), t\right) + h(\mathcal{R}'(x)) = C\Phi^W(x,t) + h_W(x).$$

Therefore Φ^W satisfies Δ_2 condition.

(2) Now assume that Φ satisfies the condition (A1) with the constant β and let $N = \lceil 2C_1C_2 \rceil$, where constants $C_1,\ C_2$ are from Lemma 3.4.6. We will show that Φ^W satisfies condition (A1).

Recall that, if Φ satisfies condition (A1), in our case on set $W_{(0,1)}$, then there exist $\beta, \delta \in (0,1)$ such that for all open balls B with $|B| < \delta$ and almost all

$x, y \in B \cap W_{(0,1)}$ we have

$$\beta \Phi^{-1}(x,t) \leq \Phi^{-1}(y,t),$$

where $t \in \left[1, \frac{1}{|B|}\right]$.

Let $S_\Phi \subset W$ be the set of measure 0 such that the above inequality holds for all $x, y \in \left(B \cap W_{(0,1)}\right) \setminus S_\Phi$, provided that B is an open ball with $|B| < \delta$. Note that, since for all $x \in W$ we have $|\det J_{\mathcal{R}'}(x)| = 1$, it follows that the set $\mathcal{R}'(S_\Phi)$ is also of measure 0. We will show now that for any open ball B with $|B| < \delta$ and any $x, y \in (W_{(-1,0)} \cup W_{(0,1)}) \cap B \setminus (S_\Phi \cup \mathcal{R}'(S_\Phi))$ we have

$$\beta^{N+2} \left(\Phi^W\right)^{-1}(x,t) \leq \left(\Phi^W\right)^{-1}(y,t)$$

for $t \in \left[1, \frac{1}{|B|}\right]$. In view of $|S \cup S_\Phi \cup \mathcal{R}'(S_\Phi)| = 0$ we will conclude that Φ^W satisfies (A1) condition.

Take any open ball B with $|B| < \delta$ and let $x, y \in (W_{(-1,0)} \cup W_{(0,1)}) \cap B \setminus (S_\Phi \cup \mathcal{R}'(S_\Phi))$. We have three cases

1^0 $x, y \in W_{(0,1)}$,

2^0 $x, y \in W_{(-1,0)}$,

3^0 $x \in W_{(0,1)}$ and $y \in W_{(-1,0)}$.

In the first case, for any $t \geq 0$ we have that $\left(\Phi^W\right)^{-1}(x,t) = \Phi^{-1}(x,t)$ and $\left(\Phi^W\right)^{-1}(y,t) = \Phi^{-1}(y,t)$ and so, since Φ satisfies (A1) and $N, N_1 \in \mathbb{N}$ we have

$$\begin{aligned}\beta^{N+N_1} \left(\Phi^W\right)^{-1}(x,t) = \beta^{N+N_1}\Phi^{-1}(x,t) \leq \beta^{N+N_1-1}\Phi^{-1}(y,t) < \Phi^{-1}(y,t) \\ = \left(\Phi^W\right)^{-1}(y,t).\end{aligned}$$

For the second case, by Lemma 3.4.6 there exists a path $\gamma_{x,y} : [0,1] \to W$ such that $\gamma_{x,y}(0) = x$ and $\gamma_{x,y}(1) = y$ and $\gamma_{x,y}(s) \in W_{(-1,0)}$ for each $s \in [0,1]$. By the same lemma we have that

$$\gamma_{\mathcal{R}'(x),\mathcal{R}'(y)}(s) = \mathcal{R}'(\gamma_{x,y}(s)) \in W_{(0,1)}$$

for each $s \in [0,1]$. Therefore the points $\mathcal{R}'(x)$ and $\mathcal{R}'(y)$ are connected by the path $\gamma_{\mathcal{R}'(x),\mathcal{R}'(y)}$ that is totally contained in $W_{(0,1)}$. Moreover by Lemma 3.4.6, we have that for any $x, y \in W$,

$$l(\gamma_{\mathcal{R}'(x),\mathcal{R}'(y)}) \leq C_1 C_2 |x - y|.$$

Since both $x, y \in B$ we have that

$$|x - y| \leq 2\sigma_d^{-1/d} |B|^{1/d},$$

and so

$$l(\gamma_{\mathcal{R}'(x),\mathcal{R}'(y)}) \leq 2C_1 C_2 \sigma_d^{-1/d} |B|^{1/d}.$$

Hence the path $\gamma_{\mathcal{R}'(x),\mathcal{R}'(y)}$ can be covered with a chain of $N = \lceil 2C_1 C_2 \rceil$ open balls $B_1, \ldots, B_N$, each of measure $|B|$, such that each two successive balls B_i, B_{i+1} have non-empty intersection and $\mathcal{R}'(x) \in B_1$, $\mathcal{R}'(y) \in B_N$.

Set $x_1 = \mathcal{R}'(x)$ and $x_{N+1} = \mathcal{R}'(y)$, and for each $i = 2, \ldots, N$ choose a point $x_i \in B_{i-1} \cap B_i \cap W_{(0,1)} \setminus S_\Phi$. We observe that for each $i = 1, \ldots, N$ we have $x_i, x_{i+1} \in B_i$. Now notice that $\left(\Phi^W\right)^{-1}(x,t) = \Phi^{-1}(\mathcal{R}'(x), t)$ and $\left(\Phi^W\right)^{-1}(y,t) = \Phi^{-1}(\mathcal{R}'(y), t)$. Finally, take any $t \in \left[1, \frac{1}{|B|}\right]$ and since for any $i = 1, \ldots, N$ we have $x_i, x_{i+1} \in B_i$ and $|B_i| = |B|$, by successive N times application of condition (A1) we arrive at

$$\begin{aligned}\beta^{N+N_1}(\Phi^W)^{-1}(x,t) = \beta^{N+N_1}\Phi^{-1}(\mathcal{R}'(x),t) = \beta^{N+N_1}\Phi^{-1}(x_1,t) &\le \beta^{N+N_1-1}\Phi^{-1}(x_2,t)\\ &\le \beta^{N+N_1-2}\Phi^{-1}(x_3,t) \le \dots\\ &\le \beta^{N+N_1-i}\Phi^{-1}(x_{i+1},t) \le \beta^{N+N_1-i-1}\Phi^{-1}(x_{i+2},t) \le \dots\\ &\le \beta^{N_1}\Phi^{-1}(x_{N+1},t) = \beta^{N_1}\Phi^{-1}(\mathcal{R}'(y),t) < \left(\Phi^W\right)^{-1}(y,t).\end{aligned}$$

For the third case let us take the path $\gamma_{x,y}$ from Lemma 3.4.6 (see (3.64)). Since $x \in B \cap W_{(0,1)}$, and $y \in B \cap W_{(-1,0)}$ there exist $t_x \in (0,1)$ and $t_y \in (-1,0)$ such that $x = (x', \mathfrak{f}(x') + At_x)$ and $y = (y', \mathfrak{f}(y') + At_y)$. Hence, by continuity of $\gamma_{x,y}$ there exists $s_0 \in [0,1]$ such that $((1-s_0)x' + s_0 y', \mathfrak{f}((1-s_0)x' + s_0 y')) = \gamma_{x,y}(s_0) \in S$, let us denote $z = \gamma_{x,y}(s_0)$. By Lemma 3.4.6 we have for any $x, y \in W$,

$$l(\gamma_{x,y}) \le C_1|x-y|,$$

and since $x, y \in B$ we conclude that

$$l(\gamma_{x,y}) \le C_1|x-y| \le 2C_1\sigma_d^{-1/d}|B|^{1/d}.$$

Hence, the path $\gamma_{x,y}$ can be covered by a chain of $N_1 = \lceil 2C_1 \rceil$ open balls, $B_1, \dots, B_{N_1}$ such that each two consecutive balls B_i, B_{i+1} have a non-empty intersection. Let K be the smallest integer such that $z \in B_K$, it follows that we have that $K \le N_1$. Consider now the path $\gamma_{z,y}$. Since $z \in S$ we have that $z = (z', \mathfrak{f}(z'))$, and for any $s \in [0,1]$,

$$\gamma_{z,y}(s) = ((1-s)z' + sy', \mathfrak{f}((1-s)z' + sy') + At_y s).$$

Since $y \in W_{(-1,0)}$ we conclude that $\gamma_{z,y}(s) \in W_{(-1,0)}$ for $s \in (0,1]$, but

$\gamma_{z,y}(0) = z \in S$ and $S \cap W_{(-1,0)} = \emptyset$. Consequently for $s \in [0,1]$,

$$\gamma_{z,y}(s) \in S \cup W_{(-1,0)} \text{ and } \mathcal{R}'(\gamma_{z,y}(s)) \in S \cup W_{(0,1)}.$$

On the other hand, since $z \in S$ and $z = \gamma_{x,y}(s_0)$ we have that

$$(1 - s_0)At_x + s_0 At_y = 0. \tag{3.68}$$

Hence,

$$\begin{aligned} z = \gamma_{x,y}(s_0) &= ((1-s_0)x' + s_0 y', \mathfrak{f}((1-s_0)x' + s_0 y') + (1-s_0)At_x + s_0 At_y) \\ &= ((1-s_0)x' + s_0 y', \mathfrak{f}((1-s_0)x' + s_0 y')), \end{aligned}$$

and so for $s \in [0,1]$ we have

$$\begin{aligned} \gamma_{z,y}(s) &= ((1-s)z' + sy', \mathfrak{f}((1-s)z' + sy') + sAt_y) \\ &\overset{(3.68)}{=} ((1-s)((1-s_0)x' + s_0 y') + sy', \\ &\quad \mathfrak{f}((1-s)((1-s_0)x' + s_0 y') + sy') + (1-s)((1-s_0)At_x + s_0 At_y) + sAt_y) \\ &= ((1-(s+s_0-ss_0))x' + (s+s_0-ss_0)y', \\ &\quad \mathfrak{f}((1-(s+s_0-ss_0))x' + (s+s_0-ss_0)y') \\ &\quad + (1-(s+s_0-ss_0))At_x + (s+s_0-ss_0)At_y) \\ &= \gamma_{x,y}(s+s_0-ss_0) = \gamma_{x,y}((1-s)s_0+s). \end{aligned}$$

Since $s, s_0 \in [0,1]$ we conclude that $((1-s)s_0+s) \in [s_0,1]$ and so

$$\gamma_{z,y}([0,1]) = \gamma_{x,y}([s_0,1]).$$

Clearly $\gamma_{z,y}$ is a part of the path $\gamma_{x,y}$. By Lemma 3.4.6 (5), we have that

$$l(\mathcal{R}'(\gamma_{z,y})) \leq l(\mathcal{R}'(\gamma_{x,y})) \leq 2C_1C_2\sigma_d^{-1/d}|B|^{1/d}.$$

Similarly to the second case, we can find a chain of $N = \lceil 4C_1C_2 \rceil$ open balls $B'_{K+1}, \ldots, B'_{N+K}$, that covers the path $\mathcal{R}'(\gamma_{z,y}) = \gamma_{z,\mathcal{R}'(y)}$, each of measure $|B|$ such that $z \in B'_{K+1}$, $\mathcal{R}'(y) \in B'_{N+K}$ and for $i = K+1, \ldots, N+K-1$ each two successive balls B'_i, B'_{i+1} have non-empty intersection. Since $z \in S \cap B_K \cap B'_{K+1}$ there exists a point $x_{K+1} \in W_{(0,1)} \cap B_K \cap B'_{K+1} \setminus S_\Phi$. Let now $x_1 = x$ and for each $i = 2, \ldots, K$ choose a point $x_i \in B_{i-1} \cap B_i \cap W_{(0,1)} \setminus S_\Phi$, and for $i = K+2, \ldots, N+K$ we choose a point $x_i \in B'_{i-1} \cap B'_i \cap W_{(0,1)} \setminus S_\Phi$ and finally we set $x_{N+K+1} = \mathcal{R}'(y)$. We notice that $(\Phi^W)^{-1}(x,t) = \Phi^{-1}(x,t)$ and $(\Phi^W)^{-1}(y,t) = \Phi^{-1}(\mathcal{R}'(y),t)$. Now we take any $t \in \left[1, \frac{1}{|B|}\right]$ and argue as in the second case,

$$\begin{aligned}
\beta^{N+N_1}(\Phi^W)^{-1}(x,t) = \beta^{N+N_1}\Phi^{-1}(x_1,t) &\leq \beta^{N+K}\Phi^{-1}(x_1,t) \leq \beta^{N+K-1}\Phi^{-1}(x_2,t) \\
&\leq \beta^{N+K-2}\Phi^{-1}(x_3,t) \leq \cdots \leq \beta^{N+K-i}\Phi^{-1}(x_{i+1},t) \\
&\leq \beta^{N+K-i-1}\Phi^{-1}(x_{i+2}.t) \leq \ldots \overset{i=N+K-1}{\leq} \Phi^{-1}(x_{N+K+1},t) \\
&= \Phi^{-1}(\mathcal{R}'(y),t) = \left(\Phi^W\right)^{-1}(y,t).
\end{aligned}$$

Since all three cases are covered this ends the proof. □

Lemma 3.4.8. *Let V, W be open sets in $\mathbb{R}^d$ such that there exists a rigid motion T on $\mathbb{R}^d$ with $T(V) = W$. Let Φ be a MO function on V and define $\Phi_W := \Phi \circ T^{-1}$. Then Φ_W is a MO function on W and*

(1) *if Φ satisfies Δ_2 condition, then so does Φ_W,*

(2) *if Φ satisfies* (A1)*, then Φ_W also satisfies* (A1)*,*

(3) *the operator* $G_T : W^{1,\Phi_W}(W) \to W^{1,\Phi}(V)$*, defined by the formula*

$$(G_T u)(x) = u(T(x)),$$

for $u \in W^{1,\Phi_W}$ *and* $x \in W$*, is an isomorphism.*

Proof. Let V, W and T be as in the assumption. Recall that, since T is a rigid motion, we have

$$T(x) = R(x) + c,$$

for every $x \in \mathbb{R}^d$, where $c \in \mathbb{R}^d$ and the map $R : \mathbb{R}^d \to \mathbb{R}^d$ is a rotation about the origin and is given by the formula

$$R(x) = R \cdot x = Rx,$$

for $x \in \mathbb{R}^d$. Here $R = (r_{ij})_{i,j=1}^d$ is a rotation matrix and the multiplication $\cdot$ is the usual matrix multiplication and $c \in \mathbb{R}^d$. Hence, for every $x \in \mathbb{R}^d$,

$$J_T(x) = R.$$

Since R is a rotation matrix, by definition (see [22, p. 39]),

$$\det J_T(x) = 1.$$

Now taking any $x \in W$ and letting $y = T^{-1}(x)$, we have $y \in V$. For any $t \geq 0$, $x \in W$,

$$\Phi_W(x, t) = \Phi(T^{-1}(x), t) = \Phi(y, t).$$

Since Φ is a MO function on V and T is continuous, we conclude that Φ_W is a MO function on W.

Assume now that $\Phi \in \Delta_2$. Then there exist $C > 0$ and a non-negative function $h \in L^1(V)$, such that for a.a $x \in V$ and $t \geq 0$ we have

$$\Phi(x, 2t) \leq C\Phi(x, t) + h(x).$$

Let us define h_W by the formula

$$h_W(x) = h(T^{-1}(x))$$

for $x \in W$. We have

$$\int\limits_W h(T(x))dx = \int\limits_V h(x)|\det J_T(x)|dx = \int\limits_V h(x)dx,$$

hence h_W is an non-negative element of $L^1(W)$. For any $x \in W$ and $y = T^{-1}(x)$, then we have $y \in V$ and

$$\begin{aligned}\Phi_W(x, 2t) &= \Phi(T^{-1}(x), 2t) = \Phi(y, 2t) \leq C\Phi(y, t) + h(y) \\ &= C\Phi(T^{-1}(x), t) + h(T^{-1}(x)) = C\Phi_W(x, t) + h_W(x).\end{aligned}$$

We conclude that Φ_W satisfies Δ_2.

Assume now that Φ satisfies (A1) condition. Then there exist constants $\beta, \delta \in (0, 1)$ such that for all open balls B with $|B| < \delta$ and almost every $x, y \in B \cap V$ we have for all $t \in \left[\Phi^{-1}(y, 1), \Phi^{-1}\left(y, \frac{1}{|B|}\right)\right]$,

$$\Phi(x, \beta t) \leq \Phi(y, t).$$

Take any ball B with $|B| < \delta$. Since T is a rigid motion $T^{-1}(B)$ is also a ball and $|T^{-1}(B)| = |B|$. Let us take any $x, y \in B \cap W$, then $T^{-1}(x), T^{-1}(y) \in T^{-1}(B) \cap V$

and for all $t \geq 0$, we have $\Phi_W^{-1}(y,t) = \Phi^{-1}(T^{-1}(y),t)$. For any $t \in \left[\Phi_W^{-1}(y,1), \Phi_W^{-1}\left(y, \frac{1}{|B|}\right)\right]$, we have

$$\Phi_W(x,\beta t) = \Phi(T^{-1}(x),\beta t) \leq \Phi(T^{-1}(y),t) = \Phi_W(y,t)$$

and so Φ_W satisfies (A1) condition.

Finally, let us show that G_T is an isomorphism between $W^{1,\Phi_W}(W)$ and $W^{1,\Phi}(V)$. Take any $u \in W^{1,\Phi}(W)$ and $\lambda > 0$ such that $I_{\Phi_W}(\lambda u) < \infty$. Then

$$\begin{aligned} I_\Phi(\lambda G_T u) &:= \int_V \Phi(x, \lambda|u(T(x))|)dx = \int_W \Phi(T^{-1}(x), \lambda|u(x)|)|\det J_{T^{-1}}(x)|dx \\ &= \int_W \Phi_W(x, \lambda|u(x)|)dx = I_{\Phi_W}(\lambda u) < \infty. \end{aligned}$$

Hence $\|G_T u\|_\Phi = \|u\|_{\Phi_W}$ and so G_T is a linear isometry between $L^\Phi(V)$ and $L^{\Phi_W}(W)$.

We will show now that $G_T u$ is a weakly differentiable function on V. First take any $v \in C_C^\infty(V)$. Since for all $x \in \mathbb{R}^d$ we have $J_{T^{-1}}(x) = R^{-1}$ and it is well known that an inverse of a rotation matrix R is its transpose, $R^{-1} = R^T$, we have

$$\partial_i(v \circ T^{-1})(x) = \sum_{j=1}^d \partial_j v(T^{-1}(x))\partial_i T_j^{-1}(x) = \sum_{j=1}^d \partial_j v(T^{-1}(x)) r_{ij},$$

$i = 1, \ldots, d$ and $x \in W$, where $(r_{ij})_{i,j}^d$ is a rotation matrix. Hence we have $\nabla(v \circ T^{-1})(x) = R(\nabla v(T^{-1}(x)))$ and so $\nabla v(T^{-1}(x)) = R^T(\nabla(v \circ T^{-1})(x))$, in view of $R^{-1} = R^T$. Consequently, for each $i = 1, \ldots, d$ we have $\partial_i v(T^{-1}(x)) = \sum_{j=1}^d \partial_j(v \circ T^{-1})(x) r_{ji}$. Going back to $G_T u$, u is weakly differentiable, and so for each $v \in C_C^\infty(V)$ and for $i = 1, \dots, d$ we have

$$\begin{aligned}\int_V G_T u(x)\partial_i v(x)dx &= \int_V u(T(x))\partial_i v(x)dx = \int_W u(x)\partial_i v(T^{-1}(x))|\det J_T(x)|dx \\ &= \int_W u(x)\partial_i v(T^{-1}(x))dx = \int_W u(x)\sum_{j=1}^d \partial_j(v \circ T^{-1})(x) r_{ji} dx \\ &= -\int_W \left(\sum_{j=1}^d \partial_j u(x) r_{ji}\right)(v \circ T^{-1})(x)dx \\ &= -\int_V \left(\sum_{j=1}^d \partial_j u(T(x)) r_{ji}\right) v(x)dx.\end{aligned}$$

Hence $G_T u$ is weakly differentiable on V and for each $i = 1, \dots, d$ we have

$$\partial_i(G_T u)(x) = \sum_{j=1}^d \partial_j u(T(x)) r_{ji} = \sum_{j=1}^d (\partial_j u \circ T)(x) r_{ji},$$

for a.a $x \in V$. Notice that, since R is a rotation matrix we have that $|r_{ji}| \leq 1$ and so for $i = 1, \dots, d$, a.a. $x \in V$,

$$|\partial_i(G_T u)(x)| \leq \sum_{j=1}^d |(\partial_j u \circ T)(x)|.$$

Now let $\lambda > 0$ be such that for all $j = 1, \dots, d$ we have $I_{\Phi_W}(\lambda \partial_j u) < \infty$. For any $i = 1, \dots, d$,

$$\begin{aligned}I_\Phi\left(\frac{\lambda}{d}\partial_i G_T u\right) &\leq I_\Phi\left(\frac{\lambda}{d}\sum_{j=1}^d |(\partial_j u) \circ T|\right) \leq \frac{1}{d}\sum_{j=1}^d I_\Phi(\lambda|(\partial_j u) \circ T|) \\ &= \frac{1}{d}\sum_{j=1}^d \int_V \Phi(x, \lambda|\partial_j u(T(x))|)dx = \frac{1}{d}\sum_{j=1}^d \int_W \Phi(T^{-1}(x), \lambda|\partial_j u(x)|)dx \\ &= \frac{1}{d}\sum_{j=1}^d I_{\Phi_W}(\lambda \partial_j u) < \infty.\end{aligned}$$

Taking $\lambda = 1/\left(\sum_{j=1}^{d} \|\partial_j u\|_{\Phi_W}\right)$ we arrive at

$$I_\Phi\left(\frac{\lambda}{d}\partial_i G_T u\right) \leq 1.$$

Hence, we have that $\|\partial_i G_T u\|_\Phi \leq d \sum_{j=1}^{d} \|\partial_j u\|_{\Phi_W}$ for $i = 1, \ldots, d$ and so

$$\|G_T u\|_{W^{1,\Phi}(V)} \leq (d^2 + 1)\|u\|_{W^{1,\Phi_W}(W)}.$$

We conclude that G_T is continuous. Notice also that

$$G_T^{-1} = G_{T^{-1}}$$

and since T^{-1} is a rigid motion, the same reasoning as above leads us to conclude that G_T^{-1} is also continuous, and thus G_T is an isomorphism. □

Theorem 3.4.9. *(Local extension) Let* Ω *be an open set in* $\mathbb{R}^d$ *such that its boundary is of class* C^1*. Let* Φ *be a MO function on* Ω *satisfying both* (A1) *and* Δ_2 *property. For any* x_0 *let* $(V, \{V_{(t_0,t_1)}\}_{(t_0,t_1)\subset(-1,1)}, \mathcal{R})$ *and* $(W, \{W_{(t_0,t_1)}\}_{(t_0,t_1)\subset(-1,1)}, \mathcal{R}')$ *be the triples from Theorem 3.4.4. Let also* T *be the rigid motion from the same theorem. Define the function* $\widetilde{\Phi} : W_{(0,1)} \times [0,\infty) \to [0,\infty)$ *by the formula*

$$\widetilde{\Phi}(y,t) = \Phi(T^{-1}(y), t),$$

where $y \in W_{(0,1)}$ *ant* $t \geq 0$*. We extend* $\widetilde{\Phi}$ *to* $\widetilde{\Phi}^W$ *on* W *as in Lemma 3.4.7 and define the function* $\Phi_V : V \times [0,\infty) \to [0,\infty)$ *via the formula*

$$\Phi_V(x,t) = \widetilde{\Phi}^W(T(x), t),$$

for $t \geq 0$ *and* $x \in V$*. Then, for any compactly supported* $u \in W^{1,\Phi}(\Omega \cap V)$*, there* *exists an extension* $\widetilde{u}$ *defined by the formula*

$$\widetilde{u}(x) := \begin{cases} u(x) & x \in V_{(0,1)} \\ 0 & x \in \mathrm{bd}(\Omega) \cap V \\ u(\mathcal{R}(x)) & x \in V_{(-1,0)}, \end{cases}$$

satisfying $\widetilde{u} \in W^{1,\Phi_V}(V)$ *and* $\widetilde{u}$ *is compactly supported.*

Proof. Fix $x_0 \in \mathrm{bd}(\Omega)$. Let $(V, \{V_{(t_0,t_1)}\}_{(t_0,t_1)\subset(-1,1)}, \mathcal{R})$, $(W, \{W_{(t_0,t_1)}\}_{(t_0,t_1)\subset(-1,1)}, \mathcal{R}')$ be the triples from Theorem 3.4.4 and T the rigid motion from the same theorem. By Lemma 3.4.8 the function $\widetilde{\Phi}$ satisfies both Δ_2 and (A1) conditions. By Lemma 3.4.7 the function $\widetilde{\Phi}^W$ also satisfies Δ_2 and (A1) conditions. Moreover for $y \in W$ and $t \geq 0$,

$$\widetilde{\Phi}^W(y,t) = \begin{cases} \Phi(T^{-1}(y), t) & y \in W_{(0,1)} \\ 0 & y \in S \\ \Phi(T^{-1}(R'(y)), t) & y \in W_{(-1,0)}. \end{cases}$$

By construction of V and W, the rigid motion T maps V onto W. Therefore any $x \in V$ can be uniquely written as $x = T^{-1}(y)$ for some $y \in W$. Since $\mathcal{R} = T^{-1} \circ \mathcal{R}' \circ T$, we arrive at

$$\Phi_V(x,t) = \begin{cases} \Phi(x,t) & x \in V_{(0,1)} \\ 0 & x \in \mathrm{bd}(\Omega) \cap V \\ \Phi(\mathcal{R}(x), t) & x \in V_{(-1,0)}, \end{cases}$$

where $x \in V$, and $t \geq 0$.

Taking any compactly supported $u \in W^{1,\Phi}(\Omega \cap V)$, by the fact that Φ satisfies (A1) and by Lemma 3.2.9 we have that $u \in L^1(\Omega \cap V)$ and for each $i = 1, \ldots, d$,

$\partial_i u \in L^1(\Omega \cap V)$. Therefore $u \in W^{1,1}(\Omega)$. We define $g : W_{(0,1)} \to \mathbb{C}$ by the formula

$$g(y) = u(T^{-1}(y)), \text{ for } y \in W_{(0,1)}.$$

In view of $V \cap \Omega = V_{(0,1)}$ and the fact that T is a rigid motion and $T(V_{(0,1)}) = W_{(0,1)}$, by Theorem 3.4.2 we conclude that $g \in W^{1,1}(W_{(0,1)})$. Hence by Lemma 3.4.5 the function

$$\widetilde{g}(y) = \begin{cases} g(y) & y \in W_{(0,1)} \\ 0 & y \in W \setminus (W_{(-1,0)} \cup W_{(0,1)}) \\ g(\mathcal{R}'(y)) & y \in W_{(-1,0)} \end{cases}$$

is an element of $W^{1,1}(W)$ and for a.a $y \in W$ we have

$$\partial_i \widetilde{g}(y) = \begin{cases} \partial_i g(y) & (y', y_d) \in W_{(0,1)},\ i = 1, \ldots, d \\ (\partial_i g)(\mathcal{R}'(y)) + 2(\partial_d g)(\mathcal{R}'(y))\partial_i \mathfrak{f}(y') & (y', y_d) \in W_{(-1,0)}\ i = 1, \ldots, d-1 \\ -(\partial_d g)(\mathcal{R}'(y)) & (y', y_d) \in W_{(-1,0)},\ i = d, \end{cases}$$

where $\mathfrak{f}$ is the function associated with x_0 via the definition of bd(Ω) being of class C^1 (Definition 3.4.3). By (3.26) we have that $|\partial_i \mathfrak{f}(y')| < C$ for some constant $C > 0$, all $i = 1, \ldots, d-1$, and any $y = (y', y_d) \in W$. Therefore, for any $\lambda > 0$ and

$i = 1, \dots, d-1$, using a substitution for $\mathcal{R}'(y)$,

$$\begin{aligned} I_{\widetilde{\Phi}^W}(\lambda \partial_i \widetilde{g}) &\leq \int\limits_{W_{(0,1)}} \widetilde{\Phi}(y, \lambda|\partial_i g(y)|)dy + \frac{1}{2} \int\limits_{W_{(-1,0)}} \widetilde{\Phi}(\mathcal{R}'(y), 2\lambda|\partial_i g(\mathcal{R}'(y))|)dy \\ &+ \frac{1}{2} \int\limits_{W_{(-1,0)}} \widetilde{\Phi}(\mathcal{R}'(y), 4C\lambda|\partial_d g(\mathcal{R}'(y))|)dy = \int\limits_{W_{(0,1)}} \widetilde{\Phi}(y, \lambda|\partial_i g(y)|)dy \\ &+ \frac{1}{2} \int\limits_{W_{(0,1)}} \widetilde{\Phi}(y, 2\lambda|\partial_i g(y)|)dy + \frac{1}{2} \int\limits_{W_{(0,1)}} \widetilde{\Phi}(y, 4C\lambda|\partial_d g(y)|)dy \\ &= \int\limits_{V_{(0,1)}} \Phi(x, \lambda|\partial_i g(T(x))|)dx \\ &+ \frac{1}{2} \int\limits_{V_{(0,1)}} \Phi(x, 2\lambda|\partial_i g(T(x))|)dx + \frac{1}{2} \int\limits_{V_{(0,1)}} \Phi(x, 4C\lambda|\partial_d g(T(x))|)dx. \end{aligned}$$

Let $T(x) = R(x) + c$, where $R = (r_{ij})_{i,j=1}^d$ is a rotation matrix and $c \in \mathbb{R}^d$. By definition of g for $x \in V$,

$$\partial_i g(T(x)) = \sum_{j=1}^d \partial_j u(x) r_{ji},$$

for all $i = 1, \dots, d$. Since R is a rotation matrix we have $|r_{ji}| \leq 1$ for any $1 \leq i, j \leq d$. Hence, by Δ_2 condition, for $i = 1, \dots, d-1$,

$$\begin{aligned} I_{\widetilde{\Phi}^W}(\lambda \partial_i \widetilde{g}) &\leq \frac{1}{d} \sum_{j=1}^d \int_{V_{(0,1)}} \Phi(x, d\lambda|\partial_j u(x)|)dx + \frac{1}{2d} \sum_{j=1}^d \int_{V_{(0,1)}} \Phi(x, 2d\lambda|\partial_j u(x)|)dx \\ &+ \frac{1}{2d} \sum_{j=1}^d \int_{V_{(0,1)}} \Phi(x, 4Cd\lambda|\partial_j u(x)|)dx < \infty. \end{aligned}$$

For $i = d$ similarly, we have

$$I_{\widetilde{\Phi}^W}(\lambda \partial_d \widetilde{g}) \leq \frac{1}{d} \sum_{j=1}^d \int_{V_{(0,1)}} \Phi(x, d\lambda|\partial_j u(x)|)dx + \frac{1}{d} \sum_{j=1}^d \int_{V_{(0,1)}} \Phi(x, d\lambda|\partial_j u(x)|)dx < \infty.$$

Since $\mathcal{R}'$ is a continuous involution it maps compact sets to compact sets. Thus $\widetilde{g}$ is compactly supported. Notice now that

$$\widetilde{u} = \widetilde{g} \circ T = G_T \widetilde{g},$$

where G_T is the operator from Lemma 3.4.8. Hence we conclude that $\widetilde{u} \in W^{1,\Phi_V}(V)$. By the fact that T is a rigid motion we also notice that $\widetilde{u}$ is compactly supported. This completes the proof.

$\square$

Theorem 3.4.10. *(Partition of unity)([2, Theorem 3.15, p. 65]) Let A be a subset of $\mathbb{R}^d$ and $\mathcal{U}$ be an open covering of A. A family of functions $\{\psi_U\}_{U\in\mathcal{U}} \subset C_C^\infty(\mathbb{R}^d)$ is called a partition of unity associated with the covering $\mathcal{U}$ of A, if*

1. *For every $U \in \mathcal{U}$ and $x \in \mathbb{R}^d$, $0 \leq \psi_U(x) \leq 1$.*
2. *For any compact subset $K \subset A$ all but finitely many ψ_U vanish identically on K.*
3. *For any $U \in \mathcal{U}$ we have $\operatorname{supp} \psi_U \subset U$.*
4. *For every $x \in A$, $\sum\limits_{U\in\mathcal{U}} \psi_U(x) = 1$.*

The theorem below is the main result of this section.

Theorem 3.4.11. *Let Ω be an open set in $\mathbb{R}^d$ such that its boundary is of class C^1. Let Φ be a MO function on Ω satisfying both the (A1) and Δ_2 property. Then the set of restrictions of functions from $C_C^\infty(\mathbb{R}^d)$ to Ω is dense in $W^{1,\Phi}(\Omega)$.*

Proof. Let Φ, Ω be as in the assumption. Take any $f \in W^{1,\Phi}(\Omega)$. Let $s : \mathbb{R}^d \to [0,1]$ be a smooth compactly supported function such that $s(x) = 1$ for $|x| \leq 1$ and $s(x) = 0$ for $|x| > 2$. For $R > 0$ define $s_R(x) = s\left(\frac{x}{R}\right)$. For any $R > 0$ recall that

$$f_R := f s_R$$

and notice that f_R has compact support. Since Φ satisfies Δ_2, by Corollary 3.1.9,

$$\lim_{R\to\infty} \|f - f_R\|_{W^{1,\Phi}} = 0.$$

Hence it suffices to show that the claim of the theorem holds for compactly supported $f \in W^{1,\Phi}(\Omega)$. We have two cases.

1^0 $\operatorname{ess\,supp} f \subset \Omega$.

2^0 $\operatorname{ess\,supp} f \cap \operatorname{bd}(\Omega) \neq \emptyset$.

In the first case, the assumptions of Corollary 3.3.5 are satisfied. Hence there exists a sequence $\{u_n\}_{n=1}^{\infty} \subset C_C^{\infty}(\Omega) \subset C_C^{\infty}(\mathbb{R}^d)$ such that

$$\lim_{n\to\infty} \|f - u_n\|_{W^{1,\Phi}(\Omega)} = 0,$$

so the claim follows.

In the second case we proceed as follows. Define the set

$$(\operatorname{bd}(\Omega))_f = \operatorname{bd}(\Omega) \cap \operatorname{ess\,supp} f.$$

Clearly $(\operatorname{bd}(\Omega))_f$ is compact. For each $x \in (\operatorname{bd}(\Omega))_f$ let $(V, \{V_{(t_0,t_1)}\}_{(t_0,t_1)\subset(-1,1)}, \mathcal{R})$ be the triple from Theorem 3.4.4. We define now $V_x = V$. By compactness of $(\operatorname{bd}(\Omega))_f$, there exists a natural number K and points $x_k \in (\operatorname{bd}(\Omega))_f$, for $k = 1, \ldots, K$, such that

$$(\operatorname{bd}(\Omega))_f \subset \bigcup_{k=1}^{K} V_{x_k}.$$

For each $k = 1, \ldots, K$ define $V^k = V_{x_k}$, then the set $\operatorname{ess\,supp} f \setminus \bigcup_{k=1}^{K} V^k$ is a compact subset of Ω. Hence, there exists an open set V^0 such that $\overline{V^0}$ is compact and $\overline{V^0} \subset \Omega$ and $\operatorname{ess\,supp} f \setminus \bigcup_{k=1}^{K} V^k \subset V^0$. We conclude that $\operatorname{ess\,supp} f \subset \bigcup_{k=0}^{K} V^k$. Let $\{\psi_k\}_{k=0}^{K}$ be the partition of unity associated with the covering $\{V^k\}_{k=0}^{K}$ of $\operatorname{ess\,supp} f$.

For each $k = 0, \ldots, K$ we define

$$f^k = f\psi_k.$$

Clearly $f = \sum_{k=0}^K f^k$ on $\operatorname{ess\,supp} f$. Notice that
$f - \sum_{k=0}^K f_k = 0$ a.e. on $\Omega \setminus \operatorname{ess\,supp} f$. Hence

$$f = \sum_{k=0}^{K} f^k \text{ a.e. on } \Omega.$$

Also, for each $k = 0, \ldots, K$ the function f^k is an element of $W^{1,\Phi}(\Omega)$. Indeed, for $i = 1, \ldots, d$ and for a.e. $x \in \Omega$,

$$|\partial_i f^k(x)| = |\partial_i f(x)\psi_k(x) + f(x)\partial_i\psi_k(x)| \leq |\partial_i f(x)| + C_{i,k}|f(x)|,$$

where $C_{i,k} = \sup_{x\in\mathbb{R}^d} |\partial_i\psi_k(x)| < \infty$. Therefore, for each $k = 0, \ldots, K$ and $i = 1, \ldots, d$ we have

$$\begin{aligned} I_\Phi(\partial_i f^k) &= \int_\Omega \Phi(x, |\partial_i f_k(x)|)dx \leq \int_\Omega \Phi(x, |\partial_i f(x)| + C_{i,k}|f(x)|)dx \\ &\leq \frac{1}{2}\int \Phi(x, 2|\partial_i f(x)|)dx + \frac{1}{2}\int_\Omega \Phi(x, 2C_{i,k}|f(x)|)dx < \infty, \end{aligned}$$

by Δ_2 condition.

Since $\operatorname{ess\,supp} f^0 \subset \overline{V^0} \subset \Omega$, f^0 is a compactly supported element of $W^{1,\Phi}(\Omega)$, so it satisfies the assumption of case 1^0 and so there exists a sequence $\{u_{n,0}\}_{n=1}^\infty \subset C_C^\infty(\mathbb{R}^d)$ such that

$$\lim_{n\to\infty} \|f^0 - u_{n,0}\|_{W^{1,\Phi}(\Omega)} = 0. \tag{3.69}$$

Fix $k \in \{1, \ldots, K\}$, let $\{V^k_{(t_0,t_1)}\}_{(t_0,t_1)\subset(-1,1)}$ be the family of open sets from Theorem

3.4.4 and $\mathcal{R}_k$ be the map from the same theorem, both associated with the set V^k. Define

$$\Phi_k : V^k \times [0,\infty) \to [0,\infty)$$

by the formula

$$\Phi_k(x,t) = \begin{cases} \Phi(x,t) & x \in V^k_{(0,1)} \\ 0 & x \in \mathrm{bd}(\Omega) \cap V^k \\ \Phi(\mathcal{R}_k(x),t) & x \in V^k_{(-1,0)}. \end{cases}$$

By Lemma 3.4.7 each function Φ_k is a MO function on V^k and satisfies (A1) and Δ_2 property. We proceed by extending f^k to V^k. Define

$$\widetilde{f^k}(x) := \begin{cases} f^k(x) & x \in V^k_{(0,1)} \\ 0 & x \in \mathrm{bd}(\Omega) \cap V^k \\ f^k(\mathcal{R}_k(x)) & x \in V^k_{(-1,0)}. \end{cases}$$

By Theorem 3.4.9 we conclude that $\widetilde{f^k}$ is compactly supported and $\widetilde{f^k} \in W^{1,\Phi_k}(V^k)$. By Collorary 3.3.5, there exists a sequence $\{u_{k,n}\}_{n=1}^{\infty} \subset C_C^{\infty}(V^k)$ such that

$$\lim_{n\to\infty} \|\widetilde{f^k} - u_{n,k}\|_{W^{1,\Phi_k}(V^k)} = 0. \tag{3.70}$$

Moreover, notice that if we extend $u_{n,k}$ to be 0 in $\mathbb{R}^d \setminus \operatorname{ess\,supp} u_{n,k}$, this extension is of the class $C_C^{\infty}(\mathbb{R}^d)$. Then for any $\lambda > 0$ and multi-index α with $|\alpha| \leq 1$ and

$k = 1, \ldots, K$ we have

$$
\begin{aligned}
&I_\Phi(\lambda \partial^\alpha (f^k - u_{n,k|\Omega})) = \int_\Omega \Phi(x, \lambda |\partial^\alpha f^k(x) - \partial^\alpha u_{n,k}(x)|) dx \\
&= \int_{V^k_{(0,1)}} \Phi(x, \lambda |\partial^\alpha f^k(x) - \partial^\alpha u_{n,k}(x)|) dx \\
&\leq \int_{V^k_{(0,1)}} \Phi(x, \lambda |\partial^\alpha f^k(x) - \partial^\alpha u_{n,k}(x)|) dx + \int_{V^k_{(-1,0)}} \Phi(\mathcal{R}(x), \lambda |\partial^\alpha \widetilde{f^k}(x) - \partial^\alpha u_{n,k}(x)|) dx \\
&= \int_{V^k} \Phi_k(x, \lambda |\partial^\alpha \widetilde{f^k}(x) - \partial^\alpha u_{n,k}(x)|) dx = I_{\Phi_k}(\lambda \partial^\alpha (\widetilde{f^k} - u_{n,k})).
\end{aligned}
$$

Hence we have for $k = 1, \ldots, K$,

$$
\|f^k - u_{n,k|\Omega}\|_{W^{1,\Phi}(\Omega)} \leq \|\widetilde{f^k} - u_{n,k}\|_{W^{1,\Phi_k}(V^k)}. \tag{3.71}
$$

Define now $u_n = \sum_{k=0}^{K} u_{n,k}$, $u_n \in C_C^\infty(\mathbb{R}^d)$ and by (3.69), (3.70), (3.71) we have

$$
\|f - u_{n|\Omega}\|_{W^{1,\Phi}(\Omega)} \leq \sum_{k=0}^{K} \|f^k - u_{n,k|\Omega}\|_{W^{1,\Phi}(\Omega)} \leq \sum_{k=0}^{K} \|\widetilde{f^k} - u_{n,k}\|_{W^{1,\Phi_k}(V^k)}
$$

and since the last expression goes to 0 as $n \to \infty$, we conclude that $\{u_n\}_{n=1}^\infty$ is the desired sequence. □

CHAPTER 4

UNIFORM CONVEXITY IN MUSIELAK ORLICZ SOBOLEV SPACES

4.1 Introduction

The main goal of this chapter is to study some geometric properties of Sobolev spaces $W^{1,\Phi}$ induced by Musielak Orlicz spaces L^{Φ}, where Φ is a Musielak Orlicz function. We will present criteria on existence of subspaces in $W^{1,\Phi}$ isomorphic to ℓ^{∞} and ℓ^{1}. The main result is a characterization of uniform convexity of $W^{1,\Phi}$ in terms of the properties of Φ. We will also provide conditions for the space to be reflexive, superreflexive and B-convex. In particular we obtain characterizations of those properties in the variable exponent Sobolev space $W^{1,p(\cdot)}$.

The concept of uniform convexity of a Banach space X was first introduced by James A. Clarkson in 1936 where the author showed the uniform convexity of ℓ^p and L^P spaces for $1 < p < \infty$. Other authors had studied this property in many other instances of Banach spaces (see for example [38][33]).

Let us denote $\mathbb{R}$ to be the set of real numbers, $\mathbb{N}$ the set of natural numbers and $\mathbb{R}_+ = [0, \infty)$. Let (Ω, Σ, μ) be a measurable complete space, where Σ is a σ-algebra of subsets of Ω and μ is a σ-finite, complete measure on Σ. A function $\Phi : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ is called a *Musielak Orlicz function* (*MO function* for short) if it satisfies the following conditions

(i) For a.a. $x \in \Omega$, $t \to \Phi(x, t)$ is convex for $t \in \mathbb{R}_+$ and $\Phi(x, 0) = 0$.

(ii) For all $t \in \mathbb{R}^+$, the function $x \to \Phi(x, t)$ is μ measurable.

By $L^0 = L^0(\Omega)$ define all μ-measurable complex valued functions on Ω. Given Musielak-Orlicz function Φ, the *Musielak-Orlicz space* (*MO-spaces*), called also *generalized Orlicz space* L^{Φ}, consists of all functions $f \in L^0$ such that for some

$\lambda > 0$,

$$I_\Phi(\lambda f) = \int_\Omega \Phi(x, \lambda|f(x|)\, d\mu(x) < \infty.$$

By Φ^* denote the *complementary function* to Φ, that is

$$\Phi^*(x,t) = \sup_{s\geq 0}\{st - \Phi(x,s)\}, \quad a.a.\ x \in \Omega,\ t \geq 0.$$

It follows the *Young inequality*,

$$st \leq \Phi(x,s) + \Phi^*(x,t), \qquad a.a.\ x \in \Omega \quad s,t \geq 0.$$

It is well known and standard to check that $\Phi^* : \Omega \times \mathbb{R}_+ \to [0,\infty]$ and satisfies conditions (i), (ii) above. The function Φ^* is a Musielak-Orlicz function with extended values in $[0,\infty]$. We also have $\Phi^{**} = \Phi$. Recall that for all $t \geq 0$, a.a. $x \in \Omega$,

$$t \leq (\Phi^*)^{-1}(x,t)\Phi^{-1}(x,t) \leq 2t. \tag{4.1}$$

For a function $\Phi : \Omega \times \mathbb{R}_+ \to [0,\infty]$ we define the *generalized inverse* of Φ by the formula

$$\Phi^{-1}(x,t) = \inf\{u : \Phi(x,u) \geq t\}.$$

Note that, if Φ is a MO function then

(1) $\Phi(x, \Phi^{-1}(, x, t)) = \Phi^{-1}(x, \Phi(x,t)) = t$, provided $0 < \Phi(x,t) < \infty$.

(2) $\Phi(x, \Phi^{-1}(x,t)) = \Phi^{-1}(x, \Phi(x,t)) = 0$, for $t \geq 0$ such that $\Phi(x,t) = 0$.

(3) $\Phi(x, \Phi^{-1}(x,t)) = b_{\Phi(x,\cdot)}$ for $t \geq 0$ such that $\Phi(x,t) = \infty$,
where $b_{\Phi(x,\cdot)} = \inf\{t \geq 0 : \Phi(x,t) = \infty\}$.

Moreover, recall that Φ all $t \geq 0$, a.a. $x \in \Omega$,

$$t \leq (\Phi^*)^{-1}(x,t)\Phi^{-1}(x,t) \leq 2t. \tag{4.2}$$

Indeed, the second inequality follows from Young inequality. Take any $x \in \Omega$ and $t \geq 0$ and let $u = \Phi^{-1}(x,t)$, $v = (\Phi^*)^{-1}(x,t)$, we have

$$(\Phi^*)^{-1}(x,t)\Phi^{-1}(x,t) = uv \leq \Phi(x,u) + \Phi^*(x,v) \leq 2t.$$

For the first inequality, recall that

$$\Phi(x,t) = \int_0^t P(x,s)ds, \quad \Phi^*(x,t) = \int_0^t P^{-1}(x,s)ds$$

Where $P(x,\cdot)$ is the right derivative of $\Phi(x,\cdot)$. For any $x \in \Omega$ and $u > 0$ such that $\Phi(x,u) = \int_0^u P(x,s)ds \geq t$ we have

$$\frac{t}{u} \leq \frac{\Phi(x,u)}{u} = \frac{1}{u}\int_0^u P(x,s)ds \leq \frac{P(x,u)u}{u} = P(x,u).$$

On the other hand

$$\Phi^*\left(x, \frac{t}{u}\right) = \int_0^{\frac{t}{u}} P^{-1}(x,s)ds \leq P^{-1}\left(x, \frac{t}{u}\right)\frac{t}{u} \leq P^{-1}(x, P(x,u))\frac{t}{u} \leq t.$$

Therefore we have

$$u(\Phi^*)^{-1}(x,t) \geq t,$$

taking the infimum of both sides over $u > 0$ such that $\Phi(x,u) = \int_0^u P(x,s)ds \geq t$ we arrive at

$$\Phi^{-1}(x,t)(\Phi^*)^{-1}(x,t) = \inf\{u : \Phi(x,u) \geq t\}(\Phi^*)^{-1}(x,t) \geq t.$$

Two standards norms on L^Φ are considered. The *Luxemburg norm*

$$\|f\|_\Phi = \inf\{\lambda > 0 : I_\Phi(f/\lambda) \le 1\},$$

and the *Orlicz norm*

$$\|f\|_\Phi^0 = \sup_{I_{\Phi^*}(g)\le 1} \int_\Omega f(x)g(x)d\mu(x) = \sup_{I_{\Phi^*}(g)\le 1} \int_\Omega fg\, d\mu.$$

The norms are equivalent, and in fact $\|f\|_\Phi \le \|f\|_\Phi^0 \le 2\|f\|_\Phi$.

In view of the simple observation that $I_\Phi(f) \le 1$ if and only if $\|f\|_\Phi \le 1$, the following Hölder inequalities are satisfied for any $f \in L^\Phi$, $g \in L^{\Phi^*}$,

$$\left|\int_\Omega f\, g\, d\mu\right| \le \|f\|_\Phi^0\, \|g\|_{\Phi^*}, \qquad \left|\int_\Omega f\, g\, d\mu\right| \le \|f\|_\Phi\, \|g\|_{\Phi^*}^0.$$

Recall that $(X, \|\cdot\|_X)$ is a Banach function space if $X \subset L^0$, and if $f \in L^0$, $g \in X$ and $|f| \le |g|$ then $f \in X$ and $\|f\|_X \le \|g\|_X$. We say that a Banach function space $(X, \|\cdot\|_X)$ has the *Fatou property*, if for any $0 \le f_n \uparrow f$ a.e., $f_n \in X$, $f \in L^0$ and $\sup_n \|f_n\|_X < \infty$ then $f \in X$ and $\|f_n\|_X \uparrow \|f\|_X$ as $n \to \infty$. An element $f \in X$ is called order continuous whenever for any $0 \le f_n \le f$ with $f_n \downarrow 0$ a.e., we have $\|f_n\|_X \downarrow 0$. Letting X_a be the set of all order continuous elements from X, and X_b be the closure of all simple functions from X, we have $X_a \subset X_b$. Let X^* be the dual space to X. The Köthe dual space of X is the following subspace of L^0,

$$X' = \left\{ f \in L^0 : \|g\|_{X'} = \sup\left\{ \int_\Omega fg\, d\mu \ :\ \|f\|_X \le 1 \right\} < \infty \right\}.$$

The space X' equipped with the norm $\|\cdot\|_{X'}$ is a Banach function space satisfying the Fatou property. If $X_a = X_b$ and X has the Fatou property then $(X_a)^*$ is isometrically isomorphic to X'. In this case $X^* \simeq X' \oplus X_s^*$, where the symbol $\simeq$

denotes linear isometry, and $X_s^* = X_a^\perp$ is the set of all singular functionals that is the set of $S \in X^*$ such that $S(f) = 0$ for every $f \in X_a$.

Recall some results in MO-spaces which we will need later.

The space L^Φ with either norm is a Banach function lattice satisfying the Fatou property. We say that MO function Φ satisfies *condition* Δ_2 ($\Phi \in \Delta_2$) if there exist $K > 0$ and a non-negative function $h \in L^1$ such that

$$\Phi(x, 2t) \leq K\Phi(x, t) + h(x), \quad a.a.\ x \in \Omega, \quad t \geq 0.$$

The growth condition Δ_2 of Φ plays an important role in the theory of MO-spaces as we will see further.

Theorem 4.1.1. *[49, Theorem 7.6,Theorem 8.14][5, Proposition 3.10, Theorem 3.13]*

Let Φ be a MO function. The following properties are equivalent.

(i) *The space L^Φ is order continuous that is $L^\Phi = (L^\Phi)_a$.*

(ii) *The modular convergence $I_\Phi(u) \to 0$ is equivalent to norm convergence $\|u\|_\Phi \to 0$.*

(iii) *Φ satisfies Δ_2.*

Theorem 4.1.2. *[49, Theorem 7.10] The MO-space $L^\Phi(\Omega)$ is separable if and only if the measure μ is separable and Φ satisfies Δ_2.*

Lemma 4.1.3. *[41, Lemma 4.7] [39, page 64] For any MO-function Φ there exists an increasing sequence of measurable sets $\Omega_i \subset \Omega$ such that $\mu(\Omega_i) < \infty$ for all $i \in \mathbb{N}$, $\mu(\Omega \setminus \cup_{i=1}^\infty \Omega_i) = 0$, and $\sup_{x \in \Omega_i} \Phi(x, t) < \infty$ for all $t \geq 0$ and $i \in \mathbb{N}$. Consequently $(L^\Phi)_a = (L^\Phi)_b$.*

Theorem 4.1.4. (i) *[41, Theorem A4] [49] For any MO function,* $(L^\Phi, \|\cdot\|_\Phi)' = (L^{\Phi^*}, \|\cdot\|^0_{\Phi^*})$ *and* $(L^\Phi, \|\cdot\|^0_\Phi)' = (L^{\Phi^*}, \|\cdot\|_{\Phi^*})$.
(ii) *[53]*

$$(L^\Phi, \|\cdot\|_\Phi)^* \simeq (L^{\Phi^*}, \|\cdot\|^0_{\Phi^*}) \oplus (L^\Phi)_s.$$

(iii) *[49] The MO-space* L^Φ *is reflexive if and only if both* Φ *and* Φ^* *satisfy condition* Δ_2.

Given MO-functions Φ_i, $i = 1, 2$, the symbol $\Phi_2 \prec \Phi_1$ denotes that there exist a constant $K > 0$ and a non-negative function $h \in L^1$ such that

$$\Phi_2(x, Kt) \leq \Phi_1(x, t) + h(t), \qquad a.a.\ x \in \Omega,\ t \geq 0.$$

We say that Φ_1 and Φ_2 are *equivalent* if $\Phi_2 \prec \Phi_1$ and $\Phi_1 \prec \Phi_2$. It is standard to show that equivalence of two MO-functions preserves condition Δ_2.

For a proof of the next theorem in general setting of Banach function spaces see [5, Proposition 2.10, p. 13]. In the book [5], the authors made additional assumptions on the function norm, among others the Fatou property. Since MO-spaces poses this property, and the other assumptions could be avoided, the general results on function spaces in this book apply also to MO-spaces.

Theorem 4.1.5. *[49] Given MO-functions* Φ_i, $i = 1, 2$, $L^{\Phi_1} \subset L^{\Phi_2}$ *if and only if* $\Phi_2 \prec \Phi_1$. *The embedding of the spaces* $L^{\Phi_1} \subset L^{\Phi_2}$ *is automatically bounded. Consequently* $L^{\Phi_1} = L^{\Phi_2}$ *as sets with equivalent norms if and only if* Φ_1 *is equivalent to* Φ_2.

The important class of Musielak-Orlicz spaces are *Nakano spaces*, recently also called *variable exponent Lebesgue spaces*. They are defined in two equivalent versions. Let $1 \le p(x) < \infty$ a.e. be a measurable function and let

$$\Phi(x,t) = \frac{t^{p(x)}}{p(x)} \quad \text{or} \quad \widetilde{\Phi}(x,t) = t^{p(x)}, \quad x \in \Omega,\ t \ge 0.$$

Then $L^{p(\cdot)} = L^{\Phi}$ and $\widetilde{L}^{p(\cdot)} = L^{\widetilde{\Phi}}$ are variable exponent Lebesgue spaces. For $1 < p(x) < \infty$ a.e. in Ω, $t \ge 0$,

$$\Phi^*(x,t) = \frac{t^{q(x)}}{q(x)} \quad \text{and} \quad \widetilde{\Phi}^*(x,t) = p(x)^{-\frac{q(x)}{p(x)}} \frac{t^{q(x)}}{q(x)},$$

where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. We have $L^{p(\cdot)} = \widetilde{L}^{p(\cdot)}$ as sets with equivalent norms. Let further

$$p^+ = \operatorname*{ess\,sup}_{x\in\Omega} p(x) \quad \text{and} \quad p^- = \operatorname*{ess\,inf}_{x\in\Omega} p(x).$$

In order to avoid confusion, in the sequel by variable exponent MO-function we will always mean $\Phi(x,t) = \frac{t^{p(x)}}{p(x)}$, a.e. $x \in \Omega$, $t \ge 0$, and the variable exponent Lebesgue space, the space $L^{p(\cdot)}$ equipped with the Luxemburg norm.

Theorem 4.1.6. *The variable exponent MO-function $\Phi(x,t) = t^{p(x)}/p(x)$, $1 \le p(x) < \infty$, for a.a. $x \in \Omega$, $t \ge 0$, satisfies Δ_2 if and only if $p^+ < \infty$. Its complement function Φ^* satisfies Δ_2 if and only if $p^- > 1$.*

Proof. Assume that $\Phi(x,t) = t^{p(x)}/p(x)$ satisfies Δ_2. There exists $C \ge 2$ and an positive, integrable function h such that for a.a. $x \in \Omega$ and $t \ge 0$ we have

$$2^{p(x)} \frac{t^{p(x)}}{p(x)} = \Phi(x,2t) \le C\Phi(x,t) + h(x) = C\frac{t^{p(x)}}{p(x)} = h(x).$$

Hence,

$$(2^{p(x)} - C)\frac{t^{p(x)}}{p(x)} \le h(x).$$

If $p(x) = 1$ for a.e. $x \in \Omega$, then clearly Φ satisfies Δ_2 condition. Now if $p(x) \ne 1$ a.e., then for a.e $x \in \Omega$ it follows that $\sup\limits_{t>0} \frac{t^{p(x)}}{p(x)} = \infty$ for a.e. $x \in \Omega$. Therefore we conclude that

$$2^{p(x)} \leq C.$$

Now for a.a. $x \in \Omega$ we have $p(x) \leq \log_2 C$ and so $p^+ \leq \log_2 C < \infty$. If on the other hand $p^+ < \infty$, then for a.a $x \in \Omega$ and any $t \geq 0$ we have

$$\Phi(x, 2t) = 2^{p(x)} \frac{t^{p(x)}}{p(x)} \leq 2^{p^+} \frac{t^{p(x)}}{p(x)} = 2^{p+} \Phi(x, t),$$

i.e. Φ satisfies Δ_2.

As for the second assertion, notice that $\Phi^*(x, t) = t^{q(x)}/q(x)$, where $q(x) = p(x)/(p(x) - 1)$. By the first part of the proof $\Phi^*(x, t)$ satisfies Δ_2 if and only if $q^+ < \infty$. Notice that, by the fact that $u/(u - 1)$ is an decreasing function on $(0, \infty)$ and $\lim_{u \to 1^+} u/(u - 1) = \infty$, for a.a. $x \in \Omega$ we have

$$q(x) = \frac{p(x)}{p(x) - 1} \leq \frac{p^-}{p^- - 1}.$$

Therefore we conclude that $q^+ < \infty$ if and only if $p^- > 1$. □

Let $\Omega = (\alpha, \beta)$ with $-\infty < \alpha \leq \beta < \infty$ with the Lebesgue measure μ. By $L^1_{loc} = L^1_{loc}(\Omega)$ denote the set of all *locally integrable* functions on Ω, that is all functions $f \in L^0$ such that $\int_K |f| \, d\mu < \infty$ for every compact set $K \subset \Omega$. A function $f \in L^1_{loc}$ is *weakly differentable* if there exists a function $f' \in L^1_{loc}$ such that for every $u \in C^\infty_c(\Omega)$ we have

$$\int_\alpha^\beta f(x)u'(x)dx = -\int_\alpha^\beta f'(x)u(x)dx.$$

The Sobolev space $W^{1,\Phi} = W^{1,\Phi}(\Omega)$, consists of all $f \in L^1_{loc}(\Omega)$ such that their weak derivative f' exists and

$$\|f\|_{1,\Phi} = \|f\|_\Phi + \|f'\|_\Phi < \infty. \tag{4.3}$$

If $f : \Omega \to \mathbb{R}$ is absolutely continuous on every compact subinterval of Ω, then f is weakly differentiable and f' coincides with the classical derivative of f a.e. [36, Theorem 7.16].

In the case of variable exponent MO-function, the variable exponent Sobolev space is denoted by $W^{1,p(\cdot)}$. The space $W^{1,\Phi}$ equipped with the norm $\|\cdot\|_{1,\Phi}$ is a complete space [26, Theorem 6. 1.4], in the case of $W^{1,p(\cdot)}$ see also ([12, Theorem 6.6], [17, Theorem 8.1.6]).

Anytime further we use Sobolev spaces $W^{1,\Phi}$, they are defined on the finite interval (α,β) equipped with the Lebesgue measure.

4.2 Integral Operators in L^{Φ}

Let (Ω,Σ,μ) be a σ-finite measure space. For MO functions Φ_1,Φ_2 define the function

$$\phi((x,y),t)=\Phi_2(x,\Phi_1^*(y,t)),\quad (x,y)\in\Omega\times\Omega,\ t\geq 0. \tag{4.4}$$

Observe first that Φ_1^* can achieve infinite values. Despite that this function posses all other properties of MO-function. Moreover for such functions the Young inequality is also satisfied. We will adopt the convention that if $\Phi_1^*(y,t)=\infty$ then $\phi((x,y),t)=\Phi_2(x,\infty)=\infty$. Therefore $\phi:(\Omega\times\Omega)\times\mathbb{R}_+\to[0,\infty]$ is a Musielak-Orlicz function on $\Omega\times\Omega$ with extended values. Denote by L^{ϕ} the Musielak-Orlicz space as a subspace of $L^0(\Omega\times\Omega)$. By $\|\cdot\|_{\phi}$ and $\|\cdot\|_{\phi}^0$ we mean the Luxemburg and Orlicz norm on L^{ϕ}, respectively.

Lemma 4.2.1. *Let $\Phi_i, i=1,2,$ be MO-functions on Ω, where $\mu(\Omega)<\infty$. Assume $\int_\Omega\Phi_2(x,b)d\mu(x)<\infty$ for some $b>0$. Let $\psi:(\Omega\times\Omega)\times\mathbb{R}_+\to[0,\infty]$ be such that*

$$\psi((x,y),t)=\Phi_2(x,\Phi_1^*(y,t)),$$

and let $\phi = \psi^*$. *Then there exists* $l > 0$ *such that whenever* $u \in L^{\Phi_1}$, $v \in L^{\Phi_2}$ *and*

$$w(x,y) = u(y)v(x), \quad x, y \in \Omega,$$

then

$$\|w\|_\phi^0 \le l\, \|u\|_{\Phi_1} \|y\|_{\Phi_2^*}.$$

Proof. By Young's inequality applied to Φ_1 and Φ_1^* and the measure $\frac{|v(x)|}{\|v\|_{\Phi_2^*}} d\mu(x)$,

$$\begin{aligned}
&\left| \int_\Omega \int_\Omega bw(x,y)g(x,y)\, d\mu(x) d\mu(y) \right| \\
&\le \|u\|_{\Phi_1} \|v\|_{\Phi_2^*}\, b \int_\Omega \int_\Omega |g(x,y)| \frac{|u(y)||v(x)|}{\|u\|_{\Phi_1} \|v\|_{\Phi_2^*}}\, d\mu(x) d\mu(y) \\
&\le \|u\|_{\Phi_1} \|v\|_{\Phi_2^*}\, b \int_\Omega \left[\int_\Omega \Phi_1^*(y, |g(x,y)|) \frac{|v(x)|}{\|v\|_{\Phi_2^*}}\, d\mu(x) \right] d\mu(y) \\
&+ \|u\|_{\Phi_1} \|v\|_{\Phi_2} \int_\Omega \int_\Omega \left[\Phi_1 \left(y, \frac{|u(y)|}{\|u\|_{\Phi_1}} \right) \frac{|v(x)|}{\|v\|_{\Phi_2^*}}\, d\mu(x) \right] d\mu(y) = \|u\|_{\Phi_1} \|v\|_{\Phi_2^*} (A + B).
\end{aligned}$$

Applying now to term A, Young's inequality to Φ_2 and Φ_2^* and the obvious fact $I_{\Phi_2^*}(v/\|v\|_{\Phi_2^*}) \le 1$, we get

$$\begin{aligned}
A &= b \int_\Omega \left[\int_\Omega \Phi_1^*(y, |g(x,y)|) \frac{|v(x)|}{\|v\|_{\Phi_2^*}}\, d\mu(x) \right] d\mu(y) \\
&\le b \int_\Omega \left[\int_\Omega \Phi_2(x, \Phi_1^*(y, |g(x,y)|))\, d\mu(x) + \int_\Omega \Phi_2^* \left(x, \frac{|v(x)|}{\|v\|_{\Phi_2^*}} \right) d\mu(x) \right] d\mu(y) \\
&\le b \int_\Omega \int_\Omega \psi((x,y), |g(x,y)|)\, d\mu(x) d\mu(y) + b\mu(\Omega).
\end{aligned}$$

By Young's inequality applied to B twice, Fubini's theorem and $I_{\Phi_1}(u/\|u\|_{\Phi_1}) \le 1$,

$$\begin{aligned}
B &= \int_\Omega \left[\int_\Omega \Phi_1 \left(y, \frac{|u(y)|}{\|u\|_{\Phi_1}} \right) d\mu(y) \right] b \frac{|v(x)|}{\|v\|_{\Phi_2^*}}\, d\mu(x) \le \int_\Omega b \frac{|v(x)|}{\|v\|_{\Phi_2^*}}\, d\mu(x) \\
&\le \int_\Omega \Phi_2^* \left(x, \frac{|v(x)|}{\|v\|_{\Phi_2^*}} \right) d\mu(x) + \int_\Omega \Phi_2(x,b)\, d\mu(x) \le 1 + \int_\Omega \Phi_2(x,b)\, d\mu(x).
\end{aligned}$$

By the above,

$$\left|\int_\Omega \int_\Omega bw(x,y)g(x,y)\,d\mu(x)d\mu(y)\right|\,d\mu(x)d\mu(y) \le \|u\|_{\Phi_1}\|v\|_{\Phi_2^*}(A+B)$$
$$\le \|u\|_{\Phi_1}\|v\|_{\Phi_2^*} b\int_\Omega \int_\Omega \psi((x,y),|g(x,y)|)\,d\mu(x)d\mu(y)$$
$$+\|u\|_{\Phi_1}\|v\|_{\Phi_2^*}\left(b\mu(\Omega)+1+\int_\Omega \Phi_2(x,b)\,d\mu(x)\right)$$
$$=\|u\|_{\Phi_1}\|v\|_{\Phi_2^*}\left(bI_\psi(g)+b\mu(\Omega)+1+\int_\Omega \Phi_2(x,b)\,d\mu(x)\right).$$

Finally,

$$\|w\|_\phi^0 = \sup_{I_\psi(g)\le 1}\left|\int_\Omega \int_\Omega w(x,y)g(x,y)\,d\mu(x)d\mu(y)\right| \le l\,\|u\|_{\Phi_1}\|v\|_{\Phi_2^*},$$

where $l = 1+\mu(\Omega)+\frac{1}{b}\left(1+\int_\Omega \Phi_2(x,b)\,d\mu(x)\right) < \infty$ by assumption.

□

Theorem 4.2.2. *Let Φ_i, $i=1,2$, be MO-functions on Ω. Assume $\phi : (\Omega\times\Omega)\times\mathbb{R}_+ \to \mathbb{R}_+$ is a MO function and there exists $l>0$ such that if $u\in L^{\Phi_1}$, $v\in L^{\Phi_2}$ and $w(x,y)=u(y)v(x)\in L^\phi$ then*

$$\|w\|_\phi^0 \le l\|u\|_{\Phi_1}\|v\|_{\Phi_2^*}.$$

Let A be an integral operator for $u\in L^0$,

$$Au(x) = \int_\Omega k(x,y)u(y)\,d\mu(y), \quad x\in\Omega,$$

where the kernel $k(x,y)\in L^{\phi^*}$. Then $A : L^{\Phi_1}\to L^{\Phi_2}$ is bounded.

Proof. By Hölder's inequality, for $u \in L^{\Phi_1}$, $v \in L^{\Phi_2^*}$,

$$\begin{aligned}\left|\int_\Omega Au(x)v(x)\,d\mu(x)\right| &= \left|\int_\Omega\int_\Omega k(x,y)u(y)v(x)\,d\mu(x)d\mu(y)\right|\\ &= \left|\int_\Omega\int_\Omega k(x,y)w(x,y)\,d\mu(x)d\mu(y)\right|\\ &\le \|k\|_{\phi^*}\|w\|_\phi^0 \le l\|k\|_{\phi^*}\|u\|_{\Phi_1}\|v\|_{\Phi_2^*}.\end{aligned}$$

Recall that $I_{\Phi_2^*}(v) \le 1$ if and only if $\|v\|_{\Phi_2^*} \le 1$. Consequently,

$$\|Au\|_{\Phi_2}^0 = \sup_{I_{\Phi_2^*}(v)\le 1}\left|\int_\Omega Au(x)v(x)\,d\mu(x)\right| \le l\|k\|_{\phi^*}\|u\|_{\Phi_1},$$

and A is bounded.

□

Corollary 4.2.3. *Let Φ_i, $i = 1, 2$, be MO-functions on Ω and $\mu(\Omega) < \infty$. Assume there exists $b > 0$ such that $\int_\Omega \Phi_2(x,b)\,d\mu(x) < \infty$. Let $\psi : (\Omega\times\Omega)\times\mathbb{R}_+ \to \mathbb{R}_+$ be such that*

$$\psi((x,y),t) = \Phi_2(x, \Phi_1^*(y,t)).$$

If $k(x,y) \in L^\psi$, then the operator $Au(x) = \int_\Omega k(x,y)u(y)\,d\mu(y)$ is bounded from L^{Φ_1} to L^{Φ_2}.

Proof. By Lemma 4.2.1, there is $l > 0$ such that for any $u \in L^{\Phi_1}$, $v \in L^{\Phi_2^*}$, and $w(x,y) = u(y)v(x)$ we have $\|w\|_\phi^0 \le l\|u\|_{\Phi_1}\|v\|_{\Phi_2^*}$, where $\phi = \psi^*$. Consequently, the assumptions of Theorem 4.2.2 are satisfied and A is bounded. □

The next result is immediate consequence of Corollary 4.2.3.

Corollary 4.2.4. *Let Φ be a MO function on Ω with $\mu(\Omega) < \infty$ and such that $\int_\Omega \Phi(x,b)\,d\mu(x) < \infty$ for some $b > 0$. Let*

$$\psi((x,y),t) = \Phi(x, \Phi^*(y,t)), \quad x,y \in \Omega,\ t \ge 0.$$

If $k(x,y) \in L^{\psi}$ *then* $Au(x) = \int_\Omega k(x,y)u(y)\,d\mu(y)$ *is bounded from* L^Φ *to* L^Φ*.*

Lemma 4.2.5. *Given a MO function* Φ *on* Ω*, we have that* $\operatorname{ess\,inf}_{x\in\Omega} \Phi(x,a) > 0$ *for some* $a > 0$ *if and only if there exists* $c > 0$ *with* $\operatorname{ess\,sup}_{x\in\Omega} \Phi^*(x,c) < \infty$*.*

Proof. If $\operatorname{ess\,inf}_{x\in\Omega} \Phi(x,a) > 0$ for some $a > 0$, then $\Phi(x,a) \geq M > 0$ for a.a. $x \in \Omega$ and some M. Hence $a \geq \Phi^{-1}(x,M)$. Thus in view of (4.2), $(\Phi^*)^{-1}(x,M) \geq \frac{M}{\Phi^{-1}(x,M)} \geq \frac{M}{a}$. Therefore $\infty > M \geq \operatorname{ess\,sup}_{x\in\Omega} \Phi^*(x,c)$ with $c = \frac{M}{a}$. In the opposite direction the proof goes in a similar way.

□

Definition 4.2.6. *Let* $\Omega = (\alpha,\beta)$*,* $-\infty < \alpha < \beta < \infty$*, equipped with the Lebesgue measure. We say that* Φ *satisfies condition* (V) *if for some* $a, b > 0$*,*

$$\int_\alpha^\beta \Phi(x,b)\,dx < \infty \quad \textit{and} \quad \operatorname*{ess\,inf}_{x\in\Omega} \Phi(x,a) > 0. \tag{V}$$

Theorem 4.2.7. *Let* $\Omega = (\alpha,\beta)$*,* $-\infty < \alpha < \beta < \infty$*, equipped with the Lebesgue measure. Assume* Φ *is a MO function on* Ω *satisfying the condition* (V)*. Then the Voltera operator*

$$Au(x) = \int_\alpha^x u(y)\,dy,$$

$x \in (\alpha,\beta)$ *is bounded on* L^Φ*.*

Proof. By Lemma 4.2.5 we get $\operatorname{ess\,sup}_{x\in\Omega} \Phi^*(x,c) < \infty$ for some $c > 0$. Now by convexity of Φ^*, for all $n \in \mathbb{N}$, $\operatorname{ess\,sup}_{x\in\Omega} \Phi^*\left(x, \frac{c}{n}\right) \leq \frac{1}{n} \operatorname{ess\,sup}_{x\in\Omega} \Phi^*(x,c)$. Thus for some $c_1 > 0$,

$$\operatorname*{ess\,sup}_{x\in\Omega} \Phi^*(x,c_1) \leq b.$$

Setting $k(x,y) = \chi_{(\alpha,x)}(y)$, $x, y \in (\alpha,\beta)$, we have $Au(x) = \int_\Omega k(x,y)u(y)\,dy$. Let ψ

be as in Corollary 4.2.4. Then

$$I_\psi(dk) = \int_\alpha^\beta \int_\alpha^\beta \psi((x,y), c_1\chi_{(\alpha,x)}(y))\,dxdy = \int_\alpha^\beta \int_\alpha^\beta \Phi(x, \Phi^*(y, c_1\chi_{(\alpha,x)}(y))\,dxdy$$
$$\leq \int_\alpha^\beta \int_\alpha^\beta \Phi(x,b)\,dxdy = (\beta-\alpha)\int_\alpha^\beta \Phi(x,b)\,dxdy < \infty.$$

Hence the kernel $k \in L^\psi$. Finally by Corollary 4.2.4, the Voltera operator is bounded on L^Φ.

□

Corollary 4.2.8. *Let* $1 \leq p(x) < \infty$ *a.e. on* (α, β)*, where* $-\infty < \alpha < \beta < \infty$*. Then the Voltera operator is bounded on* $L^{p(\cdot)}$*.*

Proof. Since for variable exponent MO-function, $\Phi(x,1) = 1/p(x) \leq 1$ and $\Phi^*(x,1) = 1/q(x) \leq 1$ for a.a. $x \in \Omega$, so in view of Lemma 4.2.5, the conditions in Theorem 4.2.7 are satisfied and the conclusion holds. □

Corollary 4.2.9. *Let*

$$\Phi(x,t) = t^{p(x)} + a(x)t^{q(x)},$$

where a, p, q *are measurable functions on* (α, β), $-\infty < \alpha < \beta < \infty$*, such that* $a(x) \geq 0$ *a.e. and* $1 \leq p(x) < \infty$, $1 \leq q(x) < \infty$ *a.e.. If* $a \in L^1$ *then the Voltera operator is bounded on* L^Φ*.*

Proof. We have

$$\int_\alpha^\beta \Phi(x,1)\,dx = (\beta-\alpha) + \int_\alpha^\beta a\,dx < \infty,$$

and clearly $\operatorname{ess\,inf}_{x\in\Omega} \Phi(x,1) \geq 1$. In view of Theorem 4.2.7 we conclude the proof.

□

Now we consider double phase functionals,

$$\Phi_1(x,t) = t^p + a(x)t^q, \quad \Phi_2(x,t) = t^p + a(x)t^q \log(e+t), \quad \Phi_3(x,t) = (t-1)_+^p + a(x)(t-1)_+^q$$

with $1 \leq p < q < \infty$ and $a \geq 0$ a.e. Ω. The functions Φ_1 an Φ_2 are MO functions while Φ_3 is an extended value MO function. Notice that to compute the explicit formulas for the conjugates of Φ_i, for $i = 1, 2, 3$ one would need to find an inverse of $\frac{\partial_t \Phi(x,t)}{\partial t}$. For general values of p, q it is not possible to find this inverse algebraically and so it is not possible to give explicit, closed form formulas for conjugates. However one can find explicit functions that are equivalents to those conjugates. We have the following proposition.

Proposition 4.2.10. *Let $1 \leq p \leq q < \infty$ and $a(x) \geq 0$ for a.a $x \in \Omega$, $\mu(\Omega) < \infty$. For the double phase functionals $\Phi_1(x,t) = t^p + a(x)t^q$, $\Phi_2(x,t) = t^p + a(x)t^q \log(e+t)$, $\Phi_3(x,t) = (t-1)_+^p + a(x)(t-1)_+^q$ the following statements are true.*

(1) *For $i = 1, 2, 3$ the functional Φ_i satisfies the (V) condition, provided that $\int\limits_\Omega a(x) d\mu(x) < \infty$.*

(2) *The functionals Φ_1, Φ_2 satisfy Δ_2 condition.*

(3) *The functional Φ_3 satisfies Δ_2 condition, provided that $\int_\Omega a(x) d\mu(x) < \infty$.*

(4) *If $\int_\Omega a(x) d\mu(x) < \infty$ then functions Φ_1 and Φ_3 are equivalent. Moreover*

$$\Psi_1(x,t) = \left(\min\left(C_p t^{p/(p-1)}, C_q a(x)^{1/(1-q)} t^{q/(q-1)}\right)\right)^{**}$$

is equivalent to Φ_1^* and Φ_3^*, where C_p and C_q are constants depending only on p and q.

(5) *The function*

$$\Psi_2(x,t)=\begin{cases}\Big(\big(a(x)e^{t/a(x)}-et-a(x)\big)\chi_{(1,\infty)}(t)\Big)^{**} & \text{if } q=1\\ \Big(\frac{q-1}{q}\frac{t^{q/(q-1)}}{(qa(x)\log(e+t))^{1/(q-1)}}\chi_{(1,\infty)}(t)\Big)^{**} & \text{if } p=1,\ q>1\\ \Big(\min\Big(C_p t^{p/(p-1)},\ \frac{q-1}{q}\frac{t^{q/(q-1)}}{(qa(x)\log(e+t))^{1/(q-1)}}\Big)\Big)^{**} & \text{if } p>1,\ q>1\end{cases}$$

is equivalent to Φ_2^*, *where* C_p *is the constant from* (4).

Proof. (1) Note that, if $\int_\Omega a(x)d\mu(x)<\infty$, then for any $t\geq 0$, $x\in\Omega$ and $i=1,2,3$,

$$\int_\Omega \Phi_i(x,2)\,d\mu(x)\leq 2^p\mu(\Omega)+2^q\log(e+2)\int_\Omega a(x)d\mu(x)<\infty,$$

On the other hand, for $i=1,2,3$ we have $\Phi_i(x,2)\geq 1$ for a.e. $x\in\Omega$, hence the double phase functionals satisfy the (V) condition provided that $\int_\Omega a(x)d\mu(x)<\infty$.

(2) For functions Φ_1 and Φ_2 the Δ_2 condition is satisfied with $h(x)\equiv 0$ and $C=2^q,\ 2^{q+1}$ respectively. Indeed, for a.e. $x\in\Omega$, $t\geq 0$,

$$\Phi_1(x,2t)=(2t)^p+a(x)(2t)^q\leq 2^q(t^p+a(x)t^q)=2^q\Phi_1(x,t),$$

$$\Phi_2(x,2t)=(2t)^p+a(x)(2t)^q\log(e+2t)\leq 2^pt^p+2^qt^qa(x)\log((e+t)^2)$$

$$\leq 2^{q+1}(t^p+a(x)t^q\log(e+t))=2^{q+1}\Phi_2(x,t).$$

(3) As for the function Φ_3 assume that $\int_\Omega a(x)d\mu(x)<\infty$, then Δ_2 condition is satisfied with $h(x)=3^p+3^qa(x)$ and $C=3^q$. Indeed, for $t\in[0,2]$ and $x\in\Omega$,

$$(2t-1)_+^p+a(x)(2t-1)_+^q\leq 3^p+3^qa(x)$$

and for $t>2$,

$$(2t-1)_+^p+a(x)(2t-1)_+^q\leq(3t-3)_+^p+a(x)(3t-3)_+^q\leq 3^q((t-1)_+^p+a(x)(t-1)_+^q).$$

Hence, for every $t \geq 0$,

$$\begin{aligned}\Phi_3(x, 2t) &= (2t-1)_+^p + a(x)(2t-1)_+^q \leq 3^q((t-1)_+^p + a(x)(t-1)_+^q) + 3^p + 3^q a(x) \\ &= 3^q \Phi_3(x,t) + h(x),\end{aligned}$$

where $h(x) = 3^p + 3^q a(x) \in L^1$.

(4) First we notice that under the assumption $\int_\Omega a(x) d\mu(x) < \infty$ the functions Φ_1 and Φ_3 are equivalent as MO functions. We remark that the notion of equivalent extended MO function is the same as for MO functions. Indeed, for $t \geq 0$ and any $x \in \Omega$,

$$\Phi_3(x,t) = (t-1)_+^p + a(x)(t-1)_+^q \leq t^p + a(x)t^q = \Phi_1(x,t). \tag{4.5}$$

On the other hand, for $t \in [0,2]$ and $x \in \Omega$ we have

$$(t/2)^p + a(x)(t/2)^q \leq 1 + a(x)$$

and for $t \geq 2$,

$$(t/2)^p + a(x)(t/2)^q \leq (t-1)_+^p + a(x)(t-1)_+^q.$$

Hence for any $t \geq 0$ and any $x \in \Omega$ we have

$$(t/2)^p + a(x)(t/2)^q \leq (t-1)_+^p + a(x)(t-1)_+^q + 1 + a(x).$$

Thus $\Phi_1\left(x, \frac{t}{2}\right) \leq \Phi_3(x,t) + h(x)$, where $h(x) = 1 + a(x) \in L^1$. Observe now that for any $t \geq 0$ and $x \in \Omega$ we have

$$\max(t^p, a(x)t^q) \leq t^p + a(x)t^q \leq 2\max(t^p, a(x)t^q).$$

Therefore it suffices to find a MO-function equivalent to conjugate of the function $\max(t^p, a(x)t^q)$. Following [32, Theorem 3] we conclude that

$$
\begin{aligned}
&(\max(t^p, a(x)t^q)^* \leq \min((t^p)^*, (a(x)t^q)^*) = \\
&\min\left((p-1)p^{p/(1-p)}t^{p/(p-1)}, (q-1)q^{q/(1-q)}a(x)^{1/(1-q)}t^{q/(q-1)}\right) \leq \\
&(\max((2t)^p, a(x)(2t)^q)^*.
\end{aligned}
$$

We define the function

$$
\Psi_1(x,t) = \left(\min\left((p-1)p^{p/(1-p)}t^{p/(p-1)}, (q-1)q^{q/(1-q)}a(x)^{1/(1-q)}t^{q/(q-1)}\right)\right)^{**},
$$

It is well known [35, Corollary 1, p.177] that Ψ_1 is the largest convex minorant of

$$
\min\left((p-1)p^{p/(1-p)}t^{p/(p-1)}, (q-1)q^{q/(1-q)}a(x)^{1/(1-q)}t^{q/(q-1)}\right),
$$

and for any $t \geq 0$ and $x \in \Omega$ we have

$$
\Psi_1(x,t) \leq \min\left((p-1)p^{p/(1-p)}t^{p/(p-1)}, (q-1)q^{q/(1-q)}a(x)^{1/(1-q)}t^{q/(q-1)}\right) \leq \Psi_1(x,2t).
$$

Therefore Ψ_1 is equivalent to both Φ_1^* and Φ_3^*, where $C_p = (p-1)p^{p/(1-p)}$ and $C_q = (q-1)q^{q/(1-q)}$.

(5) Now we find an equivalent function to Ψ_2^*. If $q = 1$, then for any $t \geq 0$ and $x \in \Omega$ we have

$$
\begin{aligned}
&\max\left(t, a(x)\int_0^t \log(e+u)du\right) \leq \max(t, a(x)t\log(e+t)) \leq t + a(x)t\log(e+t) \\
&\leq 2\max(t, a(x)t\log(e+t)) \leq 2\max\left(2t, a(x)\int_0^{2t} \log(e+u)du\right).
\end{aligned}
$$

Hence it suffices to find an equivalent to the convex conjugate of

$\max\left(t, a(x)\int_0^t \log(e+u)\right)$. Let $\varphi_1(t) = t$, then $\varphi_1^*(t) = \chi_{(1,\infty)}(t)$, for $t \geq 0$. Moreover by the techniques [44] it not difficult to compute

$$\left(a(x)\int_0^t \log(e+u)du\right)^* = \int_0^t (e^{u/a(x)} - e)du = a(x)e^{t/a(x)} - et - a(x).$$

By [32, Theorem 3] for any $t \geq 0$ and $x \in \Omega$,

$$\left(\max\left(t, a(x)\int_0^t \log(e+u)du\right)\right)^* \leq \min\left(\varphi_1^*(t), \left(a(x)\int_0^t \log(e+u)du\right)^*\right) =$$
$$\min\left(\infty\chi_{(1,\infty)}(t), a(x)e^{t/a(x)} - et - a(x)\right) = (a(x)e^{t/a(x)} - et - a(x))\chi_{(1,\infty)}(t).$$

For any $t \geq 0$ and $x \in \Omega$ we define

$$\Psi_2(x,t) = \left((a(x)e^{t/a(x)} - et - a(x))\chi_{(1,\infty)}(t)\right)^{**},$$

as before by [35, Corollary 1, p.177], Ψ is the largest convex minorant of $(a(x)e^{t/a(x)} - et - a(x))\chi_{(1,\infty)}(t)$ and for $t \geq 0$ and $x \in \Omega$ we have

$$\Psi(x,t) \leq (a(x)e^{t/a(x)} - et - a(x))\chi_{(1,\infty)}(t) \leq \Psi(x, 2t).$$

Hence Ψ is the desired function.

Proof in case $p = 1,\ q > 1$ and $p > 1, q > 1$ is similar. The only thing that we need to do is to find function equivalent to convex conjugate of

$$a(x)t^q \log(e+t).$$

Following [44, Lemma 7.2] and the proofs of [44, Theorem 3.2, 3.4] one can see that the conjugate is equivalent to

$$\frac{q-1}{q}\frac{t^{q/(q-1)}}{(qa(x)\log(e+t))^{1/(q-1)}}.$$

Now we can repeat the first part of the proof and arrive at the desired conclusion. □

Proposition 4.2.11. *Let*

$$\Phi(x,t) = t^{p(x)} + a(x)t^{q(x)},$$

where a, p, q are measurable functions on Ω, $a(x) \geq 0$ a.e. and $1 \leq p(x) \leq q(x) < \infty$ a.e.. Assume also

$$1 \leq p^+ = \operatorname*{ess\,sup}_{x\in\Omega} p(x) < q^- = \operatorname*{ess\,inf}_{x\in\Omega} q(x) < \infty.$$

Then Φ satisfies condition Δ_2 if and only if $p^+ < \infty$ and $q^+|_{\Omega_1} = \operatorname{ess\,sup}_{x\in\Omega_1} q(x) < \infty$, where $\Omega_1 = \operatorname{ess\,supp} a$.

Proof. If $p^+ < \infty$ and $q^+|_{\Omega_1} < \infty$ then

$$\Phi(x,2t) = 2^{p(x)}t^{p(x)} + 2^{q(x)}a(x)t^{q(x)} \leq 2^r(t^{p(x)} + a(x)t^{q(x)}) = 2^r\Phi(x,t),$$

where $r = \max(p^+, q^+|_{\Omega_1})$, and so Φ satisfies condition Δ_2.

Now assume Φ satisfies Δ_2 and $p^+ = \infty$ or $q^+|_{\Omega_1} = \infty$. Then by definition of Δ_2, there exists $c > 0$ such hat the function

$$h_c(x) = \sup_{t\geq 0}(\Phi(x,2t) - c\Phi(x,t))$$

belongs to L^1. We shall consider two cases.

Case 1^0. Let $p^+ = q^+|_{\Omega_1} = \infty$. Define

$$A_n = \{x : p(x) > n\}, \quad n \in \mathbb{N}.$$

By assumptions for every $n \in \mathbb{N}$, $|A_n| > 0$, and for a.e. $x \in A_n$,

$$q(x) > p(x) > n.$$

Choose $n \in \mathbb{N}$ such that $2^n > c$. Then for $x \in A_n$, $t \geq 0$,

$$\begin{aligned}\Phi(x, 2t) - c\Phi(x, t) &\geq 2^n t^{p(x)} + a(x)2^n t^{q(x)} - ct^{p(x)} - ca(x)t^{q(x)} \\ &= (2^n - c)(t^{p(x)} + a(x)t^{q(x)}) = (2^n - c)\Phi(x, t).\end{aligned}$$

Hence $h_c(x) = \infty$ for every $x \in A_n$ and $c < 2^n$. Therefore h_c is not integrable over Ω, and by monotonicity of h_c with respect to c, $h_c \notin L^1(\Omega)$ for every $c > 0$. Consequently, Φ does not satisfy Δ_2.

Case 2^0. Let $p^+ < \infty$ and $q^+|_{\Omega_1} = \infty$. We can assume that $c > 2^{p+}$. Thus

$$2^{p(x)} - c < 0 \quad a.e.\ x \in \Omega.$$

Let $x \in \Omega_1$. Then $a(x) > 0$. Hence $p(x) - q(x) < 0$. Therefore there exists $T_x > 2$ such that for all $t > T_x$,

$$t^{p(x)-q(x)} < a(x).$$

For a.e. $y \in \Omega$, we have $\Phi(y, 2t) - c\Phi(y, t) = t^{q(y)}[(2^{p(y)} - c)t^{p(y)-q(y)} + (2^{q(y)} - c)a(y)]$.

Hence for a fixed $t > T_x$,

$$\begin{aligned}\Phi(x,2t) - c\Phi(x,t) &= t^{q(x)}[(2^{p(x)} - c)\, t^{p(x)-q(x)} + (2^{q(x)} - c)\, a(x)] \\ &\geq t^{q(x)}[(2^{p(x)} - c)\, a(x) + (2^{q(x)} - c)\, a(x)] \\ &= t^{q(x)} a(x)(2^{p(x)} + 2^{q(x)} - 2c) > t^{q(x)} a(x)(2^{q(x)} - 2c).\end{aligned}$$

Let for $n \in \mathbb{N}$,

$$B_n = \{x \in \Omega_1 : \ q(x) > n\}.$$

Since $q^+|_{\Omega_1} = \infty$, there is N such that for all $n > N$, $|B_n| > 0$ and $2^n > 2c$. Therefore

$$t^{q(x)} a(x)(2^{q(x)} - 2c) \geq t_x^{q(x)} a(x)(2^n - 2c).$$

Thus for any $x \in B_n$ and $t_x > T_x$,

$$\sup_{t \geq 0} [t^{q(x)} a(x)\, (2^{q(x)} - 2c)] \geq t_x^{q(x)} a(x)\, (2^n - 2c).$$

Since t_x can be taken arbitrary big it follows, a.e. $x \in B_n$

$$\sup_{t \geq 0} [t^{q(x)} a(x)\, (2^{q(x)} - 2c)] = \infty.$$

Hence $h_c(x) = \infty$ for every $x \in B_n$. Therefore, if $c < 2^n$, then h_c is not integrable, but n can be chosen arbitrary large, so for every $c > 0$, $h_c \notin L^1$.

□

4.3 Copy of ℓ^∞

In this part we give conditions on Φ in order to $W^{1,\Phi}$ to contain a subspace isomorphic to ℓ^∞.

Proposition 4.3.1. *[39, 40] Let (Ω, Σ, μ) be non-atomic measure space. Then a*

MO function Φ does not satisfy condition Δ_2 if and only if there exists a sequence of bounded and non-negative functions $f_n \in L^\Phi$ such that $f_n \wedge f_m = 0$ for $n \neq m$, $I_\Phi(f_n) \leq 1/2^n$ and $\|f_n\|_\Phi = 1$ for all $n \in \mathbb{N}$. Consequently,

$$\left\| \sum_{n=1}^{\infty} f_n \right\|_\Phi = \|f_n\|_\Phi = 1$$

for all $n \in \mathbb{N}$.

The next theorem results directly from Proposition 4.3.1.

Theorem 4.3.2. *Let the measure space (Ω, Σ, μ) be separable and non-atomic. A Musielak Orlicz space L^Φ contains an isomorphic copy of ℓ^∞ if and only if Φ does not satisfy condition Δ_2.*

Proof. If $\Phi \notin \Delta_2$, then taking any element $a = \{a_n\}_{n=1}^\infty \in \ell^\infty$ and the sequence $\{f_n\}_{n=1}^\infty$ from Proposition 4.3.1, we get for every $n \in \mathbb{N}$,

$$|a_n| = \|a_n f_n\|_\Phi \leq \left\| \sum_{n=1}^{\infty} a_n f_n \right\|_\Phi \leq \|a\|_\infty.$$

It follows

$$\left\| \sum_{n=1}^{\infty} a_n f_n \right\|_\Phi = \|a\|_\infty,$$

and in fact ℓ^∞ is an isometric isomorphic subspace of L^Φ.

Now assume opposite that L^Φ has a subspace isomorphic to ℓ^∞. Then L^Φ can not be separable, and so $\Phi \notin \Delta_2$ by Theorem 4.1.2.

□

Corollary 4.3.3. *Let the measure space (Ω, Σ, μ) be separable and non-atomic. The variable exponent Lebesgue space $L^{p(\cdot)}$ contains an isomorphic subspace to ℓ^∞ if and only if $p^+ = \infty$.*

Proof. By Theorem 4.1.6, the variable exponential function satisfies Δ_2 if and only if $p^+ < \infty$. Therefore the proof is completed by application of Theorem 4.3.2. $\square$

Theorem 4.3.4. *Let Φ be a MO function on (α, β), $\infty < \alpha < \beta < \infty$ with the Lebesgue measure. If $W^{1,\Phi}$ contains a subspace isomorphic to ℓ^∞ then Φ does not satisfy condition Δ_2.*

Proof. Let $A = \{(f, f') : f \in W^{1,\Phi}\}$ be a subspace of the product $L^\Phi \times L^\Phi$. If $L^\Phi \times L^\Phi$ is equipped with the norm $\|(\cdot, \cdot\cdot)\|_{L^\Phi \times L^\Phi} = \|\cdot\|_\Phi + \|\cdot\cdot\|_\Phi$, then the space $W^{1,\Phi}$ is isometrically isomorphic to A. Notice that $L^\Phi \times L^\Phi$ is isomorphic to the MO space $L^{\overline{\Phi}}(\Omega \times \{1,2\})$ where $\overline{\Phi} : \Omega \times \{1,2\} \times [0,\infty) \to [0,\infty)$ is defined as

$$\overline{\Phi}(x, y, t) = \Phi(x,t)\chi_{\{1\}}(y) + \Phi(x,t)\chi_{\{2\}}(y),$$

for $x \in \Omega,\ t \in \{1,2\}$ and $t \geq 0$. Indeed, the operator $T : L^\Phi \times L^\Phi \to L^{\overline{\Phi}}(\Omega \times \{1,2\})$ defined by

$$(T(f_1, f_2))(x,y) = f_1(x)\chi_{\{1\}}(y) + f_2(x)\chi_{\{2\}}(y)$$

for $x \in \Omega$, $y \in \{1,2\}$, is clearly a linear bijection and its inverse is given by

$$(T^{-1}f)(x) = (f(x,1), f(x,2)).$$

Moreover, taking any $(f_1, f_2) \neq (0,0)$ where $(f_1, f_2) \in L^\Phi \times L^\Phi$,

$$\begin{aligned} &I_{\overline{\Phi}}\left(\frac{T(f_1,f_2)}{2\|(f_1,f_2)\|_{L^\Phi \times L^\Phi}}\right) \\ &= \int\limits_\Omega \Phi\left(x, \frac{|f_1(x)|}{2(\|f_1\|_\Phi + \|f_2\|_\Phi)}\right) dx + \int\limits_\Omega \Phi\left(x, \frac{|f_2(x)|}{2(\|f_1\|_\Phi + \|f_2\|_\Phi)}\right) dx \leq 1. \end{aligned}$$

Therefore, for every $(f_1, f_2) \in L^\Phi \times L^\Phi$ we have

$$\|T(f_1, f_2)\|_{L^{\overline{\Phi}}} \leq 2\|(f_1, f_2)\|_{L^\Phi \times L^\Phi}.$$

On the other hand, if $0 \neq f \in L^{\overline{\Phi}}(\Omega \times \{1, 2\})$, then for $j = 1, 2$,

$$I_\Phi\left(\frac{f(\cdot, j)}{\|f\|_{\overline{\Phi}}}\right) = \int\limits_\Omega \Phi\left(x, \frac{f(x, j)}{\|f\|_{\overline{\Phi}}}\right) dx$$
$$\leq \int\limits_\Omega \Phi\left(x, \frac{f(x, 1)}{\|f\|_{\overline{\Phi}}}\right) dx + \int\limits_\Omega \Phi\left(x, \frac{f(x, 2)}{\|f\|_{\overline{\Phi}}}\right) dx = I_{\overline{\Phi}}\left(\frac{f}{\|f\|_{\overline{\Phi}}}\right) \leq 1.$$

Hence for every $f \in L^{\overline{\Phi}}(\Omega \times \{1, 2\})$ we have

$$\|T^{-1} f\|_{L^\Phi \times L^\Phi} \leq 2\|f\|_{\overline{\Phi}}.$$

From this we conclude that indeed $L^\Phi \times L^\Phi$ is isomorphic to $L^{\overline{\Phi}}(\Omega \times \{1, 2\})$.

If $W^{1,\Phi}$ contains ℓ^∞ isomorphically, then $L^\Phi \times L^\Phi$ does it too and so $L^{\overline{\Phi}}$ must contain ℓ^∞, which implies that $\overline{\Phi}$ does not satisfy condition Δ_2 by Theorem 4.3.2. Now we argue by contradiction. If Φ satisfies Δ_2, then there exist a constant $C > 0$ and a non-negative function $h \in L^1(\Omega)$ such that for any $t \geq 0$ and a.a. $x \in \Omega$,

$$\Phi(x, 2t) \leq C\Phi(x, t) + h(x).$$

Hence for any $t \geq 0$ and $a.a$ $(x, y) \in \Omega \times \{1, 2\}$ we have

$$\overline{\Phi}(x, y, 2t) = \Phi(x, 2t)\chi_{\{1\}}(y) + \Phi(x, 2t)\chi_{\{2\}}(y) \leq$$
$$(C\Phi(x, t) + h(x))\chi_{\{1\}}(y) + (C\Phi(x, t) + h(x))\chi_{\{2\}}(y) =$$
$$C\overline{\Phi}(x, y, t) + h(x)(\chi_{\{1\}}(y) + \chi_{\{2\}}(y)).$$

The function H given by the formula $H(x, y) = h(x)(\chi_{\{1\}}(y) + \chi_{\{2\}}(y))$ is a non-negative element of $L^1(\Omega \times \{1,2\})$. Therefore $\overline{\Phi}$ satisfies the Δ_2 condition, a contradiction.

□

Theorem 4.3.5. *Let $\Omega = (\alpha, \beta)$ where $-\infty < \alpha < \beta < \infty$ and Φ be a MO function on Ω satisfying condition* (V)*. If Φ does not satisfy condition Δ_2 then the Sobolev space $W^{1,\Phi}$ contains a subspace isomorphic to ℓ^∞.*

Proof. Let $\{f_k\} \subset L^\Phi$ satisfy the hypothesis of Proposition 4.3.1. Since they are bounded on Ω, so $f_k \in L^1(\Omega)$. Define

$$g_k(x) = \int_\alpha^x f_k \, dy, \quad x \in (\alpha, \beta), \; k \in \mathbb{N}.$$

We have that $g_k \in W^{1,\Phi}$. Indeed $g_k' = f_k \in L^\Phi$, and by the assumption (V), the Voltera operator is bounded on L^Φ, and so $\|g_k\|_\Phi \le l\|f_k\|_\Phi < \infty$ for some constant l. Moreover for every $k \in \mathbb{N}$,

$$1 = \|f_k\|_\Phi \le \|g_k\|_{1,\Phi} = \|f_k\|_\Phi + \|g_k\|_\Phi \le (1 + l)\|f_k\|_\Phi = 1 + l.$$

Analogously, for every $m \in \mathbb{N}$,

$$\begin{aligned} 1 \le \left\| \sum_{k=1}^m g_k \right\|_{1,\Phi} &\le \left\| \left(\sum_{k=1}^m g_k \right)' \right\|_\Phi + \left\| \sum_{k=1}^m g_k \right\|_\Phi \\ &= \left\| \sum_{k=1}^m f_k \right\|_\Phi + \left\| \sum_{k=1}^m g_k \right\|_\Phi \le (1 + l) \left\| \sum_{k=1}^m f_k \right\|_\Phi \le 1 + l. \end{aligned} \tag{4.6}$$

Hence $\sum_{k=1}^\infty g_k \in W^{1,\Phi}$. Notice that $g_k \ge 0$ and $f_k \ge 0$. Therefore in view of (4.6), for every element $a = (a_k) \in \ell^\infty$, $m \in \mathbb{N}$,

$$\left\|\sum_{k=1}^m a_k g_k\right\|_{1,\Phi} = \left\|\sum_{k=1}^m a_k g'_k\right\|_{\Phi} + \left\|\sum_{k=1}^m a_k g_k\right\|_{\Phi} \le \left\|\sum_{k=1}^m |a_k|\,|g'_k|\right\|_{\Phi} + \left\|\sum_{k=1}^m |a_k|\,|g_k|\right\|_{\Phi} \tag{4.7}$$

$$= \left\|\sum_{k=1}^m |a_k| f_k\right\|_{\Phi} + \left\|\sum_{k=1}^m |a_k| g_k\right\|_{\Phi} \le \|a\|_\infty \left(\left\|\sum_{k=1}^m f_k\right\|_{\Phi} + \left\|\sum_{k=1}^m g_k\right\|_{\Phi}\right)$$

$$\le (1+l)\|a\|_\infty \left\|\sum_{k=1}^m f_k\right\|_{\Phi} \le (1+l)\|a\|_\infty.$$

On the other hand for every $m, k \in \mathbb{N}$,

$$\left\|\sum_{k=1}^m a_k g_k\right\|_{1,\Phi} \ge \left\|\sum_{k=1}^m a_k f_k\right\|_{\Phi} = \left\|\sum_{k=1}^m |a_k|\,|f_k|\right\|_{\Phi} \ge \|\,|a_k|\,|f_k|\|_\Phi = |a_k|\,\|f_k\|_\Phi = |a_k|.$$

Hence for every $m \in \mathbb{N}$,

$$\left\|\sum_{k=1}^m a_k g_k\right\|_{1,\Phi} \ge \|a\|_\infty. \tag{4.8}$$

Combining (4.7) and (4.8) we have that ℓ^∞ is an isomorphic copy in $W^{1,\Phi}$.

□

Corollary 4.3.6. *Let $\Omega = (\alpha, \beta)$ where $-\infty < \alpha < \beta < \infty$. The variable exponent Sobolev space $W^{1,p(\cdot)}$ contains an isomorphic subspace to ℓ^∞ if and only if $p^+ = \infty$.*

Proof. By Corollary 4.2.8, the variable exponent function satisfies condition (V). Thus the proof is an immediate consequence of Theorems 4.3.4 and 4.3.5. □

4.4 Copy of ℓ^1

Theorem 4.4.1. *Let (Ω, Σ, μ) be a non-atomic and separable measure space. MO space L^Φ contains an isomorphic copy of ℓ^1 if and only if Φ or Φ^* do not satisfy condition Δ_2.*

Proof. If Φ does not satisfy condition Δ_2 then L^Φ contains isomorphically ℓ^∞ by Theorem 4.3.2, and so ℓ^1 is an isomorphic subspace of L^Φ [7, Corollary 6.8.]. If Φ^* does not satisfy Δ_2 then L^{Φ^*} contains an isomorphic subspace of ℓ^∞ again by Theorem 4.3.2. Now applying Theorem 4.1.4, the dual space $(L^\Phi)^* \simeq L^{\Phi^*} \oplus S$ must also contain an isomorphic copy of ℓ^∞. Finally by general result in Banach spaces [47, Proposition 2.e.8], the space L^Φ must contain a subspace isomorphic to ℓ^1.

If L^Φ has a subspace isomorphic to ℓ^1, then it can not be reflexive, and so by Theorem 4.1.4, Φ or Φ^* does not satisfy Δ_2. □

Corollary 4.4.2. *Let (Ω, Σ, μ) be a non-atomic and separable measure space. The variable exponent Lebesgue space $L^{p(\cdot)}$ contains an isomorphic subspace to ℓ^1 if and only if $p^+ = \infty$ or $p^- = 1$.*

Proof. The proof follows from Theorems 4.1.6 and 4.4.1. □

Theorem 4.4.3. *Let $\Omega = (\alpha, \beta)$ where $-\infty < \alpha < \beta < \infty$. If a MO function Φ and its conjugate Φ^* satisfy condition Δ_2 then $W^{1,\Phi}$ does not contain an isomorphic subspace of ℓ^1.*

Proof. If both Φ and Φ^* satisfy condition Δ_2 then the space L^Φ is reflexive by Theorem 4.1.4. Hence $W^{1,\Phi}$ is reflexive too. Therefore it can not contain a subspace ℓ^1. □

Theorem 4.4.4. *Let $\Omega = (\alpha, \beta)$ where $-\infty < \alpha < \beta < \infty$. Let Φ be MO function satisfying condition* (V)*. If $W^{1,\Phi}$ does not contain a subspace isomorphic to ℓ^1 then both Φ and Φ^* satisfy Δ_2.*

Proof. Let first Φ do not satisfy Δ_2. By Theorem 4.3.5, ℓ^∞ is an isomorphic copy in $W^{1,\Phi}$. Thus ℓ^1 is contained isomorphically in ℓ^∞ by [7, Corollary 6.8.] and so in $W^{1,\Phi}$.

Assume now that Φ^* does not satisfy Δ_2. Then in view of Proposition 4.3.1, there exists a sequence of bounded and non-negative functions $\{f_k\} \subset L^{\Phi^*}$ such that $f_k \wedge f_i = 0$, $k \neq i$, and

$$\left\| \sum_{k=1}^{\infty} f_k \right\|_{\Phi^*} = \|f_k\|_{\Phi^*} = 1, \quad k \in \mathbb{N}. \tag{4.9}$$

Hence

$$1 = \|f_k\|_{\Phi^*} \leq \|f_k\|^0_{\Phi^*} \leq 2\|f_k\|_{\Phi^*} = 2, \quad k \in \mathbb{N}. \tag{4.10}$$

By the definition of the Orlicz norm $\|\cdot\|^0_{\Phi^*}$ for $\epsilon > 0$, each $k \in \mathbb{N}$, there exists a non-negative, bounded function $g_k \in L^\Phi$ such that $\operatorname{ess\,supp} g_k \subset \operatorname{ess\,supp} f_k$, $I_\Phi(g_k) \leq 1$ and

$$\|f_k\|^0_{\Phi^*} \leq \frac{\epsilon}{2^k} + \int_\Omega f_k\, g_k\, dx. \tag{4.11}$$

Define

$$h_k(x) = \int_\alpha^x g_k\, dy, \quad x \in (\alpha, \beta),\ k \in \mathbb{N}.$$

Thus its derivative $h_k' = g_k \in L^\Phi$. Moreover, by the boundedness of the Voltera operator $\|h_k\|_\Phi \leq l\|g_k\|_\Phi \leq l$, which implies that h_k also belongs to L^Φ.

Clearly for every $k \in \mathbb{N}$,

$$\|h_k\|_{1,\Phi} = \|g_k\|_\Phi + \|h_k\|_\Phi \leq 1 + l.$$

Hence for all $a = (a_k) \in \ell^1$, $m \in \mathbb{N}$,

$$\left\| \sum_{k=1}^{m} a_k\, h_k \right\|_{1,\Phi} \leq \sum_{k=1}^{m} |a_k|\, \|h_k\|_{1,\Phi} \leq (1+l)\|a\|_1.$$

On the other hand,

$$\left\|\sum_{k=1}^{m} a_k\, h_k\right\|_{1,\Phi} = \left\|\sum_{k=1}^{m} a_k\, h_k\right\|_{\Phi} + \left\|\sum_{k=1}^{m} a_k\, h'_k\right\|_{\Phi} \geq \left\|\sum_{k=1}^{m} a_k\, g_k\right\|_{\Phi},$$

and by Hölder's inequality and by (4.9), (4.10),

$$\frac{1}{2}\int_{\Omega}\left(\sum_{k=1}^{m} a_k\, g_k\right)\left(\sum_{k=1}^{m}(\text{sign } a_k)\, f_k\right)\, dx \leq \left\|\sum_{k=1}^{m} a_k\, g_k\right\|_{\Phi}\left\|\sum_{k=1}^{m} f_k\right\|_{\Phi^*} \leq \left\|\sum_{k=1}^{m} a_k\, g_k\right\|_{\Phi}.$$

Therefore in view of $\operatorname{ess\,supp} g_k \subset \operatorname{ess\,supp} f_k$ and of that f_k are disjoint and of (4.10), (4.11), we get

$$\begin{aligned}
\left\|\sum_{k=1}^{m} a_k\, h_k\right\|_{1,\Phi} &\geq \frac{1}{2}\int_{\Omega}\left(\sum_{k=1}^{m} a_k\, g_k\right)\left(\sum_{k=1}^{m}(sign\, a_k)\, f_k\right)\, dx = \frac{1}{2}\int_{\Omega}\sum_{k=1}^{m}|a_k|\, f_k\, g_k\, dx \\
&= \frac{1}{2}\sum_{k=1}^{m}|a_k|\int_{\Omega} f_k\, g_k\, dx \geq \frac{1}{2}\sum_{k=1}^{m}|a_k|\,(\|f_k\|^0_{\Phi^*} - \frac{\epsilon}{2^k}) \\
&\geq \frac{1}{2}\left(\sum_{k=1}^{m}|a_k| - \sum_{k=1}^{m}\frac{\epsilon}{2^k}\right) = \frac{1}{2}\|a\|_1 - \sum_{k=1}^{m}\frac{\epsilon}{2^{k+1}} \geq \frac{1}{2}(\|a\|_1 - \epsilon).
\end{aligned}$$

Since ϵ was arbitrary, combining the above inequalities we get

$$\frac{1}{2}\|a\|_1 \leq \left\|\sum_{k=1}^{\infty} a_k\, h_k\right\|_{1,\Phi} \leq (1+l)\|a\|_1,$$

which shows that $W^{1,\Phi}$ contains an isomorphic subspace of ℓ^1 and completes the proof.

□

Corollary 4.4.5. *Let* $\Omega = (\alpha, \beta)$ *where* $-\infty < \alpha < \beta < \infty$. *The space* $W^{1,p(\cdot)}$ *does not contain isomorphic copy of* ℓ^1 *if and only if* $1 < p^- \leq p^+ < \infty$.

Proof. We observe first that the Voltera operator is bounded on $L^{p(\cdot)}$ by Corollary 4.2.8. Therefore the conclusion follows by Theorem 4.1.6 and Theorems 4.4.3 and

4.4.4. □

4.5 Reflexivity of $W^{1,\Phi}$

In this short section we always assume that $\Omega = (\alpha, \beta)$, where $-\infty < \alpha < \beta < \infty$, is equipped with the Lebesgue measure.

Theorem 4.5.1. *If Φ and Φ^* satisfy condition Δ_2 then $W^{1,\Phi}$ is reflexive.*

Let Φ be MO-function satisfying condition (V)*. If the space $W^{1,\Phi}$ is reflexive then both Φ and Φ^* satisfy condition Δ_2.*

Proof. If both Φ and Φ^* satisfy Δ_2 then the MO-space L^Φ is reflexive by Theorem 4.1.4 (iii), and so is $W^{1,\Phi}$.

Let assume now condition (V) that is the Voltera operator is bounded on L^Φ. If $W^{1,\Phi}$ is reflexive then it can not contain isomorphic copy of ℓ^∞. Therefore by Theorem 4.3.5, Φ satisfies Δ_2. Similarly $W^{1,\Phi}$ can not contain an isomorphic copy of ℓ^1, and thus by Theorem 4.4.4, Φ^* also satisfies Δ_2.

□

4.6 Uniform convexity of $W^{1,\Phi}$

Let $(X, \|\cdot\|)$ be a Banach space equipped with the norm $\|\cdot\|$. We denote by $S(X)$ and $B(X)$ the unit sphere and unit ball of X respectively. We say the X is *strictly convex* (SC) whenever $\|\frac{x+y}{2}\| < 1$ for any $x, y \in S(X)$, $x \neq y$. Recall that X is *uniformly convex* (UC) if

$$\forall \epsilon \in (0,1)\ \exists \delta > 0\ \forall x, y \in B(X)\ \|x - y\| > \epsilon \Rightarrow \left\| \frac{x+y}{2} \right\| < 1 - \delta.$$

It is not difficult to show that equivalently X is (UC) whenever for any sequences $\{x_n\}, \{y_n\} \subset B(X), \|x_n + y_n\| \to 2$ implies $\|x_n - y_n\| \to 0$ as $n \to \infty$.

A MO function Φ is called *strictly convex* if for a.e. $x \in \Omega$, the function $t \mapsto \Phi(x,t)$ is strictly convex, that is

$$\exists A \subset \Omega,\ \mu(A) = 0\ \forall x \in \Omega \setminus A, \forall u \neq v \in \mathbb{R}_+\ \forall \lambda \in (0,1)$$
$$\Phi(x, \lambda u + (1-\lambda)v) < \lambda \Phi(x,u) + (1-\lambda)\Phi(x,v).$$

Following [33] we say that a MO function Φ is *uniformly convex* (UC) if

$$\forall \epsilon > 0\ \exists \delta > 0\ \exists\ 0 \leq f \in L^0,\ \int_\Omega \Phi(x, f(x))\, dx \leq \epsilon,$$

and whenever $\forall u, v \geq 0\ \forall a.a.\ x \in \Omega\ |u - v| \geq \epsilon \max\{u, v\}$ and $|u - v| \geq f(x)$,

then $\Phi\left(x, \frac{u+v}{2}\right) \leq \frac{1-\delta}{2}(\Phi(x,u) + \Phi(x,v))$.

It is not difficult to show that Φ is uniformly convex if and only if for every $\epsilon > 0$,

$$\lim_{c \to 0} \int_\Omega \Phi(x, P_{\epsilon,c}(x))\, dx = 0, \tag{4.12}$$

where

$$P_{\epsilon,c}(x) = \sup\{u - v : (u,v) \in E_{\epsilon,c,x}\}, \quad \text{with} \tag{4.13}$$
$$E_{\epsilon,c,x} = \{(u,v) : u, v \geq 0, |u - v| > \epsilon \max\{u,v\},\ \text{and}$$
$$\Phi\left(x, \frac{u+v}{2}\right) > \frac{1}{2}(1-c)(\Phi(x,u) + \Phi(x,v))\}.$$

First version of uniform convexity for an Orlicz function φ was defined in the doctoral thesis of W.A.J. Luxemburg [48]. He showed that L^φ is uniformly convex if φ is uniformly convex. The necessity of this condition was shown later in [38] in case of non-atomic infinite measure, where two new versions of uniform convexity have been introduced, one for the case of non-atomic finite measure, and another one for counting measure. Combining all three conditions in the case of

MO-function results in the present form [8] where in the definition we have a function f with small modular $I_\Phi(f) \leq \epsilon$. If $f \equiv 0$ on Ω then in fact the inequality defining uniform convexity of Φ is the same condition as in the original paper [48] uniform for every parameter x.

The definition of uniform convexity of Φ seems to be quite complicated, but in the next remark is explained that this condition not only implies strict convexity of Φ, but something more, a sort of uniform strict convexity. In fact it can be expressed by the uniform inequalities of ratios of its derivatives. This was first observed in [3] for Orlicz functions, and later in [34] for MO-functions.

Remark 4.6.1. (1) *One can show that a uniformly convex function Φ is strictly convex on $\mathbb{R}_+$ for a.e. $x \in \Omega$.*

(2) *The condition for Φ being uniformly convex can be expressed in terms of its derivative. Let $\Phi'(x, \cdot)$ denote the right derivative of $\Phi(x,t)$ with respect to $t > 0$. Then Φ is uniformly convex with $f \equiv 0$ if and only if for every $\epsilon > 0$ there exists a constant $k_\epsilon > 1$ such that*

$$\Phi'(x, (1+\epsilon)t) \geq k_\epsilon \Phi'(x,t)$$

for a.a. $x \in \Omega$, every $t \geq 0$.

Characterization of UC of L^Φ was considered in [33] and later in [8]. Partial results on uniform convexity of L^Φ or $L^{p(\cdot)}$ have been given in Section 2.4 in [17]. Recall the complete characterization of UC in L^Φ.

Theorem 4.6.2. *[8, Theorem 5.15] Let (Ω, Σ, μ) be a non-atomic separable measure space. The MO space $(L^\Phi, \|\cdot\|_\Phi)$ is uniformly convex if and only if* (1) *$\Phi \in \Delta_2$,* (2) *$\Phi \in UC$.*

Lemma 4.6.3. *Let (Ω, Σ, μ) be a σ-finite measure space. Let $\Phi(x,t) = \frac{t^{p(x)}}{p(x)}$, where $1 < p^- \leq p^+ < \infty$. Then Φ is uniformly convex.*

Proof. We will show that for every $a \in (0,1)$, there is $\delta(a) \in (0,1)$ such that for all $t \geq 0$ and all $b \in [0,a]$ the following inequality is satisfied for a.a. $x \in \Omega$,

$$\Phi\left(x, \frac{1+b}{2}t\right) \leq (1-\delta(a))\frac{\Phi(x,t)+\Phi(x,bt)}{2}. \tag{4.14}$$

The above inequality is equivalent to the following condition $\forall \epsilon > 0\ \exists \delta > 0$ such that

$$\forall u,v \geq 0 \quad |u-v| \geq \epsilon \max\{u,v\} \quad \Rightarrow \quad \Phi\left(x, \frac{u+v}{2}\right) \leq \frac{1-\delta}{2}(\Phi(x,u)+\Phi(x,v))$$

for a.a. $x \in \Omega$ [34, 38, 49]. The latter inequality coincides with uniform convexity of Φ where $f \equiv 0$. Thus we will show that $\frac{t^{p(x)}}{p(x)}$ is uniformly convex. In the case of our Φ the condition (4.14) is equivalent to

$$\left(\frac{1+s}{2}\right)^{p(x)} \leq (1-\delta(a))\frac{1+s^{p(x)}}{2}$$

for any $s \in [0,a]$. In order to prove the above inequality observe that since $p(x) > 1$ for a.e. $x \in \Omega$, so for all $s \in [0,1)$ and a.a. $x \in \Omega$,

$$\left(\frac{1+s}{2}\right)^{p(x)} < \frac{1+s^{p(x)}}{2}. \tag{4.15}$$

Therefore for every $s \in [0,1)$, and a.a. $x \in \Omega$,

$$f(x,s) = \frac{\left(\frac{1+s}{2}\right)^{p(x)}}{\frac{1+s^{p(x)}}{2}} < 1.$$

It is easy to check that $f(x,s)$ is strictly increasing for $s \in [0,1)$ a.e. in Ω. If $a \in (0,1)$ then for every $0 \leq b < a$ we have $f(x,b) < f(x,a)$. Thus it is enough to show that there exists $\delta(a) \in (0,1)$ such that $f(x,a) \leq 1-\delta(a)$ for a.a. $x \in \Omega$. If this is not true we will find a sequence $\{x_n\} \subset \Omega$ with $\lim_n f(x_n,a) = 1$. By the assumption $p(x_n) \in [p^-,p^+]$, so there exist a subsequence $\{n_i\}$ and $p \in [p^-,p^+]$ such that $p = \lim_{i\to\infty} p(x_{n_i})$. Hence $\lim_{i\to\infty} f(x_{n_i}, a) = 1$, and so

$$\lim_{i\to\infty} f(x_{n_i}, a) = \frac{\left(\frac{1+a}{2}\right)^p}{\frac{1+a^p}{2}} = 1,$$

which is a contradiction with (4.15) and the proof is done by Theorem 4.6.2.

□

Partial results of the next corollary are known, but here we present a complete criterion of UC of $L^{p(\cdot)}$.

Corollary 4.6.4. *Let (Ω, Σ, μ) be a non-atomic separable measure space. The variable exponent Lebesgue space $L^{p(\cdot)}$ is uniformly convex if and only if $1 < p^- \leq p^+ < \infty$.*

Proof. If $1 < p^- \leq p^+ < \infty$ then $\Phi(x,t) = \frac{t^{p(x)}}{p(x)}$ satisfies Δ_2 and is uniformly convex by Lemma 4.6.3. Thus the space is uniformly convex in view of Theorem 4.6.2. On the other hand uniform convexity of $L^{p(\cdot)}$ implies reflexivity of the space, and so it cannot have an isomorphic subspace of ℓ^1, and thus $1 < p^- \leq p^+ < \infty$ (see Corollary 4.4.2).

□

The next theorem is the main result of this section.

Theorem 4.6.5. *Let $\Omega = (\alpha, \beta)$ where $-\infty < \alpha < \beta < \infty$. Let Φ be an MO function and $W^{1,\Phi}$ be the Sobolev space equipped with the norm $\|f\|_{1,\Phi} = \|f\|_\Phi + \|f'\|_\Phi$, $f \in W^{1,\Phi}$.*

If Φ satisfies condition Δ_2 and Φ is uniformly convex then the space $W^{1,\Phi}$ is uniformly convex.

If in addition Φ satisfies (V), then Δ_2 and uniform convexity of Φ are also necessary conditions for the space $W^{1,\Phi}$ to be uniformly convex.

Proof. By Theorem 4.6.2 if Φ satisfies condition Δ_2 and Φ is uniformly convex then the space L^Φ is uniformly convex. Therefore $L^\Phi \times L^\Phi$ equipped with norm (4.3) is also uniformly convex, and so is $W^{1,\Phi}$.

Let now $W^{1,\Phi}$ be uniformly convex. Then the space $W^{1,\Phi}$ is reflexive, and so it can not have a subspace isomorphic to ℓ^∞. By Theorem 4.3.5, Φ needs to satisfy condition Δ_2.

Thus assume that $\Phi \in \Delta_2$ and $\Phi \notin UC$. It follows by (4.12) that there exist $\epsilon > 0$ and a sequence $\{c_k\} \subset (0,1)$ with $\lim_{k\to\infty} c_k = 0$ and

$$\lim_{k\to\infty} \int_\Omega \Phi(x, P_{\epsilon,c_k}(x))\, dx > 0.$$

Hence there are $\delta > 0$, $N > 0$ such that for all $k > N$,

$$\int_\Omega \Phi(x, P_{\epsilon,c_k}(x))\, dx > 2\delta. \tag{4.16}$$

In view of (4.13) for every $c \in (0,1)$ there exists a sequence $\{u_k^c, v_k^c\}$ of non-negative measurable functions satisfying the following conditions,

$$\forall a.a. x \in \Omega,\ |u_k^c(x) - v_k^c(x)| \uparrow P_{\epsilon,c}(x) \ \text{ if } \ k \to \infty, \tag{4.17}$$

If $u_k^c(x) \neq v_k^c(x)$ then $\Phi(x, |u_k^c(x) - v_k^c(x)|) \geq \max\{\Phi(x, \epsilon u_k^c(x)), \Phi(x, \epsilon v_k^c(x))\}$ and $\Phi\left(x, \dfrac{|u_k^c(x) + v_k^c(x)|}{2}\right) > \dfrac{1-c}{2}(\Phi(x, u_k^c(x)) + \Phi(x, v_k^c(x)))$.

By (4.16) and (4.17), for every $k \in \mathbb{N}$, there exsts $j_k \in \mathbb{N}$ such that for all $k \in \mathbb{N}$,

$$\int_\Omega \Phi(x, |u_{j_k}^{c_k}(x) - v_{j_k}^{c_k}(x)|)\, dx > \frac{1}{2}\int_\Omega \Phi(x, P_{\epsilon,c_k}(x))\, dx > \delta.$$

Letting $u_k = u_{j_k}^{c_k}$, $v_k = v_{j_k}^{c_k}$ and applying the above inequalities we obtain for every

$k \in \mathbb{N}$,

$$I_\Phi(u_k - v_k) = \int_\Omega \Phi(x, |u_k(x) - v_k(x)|\, dx > \delta, \tag{4.18}$$

and if $u_k(x) \neq v_k(x)$ then

$$\Phi(x, |u_k(x) - v_k(x)|) > \max\{\Phi(x, \epsilon u_k(x), \Phi(x, \epsilon v_k(x)\}, \tag{4.19}$$

$$\Phi\left(x, \frac{1}{2}(u_k(x) + v_k(x))\right) > \frac{1-c_k}{2}(\Phi(x, u_k(x) + \Phi(x, v_k(x))). \tag{4.20}$$

In view of Δ_2, by Theorem 4.1.1, there exists $\gamma \in (0, \delta)$ such that for all $u \in L^\Phi$,

$$I_\Phi(u) < \gamma \Rightarrow \|u\|_\Phi < \epsilon. \tag{4.21}$$

By (4.18) for every $k \in \mathbb{N}$ we find the sets E_k satisfying

$$E_k \subset \{x \in \Omega : u_k(x) \neq v_k(x)\} \quad \text{and} \quad \int_{E_k} \Phi(x, |u_k(x) - v_k(x)|)\, dx = \gamma.$$

By (4.19), if $x \in E_k$ then $u_k(x) \neq v_k(x)$ and so $\Phi(x, |u_k(x) - v_k(x)) > \Phi(x, \epsilon u_k(x))$. Hence

$$\int_{E_k} \Phi(x, \epsilon u_k(x))\, dx \leq \int_{E_k} \Phi(x, |u_k(x) - v_k(x)|)\, dx = \gamma. \tag{4.22}$$

It follows in view of (4.21) that $\|\epsilon u_k \chi_{E_k}\|_\Phi < \epsilon$. Consequently and by symmetry, for all $k \in \mathbb{N}$,

$$\|u_k \chi_{E_k}\|_\Phi \leq 1 \quad \text{and} \quad \|v_k \chi_{E_k}\|_\Phi \leq 1.$$

Let for each $k \in \mathbb{N}$,

$$\Omega_k = \{x \in E_k : \Phi(x, u_k(x)) \geq \Phi(x, v_k(x))\} \quad \text{and} \quad \widetilde{\Omega}_k = E_k \setminus \Omega_k.$$

Set also

$$\alpha_k = \int_{\Omega_k} (\Phi(x, u_k(x)) - \Phi(x, v_k(x)))\, dx \quad \text{and} \quad \widetilde{\alpha}_k = \int_{\widetilde{\Omega}_k} (\Phi(x, v_k(x)) - \Phi(x, u_k(x)))\, dx.$$

Clearly, α_k, $\widetilde{\alpha}_k > 0$. Therefore for each $k \in \mathbb{N}$ there exists $F_k \subset \Omega_k$ satisfying

$$\int_{F_k} (\Phi(x, u_k(x)) - \Phi(x, v_k(x)))\, dx = \frac{\alpha_k}{2}.$$

Hence

$$\int_{F_k} (\Phi(x, u_k(x)) - \Phi(x, v_k(x)))\, dx = \int_{\Omega_k \setminus F_k} (\Phi(x, u_k(x)) - \Phi(x, v_k(x)))\, dx,$$

which implies that

$$\int_{F_k} \Phi(x, v_k(x))\, dx + \int_{\Omega_k \setminus F_k} \Phi(x, u_k(x))\, dx = \int_{\Omega_k \setminus F_k} \Phi(x, v_k(x))\, dx + \int_{F_k} \Phi(x, u_k(x))\, dx.$$

Analogously for every $k \in \mathbb{N}$ tere exists $\widetilde{F}_k \subset \widetilde{\Omega}_k$ such that

$$\int_{\widetilde{F}_k} \Phi(x, v_k(x))\, dx + \int_{\widetilde{\Omega}_k \setminus \widetilde{F}_k} \Phi(x, u_k(x))\, dx = \int_{\widetilde{\Omega}_k \setminus \widetilde{F}_k} \Phi(x, v_k(x))\, dx + \int_{\widetilde{F}_k} \Phi(x, u_k(x))\, dx.$$

Set now

$$\begin{aligned} \hat{x}_k &= u_k \chi_{F_k \cup (\widetilde{\Omega}_k \setminus \widetilde{F}_k)} + v_k \chi_{\widetilde{F}_k \cup (\Omega_k \setminus F_k)}, \\ \hat{y}_k &= v_k \chi_{F_k \cup (\widetilde{\Omega}_k \setminus \widetilde{F}_k)} + u_k \chi_{\widetilde{F}_k \cup (\Omega_k \setminus F_k)}. \end{aligned}$$

We have $[F_k \cup (\widetilde{\Omega}_k \setminus \widetilde{F}_k)] \cup [\widetilde{F}_k \cup (\Omega_k \setminus F_k)] = E_k$, $[F_k \cup (\widetilde{\Omega}_k \setminus \widetilde{F}_k)] \cap [\widetilde{F}_k \cup (\Omega_k \setminus F_k)] = \emptyset$ and $I_\Phi(\hat{x}_k) = I_\Phi(\hat{y}_k)$ for all $k \in \mathbb{N}$. Now, in view of (4.19) and (4.22),

$$\delta > \gamma = I_\Phi((u_k - v_k)\chi_{E_k}) \geq I_\Phi(\epsilon \max\{u_k, v_k\}\chi_{E_k}),$$

and so by (4.21), $\|\epsilon \max\{u_k, v_k\}\chi_{E_k}\|_\Phi \le \epsilon$. Hence $\|\max\{u_k, v_k\}\chi_{E_k}\|_\Phi \le 1$ and consequently

$$I_\Phi(\max\{u_k, v_k\}\chi_{E_k}) \le 1.$$

Since $\hat{x}_k \le \max\{u_k, v_k\}\chi_{E_k}$ and $\hat{y}_k \le \max\{u_k, v_k\}\chi_{E_k}$, we get

$$\beta_k := I_\Phi(\hat{x}_k) = I_\Phi(\hat{y}_k) \le I_\Phi(\max\{u_k, v_k\}\chi_{E_k}) \le 1.$$

Now in view of $\mu(\Omega \setminus E_k) > 0$, there exist $G_k \subset \Omega \setminus E_k$ and $\sigma_k > 0$ such that

$$\int_{G_k} \Phi(x, \sigma_k)\, dx = 1 - \beta_k.$$

Finally let

$$x_k = \hat{x}_k \chi_{E_k} + \sigma_k \chi_{E_k}, \quad y_k = \hat{y}_k \chi_{E_k} + \sigma_k \chi_{E_k}.$$

Then for all $k \in \mathbb{N}$,

$$I_\Phi(x_k) = I_\Phi(y_k) = 1. \tag{4.23}$$

Moreover by (4.22),

$$\begin{aligned} 0 < \gamma = I_\Phi((u_k - v_k)\chi_{E_k}) &= I_\Phi(x_k - y_k) = I_\Phi(\hat{x}_k - \hat{y}_k) \\ &\le I_\Phi(\max\{\hat{x}_k, \hat{y}_k\}) = I_\Phi(\max\{u_k, v_k\}\chi_{E_k}) \le 1. \end{aligned}$$

Consequently for all $k \in \mathbb{N}$,

$$0 < \gamma \le I_\Phi(x_k - y_k) \le \|x_k - y_k\|_\Phi. \tag{4.24}$$

Since $\|x_k\|_\Phi = \|y_k\|_\Phi = 1$, $\|\frac{x_k + y_k}{2}\|_\Phi \le 1$. Hence

$$\left\|\frac{x_k + y_k}{2}\right\|_\Phi \ge I_\Phi\left(\frac{x_k + y_k}{2}\right). \tag{4.25}$$

Moreover,

$$
\begin{aligned}
I_\Phi\left(\frac{x_k+y_k}{2}\right) &= I_\Phi\left(\frac{\hat{x}_k+\hat{y}_k}{2}\chi_{E_k}\right) + I_\Phi(\sigma_k\chi_{E_k}) \\
&= \int_\Omega \Phi\left(x, \frac{u_k(x)+v_k(x)}{2}\chi_{F_k\cup(\widetilde{\Omega}_k\setminus\widetilde{F}_k)}(x)\right)dx \\
&+ \int_\Omega \Phi\left(x, \frac{u_k(x)+v_k(x)}{2}\chi_{\widetilde{F}_k\cup(\Omega_k\setminus F_k)}(x)\right)dx + \int_{G_k}\Phi(x,\sigma_k)\,dx \\
&= \int_{E_k}\Phi\left(x, \frac{u_k(x)+v_k(x)}{2}\right)dx + \int_{G_k}\Phi(x,\sigma_k)\,dx \\
&\geq \frac{1-c_k}{2}\int_{E_k}(\Phi(x,u_k(x))+\Phi(x,v_k(x)))\,dx + \int_{G_k}\Phi(x,\sigma_k)\,dx \\
&= \frac{1}{2}I_\Phi(x_k) + \frac{1}{2}I_\Phi(y_k) - \frac{c_k}{2}\int_{E_k}(\Phi(x,u_k(x))+\Phi(x,v_k(x)))\,dx \\
&\geq 1-c_k \to 1 \quad \text{by} \quad (4.23),\ (4.20)
\end{aligned}
$$

when $k \to \infty$.

Combining the above and (4.24), (4.25), it follows that L^Φ is not uniformly convex.

Now we will proceed to show that $W^{1,\Phi}$ is not UC either.

Recall that a measurable function is called *simple* if it assumes finite number of values. A function $f : (\alpha, \beta) \to \mathbb{C}$ is called a *step function* if there exists a finite partition $\alpha = \alpha_0 < \alpha_1 < \cdots < \alpha_m = \beta$ and the numbers $\{a_i\}_{i=1}^m \subset \mathbb{C}$ such that $f(x) = \sum_{i=1}^m a_i\chi_{(\alpha_{i-1},\alpha_i)}(x)$, $x \in (\alpha, \beta)$.

First observe that the functions u_k^c, v_k^c satisfying (4.17) can be chosen to be simple functions. Therefore the functions x_k and y_k can be also chosen as simple functions. The next observation is that these functions can be replaced by step functions. In fact it follows from the regularity of the Lebesgue measure on $\Omega = (\alpha, \beta)$ and the assumption of Δ_2 condition. By Theorem 4.1.1, L^Φ is order continuous under the assumption of Δ_2. It implies in particular that for any $f \in L^\Phi$ and every $\epsilon > 0$ there is $\delta > 0$ such that whenever $\mu(A) < \delta$ then $\|f\chi_A\|_\Phi < \epsilon$ [5].

By regularity of the Lebesgue measure, for any measurable $A \subset \Omega$ with $\chi_A \in L^\Phi$ and any $\delta > 0$, there exist disjoint open intervals $G_1, \dots, G_m$ such that $\mu((A \setminus \cup_{i=1}^m G_i) \cup (\cup_{i=1}^m G_i \setminus A)) < \delta$. Hence $\|\chi_A - \chi_{\cup_{i=1}^m G_i}\|_\Phi = \|\chi_{(A \setminus \cup_{i=1}^m G_i) \cup (\cup_{i=1}^m G_i \setminus A)}\|_\Phi < \epsilon$. Therefore we can approximate any measurable subset of Ω by a finite union of open disjoint intervals. So any simple function can be replaced by a step function. It follows that the functions x_k and y_k can be taken as step functions. Recall that x_k and y_k are non-negative.

By the above discussion, without loss of generality, assume for every $k \in \mathbb{N}$,

$$x_k = a_{k1}\chi_{(\alpha_0,\alpha_1)} + a_{k2}\chi_{(\alpha_1,\alpha_2)} + \cdots + a_{kM_k}\chi_{(\alpha_{M_k-1},\alpha_{M_k})},$$

where $\alpha = \alpha_0 < \alpha_1 < \cdots < \alpha_{M_k} = \beta$ and $\{a_{ki}\}_{i=1}^{M_k} \subset [0, \infty)$. Similarly let

$$y_k = b_{k1}\chi_{(\beta_0,\beta_1)} + b_{k2}\chi_{(\beta_1,\beta_2)} + \cdots + b_{kJ_k}\chi_{(\beta_{J_k-1},\beta_{J_k})},$$

with $\alpha = \beta_0 < \beta_1 < \cdots < \beta_{J_k} = \beta$ and $\{b_{kj}\}_{j=1}^{J_k} \subset [0, \infty)$. Now let $\{(\gamma_{i-1}, \gamma_i)\}_{i=1}^m$ be the family of all intersections of the intervals $(\alpha_{p-1}, \alpha_p) \cap (\beta_{j-1}, \beta_j)$, $p = 1, \dots, M_k$, $j = 1, \dots, J_k$, which are not empty or reduced to one point. It is also ordered as $\alpha = \gamma_0 < \gamma_1 < \cdots < \gamma_m = \beta$. The numbers γ_i and $m \in \mathbb{N}$ depend on k. Both functions x_k and y_k are constants on every interval (γ_{i-1}, γ_i).

Let $l \in \{1, 2, \dots, m\}$ be fixed. Divide (γ_{l-1}, γ_l) into even and equal subintervals

$$(\gamma_1^l, \gamma_2^l), (\gamma_2^l, \gamma_3^l), \dots, (\gamma_{2n_l-1}^l, \gamma_{2n_l}^l),$$

such that

$$\int_{\gamma^l_{2i-1}}^{\gamma^l_{2i}} \max\{x_k, y_k\} < \frac{1}{2^k}, \qquad i = 1, \ldots, n_l,$$
$$\int_{\gamma^l_{2i}}^{\gamma^l_{2i+1}} \max\{x_k, y_k\} < \frac{1}{2^k}, \qquad i = 1, \ldots, n_l - 1.$$

Since x_k is constant on the interval (γ_{l-1}, γ_l), so

$$\int_{\gamma^l_{2i-1}}^{\gamma^l_{2i}} x_k = \int_{\gamma^l_{2i}}^{\gamma^l_{2i+1}} x_k \qquad i = 1, \ldots, n_l - 1.$$

The similar equality is true for y_k. Define on each interval (γ_{l-1}, γ_l), $l = 1, \ldots, m$,

$$\tilde{x}_k = x_k \chi_{(\gamma^l_1, \gamma^l_2)} - x_k \chi_{(\gamma^l_2, \gamma^l_3)} + \cdots + x_k \chi_{(\gamma^l_{2n_l - 1}, \gamma^l_{2n_l})} - x_k \chi_{(\gamma^l_{2n_l}, \gamma^l_{2n_l+1})},$$

$$\tilde{y}_k = y_k \chi_{(\gamma^l_1, \gamma^l_2)} - y_k \chi_{(\gamma^l_2, \gamma^l_3)} + \cdots + y_k \chi_{(\gamma^l_{2n_l - 1}, \gamma^l_{2n_l})} - y_k \chi_{(\gamma^l_{2n_l}, \gamma^l_{2n_l+1})}.$$

We defined $\tilde{x_k}$, $\tilde{y_k}$ on every (γ_{l-1}, γ_l), so they are well defined on (α, β). If $x \in (\gamma_{l-1}, \gamma_l)$ then either $x \in (\gamma^l_{2i-1}, \gamma^l_{2i})$ for some $i = 1, \ldots, n_l$ or $x \in (\gamma^l_{2i}, \gamma^l_{2i+1})$ for some $i = 1, \ldots, n_l - 1$. For $x \in (\gamma^l_{2i-1}, \gamma^l_{2i})$,

$$\left| \int_{\gamma_{l-1}}^{x} \tilde{x}_k \right| = \left| \left(\int_{\gamma^l_1}^{\gamma^l_2} x_k - \int_{\gamma^l_2}^{\gamma^l_3} x_k \right) + \cdots + \int_{\gamma^l_{2i-1}}^{x} x_k \right| \leq \left| \int_{\gamma^l_{2i-1}}^{\gamma^l_{2i}} x_k \right| < \frac{1}{2^k}.$$

For $x \in (\gamma_{2i}^l, \gamma_{2i+1}^l)$,

$$\begin{aligned}\left|\int_{\gamma_{l-1}}^{x} \tilde{x}_k\right| &= \left|\left(\int_{\gamma_1^l}^{\gamma_2^l} x_k - \int_{\gamma_2^l}^{\gamma_3^l} x_k\right) + \cdots + \left(\int_{\gamma_{2i-1}^l}^{\gamma_{2i}^l} x_k - \int_{\gamma_{2i}^l}^{x} x_k\right)\right| \\ &= \int_{\gamma_{2i-1}^l}^{\gamma_{2i}^l} x_k - \int_{\gamma_{2i}^l}^{x} x_k < \frac{1}{2^k}.\end{aligned}$$

Combining the above we get for every $k \in \mathbb{N}$ and $x \in (\gamma_{l-1}, \gamma_l)$,

$$\int_{\gamma_{l-1}}^{\gamma_l} \tilde{x}_k = 0, \qquad \left|\int_{\gamma_{l-1}}^{x} \tilde{x}_k\right| < \frac{1}{2^k}.$$

Similarly we get for every $k \in \mathbb{N}$ and $x \in (\gamma_{l-1}, \gamma_l)$,

$$\int_{\gamma_{l-1}}^{\gamma_l} \tilde{y}_k = 0, \qquad \left|\int_{\gamma_{l-1}}^{x} \tilde{y}_k\right| < \frac{1}{2^k}.$$

Since the above inequalities are satisfied for every $l = 1, 2, \ldots, m$ we obtain for every $x \in (\alpha, \beta)$,

$$\left|\int_{\alpha}^{x} \tilde{x}_k\right| < \frac{1}{2^k}, \qquad \left|\int_{\alpha}^{x} \tilde{y}_k\right| < \frac{1}{2^k}. \tag{4.26}$$

Let for $k \in \mathbb{N}$, $x \in (\alpha, \beta)$,

$$f_k(x) = \int_{\alpha}^{x} \tilde{x}_k, \qquad g_k(x) = \int_{\alpha}^{x} \tilde{y}_k.$$

Then

$$f_k'(x) = \tilde{x}_k(x), \qquad g_k'(x) = \tilde{y}_k(x),$$

for a.e. $x \in (\alpha, \beta)$. By (4.26) for every $\lambda > 0$,

$$I_\Phi(\lambda f_k) = \int_\Omega \Phi\left(x, \lambda \left|\int_{\alpha}^{x} \tilde{x}_k\right|\right) dx \le \frac{1}{2^k} \int_\Omega \Phi(x, \lambda)\, dx. \tag{4.27}$$

Now by assumption (V) (4.2.6) and Δ_2, $\int_\Omega \Phi(x, \lambda)\, dx < \infty$. Hence the right side of

(4.27) tends to zero when $k \to \infty$. It follows that

$$\|f_k\|_\Phi \to 0 \quad \text{if} \quad k \to \infty.$$

Similarly

$$\|g_k\|_\Phi \to 0 \quad \text{as} \quad k \to \infty.$$

In view of (4.23), we have for every $k \in \mathbb{N}$,

$$\|\tilde{x}_k\|_\Phi = \| \, |\tilde{x}_k| \, \|_\Phi = \|x_k\|_\Phi = 1 \quad \text{and} \quad \|\tilde{y}_k\|_\Phi = \| \, |\tilde{y}_k| \, \|_\Phi = \|y_k\|_\Phi = 1.$$

Consequently,

$$\|f_k\|_{1,\Phi} = \|f_k\|_\Phi + \|f_k'\|_\Phi = \|f_k\|_\Phi + \|\tilde{x}_k\|_\Phi \to 1,$$

$$\|g_k\|_{1,\Phi} = \|g_k\|_\Phi + \|g_k'\|_\Phi = \|g_k\|_\Phi + \|\tilde{y}_k\|_\Phi \to 1,$$

as $k \to \infty$.

Moreover $\left|\frac{\tilde{x}_k+\tilde{y}_k}{2}\right| = \frac{x_k+y_k}{2}$ for every $k \in \mathbb{N}$. Thus in view of (4.25),

$$\begin{aligned} \left\|\frac{f_k+g_k}{2}\right\|_{1,\Phi} &= \left\|\frac{f_k+g_k}{2}\right\|_\Phi + \left\|\frac{\tilde{x}_k+\tilde{y}_k}{2}\right\|_\Phi \\ &\geq \left\|\frac{x_k+y_k}{2}\right\|_\Phi \geq I_\Phi\left(\frac{x_k+y_k}{2}\right) \geq 1 - c_k \to 1, \end{aligned}$$

as $k \to \infty$. We also have by (4.24),

$$\|f_k - g_k\|_{1,\Phi} = \|f_k - g_k\|_\Phi + \|\tilde{x}_k - \tilde{y}_k\|_\Phi \geq \|\tilde{x}_k - \tilde{y}_k\|_\Phi = \|x_k - y_k\|_\Phi \geq \gamma$$

for all $k \in \mathbb{N}$. It shows that $W^{1,\Phi}$ is not uniformly convex and the proof is finished.

□

The next result follows from Theorems 4.6.2 and 4.6.5.

Corollary 4.6.6. *Let $\Omega = (\alpha, \beta)$, $\infty < \alpha < \beta < \infty$. Let Φ satisfy* (V)*. Then $L^\Phi \in UC$ if and only if $W^{1,\Phi} \in UC$. This in turn is equivalent to $\Phi \in \Delta_2$ and $\Phi \in UC$.*

Corollary 4.6.7. *Let $\Omega = (\alpha, \beta)$, $\infty < \alpha < \beta < \infty$. The variable exponent Sobolev space $W^{1,p(\cdot)}$ is uniformly convex if and only if $1 < p^- \leq p^+ < \infty$.*

Proof. It follows immediately from Corollaries 4.6.6 and 4.6.4. □

Let Φ be an Orlicz function, that is $\Phi(x,t) = \phi(t)$ for all $x \in \Omega$. If $\mu(\Omega) < \infty$ then $\Phi \in \Delta_2$ if and only if $\varphi(2u) \leq k\varphi(u)$ for all $u \geq u_0$ and some $k > 0$ and $u_0 \geq 0$. Under the same conditions $\Phi \in UC$ if and only if $\varphi \in UC$ that is

$$\forall \varepsilon > 0\ \exists \delta > 0\ \exists u_0 \geq 0 \forall u,\ v \geq u_0\ \ |u - v| \geq \max\{u, v\}$$
$$\implies \varphi\left(\frac{u+v}{2}\right) \leq \frac{1-\delta}{2}(\varphi(u) + \varphi(v)).$$

In paper [9] the authors gave a characterization of UC of $W^{1,\varphi}$ under additional assumption that φ is a N-function. Moreover, their methods are different and not applicable in the case of MO function. Notice also, that for an Orlicz function φ the condition (V) is always satisfied on $\Omega = (\alpha, \beta)$, that is the Voltera operator is bounded on L^φ (Theorem 4.2.7). By Theorem (4.6.5) and the above remarks we arrive at the following result.

Corollary 4.6.8. *Let $\Omega = (\alpha, \beta)$ where $-\infty < \alpha < \beta < \infty$. For an Orlicz function φ the Orlicz Sobolev space is $W^{1,\varphi}$ is UC if and only if $\varphi \in \Delta_2$ and $\varphi \in UC$.*

4.7 Superreflexivity and B-convexity of $W^{1,\Phi}$

A Banach space $(X, \|\cdot\|)$ is said to be *B-convex* if there exist a $\delta > 0$ and an integer $n \geq 2$ such that for any $x_1, \ldots, x_n \in X$ we can choose $\epsilon = \{\epsilon_k\}_{k=1}^n$, $\epsilon_k = \pm 1$, in such a way that

$$\left\| \frac{1}{n} \sum_{k=1}^{n} \epsilon_k x_k \right\| \leq (1-\delta) \max_{1 \leq k \leq n} \|x_k\|.$$

A Banach space X is said to be *superreflexive* if every Banach space Y which is finitely representable in X is reflexive [4]. A uniformly convex Banach space is superreflexive. Any superreflexive Banach space has a uniformly convex equivalent norm [4, Problem 11.6].

Lemma 4.7.1. *[34, Lemma 1.1.5] Let* (Ω, Σ, μ) *be a* σ*-finite measure space. Given a MO function* Φ *on* Ω *if* Φ^* *satisfies condition* Δ_2*, then there exists a uniformly convex MO function* Ψ *equivalent to* Φ*.*

Theorem 4.7.2. *Let* $\Omega = (\alpha, \beta)$*,* $\infty < \alpha < \beta < \infty$*.* Φ *satisfy condition* (V)*. Then the following conditions are equivalent.*

(i) $W^{1,\Phi}$ *is reflexive.*

(ii) $W^{1,\Phi}$ *is superreflexive.*

(iii) $W^{1,\Phi}$ *is B-convex.*

(iv) *Both* Φ *and* Φ^* *satisfy condition* Δ_2*.*

Proof. (ii) $\Rightarrow$ (i) is clear.

(i) $\Rightarrow$ (iv) If (i) is satisfied, that is the space $W^{1,\Phi}$ is reflexive, then by Theorem 4.5.1 both Φ and Φ^* satisfy Δ_2, so (iv) holds.

(iv) $\Rightarrow$ (ii) By the assumption that $\Phi^* \in \Delta_2$, in view of Lemma 4.7.1, there exists a uniformly convex function Ψ equivalent to Φ. Since $\Phi \in \Delta_2$, the function $\Psi \in \Delta_2$. Now by Theorem 4.6.2, the space $W^{1,\Psi}$ is uniformly convex and thus superreflexive [4, Problem 11.6]. Since Ψ is equivalent to Φ, the spaces $W^{1,\Phi}$ and $W^{1,\Psi}$ coincide as sets with equivalent norms (see Theorem 4.1.5). It follows that $W^{1,\Phi}$ as isomorphic to $W^{1,\Psi}$ is superreflexive.

(iv) $\Rightarrow$ (iii) By [20, Example 3 (ii), p. 118], any uniformly convex space is B-convex. By $\Phi^* \in \Delta_2$, in view of Lemma 4.7.1, there exists a uniformly convex function Ψ equivalent to Φ. Since Δ_2 is preserved by equivalence, $\Psi \in \Delta_2$. In view of Theorem 4.6.2, the space $W^{1,\Psi}$ is uniformly convex and thus is B-convex. In the same paper [20], in Corollary 6 it was proved that if Banach spaces X and Y are isomorphic, then X is B-convex if and only if Y is B-convex. It follows that $W^{1,\Phi}$ is B-convex as $W^{1,\Psi}$ and $W^{1,\Phi}$ coincide as sets with equivalent norms.

(iii) $\Rightarrow$ (iv) If $\Phi \notin \Delta_2$, then $W^{1,\Phi}$ contains a subspace isomorphic to ℓ^∞ by Theorem 4.3.5, and since ℓ^∞ is not B-convex [20, Example 3, (iv)], it contradicts the assumption of B-convexity of $W^{1,\Phi}$.

If $\Phi^* \notin \Delta_2$ then $W^{1,\Phi}$ contains a subspace isomorphic to ℓ^1 by Theorem 4.4.4. However ℓ^1 is not B-convex [20, Example 3, (iv)] or [18], and so $W^{1,\Phi}$ can not be B-convex.

$\square$

In the next result let $\Phi(x,t) = \varphi(t)$ for all $x \in \Omega$. Then φ is an Orlicz function and L^φ is an Orlicz space. For such function the condition (V) is always satisfied on $\Omega = (\alpha, \beta)$, that is the Voltera operator is bounded on L^φ (Theorem 4.2.7). In this case when $\Omega = (\alpha, \beta)$ is a finite interval, the condition Δ_2 achieves a simpler form. Namely, there exist $K > 0$, $t_0 \geq 0$ such that $\varphi(2t) \leq K\varphi(t)$ for all $t \geq t_0$. By these remarks and Theorem 4.7.2 we get the following corollary in Orlicz-Sobolev spaces.

Corollary 4.7.3. *Let $\Omega = (\alpha, \beta)$, $\infty < \alpha < \beta < \infty$ and φ be an Orlicz function. Then the following conditions are equivalent.*

(i) $W^{1,\varphi}$ *is reflexive.*

(ii) $W^{1,\varphi}$ *is superreflexive.*

(iii) $W^{1,\varphi}$ *is B-convex.*

(iv) *Both* φ *and* φ^* *satisfy condition* Δ_2.

Since the Voltera operator is always bounded on $L^{p(\cdot)}$ (Corollary 4.2.8), the next corollary is an immediate result from Theorem 4.7.2.

Corollary 4.7.4. *Let* $\Omega = (\alpha, \beta)$, $\infty < \alpha < \beta < \infty$, *and* $\Phi(x,t) = \frac{t^{p(x)}}{p(x)}$, $1 \leq p(x) < \infty$ *a.e. in* Ω. *Then the following conditions are equivalent.*

(i) $W^{1,p(\cdot)}$ *is reflexive.*

(ii) $W^{1,p(\cdot)}$ *is superreflexive.*

(iii) $W^{1,p(\cdot)}$ *is* B*-convex.*

(iv) $1 < p^- \leq p^+ < \infty$.

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