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Local Null-Controllability of a Chemotactic Model for a Bacterial Infection in a Radially-Symmetric, Chronic Wound

Stephen Guffey

LOCAL NULL-CONTROLLABILITY OF A CHEMOTACTIC MODEL FOR A BACTERIAL INFECTION IN A RADIALLY-SYMMETRIC CHRONIC WOUND

by

Stephen Guffey

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ABSTRACT

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In this work, we introduce a differential equation model for a bacterial infection in a radially symmetric chronic wound. The goal of the current work is the following: show the model produces a unique solution in a suitable setting and then determine whether one can drive the system to the origin using controllers contained in a small subdomain of the wound. The model consists of three parabolic-like partial differential equations along with a single vector-valued ordinary differential equation. Of the critical features of the model, one of the most challenging is the nonlinear coupling describing chemotactic attraction between the neutrophils and a chemoattractant distributed throughout the wound. Lower order nonlinearities, describing various coupling phenomena, present further mathematical difficulties including the potential for our system to become singular.

For the problem of well-posedness, we invoke a fixed point argument. First, we show that the system has a solution for predetermined finite time provided the initial conditions are taken sufficiently small, using maximal regularity results from the linear heat equation. We then show the model has a maximum-principle type result, namely that nonnegative initial conditions will give rise to nonnegative solutions for the aforementioned time duration. Finally, we show that the model has local-in-time existence for (potentially large) initial conditions provided these conditions are nonnegative. In this case, the corresponding solutions are also nonnegative.

For the problem of null-controllability, we first study the control of the corresponding linear problem. Using a corollary of the Closed Graph Theorem, we derive an appropriate dual system to the linear control problem as well as the so-called observability inequality. Establishing the controllability inequality is

equivalent to the controllability problem, and hence becomes our new goal. To this end, we introduce the Carleman-type estimates for the linear heat equation. These estimates are used first to establish the observability estimate using controllers on all the equations. Then, another round of estimation using the coupling in the linear equations allows us to show the observability inequality can be established for fewer controllers. Thus, we get an optimal controllability result for our system in terms of the number of required controllers. We then return to the control of the nonlinear system. For this, we introduce the state-to-control map. By showing this map is (locally) a homeomorphism through the inverse function theorem, we can deduce that the nonlinear system is locally null-controllable as well.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

In this work, we study a system of differential equations modeling a bacterial infection inside a radially symmetric, chronic wound. The goal of the current work is to give well-posedness for our version of the model and then show the model can be driven to the zero state at a prescribed final time. The model accounts for the interactions between the oxygen, neutrophils, invasive bacteria, and a chemoattractant.

1.1 Background for our Model, Keller-Segel Chemotaxis Models, and the Associated Analysis

The model we consider is a variation of the model found in the author's master's thesis [14]. This is a model of a bacterial infection in a radially symmetric, chronic wound. These wounds are estimated to affect 1.3-3 million Americans annually. The resulting treatment for these wounds is between \$5 billion and \$10 billion [27]. The (nondimensional) model studied in that work is given by

$$\begin{cases} \frac{\partial w}{\partial t} &= \frac{\partial^2 w}{\partial x^2} + \beta + \kappa G(t) - \lambda_{nw} n w - \lambda_{bw} b w - \lambda_w w \\ \frac{\partial n}{\partial t} &= D_n \frac{\partial^2 n}{\partial x^2} - \chi_n \frac{\partial}{\partial x}\left(n \frac{\partial c}{\partial x}\right) + k_{bn} b n g_n(w)\left(1-n\right) - \lambda_n \frac{n(1+h_n(w))}{1+eb} \\ \frac{\partial b}{\partial t} &= \varepsilon_b \frac{\partial^2 b}{\partial x^2} + k_b b\left(1-b\right) - b \frac{w}{K_w + w} \frac{\delta + k_{nr} n}{\lambda_{rb} b + 1} - \lambda_b b \\ \frac{\partial c}{\partial t} &= D_c \frac{\partial^2 c}{\partial x^2} + k_c b - \lambda_c c, \end{cases} \tag{1.1}$$

where w denotes the oxygen, n the neutrophils, b the invasive bacteria, and c the chemoattractant. This model was developed by R. Schugart, modeled after his study of wound angiogesis [46] with A. Friedman, R. Zhao, and C. Sen. Due to radial symmetry, the wound was taken in the spatial interval $(0, 1)$. The time duration was taken to be $(0, T)$. The function $G(t)$ represented the control of hyperbaric oxygen therapy. The functions $g_n(w), h_n(w)$ were smooth cubic splines taken to cut off the growth and decay of neutrophils if their concentration approaches 0 or 1. The boundary and initial conditions were set as

$$\begin{aligned}
&\left.\frac{\partial w}{\partial x}\right|_{x=0} = 0 && \left.\frac{\partial n}{\partial x}\right|_{x=0} = 0 \\
&w(1,t) = 1 && n(1,t) = 1 \\
&w(x,0) = 1 && n(x,0) = x^2 e^{-\left(\frac{1-x}{\varepsilon}\right)^2} \\
& && \\
&\left.\frac{\partial c}{\partial x}\right|_{x=0} = 0 && \left.\frac{\partial c}{\partial x}\right|_{x=1} = 0 \\
&b(x,0) = (1-x)^2 e^{-\left(\frac{x}{\varepsilon}\right)^2} && c(x,0) = (1-x)^2 e^{-\left(\frac{x}{\varepsilon}\right)^2}.
\end{aligned} \tag{1.2}$$

In this model, the oxygen serves multiple purposes. First, the oxygen levels determine the growth of the neutrophils as well as the invasive bacteria. Second, the ischemic wounds to be modeled are thought to arise from a lack of blood flow (and hence oxygen and nutrients) to the wound. The neutrophil equation is used to represent some of the body's defense cells, called neutrophils. As such, this variable represents the body's response to the infection. The bacteria in the model refer to an invasive species in the wound, which uses the nutrients and oxygen being transported into the wound. The bacteria release a chemical, called the chemoattractant, into the wound. One of the salient features of the model is the response of the neutrophils to the gradient of the chemoattractant. In [14], the author studied a similar model's well-posedness and an optimal control problem associated with the wound. Of particular note, our modified model will remove the artificial motility imposed in the bacterial equation in [14]. Also in the referenced work, the goal was to minimize the invasive bacteria via a single distributed control acting on the oxygen equation. To contrast, our current work will study a null controllability problem on the modified system. As we will require the zero state to be achieved, we will need to impose the controls on many more equations.

As mentioned above, one noteworthy aspect of the model considered is the presence of the chemotactic response: the attraction of neutrophils to increasing gradients of the chemoattractant released by the invasive bacteria. This process was given a mathematical model by Keller and Segel in the seminal 1970 paper [25], inspired by the morphogenesis of slime molds. There the authors showed the aggregation of cells could arise from perturbations to a uniform equilibrium state. In the 1971 paper by the same authors, [26], unicellular organisms were assumed to be moving via Brownian motion, however fluctuations of the chemical concentration of the chemoattractant would cause the organisms to favor one direction of motion. According to the authors, "when averaged over many cells, or a long time interval, a macroscopic flux is derived which is proportional to the chemical gradient." This description gives rise to the family of PDE models for chemotaxis known, collectively, as Keller-Segel models for chemotaxis. The general form of these models is given by

$$\begin{cases} u_t = \nabla \cdot (D(u,v)\nabla u - u\nabla\chi(v)), \quad \text{in } (0,T]\times\Omega \\ v_t = \Delta v + S(u,v), \quad \text{in } (0,T]\times\Omega \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, \quad \text{in } (0,T]\times\partial\Omega \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x\in\Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is the spatial region where the u cells reside, the chemoattractant is denoted by v, and $\chi(v)$ represents the chemotactic coupling between the cells and the chemical. The semi-random motility of the cells is described by $D(u,v)$. The function $S(u,v)$ represents the production and decay of the chemoattractant.

Keller-Segel models have become increasingly popular over the few decades. Although a complete description of the results corresponding to varying assumptions is outside the scope of this work, we shall mention a few of the results relevant to our study. For further information on Keller-Segel models, their analysis, and variations of the model, see the 2015 literature review by N. Bellomo, A. Bellouquid, Y. Tao, and M. Winkler [5]. In this work, the authors give a review of a select sample of models, as well as note relevant existence and blow-up results. They note asymptotic behavior for certain models, as well. Care is given to mathematical methods used to derive the relevant results. For those interested in the modeling and application of chemotaxis models, see the literature review by Painter [43]. There, the author provided numerous examples of the application of chemotaxis models to biology, pathology, ecology, as well as a novel application to describe clique formation in academia.

We mention a few of the known results on the classic Keller-Segel Model (1.3). In [47], A. Yagi studied the local solutions to the one dimensional system

$$\begin{cases} u_t = au_{xx} - (u(\chi(v))_x)_x, \quad \text{in } (0,\infty)\times(\alpha,\beta) \\ v_t = b\Delta v + cu - dv, \quad \text{in } (0,\infty)\times(\alpha,\beta) \\ \dfrac{\partial u}{\partial x}(\alpha,t) = \dfrac{\partial u}{\partial x}(\beta,t) = \dfrac{\partial v}{\partial x}(\alpha,t) = \dfrac{\partial v}{\partial x}(\beta,t) = 0, \quad \text{in } (0,\infty) \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x \in (\alpha,\beta), \end{cases} \tag{1.4}$$

where $\chi(v)$ is a smooth function such that $0 \le \chi(v) \le C(1 + 1/v)$, as well as conditions under which the solutions blow up in finite time.

In the same year, Nagai *et al.* [39] showed that the same model in two dimensions, *mutatis mutandis*, will have globally bounded solutions in the case that $\chi(v) = \chi v$ (*i.e.*, the chemotactic response is linear). In particular, they used the Trudinger-Moser inequality to show that if the initial conditions are positive functions in $H^{1+\varepsilon}(\Omega)$ for $0 < \varepsilon \le 1$, and $\int_\Omega u_0 dx < 4\pi/(c\chi)$, then (u, v) exists globally and there exists a $C > 0$ (depending on the initial conditions) such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} \leq C, \quad \forall t > 0.$$

If, moreover, Ω is a disk and (u_0, v_0) is radially symmetric, then the same conclusion holds under the assumption that $\int_\Omega u_0 dx < 8\pi/(c\chi)$.

To contrast, Herrero and Velázquez [16], [17] constructed blow-up solutions in $\mathbb{R}^2$ in the case that the domain is radially symmetric and one takes particular symmetric initial conditions. Much like in [39], this result assumes $\chi(v) = \chi v$. Herrero and Velázquez showed that the ensuing radially symmetric solutions will form singularities comparable to the delta function. This phenomena is known as chemotactic collapse, as it represents the total population concentration into a single cell. The authors furthermore describe the mechanism by which the singularities form. The blow up is shown to occur when $\int_\Omega u_0 dx \geq 8\pi/(c\chi)$. As such, these papers combine with the work of Nagai to form a characterization of when the solutions blow up in finite time for the case that the domain is radially symmetric in $\mathbb{R}^2$.

In their 2001 paper, Osaki and Yagi [40] studied the system (1.4) assuming that χ was a smooth function such that its derivatives satisfy $|\chi^{(i)}(v)| \leq C(v + \frac{1}{v})^i$. The authors found that if $u_0 \in C\left(\overline{\Omega}\right)$ and $v_0 \in \cup_{q>1} W^{1,q}(\Omega)$ are given non-negative functions, then the solution pair (u, v) is global and bounded in the sense there exists a $C > 0$ (depending on the initial conditions) such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} \leq C, \quad \forall t > 0.$$

1.2 Background to Local Null Controllability for Parabolic Systems

To introduce the relevant concepts for controllability, we mention the relevant concepts from standard differential equations material. Consider the initial value problem (IVP):

$$\begin{cases} y' + ay = g(t), t > 0 \\ y(0) = y_0. \end{cases} \tag{1.5}$$

The variation-of-parameters formula (which can also be derived via integrating factors) is given by

$$\begin{aligned} y(t) &= e^{-at}y_0 + \int_0^t e^{-a(t-s)}g(s)\ \mathrm{d}s \\ &:= y_h + y_p, \end{aligned} \tag{1.6}$$

which tells us the solution to the inhomogeneous IVP is a superposition of the contribution from the initial conditions (the homogeneous solution) and the contribution from the external forcing term (the particular solution). Thus, it is evident that the inhomogeneous term is capable of modifying the trajectory of the solution away from the unperturbed dynamics, described by the homogeneous equation.

More abstractly, the variation-of-parameters formula defines a solution map from the pair $(y_0, g(t)) \in \mathbb{R} \times \mathcal{U}$ to the output function $y(t) \in \mathcal{Y}$, where $\mathcal{U}, \mathcal{Y}$ are some appropriately chosen spaces of functions. For the purposes of the controlling (1.5), we wish to invert this solution map for a given terminal time T. In particular, given a desired pair of initial and final states $(y_0, y_T) \in \mathbb{R}^2$ at time T, we wish to determine if it is possible to find a controller $g(t) \in \mathcal{U}$ to reach y_T. In other words, we want

$$
\begin{aligned}
y_T &= y(T) \\
&\overset{(1.6)}{=} e^{-aT} y_0 + \int_0^T e^{-a(T-s)} g(s) \, \mathrm{d}s.
\end{aligned}
\tag{1.7}
$$

In the context of PDEs, the situation is similar with the appropriate modification to the possible target output. For treatment of controllability of PDEs, several texts are available. Those new to controllability would benefit from the introduction found in Zabczyk [49]. For those interested in an introduction to modern PDE theory alongside control theory, see Bensoussan *et al.* [6]. Finally, those interested in more thorough treatments - including controls acting on the boundary - are referred to Lasiecka and Triggiani's monographs [29], [30] as well as those by Lions and Magenes [33], [34], [35]. We follow the approach in [29], based on the theory of semigroups. For an introduction to modern PDEs via semigroup theory, see Pazy[42].

We suppose our partial differential equation system can be written abstractly as

$$
\begin{cases}
y' + Ay = g(t), t > 0 \\
y(0) = y_0,
\end{cases}
\tag{1.8}
$$

with $\mathcal{D}(A) \subseteq \mathcal{Y}$ is the domain of A in $\mathcal{Y}$, $y_0 \in \mathcal{D}(A)$, A generates a strongly continuous semigroup e^{At} on $\mathcal{D}(A)$. Here $\mathcal{Y}$ is a linear space of functions over the spatial domain $\Omega \subset \mathbb{R}^n$. The semigroup is a family of operators, parameterized by t, mapping $\mathcal{D}(A)$ into $\mathcal{Y}$. If such a semigroup exists, one may write the so-called mild solution to (1.8) in the variation-of-parameters form as

$$
y(t) = e^{-At} y_0 + \int_0^t e^{-A(t-s)} g(s) \, \mathrm{d}s. \tag{1.9}
$$

As noted in Zabczyk [49], the issue of controllability is more subtle in this (infinite dimensional) setting. In particular, when $-A$ generates an analytic semigroup on $\mathcal{Y}$ (as will be our case), it is known that if $y_0 \in \mathcal{Y}$, then $e^{-At}y_0 \in \bigcap_n \mathcal{D}((-A)^n)$ for $t > 0$ [42]. This means that, regardless of the starting regularity, the image will be arbitrarily smooth in the spatial variables. In light of this fact, it would not be natural to ask for controllability to arbitrary elements of $\mathcal{Y}$. Instead, one can ask the solution to be driven to the steady states. One can also look for approximate controllability, being driven to an ε-neighborhood of a given state of $\mathcal{Y}$.

With regard to our controllability problem, we wish to study local null controllability of our system. This means that, given $T > 0$ and $\omega \subset \Omega$ we wish to impose controllers u whose supports are contained in ω and which force $y(T) = 0$ in $\mathcal{Y}$. Letting B denote the restriction of a function to ω, we can update (1.8) with $g(t) = Bu(t)$ to the controlled initial-value problem

$$\begin{cases} y' + Ay = Bu(t), t > 0 \\ y(0) = y_0. \end{cases} \tag{1.10}$$

To say that we want $y(T) = 0$ in $\mathcal{Y}$ is to say that we wish our mild solution to satisfy

$$0 = e^{-At}y_0 + \int_0^t e^{-A(t-s)}Bu(s)\ \mathrm{d}s. \tag{1.11}$$

This means we wish the convolution integral operator to map into the range of e^{-AT}. As was noted in the work [31] by I. Lasiecka and R. Triggiani, obtaining this result is equivalent (by corollary to the Closed Graph Theorem) to the observability inequality,

$$\|\varphi(0)\|^2_{\mathcal{Y}^*} \leq C \int_0^T \|B^*\varphi(t)\|^2_{\mathcal{U}^*} dt, \tag{1.12}$$

with controls in $L^2(0,T;\mathcal{U})$. In the above, $*$ denotes the topological dual and C is a positive constant. The variable φ is a solution to the adjoint problem

$$\begin{cases} \varphi_t - A^*\varphi &= 0, \\ \varphi(T) &= \varphi_T \in \mathcal{Y}^*. \end{cases} \tag{1.13}$$

This abstract approach to control of PDEs was first presented systemically in the survey paper by Russell [44].

The observability inequality (1.12) is often interpreted in terms of an inverse problem. This inequality allows us to recover the initial conditions $\varphi(0)$ from the observation of φ over $(0,T]$. In particular, if one observes $B^*\varphi(t) = 0$, (1.12) immediately implies that $\varphi(0) = 0$. In addition, by the linearity of the dual problem, if one has another (approximate) solution $\tilde{\varphi}$, then the difference $\varphi - \tilde{\varphi}$ will also solve the dual problem. Thus, this difference would also satisfy the observability inequality. If we observe $B^*(\varphi - \tilde{\varphi}) = 0$, then we would have $\varphi(0) - \tilde{\varphi}(0) = 0$ and so $\varphi(0) = \tilde{\varphi}(0)$, meaning we have recovered the initial conditions.

When attempting to show inequalities of the form (1.12), one often appeals to Carleman-type estimates. These estimates were first introduced for establishing unique continuation properties by T. Carleman in 1939, for elliptic equations in two variables [7]. These estimates were extended in the volumes by Hörmander, [19], [20], [21], and [22].

For the application to controllability of parabolic-like systems, the first results were due to Lebeau and Robbino as well as Fursikov and Imanuvilov. For the former pair, see [32]. In this paper, they studied the null controllability to the linear heat equation directly for localized L^2 controls. The latter pair developed the Carleman-like estimates for rather general parabolic equations and applied these estimates using both localized controllers and boundary controls. See the lecture notes [13] as well as the paper by Imanuvilov [24]. For an overview of the use of

Carleman estimates in the control of parabolic equations, including nonlinear heat equations, see the survey paper by Fernández-Cara and Guerrero, [11]. Alongside a recounting of known results, the authors recount the construction of Fursikov and Imanuvilov with a deliberate exposition.

To our knowledge, the first result for control of chemotaxis system was in Guo and Zhang in 2014 [15]. In this work, they study a modified parabolic-elliptic Keller Segel model

$$\begin{cases} u_t &= \nabla \cdot (\nabla u - \chi u \nabla v) + 1_\omega f \quad \text{in } \Omega \times (0,T) \\ 0 &= \Delta v - \gamma v + \delta u \quad \text{in } \Omega \times (0,T) \\ \dfrac{\partial u}{\partial \nu} &= \dfrac{\partial v}{\partial \nu} = 0 \quad \text{on } \Gamma \times (0,T) \\ u(x,0) &= u_0(x) \quad \text{for } x \in \Omega, \end{cases} \tag{1.14}$$

where $\Omega \subseteq \mathbb{R}^N$ and ν is the outward normal derivative. In this work, the authors show that (1.14) is locally controllable via localized $L^\infty(0,T;L^\infty(\omega))$ controllers. As the elliptic equation can be solved via elliptic theory in terms of the cell density u, one reduces the system to a single nonlinear parabolic equation. From there the authors take the standard approach via a fixed point argument. In particular, the authors study the controllability of a linearized system. After noting the relevant Carleman estimates, the authors use a variational argument to establish the local null controllability. This approach is detailed in the paper by Fernández-Cara and Guerrereo [11]. To study the nonlinear control problem, they use the Kakutani fixed point on a version of the control-to-state map, which they denote by Ψ.
Another important contribution to the control of the Keller Segel system comes

from [9]. Here, F. W. Chaves-Silva and S. Guerrero show that the system

$$\begin{cases} u_t & = \nabla \cdot (\nabla u - u\nabla v) & \text{in } \Omega \times (0,T) \\ \varepsilon v_t & = \Delta v - bv + au + 1_\omega g & \text{in } \Omega \times (0,T) \\ \dfrac{\partial u}{\partial \nu} & = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \Gamma \times (0,T) \\ u(x,0) & = u_0(x), v(x,o) = v_0(x) & \text{for } x \in \Omega, \end{cases} \tag{1.15}$$

is uniformly controllable for control $g \in L^2(0,T;H^1(\Omega))$ and initial data $u_0 \in H^1(\Omega), v_0 \in H^2(\Omega)$ satisfying certain conditions.

More recently, Okposo and Willie [41] studied the controllability of the system

$$\begin{cases} u_t & = \nabla \cdot (\nabla u - u\chi\nabla v) + 1_\omega f & \text{in } \Omega \times (0,T) \\ v_t & = \Delta v - \lambda v + au & \text{in } \Omega \times (0,T) \\ \dfrac{\partial u}{\partial \nu} & = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \Gamma \times (0,T) \\ u(x,0) & = u_0(x), v(x,0) = v_0(x) & \text{for } x \in \Omega. \end{cases} \tag{1.16}$$

In this work, the authors find their system to be locally null-controllable for initial conditions $u_0, v_0 \in L^2(\Omega)$ and a single controller $f \in L^2(0,T;L^2(\omega))$. Much like prior results, their controllability arguments focus on applying the global Carleman estimates to the linear adjoint system and applying a fixed point argument for the nonlinear system.

In what follows, we consider a modification of the system (1.1) and (1.2). Our system is comprised of four equations, three of which have diffusion, one of which is degenerate, in the sense that it is a vector-valued ordinary differential equation. We approach existence via a fixed-point argument, treating the nonlinearities as perturbations of generators of analytic semigroups via maximal regularity arguments. Our nonlinearities of the form $g(s) = \dfrac{s}{1+s}$ suggest the possibility of finite-time blow up unless we can control the sign of the solutions. This motivates both the analysis of positive solutions as well as the short-time controllability so we may obtain the desired state before blow-up occurs. A linear version of our control problem is studied, where we must introduce Carleman estimates due to having controllers restricted to compactly embedded subdomains. This provides us with a well-defined linear map from the set of controls to the solution space. For this nonlinear problem, we solve via the implicit function theorem in Banach spaces, which requires showing the control-to-state is a local homeomorphism. The result from the linear problem is crucial in establishing the continuous invertibility of the control-to-state map at the desired control state.

CHAPTER 2

WELL-POSEDNESS AND POSITIVITY OF SOLUTIONS

In this chapter, we begin by introducing the model to be studied, noting the variations between the current work and the previous variant studied in [14]. From there we quickly move into determining the well-posedness of the model, which is to say to determine whether the model admits solutions in an appropriate function space. The outline of the approach is as follows: we first search for solutions for small initial data. This will involve a fixed point map constructed from the variation-of-parameters form of the equations. By analyzing the nonlinearities in our model, we are able to show the convolution integrals map from the desired solution space into itself in a locally Lipschitz manner. From there, one is able to apply the contraction principle of Banach and find a unique fixed point. The restrictions on the initial conditions arise first from a desire to avoid potential singularities in some nonlinear terms, however further restrictions must be made to guarantee that the fixed-point map is a contraction.

From there, we wish to show that our model admits a form of maximum principle. In particular, we show that the local solutions are nonnegative whenever the initial conditions are nonnegative functions. This argument makes use out of a particular nonlinear functional, one which appears useful for more general semilinear and quasilinear parabolic equations. The purpose of this functional is to develop a Grönwall inequality associated to the solutions of the system. Using properties of the functional will directly imply nonnegativity of the solution.

Finally, we return to the existence problem to establish local-in-time existence for (potentially large) initial conditions. This is, essentially, a culmination of the previous arguments. In essence, we may start with nonnegative initial conditions, the solution will exist for some time, and over that time the solution will remain nonnegative by the previous result. From there, we may iterate the solution until the solution fails to exist or becomes negative.

2.1 Introduction to the Model

We work with a modified version of the model from [14] for describing the dynamics in a radially symmetric, shallow, chronic wound. We note that the system was nondimensionalized in the work by Guffey, and so the unknown functions are to be treated as concentration variables. The system we take is the following:

$$\begin{cases} w_t &= w_{xx} - \lambda_1 nw - \lambda_2 bw - \lambda_3 w + \lambda_4 + \chi_1 u_1 \\ n_t &= \lambda_5 n_{xx} - \lambda_6 \left(nc_x\right)_x + \lambda_7 bn\left(1-n\right) - \lambda_{16} n + \chi_2 u_2 \\ b_t &= \lambda_8 b(1-b) - b\dfrac{w}{\lambda_9 + w}\dfrac{\lambda_{10}+\lambda_{11}n}{\lambda_{12}b+1} + \chi_3 u_3 \\ c_t &= \lambda_{13}c_{xx} + \lambda_{14}b - \lambda_{15}c + \chi_4 u_4 \end{cases} \tag{2.1}$$

considered in $Q = (0,T)\times(0,1)$. Here $x = 0$ represents the center of the wound and $x = 1$ denotes the outside edge of the wound, an artifact of the nondimensionalization. For boundary conditions we take

$$\begin{cases} \left.\dfrac{\partial w}{\partial x}\right|_{x=0} = 0, \quad \left.\dfrac{\partial w}{\partial x}\right|_{x=1} = 0 \\ \left.\dfrac{\partial n}{\partial x}\right|_{x=0} = 0, \quad \left.\dfrac{\partial n}{\partial x}\right|_{x=1} = 0 \\ \left.\dfrac{\partial c}{\partial x}\right|_{x=0} = \quad \left.\dfrac{\partial c}{\partial x}\right|_{x=1} = 0 \end{cases} \tag{2.2}$$

and initial conditions $w_0, n_0, b_0, c_0 \in H^1(0,1)$. The boundary conditions at the origin are a type of compatibility condition. If those conditions are not met, the solutions may form cusps at the center of the wound, which is undesirable from a physical point of view. As for the independent variables, w denotes the concentration of oxygen in the wound, n denotes the concentration of the neutrophils (defense cells), b denotes the concentration of the invasive bacteria, and c denotes the concentration of a chemoattractant generated by the bacteria. The functions u_i denote the controls, taken to live in $L^2(0,T;L^2(0,1))$ for the parabolic-like equations, and $u_3 \in L^2(0,1;H^1(0,1))$. The coefficients λ_i are positive constants for all i, save for λ_4 which can be taken to be zero. As the signs of these terms are fixed, the expected contribution can be seen from the sign preceding any of the source terms: positive signs denote a growth factor and negative signs denote a decay of the given unknown. The functions χ_i denote characteristic functions for sets ω_i which are compactly supported subdomains of $(0,1)$ for $i = 1,2,4$ and $\omega_3 = (0,1)$.

We briefly mention the physical interpretation of each of the terms in the model. The oxygen w is assumed to diffuse throughout the wound at constant rate 1 (which is an artifact of the nondimensionalization). The oxygen is taken up at constant rates by the neutrophils and bacteria, represented by the multiplicative terms $\lambda_1 nw$ and $\lambda_2 bw$. The term $-\lambda_3 w$ represents the constant departure of oxygen from the wound. The control on the oxygen would have a physical interpretation of hyperbaric oxygen therapy, albeit localized through some yet-to-be-discovered technology. The neutrophils n are also assumed to diffuse throughout the wound, this time at a constant rate λ_5. The chemotactic coupling term $-\lambda_6 \left(nc_x\right)_x$ represents the attraction of the neutrophils towards increasing gradients of the chemoattractant c. This is the description of chemotaxis taken from the Keller-Segel model. The term $\lambda_7 bn\left(1-n\right)$ represents a modified logistic growth source terms. In the bacteria equation in b, we assume a constant rate contribution coming from logistic growth $\lambda_8 b(1-b)$. The decay of the bacteria is assumed to depend on the levels of oxygen and the concentration of neutrophils in the wound. In particular, the decay will stop if the oxygen is removed from the wound. Finally, The chemoattractant c is assumed to diffuse throughout the wound at a constant rate given by λ_{13}. The chemical is created by the bacteria at a constant rate of λ_{14} and is assumed a natural decay rate of $\lambda_{15} c$.

The careful reader will note the discrepancies between the current model and the one studied previously, listed as (1.1). We take a moment here to mention a few of the changes made. Mathematically, one of the most striking is the lack of diffusion in the bacterial equation. As noted in [14], this random motility was imposed artificially to aid in the mathematical analysis of the model. The numerical simulations in Guffey's work showed little spatial variation of the oxygen solution, and so as a simplification we took $g_n(w)$ to be constant. Similarly, we took the neutrophil decay to be directly proportional to the concentration itself. Although this was done as a simplification, the fact that $g_n(w), h_n(w)$ was a smooth spline means this term is no more complicated than the decay given in the bacterial equation. As such, any of the ensuing analysis can be carried through without any major modification. Finally, the last modification made to the model was to take zero Neumann conditions at the outside edge of the wound. This was a mathematical simplification, however both the existence results and Carleman estimates can be stated with boundary conditions split into Neumann and Dirichlet, so long as these conditions are not imposed on the same part of the boundary.

2.2 Well-Posedness for Small Initial Conditions

In this section we show that we have well-posedness for small initial conditions. First, we take note of the linear part of the problem, which corresponds to the linearization about zero. We apply the known maximal regularity for the resulting equations to obtain solutions for given inhomogeneous terms. From there, we need to incorporate the nonlinearities. To do so, we write our system in the form of a nonlinear abstract ODE system. We wish to apply a fixed point argument, and hence we construct potential solvers (for mild solutions) via the variation-of-parameters formula. Our nonlinearities are treated as nonlinear operators between suitable function spaces. To apply the contraction mapping principle, we show that these operators are well-defined and are Lipschitz. This allows us to show that we map a sufficiently small ball into itself. Applying the contraction mapping principle produces our fixed points, which constitute the local mild solutions we seek.

We start by considering the linear part of our system

$$\begin{cases} w_t &= w_{xx} + \lambda_4 + \chi_1 u_1 \\ n_t &= \lambda_5 n_{xx} - \lambda_{16} n + \chi_2 u_2 \\ b_t &= \lambda_8 b + \chi_3 u_3 \\ c_t &= \lambda_{13} c_{xx} + \lambda_{14} b - \lambda_{15} c + \chi_4 u_4 \end{cases} \tag{2.3}$$

with zero Neumann conditions. For the variables w, n, c we may invoke the following maximal regularity result, which is a consequence of (0.7), (0.9) and (0.14) from "Control Theory for Partial Differential Equations I" [29]:

Theorem 2.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded, smooth domain. Let $y_0 \in H^1(\Omega)$ and $f \in L^2(0,T;L^2(\Omega))$. Then the problem

$$\begin{cases} y_t &= \Delta y + f \quad \text{on } \Omega \times (0,T) \\ \frac{\partial y}{\partial \nu} &= 0 \quad \text{on } \Gamma \times (0,T) \\ y(x,0) &= y_0(x) \quad \text{on } \Omega \end{cases} \tag{2.4}$$

has a unique solution

$$y \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)) \tag{2.5}$$

satisfying

$$\|y\|_{L^2(0,T;H^2(\Omega))} + \|y_t\|_{L^2(0,T;L^2(\Omega))} \leq C\left(\|f\|_{L^2(0,T;L^2(\Omega))} + \|y_0\|_{L^2(\Omega)}\right). \tag{2.6}$$

This continuity condition is equivalent to saying the map S defined by

$$f(\cdot,t) \stackrel{S}{\longmapsto} \int_0^t e^{-A(t-s)} f(s)\ \mathrm{d}s \tag{2.7}$$

is a bounded map $S : L^2(0,T;L^2(\Omega)) \to L^2(0,T;D(-A)) \cap H^1(0,T,L^2(\Omega))$, where A is the realization of $(-\Delta)$ with Neumann boundary conditions and hence $-A$ generates the analytic semigroup e^{-At}. Applying the previous result to the linear system gives estimates (2.6) in the w, n, c variables for the linear part of the problem. The b equation can be dealt with using standard semigroup theory, the details of which are given later in the contraction argument. Combining these results, we have the following lemma.

Lemma 2.2. Let $T > 0$ and $w_0, n_0, b_0, c_0 \in H^1(0,1)$, $u_1, u_2, u_4 \in L^2(0,T;L^2(0,1))$ and $u_3 \in L^2(0,T;H^1(0,1))$. Then system (2.3) with zero flux conditions possesses a solution $w, n, c \in H^1(0,T;L^2(0,1)) \cap L^2(0,T;H^2(0,1))$ and $b \in H^1(0,T;H^1(0,1))$ satisfying the estimates

$$\begin{aligned}
\|w\|_{L^2(0,T;H^2(\Omega))} + \|w_t\|_{L^2(0,T;L^2(\Omega))} &\leq C\left(\|w_0\|_{L^2(\Omega))} + \|u_1\|^2_{L^2(0,T;L^2(\omega))}\right) \\
\|n\|_{L^2(0,T;H^2(\Omega))} + \|n_t\|_{L^2(0,T;L^2(\Omega))} &\leq C\left(\|n_0\|_{L^2(\Omega))} + \|u_2\|^2_{L^2(0,T;L^2(\omega))}\right) \\
\|b\|_{H^1(0,T;H^1(0,1))} &\leq C\left(\|b_0\|_{H^1(0,1)}\| + \|u_3\|_{L^2(0,T;H^1(0,1))}\right) \\
\|c\|_{L^2(0,T;H^2(\Omega))} + \|c_t\|_{L^2(0,T;L^2(\Omega))} &\leq C\big(\|c_0\|_{L^2(\Omega))} + \|b_0\|_{H^1(0,1)}\| \\
&\quad + \|u_3\|_{L^2(0,T;H^1(0,1))} + \|u_4\|^2_{L^2(0,T;L^2(\omega))}\big).
\end{aligned} \tag{2.8}$$

Our next step is to incorporate the lower-order nonlinear terms. To this end, we define the following with respect to system (2.1)

$$
\begin{aligned}
f_{1,1} &= -\lambda_1 nw \\
f_{1,2} &= -\lambda_2 bw \\
f_1 &= f_{1,1} + f_{1,2} \\
f_{2,1} &= \lambda_7 bn \\
f_{2,2} &= -\lambda_7 bn^2 \\
f_2 &= f_{2,1} + f_{2,2} \\
p &= -\lambda_6 (nc_x)_x \\
f_{3,1} &= -\lambda_8 b^2 \\
f_{3,2} &= -\lambda_{10} \frac{b}{\lambda_{12} b + 1} \frac{w}{\lambda_9 + w} \\
f_{3,3} &= -\lambda_{11} n \frac{b}{\lambda_{12} b + 1} \frac{w}{\lambda_9 + w} \\
f_3 &= f_{3,1} + f_{3,2} + f_{3,3}.
\end{aligned} \tag{2.9}
$$

With these definitions, system (2.1) becomes

$$
\begin{cases}
w_t & -w_{xx} + \lambda_3 w = f_1 \\
n_t & -\lambda_5 n_{xx} - \lambda_{16} n = p + f_2 \\
b_t & -\lambda_8 b = f_3 \\
c_t & +\lambda_{13} c_{xx} - \lambda_{14} b + \lambda_{15} c = 0.
\end{cases} \tag{2.10}
$$

Here, we treat each $f_\alpha : (0,1) \times (0,T) \to \mathbb{R}$, as compositions of functions evaluated at each point in space-time. In order to treat the system using functional-analytic techniques, we introduce the operators

$$
\begin{aligned}
A_1 z &= -\left(\frac{\partial^2}{\partial x^2} - \lambda_3 I\right) z \\
A_1 :& \mathcal{D}(A_1) \subset H^2(0,1) \to L^2(0,1) \\
A_2 &= -\left(\lambda_5 \frac{\partial^2}{\partial x^2} - \lambda_{16} I\right) z\lambda \\
A_2 :& \mathcal{D}(A_2) \subset H^2(0,1) \to L^2(0,1) \\
A_3 &= \lambda_8 I z \\
A_3 :& H^1(0,1) \to H^1(0,1) \\
A_4 z &= -\left(\lambda_{13} \frac{\partial^2}{\partial x^2} z - \lambda_{15} I\right) z \\
A_4 :& \ \mathcal{D}(A_4) \subset H^2(0,1) \to L^2(0,1) \\
A_{4,2} z &= -\lambda_{14} I z \\
A_{4,2} :& L^2(0,1) \to L^2(0,1).
\end{aligned}
$$

We may identify $\mathcal{D}(A_i) = \left\{ z \in H^2(0,1) : \left.\frac{\partial z}{\partial x}\right|_{x=0} = \left.\frac{\partial z}{\partial x}\right|_{x=1} = 0 \right\}$ for $i = 1, 2, 4$. We define the following spaces:

$$
\begin{aligned}
\mathcal{M} &= H^1(0,T; L^2(0,1)) \cap L^2(0,T; H^2(0,1)) \\
\mathcal{Z} &= [\mathcal{M}]^2 \times H^1(0,T; H^1(0,1)) \times \mathcal{M} \\
\mathcal{W} &= L^2(0,T; L^2(0,1)) \times L^2(0,T; H^1(0,1)) \times L^2(0,T; L^2(0,1))
\end{aligned}
\tag{2.11}
$$

For $t > 0$ we define $X(t) = [w(\cdot,t), n(\cdot,t), b(\cdot,t), c(\cdot,t)]^T \in \mathcal{Z}$. We then introduce the maps $F_i, P : \mathcal{Z} \to L^2(0,T; L^2(0,1))$, $i = 1, 2$ by

$$
\begin{aligned}
F_i(X(t)) &= f_i(w,n,b,c)(\cdot,t) \\
P(X(t)) &= p(n,c)(\cdot,t)
\end{aligned}
$$

as well as

$$F_3(X(t)) = f_3(w, n, b, c)(\cdot, t), F_3 : \mathcal{Z} \to L^2(0, 1; H^1(0, 1)). \tag{2.12}$$

As a preliminary step to establishing well-posedness, we note that our maps are well defined between the appropriate spaces.

Lemma 2.3. Let $T > 0$ be given. For every $0 \le t \le T$, the maps F_i, P are well-defined maps $\mathcal{Z} \to L^2(0, T; L^2(0, 1))$, $i = 1, 2$ and F_3 is well-defined as a map $\mathcal{Z} \to L^2(0, 1; H^1(0, 1))$.

Proof. We estimate each term in the following.

$$\begin{aligned}
\|f_{1,1}(n, w)\|^2_{L^2(0,T;L^2(0,1))} &= \lambda_1 \|nw\|^2_{L^2(0,T;L^2(0,1))} \\
&= \lambda_1 \int_0^T \int_0^1 |n(x,t)w(x,t)|^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\overset{H^1(0,1)\subset C([0,1])}{\le} \lambda_1 \int_0^T \sup_{x\in\overline{\Omega}} |n(x,t)|^2 \|w(\cdot,t)\|^2_{L^2(0,1)} \, \mathrm{d}t \\
&\overset{H^1(0,T;L^2(0,1))\subseteq C([0,T];L^2(0,1))}{=} \lambda_1 \|w\|^2_{C([0,T];L^2(0,1))} \times \dots \\
&\quad \dots \|n\|^2_{L^2(0,T;H^1(0,1))} \\
&\overset{R.K.}{\le} \lambda_1 e^{1/e} 4 \left(1 + \frac{1}{T}\right) \|n\|^2_{H^1(0,T;L^2(0,1))} \times \dots \\
&\quad s \dots \|w\|^2_{L^2(0,T;H^1(0,1))}.
\end{aligned} \tag{2.13}$$

Here, we used the $1D$ Sobolev embedding $W^{1,p}(I) \subset L^\infty(I)$ combining theorems 8.6 (pg. 209), 8.8 (pp. 213) in Brezis [4]. Similarly, we have

$$\begin{aligned}
\|f_{1,2}(b, w)\|^2_{L^2(0,T;L^2(0,1))} &= \lambda_2 \|bw\|^2_{L^2(0,T;L^2(0,1))} \\
&\le 4\lambda_2 e^{1/e} \left(1 + \frac{1}{T}\right) \|b\|^2_{H^1(0,T;L^2(0,1))} \|w\|^2_{L^2(0,T);H^1(0,1)}
\end{aligned} \tag{2.14}$$

Together, (2.13) and (2.14) imply that F_1 is well-defined and, in fact, bounded.

We also have

$$\begin{aligned} \|f_{2,1}(b,n)\|^2_{L^2(0,T;L^2(0,1))} &= \lambda_7 \|bn\|^2_{L^2(0,T;L^2(0,1))} \\ &\leq 4\lambda_7 e^{1/e}\left(1+\frac{1}{T}\right)\|b\|^2_{H^1(0,T;L^2(0,1))}\|n\|^2_{L^2(0,T);H^1(0,1)} \end{aligned} \tag{2.15}$$

Now we estimate the cubic term bn^2. For this, we must invoke the following interpolation inequality, which can inferred from the interpolation inequalities in Yagi [48]:

$$\|n\|_{H^\theta(0,T;H^{2-2\theta}(0,1)} \leq C\|n\|^\theta_{H^1(0,T;L^2(0,1)}\|n\|^{1-\theta}_{L^2(0,T;H^2(0,1)}. \tag{2.16}$$

With this in mind, we estimate

$$\begin{aligned} \|f_{2,2(b,n)}\|^2_{L^2(0,T;L^2(0,1))} &= \|bn^2\|^2_{L^2(0,T;L^2(0,1))} \\ &\leq \lambda_7 \int_0^T \int_0^1 |b(x,t)|^2 |n(x,t)|^4 \text{ d}x \text{ d}t \\ &\leq \lambda_7 \int_0^T \|n(\cdot,t)\|^4_{L^\infty(0,1)}\|b(\cdot,t)\|^2_{L^2(0,1)} \text{ d}t \\ &\leq \lambda_7 \|b\|^2_{C(0,T;L^2(0,1))}\|n\|^4_{L^4(0,T;L^\infty(0,1))} \\ &\overset{R.K}{\leq} \lambda_7 \|b\|^2_{C(0,T;L^2(0,1))}\|n\|^4_{H^{1/4}(0,T;L^\infty(0,1))} \\ &\overset{R.K}{\leq} C\lambda_7(1+\frac{1}{T})\|b\|^2_{H^1(0,T;L^2(0,1))}\|n\|^4_{H^{1/4}(0,T;H^{3/4}(0,1))} \\ &\overset{\theta=1/4}{\leq} C(1+\frac{1}{T})\|b\|^2_{H^1(0,T;L^2(0,1))}\|n\|_{H^1(0,T;L^2(0,1))} \times \dots \\ &\quad \cdots \|n\|^3_{L^2(0,T;H^2(0,1))}. \end{aligned} \tag{2.17}$$

Next, we deal with the term $\dfrac{b}{\lambda_{12}b+1}\dfrac{w}{\lambda_9+w}(\lambda_{10}+\lambda_{11}n)$. We will need some assumption for smallness for this term, until signs can be established. For any

$\lambda_{12} \neq 0$, if we have that $|b(\cdot,t)|_{L^\infty(0,1)} \leq \frac{1}{2\lambda_{12}}$ and $\|w(\cdot,t)\|_{L^\infty(0,1)} \leq \frac{\lambda_9}{2}$, then we have

$$\frac{1}{(1-\lambda_{12}\|b(t)\|)^2_{L\infty(0,1)}} \leq 16 \tag{2.18}$$

and

$$\frac{1}{(\lambda_9 - \|w(t)\|)^2_{L\infty(0,1)}} \leq \frac{4}{\lambda_9^2} \tag{2.19}$$

Hence we can estimate

$$\begin{aligned} &\left\|\frac{\lambda_{10} b}{\lambda_{12} b + 1}\frac{w}{\lambda_9 + w}\right\|^2_{L^2(0,T;L^2(0,1))} \\ &\leq \lambda_{10}^2 \int_0^T \frac{\|b(t)\|^2_{L^2(0,1)}\|w(t)\|^2_{L^\infty(0,1)}}{(1-\lambda_{12}\|b(t)\|_{L^\infty(0,1)})^2(\lambda_9 - \|w(t)\|_{L^\infty(0,1)})^2}\,\mathrm{d}t \\ &\leq \frac{64\lambda_{10}^2}{\lambda_9^2}\|b\|^2_{L^2(0,T;L^2(0,1))}\|w\|^2_{L^\infty(0,T;L^\infty(0,1))} \\ &\overset{R.K.}{\leq} C\frac{64\lambda_{10}^2}{\lambda_9^2}\|b\|^2_{L^2(0,T;L^2(0,1))}\|w\|^2_{H^{1/2}(0,T;H^1(0,1))} \\ &\overset{\theta=1/2}{\leq} C\frac{64\lambda_{10}^2}{\lambda_9^2}\|b\|^2_{L^2(0,T;L^2(0,1))}\|w\|_{H^1(0,T;L^2(0,1)}\|w\|_{L^2(0,T;H^2(0,1)} \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} &\left\|\lambda_{11} n \frac{b}{\lambda_{12} b + 1}\frac{w}{\lambda_9 + w}\right\|^2_{L^2(0,T;L^2(0,1))} \\ &\leq \lambda_{11}^2 \int_0^T \frac{\|b(t)\|^2_{L^2(0,1)}\|w(t)\|^2_{L^\infty(0,1)}\|n(t)\|^2_{L^\infty(0,1)}}{(1-\lambda_{12}\|b(t)\|_{L^\infty(0,1)})^2(\lambda_9 - \|w(t)\|_{L^\infty(0,1)})^2}\,\mathrm{d}t \\ &\overset{\theta=1/2}{\leq} C\frac{\lambda_{11}^2}{\lambda_9^2}\|b\|^2_{L^2(0,T;L^2(0,1))}\|w\|_{H^1(0,T;L^2(0,1)} \times \ldots \\ &\qquad \ldots \|w\|_{L^2(0,T;H^2(0,1)}\|n\|_{H^1(0,T;L^2(0,1)}\|n\|_{L^2(0,T;H^2(0,1)}. \end{aligned} \tag{2.21}$$

Similarly, for the gradient level of the $L^2(0,T;H^1(0,1))$ estimate, we have

$$
\begin{aligned}
&\left\|\frac{\lambda_{10} b_x}{(\lambda_{12} b+1)^2}\frac{w}{\lambda_9+w}\right\|^2_{L^2(0,T;L^2(0,1))}\\
&\le \lambda_{10}^2\int_0^T \frac{\|b_x(t)\|^2_{L^2(0,1)}\|w(t)\|^2_{L^\infty(0,1)}}{(1-\lambda_{12}\|b(t)\|_{L^\infty(0,1)})^4(\lambda_9-\|w(t)\|_{L^\infty(0,1)})^2}\ \mathrm{d}t\\
&\le \frac{64\lambda_{10}^2}{\lambda_9^2}\int_0^T \|b_x(t)\|^2_{L^2(0,1)}\|w(t)\|^2_{L^\infty(0,1)}\ \mathrm{d}t\\
&\le \frac{64\lambda_{10}^2}{\lambda_9^2}\|b_x\|^2_{L^\infty(0,T;L^2(0,1))}\|w\|^2_{L^2(0,T;L^\infty(0,1))}\\
&\le C\frac{\lambda_{10}^2}{\lambda_9^2}\|b\|^2_{H^1(0,T;H^1(0,1))}\|w\|^2_{L^2(0,T;H^2(0,1))}.
\end{aligned}
\tag{2.22}
$$

Similarly,

$$
\begin{aligned}
&\left\|\frac{\lambda_{10} b}{\lambda_{12} b+1}\frac{\lambda_9 w_x}{(\lambda_9+w)^2}\right\|^2_{L^2(0,T;L^2(0,1))}\\
&\le \lambda_{10}^2\lambda_9^2\int_0^T \frac{\|b(t)\|^2_{L^\infty(0,1)}\|w_x(t)\|^2_{L^2(0,1)}}{(1-\lambda_{12}\|b(t)\|_{L^\infty(0,1)})^2(\lambda_9-\|w(t)\|_{L^\infty(0,1)})^4}\ \mathrm{d}t\\
&\le 256\frac{\lambda_{10}^2}{\lambda_9^2}\|b\|^2_{L^\infty(0,T;L^\infty(0,1))}\|w_x\|^2_{L^2(0,T;L^2(0,1))}\\
&\le C\frac{\lambda_{10}^2}{\lambda_9^2}\|b\|^2_{H^1(0,T;H^1(0,1))}\|w\|^2_{L^2(0,T;H^2(0,1))},
\end{aligned}
\tag{2.23}
$$

$$
\begin{aligned}
&\left\|\lambda_{11} n\frac{b_x}{(\lambda_{12} b+1)^2}\frac{w}{\lambda_9+w}\right\|^2_{L^2(0,T;L^2(0,1))}\\
&\le C\frac{\lambda_{11}^2}{\lambda_9^2}\|b\|^2_{H^1(0,T;H^1(0,1))}\|w\|^2_{L^2(0,T;H^2(0,1))}\|n\|^2_{L^\infty(0,T;L^\infty(0,1))}\\
&\le C\frac{\lambda_{11}^2}{\lambda_9^2}\|b\|^2_{H^1(0,T;H^1(0,1))}\|w\|^2_{L^2(0,T;H^2(0,1))}\|n\|_{H^1(0,T;L^2(0,1)}\|n\|_{L^2(0,T;H^2(0,1)},
\end{aligned}
\tag{2.24}
$$

$$
\begin{aligned}
&\left\|\lambda_{11} n\frac{b}{\lambda_{12} b+1}\frac{\lambda_9 w_x}{(\lambda_9+w)^2}\right\|^2_{L^2(0,T;L^2(0,1))}\\
&\le C\frac{\lambda_{10}^2}{\lambda_9^2}\|b\|^2_{H^1(0,T;H^1(0,1))}\|w\|^2_{L^2(0,T;H^2(0,1))}\|n\|_{H^1(0,T;L^2(0,1)}\|n\|_{L^2(0,T;H^2(0,1)},
\end{aligned}
\tag{2.25}
$$

and

$$
\begin{aligned}
&\left\| \lambda_{11} n_x \frac{b}{\lambda_{12} b + 1} \frac{w}{\lambda_9 + w} \right\|^2_{L^2(0,T;L^2(0,1))} \\
&\le \lambda_{11}^2 \int_0^T \frac{\|b(t)\|^2_{L^2(0,1)} \|w(t)\|^2_{L^\infty(0,1)} \|n_x(t)\|^2_{L^\infty(0,1)}}{(1-\lambda_{12}\|b(t)\|_{L^\infty(0,1)})^2 (\lambda_9 - \|w(t)\|_{L^\infty(0,1)})^2} \,\mathrm{d}t \\
&\le C \frac{\lambda_{11}^2}{\lambda_9^2} \|b\|^2_{L^\infty(0,T;L^2(0,1)} \|w\|^2_{L^\infty(0,T;L^\infty(0,1))} \|n_x\|^2_{L^2(0,T;H^1(0,1))} \\
&\le C \frac{\lambda_{11}^2}{\lambda_9^2} \|b\|^2_{L^\infty(0,T;L^2(0,1)} \|w\|^2_{L^\infty(0,T;L^\infty(0,1))} \|n\|^2_{L^2(0,T;H^2(0,1))},
\end{aligned}
\tag{2.26}
$$

where we note that the last inequality in (2.26) can be estimated in terms of the solution space.

The above estimate shows that F_3 is a well-defined map into $L^2(0,T;H^1(0,1))$. It is our desire, however, to show that under our assumption of smallness on the data, we will also have that this mapping is continuous. As such, note the following observation from the fundamental theorem of calculus: if $f(0) = g(0)$ and $f'(x) \le g'(x)$ then $f(x) \le g(x)$. In particular we consider, for $0 < u < k/2$, the functions $f(u) = u/(k-u)$ and $g(u) = 4/k^2 u$. We have $f'(u) = k/(k-u)^2 \le 4/k^2 = g'(u)$ using similar arguments as above. Thus, $u/(k-u) \le 4/k^2 u$ for $0 \le u \le k/2$. Similarly, if $0 \le u \le 1/(2C)$ then $f(u) = u/(1-Cu) \le 1/4u = g(u)$. Hence, when $\|b(\cdot,t)\|_{L^\infty((0,1))} \le \frac{1}{2\lambda_{12}}$ and $\|w(\cdot,t)\|_{L^\infty((0,1))} \le \frac{\lambda_9}{2}$,

$$
\begin{aligned}
\frac{|b(\cdot,t)|_{L^\infty(0,1)}}{1-\lambda_{12}|b(\cdot,t)|_{L^\infty(0,1)}} &\le \frac{1}{4}|b(\cdot,t)|_{L^\infty(0,1)} \text{ and} \\
\frac{\|w(\cdot,t)\|_{L^\infty(0,1)}}{\lambda_9 - \|w(\cdot,t)\|_{L^\infty(0,1)}} &\le \frac{4}{\lambda_9^2}\|w(\cdot,t)\|_{L^\infty(0,1)}.
\end{aligned}
$$

As such,

$$
\begin{aligned}
\|f_{3,2}\|^2_{L^2(0,T;L^2(0,1))} &= \lambda_{10}^2 \int_0^T \left\| \frac{b}{\lambda_{12} b + 1} \frac{w}{\lambda_9 + w} \right\|^2_{L^2((0,1))} \, \mathrm{d}t \\
&\leq \lambda_{10}^2 \int_0^T \left\| \frac{b}{\lambda_{12} b + 1} \frac{w}{\lambda_9 + w} \right\|^2_{L^\infty((0,1))} \, \mathrm{d}t \\
&\leq \lambda_{10}^2 \int_0^T \left(\frac{|b(\cdot,t)|_{L^\infty(0,1)}}{1 - \lambda_{12}|b(\cdot,t)|_{L^\infty(0,1)}} \frac{|w(\cdot,t)|_{L^\infty(0,1)}}{\lambda_9 - |w(\cdot,t)|_{L^\infty(0,1)}} \right)^2 \, \mathrm{d}t \\
&\leq \frac{\lambda_{10}^2}{\lambda_9^4} \int_0^T \|b(\cdot,t)\|^2_{L^\infty(0,1)} \|w(\cdot,t)\|^2_{L^\infty(0,1)} \, \mathrm{d}t \\
&\leq \frac{\lambda_{10}^2}{\lambda_9^4} \|b\|^2_{L^\infty(0,T;L^\infty(0,1))} \|w\|^2_{L^2(0,T;L^\infty(0,1))} \\
&\leq \frac{\lambda_{10}^2}{\lambda_9^4} T^{1/2} \|b\|^2_{L^2(0,T;L^\infty(0,1))} \|w\|^2_{L^2(0,T;L^\infty(0,1))} \\
&\leq C \frac{\lambda_{10}^2}{\lambda_9^4} T^{1/2} \|b\|^2_{L^2(0,T;L^2(0,1))} \|w\|^2_{L^2(0,T;H^2(0,1))}
\end{aligned}
\tag{2.27}
$$

In a similar manner to these estimates, we have

$$
\begin{aligned}
&\|f_{3,2_x}\|^2_{L^2(0,T;L^2(0,1)} \\
&\leq C\lambda_{10}^2 \left(1 + \frac{1}{\lambda_{12}^2 \lambda_9^2}\right) \left(\|b\|^2_{L^2(0,T;H^1(0,1))} + \|w\|^2_{L^2(0,T;H^1(0,1)} \right).
\end{aligned}
\tag{2.28}
$$

We also have, using the argument from before,

$$
\begin{aligned}
\|f_{3,3}\|^2_{L^2(0,T;L^2(0,1)} &\leq \frac{\lambda_{11}^2}{\lambda_{12}^2} \|n\|^2_{L^2(0,T;L^2(0,1)} \\
\|f_{3,3_x}\|^2_{L^2(0,T;L^2(0,1)} &\leq \frac{\lambda_{11}^2}{\lambda_{12}^2} \|n\|^2_{L^2(0,T;H^1(0,1))} \\
&\quad + C\lambda_{10}^2 \lambda_{11}^2 \left(1 + \frac{1}{\lambda_{12}^2 \lambda_9^2}\right) \|n\|^2_{L^\infty(0,T;L^\infty(0,1))} \times \dots \\
&\qquad \dots \left(\|b\|^2_{L^2(0,T;H^1(0,1))} + \|w\|^2_{L^2(0,T;H^1(0,1)} \right).
\end{aligned}
\tag{2.29}
$$

Together, these give that F_3 is a bounded map into $L^2(0,T;H^1(0,1))$.

Next, we look at the terms in the chemoattractant term. We have

$$\begin{aligned}
\|n_x c_x\|^2_{L^2(0,T;L^2(0,1))} &\leq \lambda_6 \left(\|n_x\|^4_{L^4(0,T;L^4(0,1))} + \|c_x\|^4_{L^4(0,T;L^4(0,1))} \right) \\
&\overset{R.K.}{\leq} C\lambda_6 \left(\|n_x\|^4_{L^4(0,T;H^{1/4}(0,1))} + \frac{1}{2}\|c_x\|^4_{L^4(0,T;H^{1/4}(0,1))} \right) \\
&\overset{def}{\leq} C\lambda_6 \left(\|n\|^4_{L^4(0,T;H^{5/4}(0,1))} + \|c\|^4_{L^4(0,T;H^{5/4}(0,1))} \right) \\
&\overset{L^p\subset L^q}{\leq} C\lambda_6 T^{1/2} \left(\|n\|^4_{L^2(0,T;H^{5/4}(0,1))} + \|c\|^4_{L^2(0,T;H^{5/4}(0,1))} \right) \\
&\overset{R.K.}{\leq} C\lambda_6 T^{1/2} \left(\|n\|^4_{L^2(0,T;H^2(0,1))} + \|c\|^4_{L^2(0,T;H^2(0,1))} \right).
\end{aligned} \tag{2.30}$$

Finally, we take a look at the term nc_{xx}. We note that $n \in H^1(0,T;L^2(0,1)) \cap L^2(0,T;H^2(0,1))$, and hence by interpolation, $n \in H^{1/2+\varepsilon}(0,T;H^{1-2\varepsilon}(0,1))$ for any $0 \leq \varepsilon \leq 1/2$. By Rellich-Kondrachov, this will give $n \in L^\infty(0,T;L^\infty(0,1))$. Hence

$$\begin{aligned}
\|nc_{xx}\|^2_{L^2(0,T;L^2(0,1))} &= \lambda_6 \int_0^T \int_0^1 |n(x,t)c_{xx}(x,t)|^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\leq \lambda_6 \|n\|^2_{L^\infty(0,T;L^\infty(0,1))} \|c_{xx}\|^2_{L^2(0,T;L^2(0,1))} \\
&\leq \lambda_6 \|n\|^2_{L^\infty(0,T;L^\infty(0,1))} \|c\|^2_{L^2(0,T;H^2(0,1))} \\
&\leq C\lambda_6 \|n\|^2_{H^{1/2+\varepsilon}(0,T;H^{1-2\varepsilon}(0,1))} \|c\|^2_{L^2(0,T;H^2(0,1))} \\
&\leq C\lambda_6 \|n\|^2_{H^{1/2+\varepsilon}(0,T;H^{1-2\varepsilon}(0,1))} \|c\|^2_{L^2(0,T;H^2(0,1))} \\
&\leq C\lambda_6 \|n\|^{1+2\varepsilon}_{H^1(0,T;L^2(0,1))} \|n\|^{1-2\varepsilon}_{L^2(0,T;H^2(0,1))} \|c\|^2_{L^2(0,T;H^2(0,1))}.
\end{aligned} \tag{2.31}$$

This completes the fact that the nonlinear maps are well-defined and bounded between the necessary spaces. □

Next, we show the variation-of-parameters formula defines a mapping from a suitably chosen ball into itself. For this, recall

$$\begin{aligned} \mathcal{M} &= H^1(0,T;L^2((0,1)) \cap L^2(0,T;H^2((0,1))) \\ \mathcal{Z} &= [\mathcal{M}]^2 \times H^1(0,T;H^1(0,1)) \times [\mathcal{M}] \\ X &= (w,n,c,b)^T \end{aligned} \tag{2.32}$$

and define the balls

$$\begin{aligned} \mathcal{K}_0(R_0) &= \left\{X_0 \in [H^1(0,1)]^4 : \|X_0\|_{[H^1(0,1)]^4} < R_0\right\} \\ \mathcal{K}(R) &= \{X \in \mathcal{Z} : \|X\|_{\mathcal{Z}} < R\} . \end{aligned} \tag{2.33}$$

We note that $C = C(t,\omega)$ is a constant depending on the maximal regularity constants, the semigroups, and the final time. Furthermore, we define the maps $\mathbb{F} : \mathcal{K} \to \mathcal{Z}$ by

$$\mathbb{F}(X) = \begin{pmatrix} F_1(X) \\ P(X) + F_2(X) \\ 0 \\ F_3(X) \end{pmatrix}, \tag{2.34}$$

where F_i, P have been defined previously. Finally, given

$$\begin{aligned} X_0 &\in \mathcal{K}_0 \\ \tilde{X} = \left(\tilde{w},\tilde{n},\tilde{c},\tilde{b}\right)^T &\in \mathcal{K}, \end{aligned}$$

we define the solution map $T(\tilde{X}) = X$ via the equations

$$
\begin{aligned}
w(t) &= e^{-A_1 t} w_0 + \int_0^t e^{-A_1(t-s)} F_1(\tilde{X}(s))\ \mathrm{d}s \\
n(t) &= e^{-A_2 t} n_0 + \int_0^t e^{-A_2(t-s)} F_2(\tilde{X}(s))\ \mathrm{d}s + \int_0^t e^{-A_2(t-s)} P(\tilde{X}(s))\ \mathrm{d}s \\
b(t) &= e^{A_3 t} b_0 + \int_0^t e^{A_3(t-s)} F_3(\tilde{X}(s))\ \mathrm{d}s \\
c(t) &= e^{-A_4 t} c_0 + \int_0^t e^{-A_4(t-s)} F_4(\tilde{X}(s))\ \mathrm{d}s.
\end{aligned} \tag{2.35}
$$

Definition 2.4. We say that an ordered quadruple $[w, n, b, c]$ is a mild solution to system (2.1) with zero Neumann boundary conditions and initial conditions $[w_0, n_0, b_0, c_0] \in H^1(0,1)$ provided this quadruple is a fixed point of the maps given in equation (2.35).

In the following lemma, we show that the solution maps in (2.35) take a given ball in the solution space into itself.

Remark 2.5. Below, we take find a contraction by assuming the initial conditions are taken to be sufficiently small. This smallness condition depends on the (finite) final time T we wish to solve the system to. This dependence, although not explicitly listed, is contained in the constant C from our previous estimates. Most often this comes from invoking that for $p \geq q$, $L^p(0,T;X) \subset L^q(0,T;X)$ as a consequence of Hölder's inequality. Thus, the proof below is readily modified into choosing the final time according to the initial conditions given. This will be reiterated in the proof of theorem (2.13) below.

Lemma 2.6. Let

$$
\begin{aligned}
\tilde{C} &= C\left(4 + \lambda_1 + \lambda_2 + \lambda_6 + \lambda_7 + 2\left(\frac{\lambda_{10} + \lambda_{11}}{\lambda_9}\right)\right) \\
R_1 &= \min\left\{\frac{1}{6\tilde{C}}, \frac{1}{2\lambda_{12}}, \frac{\lambda_9}{2}\right\} \\
R_0 &\leq \frac{R_1}{2\tilde{C}}
\end{aligned}
$$

Then the map T, defined above, defines a map $\mathcal{K}(R_1) \to \mathcal{K}(R_1)$.

Proof. Let $X_0 = (w_0, n_0, c_0, b_0)^T \in \mathcal{K}_0$ and $\tilde{X} = (\tilde{w}, \tilde{n}, \tilde{c}, \tilde{b})^T \in \mathcal{K}$. We note that since we have $F_1 : \mathcal{Z} \to L^2(0,T; L^2(0,1))$ and by maximal regularity estimate (2.6) we have

$$\begin{aligned}
&\|w\|_{L^2(0,T;H^2(0,1))} + \|w_t\|_{L^2(0,T;L^2)} \\
&\le C(\|F_1(\tilde{X})\|_{L^2(0,T;L^2)} + \|w_0\|_{H^1(0,1)}) \\
&\overset{(2.13),(2.14)}{\le} C\left(\lambda_1 \|\tilde{n}\|_{H^1(0,T;L^2(0,1))} \|\tilde{w}\|_{L^2(0,T;H^2(0,1))}\right. \\
&\qquad \left. +\lambda_2 \|\tilde{b}\|_{H^1(0,T;L^2(0,1))} \|\tilde{w}\|_{L^2(0,T;H^2(0,1))}\right) \\
&\qquad + \|w_0\|_{H_0^1(0,1)} \\
&\le C\left((\lambda_1 + \lambda_2)R^2 + R_0\right).
\end{aligned} \tag{2.36}$$

Similarly, we use regularity estimate (2.6) with the equation in n and estimates (2.15), (2.17), (2.30), (2.31) to obtain

$$\begin{aligned}
&\|n\|_{L^2(0,T;H^2(0,1))} + \|n_t\|_{L^2(0,T;L^2)} \\
&\le C\left(\|F_2(\tilde{X})\|_{L^2(0,T;L^2)} + \|P(\tilde{X})\|_{L^2(0,T;L^2)} + \|n_0\|_{H^1(0,1)}\right) \\
&\le C\left(\lambda_7 \|\tilde{b}\|_{H^1(0,T;L^2(0,1)} \|\tilde{n}\|_{L^2(0,T;H^1(0,1)}\right. \\
&+\lambda_7 \|\tilde{b}\|_{H^1(0,T;L^2(0,1)} \|\tilde{n}\|^{1/2}_{H^1(0,T;L^2(0,1)} \|\tilde{n}\|^{3/2}_{L^2(0,T;H^1(0,1)} \\
&+\lambda_6 \|\tilde{n}\|^2_{L^2(0,T;H^2(0,1)} \|\tilde{b}\|^2_{L^2(0,T;H^2(0,1)} \\
&\left.+\lambda_6 \|\tilde{n}\|^{\frac{1+2\varepsilon}{2}}_{H^1(0,T;L^2(0,1)} \|\tilde{n}\|^{\frac{1-2\varepsilon}{2}}_{L^2(0,T;H^2(0,1)} \|\tilde{c}\|^2_{L^2(0,T;H^2(0,1)} + \|n_0\|_{H^1(0,1)}\right) \\
&\le C(\lambda_7 R^2 + \lambda_7 R^3 + \lambda_6 R^4 + \lambda_6 R^2 + R_0).
\end{aligned} \tag{2.37}$$

Next we consider the equation in c. Here we get a straightforward estimate by applying (2.6):

$$\begin{aligned}\|c\|_{L^2(0,T;H^2(0,1))} + \|c_t\|_{L^2(0,T;L^2)} &\leq C\left(\|b\|_{L^2(0,T;L^2)} + \|c_0\|_{H^1(0,1)}\right) \\ &\leq C\left(\|b\|_{L^2(0,T;H^1(0,1))} + \|c_0\|_{H^1(0,1)}\right)\end{aligned} \tag{2.38}$$

where we estimate the norm of b below.

In the equation for b, we do not get smoothing in the spatial regularity, and so we use elementary arguments from Pazy [42]. Since the operator $A_3 : H^1(0,1) \to H^1(0,1)$ is bounded (with constant λ_8) we know the generated semigroup $e^{A_3 t}$ is uniformly continuous. Furthermore, note that as $\tilde{b} \in H^1(0,T;H^1(0,1))$, so is $\tilde{b}^2$ from fact that H^1 is a Banach algebra in one dimension. From the estimates (2.20) – (2.26) we also have that $F_3(\tilde{X}(s)) \in H^1(0,1)$ for all $s \leq t$, meaning that the image is in the domain $\mathcal{D}(A_3)$. Since uniform continuity implies continuity, we can state from Pazy (Theorem 2.4 on pg. 4-5) that

$$\int_0^t e^{A_3 s} F_3(\tilde{X}(s)) \, \mathrm{d}s \in \mathcal{D}(A_3),$$

which is a classically differentiable function of t, since the continuity of the semigroup and F_3 means the integral can be taken in the sense of Riemann. However this will imply

$$\int_0^t e^{A_3(t-s)} F_3(\tilde{X}(s)) \, \mathrm{d}s \in H^1(0,1) \quad 0 \leq t \leq T.$$

Furthermore, from (2.10), there is an $\omega \in \mathbb{R}$ (unrelated to the set ω from the control problem) such that

$$
\begin{aligned}
\|b\|_{L^2(0,T;H^1(0,1))} &\le \|e^{A_3T}b_0\|_{H^1(0,1)} + \|\int_0^T e^{A_3(T-s)} f_{3,1}(s)\ \mathrm{d}s\|_{L^2(0,T;H^1(0,1))} \\
&+ \|\int_0^T e^{A_3(T-s)} f_{3,2}(s)\ \mathrm{d}s\|_{L^2(0,T;H^1(0,1))} \\
&+ \|\int_0^T e^{A_3(T-s)} f_{3,3}(s)\ \mathrm{d}s\|_{L^2(0,T;H^1(0,1))} \\
&\le \frac{1}{\omega}(e^{\omega T}-1)\big(\|b_0\|_{H^1(0,1)} + \|f_{3,1}\|_{L^2(0,T;H^1(0,1))} \\
&+\|f_{3,2}\|_{L^2(0,T;H^1(0,1))} + \|f_{3,3}\|_{L^2(0,T;H^1(0,1))}\big) \\
&\overset{(2.20)-(2.26)}{\le} \frac{1}{\omega}(e^{\omega T}-1)C\left(R_0 + \left(\frac{\lambda_{10}+\lambda_{11}}{\lambda_9}\right)(R^2+R^3)\right).
\end{aligned} \tag{2.39}
$$

Similarly, from the b equation, we have

$$
\begin{aligned}
\|b_t\|_{L^2(0,T;H^1(0,1))} &\le \lambda_8 \|b\|_{L^2(0,T;H^1(0,1))} + \|f_{3,1}\|_{L^2(0,T;H^1(0,1))} \\
&+ \|f_{3,2}\|_{L^2(0,T;H^1(0,1))} + \|f_{3,3}\|_{L^2(0,T;H^1(0,1))} \\
&\le \frac{1}{\omega}(e^{\omega t}-1)C(1+\lambda_8)\left(R_0 + 2\left(\frac{\lambda_{10}+\lambda_{11}}{\lambda_9}\right)(R^2+R^3)\right).
\end{aligned} \tag{2.40}
$$

Combining this with the previous estimates, we have

$$
\begin{aligned}
\|b\|_{H^1(0,T;H^1(0,1))} &\le \frac{1}{\omega}(e^{\omega T}-1)C(1+\lambda_8)\left(2R_0 + 3\left(\frac{\lambda_{10}+\lambda_{11}}{\lambda_9}\right)(R^2+R^3)\right) \\
&\le \frac{1}{\omega}(e^{\omega T}-1)C(1+\lambda_8)\left(R_0 + \left(\frac{\lambda_{10}+\lambda_{11}}{\lambda_9}\right)(R^2+R^3)\right).
\end{aligned} \tag{2.41}
$$

Combining estimates from (2.36), (2.37), (2.38) and (2.41), we have for $C = C(t,\omega)$

$$
\begin{aligned}
&\|w\|_{L^2(0,T;H^2(0,1))} + \|w_t\|_{L^2(0,T;L^2)} + \|n\|_{L^2(0,T;H^2(0,1))} + \|n_t\|_{L^2(0,T;L^2)} \\
&+ \|c\|_{L^2(0,T;H^2(0,1))} + \|c_t\|_{L^2(0,T;L^2)} + \|b\|_{H^1(0,T;H^1(0,1))} \\
&\leq C\left(4R_0 + \left(\lambda_1 + \lambda_2 + \lambda_6 + \lambda_7 + 2\left(\frac{\lambda_{10}+\lambda_{11}}{\lambda_9}\right)\right)R^2\right. \\
&\left.+ \left(\lambda_7 + 2\left(\frac{\lambda_{10}+\lambda_{11}}{\lambda_9}\right)\right)R^3 + \lambda_6 R^4\right) \\
&\leq \tilde{C}\left(R_0 + R^2 + R^3 + R^4\right).
\end{aligned} \tag{2.42}
$$

where $\tilde{C} = 4 + \lambda_1 + \lambda_2 + \lambda_6 + \lambda_7 + 2\left(\frac{\lambda_{10}+\lambda_{11}}{\lambda_9}\right)$. Noting that $R < \frac{1}{6\tilde{C}} < 1$, and recalling $R_0 \leq \frac{R}{2\tilde{C}}$ we then have

$$
\begin{aligned}
\tilde{C}\left(R_0 + R^2 + R^3 + R^4\right) &\leq \tilde{C}\left(\frac{R}{2\tilde{C}} + \frac{R}{6\tilde{C}} + \frac{R}{(6\tilde{C})^2} + \frac{R}{(6\tilde{C})^3}\right) \\
&\leq \tilde{C}\left(\frac{R}{2\tilde{C}} + \frac{R}{6\tilde{C}} + \frac{R}{6\tilde{C}} + \frac{R}{6\tilde{C}}\right) \\
&\leq R,
\end{aligned} \tag{2.43}
$$

which is what we wanted to show. □

To show contraction, we will also need to show that our maps are locally Lipshitz into the appropriate space. It is this property that will allow us to show the solution map can be taken as a contraction from $\mathcal{K}(R_1)$ into itself. We begin with the maps for equations with smoothing, namely w and n. After this, we show that the nonlinearities of the b equation are also locally Lipschitz between the requisite sets.

Lemma 2.7. The maps F_i, P are locally Lipschitz continuous as mappings $X(t) \to F_i(X(t)), P(X(t))$ from $\mathcal{K}(R_1) \to L^2(0,T;L^2(0,1))$ for $i = 1, 2$. The map F_3 is locally Lipschitz $\mathcal{K}(R_1) \to L^2(0,T;H^1(0,1))$.

Proof. Let $(w,n,c,b)^T, (\tilde{w},\tilde{n},\tilde{c},\tilde{b})^T \in \mathcal{K}(R_1)$. Then

$$
\begin{aligned}
&\|f_{1,1}(n,w) - f_{1,1}(\tilde{n},\tilde{w})\|^2_{L^2(0,T;L^2(0,1))} \\
&= \lambda_1 \|nw - \tilde{n}\tilde{w}\|^2_{L^2(0,T;L^2(0,1))} \\
&= \lambda_1 \int_0^T \int_0^1 |nw - \tilde{n}\tilde{w}|^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\leq 2\lambda_1 \left(\|\tilde{n}\|^2_{L^\infty(0,T;L^\infty(0,1))} + \|w\|^2_{L^\infty(0,T;L^\infty(0,1))}\right) \left(\|w-\tilde{w}\|^2_{\mathcal{M}} + \|n-\tilde{n}\|^2_{\mathcal{M}}\right) \\
&\leq L(R)^2 \left(\|w-\tilde{w}\|^2_{\mathcal{M}} + \|n-\tilde{n}\|^2_{\mathcal{M}}\right)
\end{aligned}
\tag{2.44}
$$

$$
\begin{aligned}
&\|f_{1,2}(b,w) - f_{1,2}(\tilde{b},\tilde{w})\|^2_{L^2(0,T;L^2(0,1))} \\
&= \lambda_2 \|bw - \tilde{b}\tilde{w}\|^2_{L^2(0,T;L^2(0,1))} \\
&\leq 2\lambda_2 \left(\|\tilde{b}\|^2_{L^\infty(0,T;L^\infty(0,1))} + \|w\|^2_{L^\infty(0,T;L^\infty(0,1))}\right) \left(\|w-\tilde{w}\|^2_{\mathcal{M}} + \|b-\tilde{b}\|^2_{H^1(0,T;H^1(0,1))}\right) \\
&\leq L(R)^2 \left(\|w-\tilde{w}\|^2_{\mathcal{M}} + \|b-\tilde{b}\|^2_{H^1(0,T;H^1(0,1))}\right)
\end{aligned}
\tag{2.45}
$$

$$
\begin{aligned}
&\|f_{2,1}(b,n) - f_{2,1}(\tilde{b},\tilde{n})\|^2_{L^2(0,T;L^2(0,1))} \\
&= \lambda_7 \|bn - \tilde{b}\tilde{n}\|^2_{L^2(0,T;L^2(0,1))} \\
&\leq 2\lambda_7 \left(\|\tilde{b}\|^2_{L^\infty(0,T;L^\infty(0,1))} + \|n\|^2_{L^\infty(0,T;L^\infty(0,1))}\right) \left(\|n-\tilde{n}\|^2_{\mathcal{M}} + \|b-\tilde{b}\|^2_{H^1(0,T;H^1(0,1))}\right) \\
&\leq L(R)^2 \left(\|n-\tilde{n}\|^2_{\mathcal{M}} + \|b-\tilde{b}\|^2_{H^1(0,T;H^1(0,1))}\right)
\end{aligned}
\tag{2.46}
$$

$$
\begin{aligned}
&\|f_{2,2}(b,n)-f_{2,2}(\tilde{b},\tilde{n})\|^2_{L^2(0,T;L^2(0,1))}\\
&=\lambda_7\|bn^2-\tilde{b}\tilde{n^2}\|^2_{L^2(0,T;L^2(0,1))}\\
&\leq \lambda_7\left(4\|b\|^2_{L^\infty(0,T;L^\infty(0,1))}(\|n\|^2_{L^\infty(0,T;L^\infty(0,1))}+\|\tilde{n}\|^2_{L^\infty(0,T;L^\infty(0,1))})\right.\\
&\left.+\|\tilde{n}\|^4_{L^\infty(0,T;L^\infty(0,1))}\right)\left(\|n-\tilde{n}\|^2_{\mathcal{M}}+\|b-\tilde{b}\|^2_{H^1(0,T;H^1(0,1)}\right)\\
&\leq L(R)^2\left(\|n-\tilde{n}\|^2_{\mathcal{M}}+\|b-\tilde{b}\|^2_{H^1(0,T;H^1(0,1)}\right)
\end{aligned}
\tag{2.47}
$$

Next, note that

$$
\begin{aligned}
u\in\mathcal{M}=H^1(0,T;L^2(0,1))\cap L^2(0,T;H^2(0,1))&\\
\implies u\in H^{1/4}(0,T;H^{3/2}(0,1))&\\
\implies u\in L^4(0,T;W^{1,4}(0,1))&\\
\implies u_x\in L^4(0,T;L^4(0,1)).&
\end{aligned}
\tag{2.48}
$$

Hence

$$
\begin{aligned}
&\|n_xc_x-\tilde{n}_x\tilde{c}_x\|^2_{L^2(0,T;L^2(0,1))}\\
&\leq \|n_x\|^2_{L^4(0,T;L^4(0,1))}\|c_x-\tilde{c}_x\|^2_{L^4(0,T;L^4(0,1))}+\|\tilde{c}_x\|^2_{L^4(0,T;L^4(0,1))}\|n_x-\tilde{n}_x\|^2_{L^4(0,T;L^4(0,1))}\\
&\leq L(R)^2\left(\|c_x-\tilde{c}_x\|^2_{\mathcal{M}}+\|n_x-\tilde{n}_x\|^2_{\mathcal{M}}\right)
\end{aligned}
\tag{2.49}
$$

and

$$
\begin{aligned}
&\|nc_{xx} - \tilde{n}\tilde{c}_{xx}\|^2_{L^2(0,T;L^2(0,1))} \\
&\leq \|n\|^2_{L^\infty(0,T;L^\infty(0,1))}\|c_{xx} - \tilde{c}_{xx}\|^2_{L^2(0,T;L^2(0,1))} + \|\tilde{c}_{xx}\|^2_{L^2(0,T;L^2(0,1))}\|n - \tilde{n}\|^2_{L^\infty(0,T;L^\infty(0,1))} \\
&\leq L(R)^2 \left(\|c_{xx} - \tilde{c}_{xx}\|^2_{\mathcal{M}} + \|n - \tilde{n}\|^2_{\mathcal{M}}\right)
\end{aligned}
\tag{2.50}
$$

For our remaining terms of the degenerate bacterial equation, we will use the properties of the maps to argue that these are Lipschitz into $L^2(0,T;H^1(0,1))$. We follow the (more general) arguments found in Yagi, reproduced here in slightly more detail. In what follows, we will first show that one can construct well-defined operators between appropriate spaces by superimposing smooth real-variable functions with the arguments taken from appropriate Sobolev spaces. We then show that, under more restrictive requirements, such maps can be smooth. As we are looking to apply these results to maps comprised of products of the form $\dfrac{s}{1+s}$, we will need to show that the product of these smooth maps are also smooth. We remark that, unlike in the case of finite dimensions, the operation of taking products is not closed in general Banach spaces. When such an operation is closed in the space, we call the Banach space a Banach algebra. What is critical for our problem is that $H^1(0,1)$ will form a Banach algebra. As such, we are critically using the spatial dimension here, among other arguments.

Lemma 2.8. Let $F:(a,b)\to R$ be C^1 as a real-variable function.Then F defines a map $u\mapsto F(u)$ between the spaces $H^1(0,1)\to H^1(0,1)$ for any u such that $u([0,1])\subset(a,b)$. Moreover, there is an increasing function ρ such that

$$
\|F(u)\|^2_{H^1(0,1)} \leq \rho(\|u\|_{H^1(0,1)})) \tag{2.51}
$$

Proof. Since $u\in H^1(0,1)$, then $u\in C([0,1])$ by Sobolev embeddings. Hence,

$u([0,1])$ is compact. Similarly, $F(u[0,1])$ would also be compact and, in particular, bounded. Hence,

$$\|F(u)\|^2_{L^2(0,1)} \leq \max_{x\in[0,1]} |F(u(x))|^2. \tag{2.52}$$

Similarly, as $F \in C^1$ and $u \in H^1$, we can write

$$\frac{\partial F(u)}{\partial x_i} = F'(u)\frac{\partial u}{\partial x_i.}$$

in the sense of L^2. Furthermore, we have

$$\left\|\frac{\partial F(u(x))}{\partial x_i}\right\|^2_{L^2((0,1)} \leq \max_{x\in[0,1]} |F'(u(x))|^2 \left\|\frac{\partial u}{\partial x_i}\right\|^2_{L^2(0,1)} \tag{2.53}$$

Combining, we see that

$$\begin{aligned} \|F(u)\|^2_{H^1((0,1))} &\leq \max_{x\in[0,1]} |F(u(x))|^2 + \max_{x\in[0,1]} |F'(u(x))|^2 \, \|u\|^2_{H^1(0,1)} \\ &\stackrel{\text{def}}{=} \rho(\|u\|_{H^1(0,1)}). \end{aligned} \tag{2.54}$$

□

Lemma 2.9. Let $F : (a,b) \to R$ be a C^2 real-variable function.Then F defines a Locally Lipschitz map $u \mapsto F(u)$ between the spaces $H^1(0,1) \to H^1(0,1)$ for any u, v such that $u([0,1]), v([0,1]) \subset (a,b)$.

Proof. We first note that, by the fundamental theorem of line integrals, we have that

$$\begin{aligned}
&\|F(u) - F(v)\|^2_{H^1(0,1)} \\
&= \left\| \int_0^1 F'(\theta u + (1-\theta)v) \, \mathrm{d}\theta (u - v) \right\|^2_{H^1(0,1)} \\
&\leq \int_0^1 \int_0^1 |F'(\theta u(x) + (1-\theta)v(x)) (u(x) - v(x))|^2 \, \mathrm{d}\theta \, \mathrm{d}x \\
&\overset{\text{F-T}}{=} \int_0^1 \int_0^1 |F'(\theta u(x) + (1-\theta)v(x)) (u(x) - v(x))|^2 \, \mathrm{d}x \, \mathrm{d}\theta \qquad (2.55) \\
&= \int_0^1 \|F'(\theta u + (1-\theta)v)u - v)\|^2_{H^1((0,1))} \, \mathrm{d}\theta \\
&\overset{*}{\leq} \int_0^1 \|F'(\theta u + (1-\theta)v)\|^2_{H^1(0,1)} \|u - v)\|^2_{H^1(0,1)} \, \mathrm{d}\theta \\
&\overset{**}{\leq} \int_0^1 \rho(\|\theta u + (1-\theta)v\|_{H^1(0,1)}) \, \mathrm{d}\theta \|u - v\|^2_{H^1(0,1)}.
\end{aligned}$$

In the step marked with $(*)$ we use the fact that $F' \in C^1$ to note that $F'(\theta u + (1-\theta)v) \in H^1$, which is a Banach algebra in one dimension. Similarly, in the step $(**)$ above we used the estimate from the previous lemma with real function F' and input function $\theta u + (1-\theta)v$ from H^1. □

Finally, we have the following.

Lemma 2.10. Suppose $F, G : H^1(0,1) \to H^1(0,1)$ are Locally Lipschitz and $F(0) = G(0) = 0$. Then the product $F(u)G(v)$ is also locally Lipschitz on the product space $H^1(0,1) \times H^1(0,1)$, where the locally Lipschitz constant is directly proportional to the product of local Lipschitz constants. In particular,

$$\begin{aligned}
&\|F(u_1)G(v_1) - F(u_2)G(v_2)\|_{H^1(0,1)} \\
&\leq 2RK_1(R)K_2(R) \left(\|v_1 - v_2\|^2_{H^1(0,1)} + \|u_1 - u_2\|^2_{H^1(0,1)} \right)^{\frac{1}{2}}
\end{aligned} \qquad (2.56)$$

Proof. Suppose $u_1, u_2, v_1, v_2 \in H^1(0,1)$ and that $F, G : H^1(0,1) \to H^1(0,1)$ are locally Lipschitz functions of single variables. In particular, suppose that there exists an $R > 0$ such that $\|u\|_{H^1(0,1)}, \|v\|_{H^1(0,1)} < R$. Then to say that F, G are locally Lipschitz means there exists constants $K_1(R)$ and $K_2(R)$ such that

$$\begin{aligned} \|F(u) - F(v)\|_{H^1(0,1)} &\leq K_1(R)\|u - v\|_{H^1(0,1)} \\ \|G(u) - G(v)\|_{H^1(0,1)} &\leq K_1(R)\|u - v\|_{H^1(0,1)}. \end{aligned} \tag{2.57}$$

We will first note that

$$\begin{aligned} \|F(u)\|_{H^1(0,1)} &= \|F(u) - F(0) + F(0)\|_{H^1(0,1)} \\ &\leq \|F(u) - F(0)\|_{H^1(0,1)} + \|F(0)\|_{H^1(0,1)} \\ &\leq K_1(R)\|u\|_{H^1(0,1)} + 0 \\ &\leq K_1(R)R \end{aligned} \tag{2.58}$$

where the second-to-last inequality follows from the assumed properties of F. Applying the same argument to G yields

$$\|G(u)\|_{H^1(0,1)} \leq K_2(R)R. \tag{2.59}$$

It then follows that

$$\begin{aligned} &\|F(u_1)G(v_1) - F(u_2)G(v_2)\|^2_{H^1(0,1)} \\ &= \|F(u_1)G(v_1) - F(u_1)G(v_2) + F(u_1)G(v_2) - F(u_2)G(v_2)\|^2_{H^1(0,1)} \\ &\overset{\text{T.I., B.A.}}{\leq} 4\|F(u_1)\|^2_{H^1(0,1)}\|G(v_1) - G(v_2)\|^2_{H^1(0,1)} \\ &+ 4\|G(v_2)\|^2_{H^1((0,1))}\|F(u_1) - F(u_2)\|^2_{H^1(0,1)} \\ &\leq 4R^2K_1^2(R)K_2^2(R)\|v_1 - v_2\|^2_{H^1(0,1)} \\ &+ 4R^2K_2^2(R)K_1^2(R)\|u_1 - u_2\|^2_{H^1(0,1)} \\ &\implies \|F(u_1)G(v_1) - F(u_2)G(v_2)\|_{H^1(0,1)} \\ &\leq 2RK_1(R)K_2(R)\left(\|v_1 - v_2\|^2_{H^1(0,1)} + \|u_1 - u_2\|^2_{H^1(0,1)}\right)^{\frac{1}{2}}. \end{aligned} \tag{2.60}$$

□

With the previous results in mind, we turn to our final nonlinear terms. For the following, we define

$$\begin{aligned} F(s) &= \frac{s}{\lambda_{12} s + 1} \\ G(s) &= \frac{s}{\lambda_9 + s}. \end{aligned} \tag{2.61}$$

When s is sufficiently bounded so that the denominators are bounded above, these functions are clearly C^2 by elementary calculus. In particular, suppose for each fixed $t > 0$ that $\|b(\cdot,t)\|_{L^\infty(0,1)} \leq \frac{1}{\lambda_{12}}$ and that $\|w(\cdot,t)\|_{L^\infty(0,1)} \leq \frac{\lambda_9}{2}$. Then by lemma (2.9) we know that, for each fixed $t > 0$, the functions

$$\begin{aligned} F(b(\cdot,t)) &= \frac{b(\cdot,t)}{\lambda_{12} b(\cdot,t) + 1} \\ G(w(\cdot,t)) &= \frac{w(\cdot,t)}{\lambda_9 + w(\cdot,t)} \end{aligned} \tag{2.62}$$

define locally Lipschitz mappings from $H^1(0,1)$ into itself. Thus, by lemma (2.10) the product will satisfy

$$\begin{aligned} &\|F(b(\cdot,t))G(w(\cdot,t)) - F(\tilde{b}(\cdot,t))G(\tilde{w}(\cdot,t))\|^2_{H^1((0,1))} \\ &\leq K^2(R)\left(\|b(\cdot,t) - \tilde{b}(\cdot,t)\|^2_{H^1((0,1))} + \|w(\cdot,t) - \tilde{w}(\cdot,t)\|^2_{H^1((0,1))}\right), \end{aligned} \tag{2.63}$$

for every $t > 0$. Replacing F, G with their definitions and integrating in time gives us that

$$\begin{aligned}
&\left\|\frac{\lambda_{10}b}{\lambda_{12}b+1}\frac{w}{\lambda_9+w}-\frac{\lambda_{10}\tilde{b}}{\lambda_{12}\tilde{b}+1}\frac{\tilde{w}}{\lambda_9+\tilde{w}}\right\|^2_{L^2(0,T;H^1(0,1))}\\
&\leq K(R)^2\left(\|b-\tilde{b}\|^2_{L^2(0,T;H^1(0,1)}+\|w-\tilde{w}\|^2_{L^2(0,T;H^1(0,1)}\right)\\
&\leq K(R)^2\left(\|b-\tilde{b}\|^2_{H^1(0,T;H^1(0,1)}+\|w-\tilde{w}\|^2_{\mathcal{M}}\right).
\end{aligned} \tag{2.64}$$

Via the same argument, we also have that

$$\begin{aligned}
&\left\|\lambda_{11}n\frac{b}{\lambda_{12}b+1}\frac{w}{\lambda_9+w}-\lambda_{11}\tilde{n}\frac{\tilde{b}}{\lambda_{12}\tilde{b}+1}\frac{\tilde{w}}{\lambda_9+\tilde{w}}\right\|^2_{L^2(0,T;H^1(0,1))}\\
&\leq L(R)^2\left(\|b-\tilde{b}\|^2_{H^1(0,T;H^1(0,1)}+\|n-\tilde{n}\|^2_{\mathcal{M}}+\|w-\tilde{w}\|^2_{\mathcal{M}}\right)
\end{aligned} \tag{2.65}$$

This completes the proof of lemma (2.7). □

Putting lemmas (2.3), (2.6), and (2.7) together we have the following result.

Proposition 2.11. For all $T_0 > 0$ there exists an $R_3 > 0$ such that for all initial data $X_0 = [w_0, n_0, b_0, c_0]^T \in [H^1(\Omega)]^4$ with $\|X_0\| < R_3$ there is a solution

$$X = [w, n, c, b]^T \in \mathcal{Z}$$

to (2.1) with zero Neumann conditions.

Proof. From lemma (2.3) and the maximal regularity theorem (2.1), we know the maps given by equation (2.35) map from $\mathcal{Z}$ into itself. From lemma (2.6) we know that, in fact, these maps map the ball $\mathcal{K}(R_1)$ into itself. Since these maps are locally Lipschitz by lemma (2.7) with the local Lipschitz constant $L(R)$ being an increasing function of the size of the initial conditions R, we may choose R_2 such that $L(R_2) < 1$. Choosing $R_3 = \min\{R_1, R_2\}$ we can have that the solution maps define a contraction from a ball $\mathcal{K}(R_3)$ into itself. Applying the contraction mapping principle yields the desired fixed points defining the mild solutions. $\square$

2.3 Positivity of Solutions and Well-Posedness for (Potentially Large) Nonnegative Initial Data

As the system is now known to have a solution for some initial data, we wish to study whether we can deduce any of the qualitative properties of the solution. Of particular interest is the question of sign. Namely, the rational-type source terms for the degenerate b equation may cause finite time blow up in that case that b, w become sufficiently negative locally. As our model has a physical nature, and the initial conditions correspond to concentrations, we would expect that all of the unknown solutions should be nonnegative. In this section, we aim to show this is - indeed - the case. In doing so, we show that the nonlinear sinks of the form $\dfrac{s}{1+s}$ are Lipschitz on the ranges of their arguments. Looking forward, this gives us an indication that we do not need to assume smallness on the initial data, merely the sign. We point out here that restricting the sign does make sense for initial data in $H^1(0,1)$, since these functions are continuous by Rellich-Kondrachov.

For this section, we simply state the main result and execute the proof. The idea of the proof comes from the examples on semilinear and quasilinear parabolic problems in the monograph by Yagi [48]. The key is to introduce, for each unknown state variable, the $C^{1,1}(\mathbb{R})$ (continuously differentiable with Lipschitz derivative) function

$$H(u) = \begin{cases} \frac{1}{2}u^2, & -\infty < u < 0 \\ 0, & 0 \le u < \infty, \end{cases} \tag{2.66}$$

along with the Nemytskii operator given by

$$\Psi(t) = \int_\Omega H(u(t,x))\ \mathrm{d}x. \tag{2.67}$$

Using the differential equations, we construct a Grönwall inequality of the form $\Psi'(t) \leq a(t)\Psi(t)$ for each state. In this inequality, the function $a(t)$ will be a function composed of the spatial norms of the unknown state functions. Applying the classical Grönwall inequality over $[0,T]$ directly implies $\Psi(t) \leq \Psi(0)e^{\int_0^t a(s)\ \mathrm{d}s}$. As $\Psi(0) = 0$ when the initial conditions are nonnegative, this must mean that $H(u) = 0$. The definition of H implies, in turn, that u is nonnegative.

Proposition 2.12. Suppose $(w_0, n_0, c_0, b_0)^T \in [H^1(0,1)]^4$ such that $0 \leq w_0(x), n_0(x), c_0(x)$ and $0 \leq b_0(x)$. Then the corresponding solution to our system (2.1) with zero flux conditions is also non-negative.

Proof. Suppose $w_0(x), n_0(x), c_0(x), b_0(x)$ are as described above. Let T_R denote the final time of existence, as determined by the parameters of the system and the initial conditions. Let w, n, c, b denote the solutions on $0 \leq t < T_R$. We introduce the function $H(u)$ as a $C^{1,1}(\mathbb{R})$ function (differentiable with Lipschitz-continuous derivative) given by

$$H(u) = \begin{cases} \frac{1}{2}u^2, & -\infty \leq u \leq 0 \\ 0, & 0 < u \leq \infty. \end{cases} \tag{2.68}$$

The derivative is then given by

$$H'(u) = \begin{cases} u, & -\infty \leq u \leq 0 \\ 0, & 0 < u \leq \infty. \end{cases} \tag{2.69}$$

We note that a quick consequence of (2.69) is the equality $H'(u)u = 2H(u)$, which we will use below. Furthermore, we note a few properties of the H function listed in Yagi. The function H defines a Lipschitz map $L^2(0,1) \to L^2(0,1)$. We also have that $u \mapsto H(u)$ is an operator $H^1(0,1)$ into itself by virtue of lemma (2.8). Furthermore, H is such that

$$\|H(u)\|_{H^1(0,1)} \leq C\|H'\|_{L^\infty}\|u\|_{H^1(0,1)} \tag{2.70}$$

and that when u is a nonsingular point for the operator H, we have

$$\frac{d}{dx}H(u) = H'(u)\frac{d}{dx}u \tag{2.71}$$

Furthermore, the function given by

$$\psi_u(t) = \int_0^1 H(u(t))\ \mathrm{d}x \tag{2.72}$$

is continuously differentiable for each $u \in C^1(a, b; L^2(0,1)$. The derivative is given by

$$\psi_u'(t) = \int_0^1 H'(u(t))u'(t)\ \mathrm{d}x. \tag{2.73}$$

and, in fact, the map $u \mapsto H'(u)$ is Lipschitz continuous. With this in mind, we use these functions in a Grönwall-based argument. The goal is to show that $\psi_u(t) \equiv 0$, which by the definition of H implies $u \geq 0$ for all t. We begin with the equation in w. Note that

$$\begin{aligned}\psi_w'(t) &= \int_0^1 H'(w(t))w'(t)\ \mathrm{d}x \\ &= \int_0^1 H'(w(t))w_{xx}(t)\ \mathrm{d}x - \lambda_1\int_0^1 H'(w(t))w(t)n(t)\ \mathrm{d}x \\ &- \lambda_2\int_0^1 H'(w(t))w(t)b(t)\ \mathrm{d}x - \lambda_3\int_0^1 H'(w(t))w(t)\ \mathrm{d}x.\end{aligned} \tag{2.74}$$

We estimate the right-hand-side in parts. To begin,

$$\begin{aligned}\int_0^1 H'(w(t))w_{xx}(t)\ \mathrm{d}x &\overset{\text{IBP}}{=} -\int_0^1 \frac{d}{dx}H'(w(t))\frac{d}{dx}w(t)\ \mathrm{d}x \\ &\overset{(2.69)}{=} -\int_0^1 \frac{d}{dx}H'(w(t))\frac{d}{dx}H'(w(t))\ \mathrm{d}x \\ &= -\int_0^1 \left|\frac{d}{dx}H'(w(t))\right|^2\ \mathrm{d}x \\ &\leq 0.\end{aligned} \tag{2.75}$$

Furthermore, by factoring $H'(w(t))w(t)$ out of the remaining terms in (2.74) and applying Hölder's equality with $H'(w(t))w(t) \in L^1$, we have

$$\begin{aligned}&\int_0^1 H'(w(t))w(t)\left(-\lambda_1 n(t) - \lambda_2 b(t) - \lambda_3\right)\ \mathrm{d}x \\ &\overset{\text{Hö}}{\leq} C\|H'(w(t))w(t)\|_{L^1(0,1)}\left(1 + \|n(t)\|_{L^\infty(0,1)} + \|b(t)\|_{L^\infty(0,1)}\right) \\ &\overset{\text{def of } H}{\leq} C\|H(w(t))\|_{L^1(0,1)}\left(1 + \|n(t)\|_{L^\infty(0,1)} + \|b(t)\|_{L^\infty(0,1)}\right) \\ &\overset{\text{R.K.}}{\leq} C\|H(w(t))\|_{L^1(0,1)}\left(1 + \|n(t)\|_{H^1(0,1)} + \|b(t)\|_{H^1(0,1)}\right) \\ &\quad C\psi_w(t)\left(1 + \|n(t)\|_{L^\infty(0,1)} + \|b(t)\|_{L^\infty(0,1)}\right) \\ &\overset{\text{R.K.}}{\leq} C\|H(w(t))\|_{L^1(0,1)}\left(1 + \|n(t)\|_{H^1(0,1)} + \|b(t)\|_{H^1(0,1)}\right) \\ &= C\psi_w(t)\left(1 + \|n(t)\|_{H^1(0,1)} + \|b(t)\|_{H^1(0,1)}\right)\end{aligned} \tag{2.76}$$

Placing the estimates from equations (2.75) and (2.76) into equation (2.74) gives

$$\psi_w'(t) \leq \psi_w(t)C\left(1 + \|n(t)\|_{H^1(0,1)} + \|b(t)\|_{H^1(0,1)}\right). \tag{2.77}$$

Via Grönwall's differential inequality, we have

$$\psi_w(t) \leq \psi(0)e^{C\int_0^t\left(1+\|n(t)\|_{H^1(0,1)}+\|b(t)\|_{H^1(0,1)}\right)\ \mathrm{d}t} \tag{2.78}$$

Both $n, b \in L^2(0,T;H^1(0,1))$ and hence $n, b \in L^1(0,T;H^1(0,1))$ for $0 \leq t \leq T_R$.

Thus, our integral above exists and is finite. As $\psi_w(0) = \int_0^1 H(u_0)\ \mathrm{d}x = 0$, this means $\psi_w(t) = 0$ for all $0 < t < T_R$. Since H is nonnegative, it must be that $H(w(t)) \equiv 0$ and hence $w(t) \geq 0$ for $0 < t < T_R$.

Similarly, noting that our solutions satisfy $\|b(t)\|_{L^\infty(0,1)} \leq \frac{1}{2\lambda_{12}}$ and $\|w(t)\| \leq \frac{\lambda_9}{2}$

$$\begin{aligned}
\psi_b'(t) &= \int_0^1 H'(b(t))b_t(t)\ \mathrm{d}x \\
&\leq \lambda_8 \int_0^1 H'(b(t)) \left(b(1-b) - b\frac{w}{\lambda_9 + w}\frac{\lambda_{10} + \lambda_{11} n}{\lambda_{12} b + 1} \right)\ \mathrm{d}x \\
&\leq C\|H'(b(t))b(t)\|_{L^1(0,1)} \left(1 + |b(t)| + \frac{|w(t)|}{\lambda_9 - |w(t)|}\frac{\lambda_{10} + \lambda_{11}|n(t)|}{1 - \lambda_{12}|b(t)|} \right) \\
&\leq C\|H'(b(t))b(t)\|_{L^1(0,1)} \times \ldots \\
&\ldots \left(1 + \|b(t)\|_{L^\infty(0,1)} + \frac{\|w(t)\|_{L^\infty(0,1)}}{\lambda_9 - \|w(t)\|_{L^\infty(0,1)}}\frac{\lambda_{10} + \lambda_{11}\|n(t)\|_{L^\infty(0,1)}}{1 - \lambda_{12}\|b(t)\|_{L^\infty(0,1)}} \right) \\
&\leq C\|H'(b(t))b(t)\|_{L^1(0,1)} \times \ldots \\
&\ldots \left(1 + \|b(t)\|_{L^\infty(0,1)} + \|w(t)\|_{L^\infty(0,1)} + \|w(t)\|^2_{L^\infty(0,1)} + \|n(t)\|^2_{L^\infty(0,1)} \right) \\
&\leq C\psi_b(t) \left(1 + \|b(t)\|_{L^\infty(0,1)} + \|w(t)\|_{L^\infty(0,1)} \right. \\
&\left. + \|w(t)\|^2_{L^\infty(0,1)} + \|n(t)\|^2_{L^\infty(0,1)} \right).
\end{aligned} \tag{2.79}$$

Again applying Grönwall and noting the convergence of the integrals (by the regularity of the solutions), we have $b(t) \geq 0$ for all $0 \leq t < T_R$.

Next we consider the unknown c. We have

$$\begin{aligned}
\psi_c'(t) &= \int_0^1 H'(c(t))c_t(t)\ \mathrm{d}x \\
&\leq \int_0^1 H'(c(t)) \left(\lambda_{13} c_{xx}(t) + \lambda_{14} b(t) - \lambda_{15} c(t) \right)\ \mathrm{d}x.
\end{aligned} \tag{2.80}$$

Similar to our argument for equation (2.75),

$$\int_0^1 H'(c(t))c_{xx}(t)\ \mathrm{d}x \leq 0. \tag{2.81}$$

In light of knowing $b(t) \geq 0$ from above and $H'(c(t)) \leq 0$ via equation (2.68), we also have

$$\int_0^1 H'(c(t))b(t) \, \mathrm{d}x \leq 0. \tag{2.82}$$

Hence

$$\begin{aligned} \psi_c'(t) &\leq C\|H'(c(t))c(t)\|_{L^1(0,1)} \\ &\leq C\psi_c(t). \end{aligned} \tag{2.83}$$

This gives $\psi_c(t) \leq \psi_c(0)e^{Ct}$, which then implies $c(t) \geq 0$ as well. Finally, we consider the non-negativity of the variable n. As we have argued previously,

$$\begin{aligned} \psi_n'(t) &= \int_0^1 H'(n(t))n_t(t) \, \mathrm{d}x \\ &= \int_0^1 H'(n(t)) \left(\lambda_5 n_{xx} - \lambda_6(nc_x)_x + \lambda_7 bn(1-n)\right) \, \mathrm{d}x. \end{aligned} \tag{2.84}$$

As before, we deal with each piece separately. To begin,

$$\begin{aligned} -\lambda_7 \int_0^1 H'(n(t))bn^2 \, \mathrm{d}x &\overset{\text{Hö}}{\leq} \lambda_6\|H'(n(t))n(t)\|_{L^1((0,1))}\|b(t)\|_{L^\infty(0,1)}\|n(t)\|_{L^\infty(0,1)} \\ &\leq C\psi_n(t)\left(\|b(t)\|^2_{L^\infty(0,1)} + \|n(t)\|^2_{L^\infty(0,1)}\right). \end{aligned} \tag{2.85}$$

Furthermore, we have

$$\begin{aligned} \lambda_5 \int_0^1 H'(n(t))n_{xx} \, \mathrm{d}x &= -\lambda_5 \int_0^1 \frac{d}{dx}h'(n(t))\frac{d}{dx}n(t) \, \mathrm{d}x \\ &= -\lambda_5 \int_0^1 \left|\frac{d}{dx}H'(n(t))\right|^2 \, \mathrm{d}x \\ &= -\lambda_5\|\frac{d}{dx}H'(n(t))\|_{L^2(0,1)}. \end{aligned} \tag{2.86}$$

Finally,

$$
\begin{aligned}
&- \lambda_6 \int_0^1 H'(n(t))(nc_x)_x \ \mathrm{d}x \\
&= \lambda_6 \int_0^1 n(t) \frac{d}{dx} H'(n(t)) \frac{d}{dx} c(t) \ \mathrm{d}x \\
&\overset{\text{Hö}}{\leq} \lambda_6 \int_0^1 n(t) \frac{d}{dx} H(n(t)) \ \mathrm{d}x \| \frac{d}{dx} c(t) \|_{L^\infty(0,1)} \\
&\overset{\text{def}}{=\leq} \lambda_6 \int_0^1 H'(n(t)) \frac{d}{dx} H'(n(t)) \ \mathrm{d}x \| \frac{d}{dx} c(t) \|_{L^\infty(0,1)} \\
&\leq \lambda_6 \| H'(n(t)) \|_{L^2(0,1)} \| \frac{d}{dx} H'(t)) \|_{L^2(0,1)} \| \frac{d}{dx} c(t) \|_{L^\infty(0,1)} \\
&\overset{\text{Peter-Paul}}{\leq} \frac{\lambda_5}{2} \| \frac{d}{dx} H'(n(t)) \|^2_{L^2(0,1)} + C \| H'(n(t)) \|^2_{L^2(0,1)} \| \frac{d}{dx} c(t) \|^2_{L^\infty(0,1)}.
\end{aligned}
\tag{2.87}
$$

Adding equations (2.86) and (2.87), dropping the resulting non-positive term, and substituting into equation (2.84), we have

$$
\begin{aligned}
&\psi_n'(t) \\
&\leq C\psi_n(t) \left(1 + \| b(t) \|_{H^1(0,1)} + \| b(t) \|^2_{L^\infty(0,1)} + \| n(t) \|^2_{L\infty(0,1)} + \| c(t) \|^2_{H^1(0,1)} \right),
\end{aligned}
\tag{2.88}
$$

where our C is independent of t. Owing to the regularity of our solutions, the parenthetical term is integrable, and hence we may apply Grönwall as before to conclude that n must also be positive, which completes our proof. □

Finally, we conclude with the main result of the chapter, namely the (short-time) existence of nonnegative solutions arising from nonnegative initial conditions.

Theorem 2.13. Again consider system (2.1) with zero Neumann conditions. Let

$$
\mathcal{K} = \{ f \in H^1(0,1) : 0 \leq f(x) \leq 1 \text{ for all } x \in (0,1) \}.
$$

Then for all initial data $X_0 = [w_0, n_0, c_0, b_0]^T \in [\mathcal{K}]^4$ there is a $T_0 > 0$ such a unique mild solution X to (2.1) with Neumann boundary conditions exists with regularity

$$\begin{aligned}X &= [w, n, c, b]^T \\ &\in [H^1(0, T_0; L^2(0,1)) \cap L^2(0, T_0; H^2(0,1)]^3 \times H^1(0, T_0; H^1(0,1)),\end{aligned}$$

and $w, n, b, c \geq 0$ on $(0,1)$ for $0 \leq t \leq T_0$.

Remark 2.14. Note that the variant of the well-posedness local-in-time but without assuming smallness of initial data is of particular interest within the context of controllability. Indeed, being able to control reaction in a short time (even approximately) will lead to global controlled solutions which are positive.

Proof. The proof of theorem (2.13) is essentially contained in the previous arguments. We apply fixed point theorem on a cone of positive solutions and on the interval $[0, T]$. We look at the maps

$$\tilde{X} = (\tilde{w}, \tilde{n}, \tilde{b}, \tilde{c})^T \mapsto X = (w, n, b, c)^T$$

given by the maps in equation (2.35):

$$\begin{aligned}w(t) &= e^{-A_1 t} w_0 + \int_0^t e^{-A_1(t-s)} F_1(\tilde{X}(s)) \, \mathrm{d}s \\ n(t) &= e^{-A_2 t} n_0 + \int_0^t e^{-A_2(t-s)} F_2(\tilde{X}(s)) \, \mathrm{d}s + \int_0^t e^{-A_2(t-s)} P(\tilde{X}(s)) \, \mathrm{d}s \\ b(t) &= e^{A_3 t} b_0 + \int_0^t e^{A_3(t-s)} F_3(\tilde{X}(s)) \, \mathrm{d}s \\ c(t) &= e^{-A_4 t} c_0 + \int_0^t e^{-A_4(t-s)} F_4(\tilde{X}(s)) \, \mathrm{d}s.\end{aligned} \tag{2.89}$$

As the coordinates of $\tilde{X} \in \mathcal{Z}$ are continuous (by Rellich-Kondrachov) with positive initial conditions, then by continuity we know that the coordinates of $\tilde{X}$ are nonnegative up until some time T_1. In light of this, we know the maps of form $\dfrac{s}{1+s}$ in the b equation are Lipschitz, and hence we may prevent finite time blow up. The previous estimates hold without modification. Finally, we may choose $T_0 = T_0(\|X_0\|_{\mathcal{Z}}$ sufficiently small to obtain contraction. This produces a fixed point $X = (w, n, b, c)^T$ solving

$$\begin{aligned}
w(t) &= e^{-A_1 t} w_0 + \int_0^t e^{-A_1(t-s)} F_1(X(s))\ \mathrm{d}s \\
n(t) &= e^{-A_2 t} n_0 + \int_0^t e^{-A_2(t-s)} F_2(X(s))\ \mathrm{d}s + \int_0^t e^{-A_2(t-s)} P(X(s))\ \mathrm{d}s \\
b(t) &= e^{A_3 t} b_0 + \int_0^t e^{A_3(t-s)} F_3(X(s))\ \mathrm{d}s \\
c(t) &= e^{-A_4 t} c_0 + \int_0^t e^{-A_4(t-s)} F_4(X(s))\ \mathrm{d}s.
\end{aligned} \tag{2.90}$$

We now apply proposition (2.12) to obtain that our mild solution must be nonnegative. □

CHAPTER 3

LOCAL NULL CONTROLLABILITY

The purpose of this chapter is to study the controllability of our system (2.1) with boundary conditions (2.2) and initial conditions in $H^1(0,1)$. As mentioned in the introduction, the dissipation from the Laplacian eliminates the possibility of controlling to arbitrary trajectories in $H^1(0,1)$. Instead, in parabolic-like systems one focuses on the ability to drive the system to zero or, generally, the uniform steady states of the system. For our problem, we focus on the ability to steer to system to the steady state $w(T,x) = n(T,x) = b(T,x) = c(T,x) = 0$. This problem is often referred to as the problem of null controllability. Although this problem is physically undesirable (one would not want to drive the oxygen in the wound to zero), it can be seen as an illustrative intermediary step in controlling the system to a state where the bacteria are driven to zero, while the rest of the variables are free to be driven to a physically reasonable equilibrium.

As is the general approach for nonlinear problems, we start be analyzing the controllability for the linear problem. In particular, taking the linear part of our system will correspond to the linearization of the system about the origin. For this, we introduce compactly supported controllers on the three PDE equations, and a distributed control on the degenerate ODE bacterial equation. Since three of the controllers are localized to compactly supported subdomains, we modify the terminology of our problem and say we wish to obtain local null controllability. A secondary goal is to eliminate as many of the controllers as possible while obtaining this main objective of driving the system to zero.

To establish local controllability is difficult, and is almost never dealt with directly. Instead, one often tries to find an equivalent so-called "dual" problem. With our usual variation-of-parameters formulation of the system, we are able to interpret the goal of local null controllability as a surjectivity condition for a convolution integral operator acting on the control. Using a corollary to the Closed Range Theorem, we derive the appropriate observability inequality in terms of the dual variables, which we label with $\varphi_i's$. For our problem, these variables are known to solve the appropriate *backwards* heat equation with given final time conditions. Together, these form the dual problem.

To obtain the observability estimate from the dual equations, we then must introduce Carleman-type estimates. The estimates we use were first derived by Fursikov and Imanuvilov in [13], and reproduced in the paper by Fernández-Cara and Guerrero [11]. In the latter of these papers, they outline the general theory behind the estimates. We note this below for clarity of exposition.

The general localized Carleman inequality has the form

$$\int_0^T \int_\Omega \rho^2 |\varphi|^2 dxdt \leq C_T \int_0^T \int_\omega \rho^2 |\varphi|^2 dxdt, \quad C_T \to \infty \text{ as } T \to \infty \tag{3.1}$$

where $\rho = \rho(x,t)$ is a continuous, positive weight function vanishing strongly at 0 and T. For parabolic problems, this function behaves like

$$\left(\frac{e^x}{t(T-t)}\right)^l e^{\frac{e^x}{t(T-t)}}, \tag{3.2}$$

with l being a parameter that adjusts according to the order of derivatives being taken, and auxiliary parameters are included to control estimates of lower order terms. We use the above inequality and properties of the weights to obtain an estimate of the form

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_\Omega |\varphi|^2 \text{ d}x \text{ d}t \leq C \int_0^T \int_\omega |\varphi|^2 \text{ d}x \text{ d}t. \tag{3.3}$$

The significance of this estimate is the following: by restricting our observation over time we are able to eliminate the weight functions from the estimate. This allows estimate (3.3) to be coupled with an energy estimate, which then leads us to the observability inequality. By duality, this result gives localized controllability of the linear problem.

From there, we move on to the controllability for the nonlinear problem. Here, we follow the work of Lagnese [28] and Bradley [3]. The approach, modified for our problem, again relies of the variation-of-parameters form of the solution. In our problem, we let $-A$ be the generator of a strongly continuous, analytic semigroup e^{-At} in on $\mathcal{D}(A) \subset [L^2(0,1)]^4$. Let our state vector be denoted by $Y(t) = (w(t), n(t), b(t), c(t))^T$ and $U(t) = (u_1(t), u_2(t), u_3(t), u_4(t))^T$ be our vector control and let B be the restriction of the control to the localized control sets. We will have the linear operator

$$\begin{aligned} \mathcal{L}_t(V) &= \int_0^t e^{-A(t-s)} V(s)\ \mathrm{d}s, \\ \mathcal{L}_t &: \left[L^2(0,t; L^2(0,1))\right]^4 \to \mathcal{Z} \end{aligned} \tag{3.4}$$

Using the variation of parameters formulation, the local null controllability of the linear problem tells us that for any $T > 0$ and any $Y_0 \in [H^1(0,1)]^4$, there is a control U such that

$$0 = e^{-AT} Y_0 + \mathcal{L}_T(BU), \tag{3.5}$$

or that the $\mathcal{L}_T$ map is boundedly surjective onto the range of e^{-AT}. We also introduce what we will call the control-to-state map, denoted $\mathcal{C}_T$, by

$$\mathcal{C}_t(U, Y(U)) = \mathcal{L}_t(BU) + \mathcal{L}_t\left(F\left(Y(U)\right)\right). \tag{3.6}$$

To establish local null controllability of the nonlinear problem is to show that, given $T > 0$ and Y_0, there exists controllers U such that

$$0 = e^{-AT}Y_0 + \mathcal{C}_T(U, Y(U)). \tag{3.7}$$

As such, we are again met by a surjectivity condition, this time for the control-to-state operator $\mathcal{C}_T$. As (3.7) is an implicit equation in U that we would like to solve, we use the implicit function theorem for Banach spaces. This will mean we show that the control-to-state operator locally defines a homeomorphism between the control space and the states that are close to zero. To establish this, we must show that $\mathcal{C}_t$ is (locally) Fréchet differentiable with $D\mathcal{C}_t(U, 0)$ well-defined and boundedly invertible. This, *a fortiori*, requires a study of the differentiability of the nonlinear maps $Y \mapsto F(Y)$.

3.1 Derivation of the Adjoint Problem and the Observability Inequality

We consider the controllability of the above system (2.1) with boundary conditions (2.2) and appropriate initial conditions. To begin, we first consider the control of the linear part of the problem, which we record again for clarity:

$$\begin{cases} w_t &= w_{xx} - \lambda_3 w + \chi_1 u_1 \\ n_t &= \lambda_5 n_{xx} - \lambda_{16} n + \chi_2 u_2 \\ b_t &= \lambda_8 b + \chi_3 u_3 \\ c_t &= \lambda_{13} c_{xx} + \lambda_{14} b - \lambda_{15} c + \chi_4 u_4. \end{cases} \tag{3.8}$$

We reiterate the relevant spaces

$$
\begin{aligned}
\mathcal{M} &= H^1(0,T;L^2(0,1)) \cap L^2(0,T;H^2(0,1)) \\
\mathcal{Z} &= [\mathcal{M}]^2 \times H^1(0,T;H^1(0,1)) \times \mathcal{M} \\
\mathcal{W} &= \left[L^2(0,T;L^2(0,1))\right]^2 \times L^2(0,T;H^1(0,1)) \times L^2(0,T;L^2(0,1))
\end{aligned}
\tag{3.9}
$$

as well as the operators

$$
\begin{aligned}
A_1 z &= -\left(\frac{\partial^2}{\partial x^2} - \lambda_3 I\right) z \\
A_1 &: \mathcal{D}(A_1) \subset H^2(0,1) \to L^2(0,1) \\
A_2 &= -\left(\lambda_5 \frac{\partial^2}{\partial x^2} - \lambda_{16}\right) z \\
A_2 &: \mathcal{D}(A_2) \subset H^2(0,1) \to L^2(0,1) \\
A_3 &= \lambda_8 I z \\
A_3 &: H^1(0,1) \to H^1(0,1) \\
A_4 z &= -\left(\lambda_{13} \frac{\partial^2}{\partial x^2} z - \lambda_{15} I\right) z \\
A_4 &: \mathcal{D}(A_4) \subset H^2(0,1) \to L^2(0,1) \\
A_{4,3} z &= -\lambda_{14} I z \\
A_{4,3} &: L^2(0,1) \to L^2(0,1).
\end{aligned}
$$

As before, we may write the linear system as

$$
\begin{bmatrix} w \\ n \\ b \\ c \end{bmatrix}_t = \begin{bmatrix} -A_1 & 0 & 0 & 0 \\ 0 & -A_2 & 0 & 0 \\ 0 & 0 & -A_3 & 0 \\ 0 & 0 & -A_{4,3} & -A_4 \end{bmatrix} \begin{bmatrix} w \\ n \\ b \\ c \end{bmatrix} + \begin{bmatrix} \chi_1 & 0 & 0 & 0 \\ 0 & \chi_2 & 0 & 0 \\ 0 & 0 & \chi_3 & 0 \\ 0 & 0 & 0 & \chi_4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \tag{3.10}
$$

$$
\overset{\text{def}}{\Longrightarrow} \; Y_t = AY + BU,
$$

where $-A$ generates a strongly continuous and analytic semigroup $e^{-A(t-s)}$ on $\mathcal{M}$, and $B : \mathcal{W} \to \mathcal{W}$ not only incorporates the action of restriction to the compact subdomains, but it is also interpreted as mapping the controls (in a copy of $\mathcal{W}$) into the space containing the domain of the semigroup (another copy of $\mathcal{W}$).

The variation-of-parameters solution to this problem is given by

$$Y(t) = e^{-At}Y_0 + \int_0^t e^{-A(t-s)}BU(s)\ \mathrm{d}s. \tag{3.11}$$

For what follows, we introduce the operators

$$\begin{aligned}
\mathcal{L}_t(Z) &= \int_0^t e^{-A(t-s)}Z(s)\ \mathrm{d}s \\
\mathcal{L}_t :& \mathcal{W} \to \mathcal{Z} \\
\mathcal{L}_T(Z) &= \int_0^T e^{-A(T-s)}Z(s)\ \mathrm{d}s \\
\mathcal{L}_T :& \mathcal{W} \to \left[H^1(0,1)\right]^4 \subset [L^2(0,1)]^4.
\end{aligned} \tag{3.12}$$

In order to be null controllable at time $T > 0$, we will require that there exists a control U such that $Y(T) = 0$. This means we desire, for $T > 0$ and Y_0 given,

$$0 = e^{-AT}Y_0 + \int_0^T e^{-A(T-s)}BU(s)\ \mathrm{d}s \tag{3.13}$$

in $[L^2(0,1)]^4$. This is to say, we want

$$\mathcal{R}\left(e^{-AT}\right) \subseteq \mathcal{R}\left(\mathcal{L}_T\right). \tag{3.14}$$

As noted in Part IV, Section 2.1 of Zabczyk [49] (alternately, see theorem 2.20 in Brezis [4]), since $\mathcal{D}\left(\mathcal{L}_T\right)$ is reflexive, condition (3.13) is equivalent (by Closed Graph

Theorem) to

$$\|e^{(-A)^*T}X\|_{[L^2(0,1)]^4} \leq C\|\mathcal{L}_T^*X\|_{\mathcal{W}}, \tag{3.15}$$

for some $C > 0$ and all $X \in [L^2(0,1)]^4$. We note that, since $(-A)$ generates a strongly continuous semigroup for $t > 0$, the generator $(-A)$ is closed and densely defined. As such, the adjoint A^* is well-defined, and it generates the adjoint semigroup, denoted by $e^{(-A)^*t}, t > 0$. Similarly, the operator $\mathcal{L}_T$ is bounded, and hence we may define the adjoint $\mathcal{L}_T^*$. Thus, the expressions in (3.15) are well-defined.

We now proceed giving interpretation to these expressions. First, we determine the form of the adjoint L_T^*, following Zabczyk. Let $U \in \mathcal{W}$ and $X \in [H^1(0,1)]^4$. Since our semigroup is strongly continuous, the integral defining $\mathcal{L}_T$ can be taken in the sense of Riemann, and hence can be expressed as the limit of a finite sum. By continuity of the inner product and the definition of the adjoints, $\mathcal{L}_T^*$ necessarily satisfies

$$\begin{aligned} \langle U, \mathcal{L}_T^*X\rangle_{\mathcal{W}} &\overset{\text{def}}{=} \langle \mathcal{L}_T U, X\rangle_{[L^2(0,1)]^4} \\ &\overset{\text{def}}{=} \left\langle \int_0^T e^{-A(T-s)}BU(s)\ \mathrm{d}s, X\right\rangle_{[L^2(0,1)]^4} \\ &\overset{\text{cont}}{=} \int_0^T \langle e^{-A(T-s)}BU(s), X\rangle_{[L^2(0,1)]^4}\ \mathrm{d}s \\ &\overset{\text{def}}{=} \int_0^T \langle U(s), B^*e^{(-A)^*(T-s)}X\rangle_{[L^2(0,1)]^4}\ \mathrm{d}s. \end{aligned} \tag{3.16}$$

Thus, we take

$$\mathcal{L}_T^*X = B^*e^{(-A)^*(T-s)}X. \tag{3.17}$$

In light of inequality (3.15) and definition (3.17), it is evident that our dual problem should be written in terms of the image of the adjoint semigroup. We define a dual variable $\Phi(s) = (\varphi_1(s), \varphi_2(s), \varphi_3(s), \varphi_4(s))^T$ by the expression

$$\Phi(s) = e^{(-A)^*(T-s)}X, \quad 0 \leq s \leq T. \tag{3.18}$$

Then, as $X \in [L^2(0,1)]^4$, we also have $\Phi(s) \in L^2(0,1)$ for $0 \leq s \leq T$. To give an interpretation to the left hand side of (3.15), we'll note that $\Phi(T) = e^{(-A)^*(T-T)}X = X$. As such, $\Phi(T)$ is some given element of $[L^2(0,1)]^4$. Finally, we wish to find the dual differential equation verified by $\Phi(s)$. If $X \in \mathcal{D}((-A)^*)$, then function $e^{(-A)^*t}X$ is differentiable (Pazy [42], corollary 1.10.6 with theorem 1.2.4(c)) with derivative $(-A)^*e^{(-A)^*t}X$. As such, we have

$$\begin{aligned} \Phi'(s) &= -(-A)^* e^{(-A)^*(T-s)} X \\ &= -(-A)^*\Phi(s). \end{aligned} \tag{3.19}$$

Collecting these facts, we see that the dual variable Φ must satisfy the following system:

$$\begin{cases} \Phi'(t) & = -(-A)^*\Phi(t) \quad 0 \leq t \leq T \\ \Phi(T) & = \Phi_T \in [L^2(0,1)]^4. \end{cases} \tag{3.20}$$

Since $(-A_i), i = 1,2,4$ is given by the action of the Laplacian (modulo translations) with zero Neumann conditions, one may readily verify with Green's formula (integration by parts) that the adjoint formula $(u, (-A_i)^*v)_{L^2(0,1)} = (A_i u, v)_{L^2(0,1)}$ necessitates that $(-A_i)^*$ is given by these same actions with the same boundary conditions. Thus, the action and the domain of the A_i operators and their adjoints are the same, *i.e.* these operators are self-adjoint. Using this information produces the following system in the dual variables:

$$\begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_3 \end{bmatrix}_t = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & A_3 & A_{4,3} \\ 0 & 0 & 0 & A_4. \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{bmatrix} \tag{3.21}$$

Finally, we give the scalar equations satisfied by the dual variables. In $(0,T)\times(0,1)$ our variables will satisfy

$$\begin{aligned}\varphi_{1,t} &= -\varphi_{1,xx} + \lambda_3\varphi_1\\ \varphi_{2,t} &= -\lambda_5\varphi_{2,xx} + \lambda_{16}\varphi_2\\ \varphi_{3,t} &= \lambda_8\varphi_3 + \lambda_{14}\varphi_4\\ \varphi_{4,t} &= -\lambda_{13}\varphi_{4,xx} + \lambda_{15}\varphi_4.\end{aligned} \tag{3.22}$$

These equations are coupled with zero Neumann conditions at $x=0, x=1$ and given final time conditions $(\varphi_{1,T},\varphi_{2,T},\varphi_{3,T},\varphi_{4,T})^T \in [L^2(0,1)]^4$ to form the so-called *dual problem* to system (2.1) with boundary conditions (2.2) and initial conditions in $L^2(0,1)$. In terms of the φ_i variables, we interpret inequality (3.15) as the *observability inequality*, written explicitly as

$$\begin{aligned}&\|\varphi_1(0)\|_{L^2(0,1)} + \|\varphi_2(0)\|_{L^2(0,1)} + \|\varphi_3(0)\|_{L^2(0,1)} + \|\varphi(0)_4\|_{L^2(0,1)}\\ &\leq C\left(\|\varphi_1\|_{L^2(0,T;L^2(\omega_1)} + \|\varphi_2\|_{L^2(0,T;L^2(\omega_2)} + \|\varphi_3\|_{L^2(0,T;L^2(0,1)}\right.\\ &\quad\left.+\|\varphi_4\|_{L^2(0,T;L^2(\omega_4)}\right)\end{aligned} \tag{3.23}$$

where we have used the definition of Φ in (3.18) to say that $e^{(-A)^*T}X = \Phi(0)$.

With the equivalent dual problem and associated observability posed, we can now move onto the issue of introducing the tools used to establish (3.23) for system (3.22), namely the associated Carleman estimates for the backwards heat equation.

3.2 Null Controllability for the Linear System, Localized Controllers

As mentioned above, to establish the observability inequality for our dual system, we must now introduce the associated Carleman estimates. The approach we follow is those laid out in [11] and [10]. First, we begin by introducing the weight functions constructed in the works of Fursikov and Imanuvilov. We apply these estimates to our dual equations to obtain a global Carleman estimate, corresponding to the control of the four equations with four controls, three of which can be localized. Then, following Okposo and Willie [41] we then multiply the φ_3 equation by φ_4 and some weight functions to obtain an estimate that will allow us to remove the control on the c equation. Finally, we use the usual arguments that allow us to obtain the observability inequality corresponding to localized null control with three controls, two of which are localized.

To this end, first we must introduce a weight function for the spatial domain, denoted η^0. This weight function is constructed in [13].

Lemma 3.1. Let $\omega_0 \subset\subset \omega$ be a nonempty, open subset. Then there exists an $\omega_0 \subset\subset \omega$ and $\eta^0 \in C^2(\overline{\Omega})$ such that $\eta^0 > 0$ in ω, $\eta^0 = 0$ on $\partial\omega$, $|\nabla \eta^0| > 0$ in $\overline{\Omega \setminus \omega}$.

With this, one then introduces the following weight functions for the dual variables. For $\lambda \geq 1$ (to be chosen later), define

$$\begin{aligned} \xi(x,t) &= \frac{e^{\lambda \eta^0(x)}}{t(T-t)} \\ \alpha(x,t) &= \frac{e^{2\lambda\|\eta^0\|_\infty} - e^{\lambda\eta^0(x)}}{t(T-t)}. \end{aligned} \tag{3.24}$$

Often one needs to use bounds for the derivatives of ξ, α. We record these facts below for reference in what follows:

$$\begin{aligned} |\xi_t| &= \left| \frac{e^{\lambda\eta^0(x)}(T-2t)}{t^2(T-t)^2} \right| \\ &\leq CT\xi^2, \end{aligned} \tag{3.25}$$

and similarly

$$|\alpha_t| \leq CT\xi^2. \tag{3.26}$$

Although we will not need them for our particular linearization, we remark that one can also obtain bounds of $\nabla\xi$ and $\Delta\xi$ similarly.

Next, we record the following global Carleman inequality for the linear heat equation with zero Neumann conditions, found in [13] and [10]. For this lemma, $Q = (0,T)\times\Omega$ denotes a smooth, open, bounded domain in $\mathbb{R}^{N+1}$. $\Sigma = (0,T)\Omega$ and n denotes the outward normal vector of Ω.

Lemma 3.2. Let $f \in L^2(Q)$ be given. There exist λ^*, σ^*, and C depending on Ω and ω such that, for any $\lambda \geq \lambda^*$ and $s \geq s^*(\lambda) = \sigma^*(e^{2\lambda\|\eta^0\|_\infty}T + T^2)$ and any $q^0 \in L^2(\Omega)$, the weak solution to

$$\begin{cases} -q_t - \Delta q = f(x,t) \quad \text{in } Q \\ \dfrac{\partial q}{\partial n} = 0, \quad \text{on } \Sigma \\ q(x,T) = q^0(x) \text{ in } \Omega \end{cases}$$

satisfies

$$\begin{aligned} &\iint_Q e^{-2s\alpha}\left((s\xi)^{-1}\left(|q_t|^2 + |\Delta q|^2\right) + s\lambda^2\xi|\nabla q|^2 + s^3\lambda^4\xi^3|q|^2\right)\,\mathrm{d}x\,\mathrm{d}t \\ &\leq C\left(e^{-2s\alpha}|f|^2\,\mathrm{d}x\,\mathrm{d}t + s^3\lambda^4\iint_{\omega\times(0,T)} e^{-2s\alpha}\xi^3|q|^2\,\mathrm{d}x\,\mathrm{d}t\right). \end{aligned} \tag{3.27}$$

We being our study of (3.22) with the following result, which establishes a localized Carleman-type inequality for the linearized system. For the following, we let $Q = (0,1)\times(0,T)$ and $Q_{\omega_0} = \omega_0 \times (0,T)$.

Proposition 3.3. For each $T > 0$ and ω compactly supported in $(0,1)$, there exists λ_0, s_0 such that $\forall s \geq s_0, \lambda \geq \lambda_0$ and $\varphi_{1,T}, \varphi_{2,T}, \varphi_{3,T}, \varphi_{4,T} \in L^2(0,1)$, the associated solution $(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ to (3.22) satisfies

$$I_0(s,\lambda;\varphi_1) + I_0(s,\lambda;\varphi_2) + I_0(s,\lambda;\varphi_4) \leq C\left(\iint_{Q_{w_0}} \lambda^4 s^3 \xi^2 e^{-2s\alpha}\left(|\varphi_1|^2 + |\varphi_2|^2\right)\,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_{\omega_0}} \lambda^4 s^9 \xi^9 e^{-2s\alpha}|\varphi_3|^2\,\mathrm{d}x\mathrm{d}t + \iint_{Q_{\omega_0}} \lambda^4 s^3 \xi^3 e^{-2s\alpha}|\varphi_3|^2\,\mathrm{d}x\,\mathrm{d}t\right), \tag{3.28}$$

where

$$I_0(s,\lambda,q) = \iint_Q e^{-2s\alpha}\left((s\xi)^{-1}\left(|q_t|^2 + |\Delta q|^2\right) + s\lambda^2\xi|\nabla q|^2 + s^3\lambda^4\xi^3|q|^2\right)\,\mathrm{d}x\,\mathrm{d}t \tag{3.29}$$

Remark 3.4. The use of a single set ω to control all equations is not essential. The same result will hold for $\omega_1, \omega_2, \omega_4$ compactly embedded in $(0,1)$, however one must apply the Carleman estimates with respect to the subsets where each controller is supported. When we (eventually) remove one of the controls, no change is required, since the controller that remains will be exerted over the entire interval $(0,1)$.

Remark 3.5. Those familiar with Carleman estimates will be surprised by the apparent restricted observation of the φ_3 variable, since the corresponding state equation is an ODE, and hence we do not have Carleman-type estimates. As will be shown, this restriction comes from wanting to remove the global estimate on φ_4. We do so by multiplication by a truncation function (among other terms), which then leads to the inequality as in the statement of theorem (3.3). This will not lead to localized controls as, we will see in the null-controllability result, it is the *energy estimates* along with the lack of Carleman-type estimates for the ODE that force us to exert the control brutally on the state corresponding to φ_3.

Proof. Here we may apply lemma (3.2) to each the $\varphi_1, \varphi_2, \varphi_4$ equations to obtain

$$
\begin{aligned}
I_0(s,\lambda;\varphi_1)+I_0(s,\lambda;\varphi_2)+I_0(s,\lambda;\varphi_4) &\overset{\text{def}}{=} \iint_Q (s\xi)^{-1} e^{-s\alpha}\left(|\varphi_{1_t}|^2+|\Delta\varphi_1|^2\right)\, \mathrm{d}x\ \mathrm{d}t \\
&+\iint_Q \lambda^2 (s\xi)\, e^{-2s\alpha}\left(|\nabla\varphi_1|^2\right)\, \mathrm{d}x\ \mathrm{d}t+\iint_Q \lambda^4 (s\xi)^3 e^{-2s\alpha}\left(|\varphi_1|^2\right)\, \mathrm{d}x\ \mathrm{d}t \\
&+\iint_Q (s\xi)^{-1} e^{-2s\alpha}\left(|\varphi_{2_t}|^2+|\Delta\varphi_2|^2\right)\, \mathrm{d}x\ \mathrm{d}t \\
&+\iint_Q \lambda^2 (s\xi)\, e^{-2s\alpha}\left(|\nabla\varphi_2|^2\right)\, \mathrm{d}x\ \mathrm{d}t+\iint_Q \lambda^4 (s\xi)^3 e^{-2s\alpha}\left(|\varphi_2|^2\right)\, \mathrm{d}x\ \mathrm{d}t \\
&+\iint_Q (s\xi)^{-1} e^{-2s\alpha}\left(|\varphi_{4_t}|^2+|\Delta\varphi_4|^2\right)\, \mathrm{d}x\ \mathrm{d}t \\
&+\iint_Q \lambda^2 (s\xi)\, e^{-2s\alpha}\left(|\nabla\varphi_4|^2\right)\, \mathrm{d}x\ \mathrm{d}t+\iint_Q \lambda^4 (s\xi)^3 e^{-2s\alpha}\left(|\varphi_4|^2\right)\, \mathrm{d}x\ \mathrm{d}t \\
&\overset{(3.2)}{\le} C\Bigg(\iint_Q e^{-2s\alpha}|\lambda_3\varphi_1|^2\, \mathrm{d}x\ \mathrm{d}t+\iint_Q e^{-2s\alpha}|\lambda_{16}\varphi_2|^2\, \mathrm{d}x\ \mathrm{d}t \\
&+\iint_Q e^{-2s\alpha}|\lambda_{15}\varphi_4|^2\, \mathrm{d}x\ \mathrm{d}t+\lambda^4\iint_{Q_\omega}(s\xi)^3 e^{-2s\alpha}|\varphi_1|^2\, \mathrm{d}x\ \mathrm{d}t \\
&+\lambda^4\iint_{Q_\omega}(s\xi)^3 e^{-2s\alpha}|\varphi_2|^2\, \mathrm{d}x\ \mathrm{d}t+\lambda^4\iint_{Q_\omega}(s\xi)^3 e^{-2s\alpha}|\varphi_3|^2\, \mathrm{d}x\ \mathrm{d}t\Bigg).
\end{aligned}
\tag{3.30}
$$

At this point we want to absorb the global integrals on the right-hand-side into the $|q|^2$-level terms of the I_0 integrals. To begin, we wish to have

$$
\frac{1}{2C}\iint_Q s^3\lambda^4\xi^3 e^{-2s\alpha}|\varphi_1|^2\ \mathrm{d}\ \mathrm{d}t \ge \iint_{Q_\omega}\lambda_3^2 e^{-2s\alpha}|\varphi_1|^2\ \mathrm{d}\ \mathrm{d}t. \tag{3.31}
$$

Since our terms are nonnegative, it suffices to require that

$$
s^3\lambda^4\xi^3 \ge 2C\lambda_3^2. \tag{3.32}
$$

Recalling the definition of ξ from (3.24), we see one can achieve this by requiring

$$s^3\lambda^4 e^{3\lambda\|\eta^0\|_\infty} \geq 2Ct^3(t-T)^3\lambda_3^2. \tag{3.33}$$

As $\lambda \geq 1$, we know that

$$s^3\lambda^4 e^{3\lambda\|\eta^0\|_\infty} \geq s^3 e^{3\|\eta^0\|_\infty}. \tag{3.34}$$

Furthermore, as $0 \leq t \leq T$, we have

$$t(T-t) \leq \frac{T^2}{4}. \tag{3.35}$$

Combining these inequalities, we take

$$s \geq \left(\frac{T^6 C\lambda_3^2}{32e^{3\|\eta^0\|_\infty}}\right)^{1/3} \tag{3.36}$$

and thereby obtain (3.31). We argue similarly for the remaining global terms. If we also choose

$$s \geq \left(\frac{T^6 C\lambda_{16}^2}{32e^{\|\eta^0\|_\infty}}\right)^{1/3} \tag{3.37}$$

then will have

$$\frac{1}{2C}\iint_Q s^3\lambda^4\xi^3 e^{-2s\alpha}|\varphi_2|^2 \text{ d dt} \geq \iint_{Q_\omega} \lambda_{16}^2 e^{-2s\alpha}|\varphi_2|^2 \text{ d d}t. \tag{3.38}$$

Finally, we further restrict s so that

$$s \geq \left(\frac{T^6 C\lambda_{16}^2}{32e^{\|\eta^0\|_\infty}}\right)^{1/3}, \tag{3.39}$$

and, as a consequence, we obtain

$$\frac{1}{2C}\iint_Q s^3\lambda^4\xi^3 e^{-2s\alpha}|\varphi_4|^2 \text{ d}x \text{ d}t \geq \iint_{Q_\omega} \lambda_{15}^2 e^{-2s\alpha}|\varphi_4|^2 \text{ d}x \text{ d}t. \tag{3.40}$$

Together, we must choose

$$s \geq \left(\frac{T^6 C \max\{\lambda_3^2, \lambda_{15}^2, \lambda_{16}^2\}}{32 e^{\|\eta^0\|_\infty}} \right)^{1/3}. \tag{3.41}$$

Doing this allows us to write

$$\begin{aligned}
& I_0(s,\lambda;\varphi_1) + I_0(s,\lambda;\varphi_2) + I_0(s,\lambda;\varphi_4) \\
& \overset{(3.30)}{\leq} C \left(\iint_Q e^{-2s\alpha} |\lambda_3 \varphi_1|^2 \,\mathrm{d}x\,\mathrm{d}t + \iint_Q e^{-2s\alpha} |\lambda_{16} \varphi_2|^2 \,\mathrm{d}x\,\mathrm{d}t \right. \\
& + \iint_Q e^{-2s\alpha} |\lambda_{15} \varphi_4|^2 \,\mathrm{d}x\,\mathrm{d}t + \lambda^4 \iint_{Q_\omega} (s\xi)^3 e^{-2s\alpha} |\varphi_1|^2 \,\mathrm{d}x\,\mathrm{d}t \\
& \left. + \lambda^4 \iint_{Q_\omega} (s\xi)^3 e^{-2s\alpha} |\varphi_2|^2 \,\mathrm{d}x\,\mathrm{d}t + \lambda^4 \iint_{Q_\omega} (s\xi)^3 e^{-2s\alpha} |\varphi_3|^2 \,\mathrm{d}x\,\mathrm{d}t \right) \\
& \overset{(3.31),(3.38),(3.40)}{\leq} \frac{1}{2} \left(\iint_Q s^3 \lambda^4 \xi^3 e^{-2s\alpha} |\varphi_1|^2 \,\mathrm{d}x\,\mathrm{d}t + \iint_Q s^3 \lambda^4 \xi^3 e^{-2s\alpha} |\varphi_2|^2 \,\mathrm{d}x\,\mathrm{d}t \right. \\
& \left. + \iint_Q s^3 \lambda^4 \xi^3 e^{-2s\alpha} |\varphi_4|^2 \,\mathrm{d}\,\mathrm{d}t \right) + C \left(\lambda^4 \iint_{Q_\omega} (s\xi)^3 e^{-2s\alpha} |\varphi_1|^2 \,\mathrm{d}x\,\mathrm{d}t \right. \\
& \left. + \lambda^4 \iint_{Q_\omega} (s\xi)^3 e^{-2s\alpha} |\varphi_2|^2 \,\mathrm{d}x\,\mathrm{d}t + \lambda^4 \iint_{Q_\omega} (s\xi)^3 e^{-2s\alpha} |\varphi_3|^2 \,\mathrm{d}x\,\mathrm{d}t \right).
\end{aligned} \tag{3.42}$$

Subtracting the global terms to the left-hand side and multiplying by a factor of two yields

$$\begin{aligned}
& I_0(s,\lambda;\varphi_1) + I_0(s,\lambda;\varphi_2) + I_0(s,\lambda;\varphi_4) \\
& \leq C \left(\lambda^4 \iint_{Q_\omega} (s\xi)^3 e^{-2s\alpha} |\varphi_1|^2 \,\mathrm{d}x\,\mathrm{d}t + \lambda^4 \iint_{Q_\omega} (s\xi)^3 e^{-2s\alpha} |\varphi_2|^2 \,\mathrm{d}x\,\mathrm{d}t \right. \\
& \left. + \lambda^4 \iint_{Q_\omega} (s\xi)^3 e^{-2s\alpha} |\varphi_4|^2 \,\mathrm{d}x\,\mathrm{d}t \right).
\end{aligned} \tag{3.43}$$

Now we wish to eliminate the φ_4 term above by estimating in terms of φ_3. It will be sufficient to show

$$\lambda^4 C \iint_{Q_\omega} s^3\xi^3 e^{-2s\alpha}|\varphi_4|^2 \,\mathrm{d}x\,\mathrm{d}t \le \iint_Q |\varphi_3|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

With this in mind, let $\omega_0 \subset\subset \omega$ be a compactly supported subdomain of ω_0. We introduce a truncation function $\zeta(x) \in C_0^\infty(0,1)$ satisfying

$$\begin{aligned} \zeta(x) \equiv 1, \forall x \in \partial\omega_0, \quad 0 \le \zeta(x) \le 1, \forall x \in \omega, \quad \zeta(x) = 0, \forall x \in (0,1)\setminus\overline{\omega} \\ \frac{\frac{d^2\zeta}{dx^2}}{\zeta^{1/2}} \in L^\infty(0,1), \quad \frac{\frac{d}{dx}\zeta}{\zeta^{1/2}} \in L^\infty(0,1). \end{aligned} \tag{3.44}$$

Here ζ will act to localize the integrals to ω_0, which we take compact in ω so that we can allow for a smooth truncation and keep any spillover contained in ω.

Multiply the φ_3 equation by$\lambda^4 s^3\xi^2 e^{-2s\alpha}\zeta\varphi_4$ and integrate over the space-time region Q. This results in

$$\begin{aligned} &\iint_Q (\varphi_{3_t} - \lambda_8\varphi_3 - \lambda_{14}\varphi_4)\,\lambda^4 s^3\xi^3 e^{-2s\alpha}\zeta\varphi_4 \,\mathrm{d}x\,\mathrm{d}t = 0 \\ \implies \iint_Q \lambda_{14}\lambda^4 s^3\xi^3 e^{-2s\alpha}\zeta|\varphi_4|^2 \,\mathrm{d}x\,\mathrm{d}t = &\underbrace{\iint_Q (\varphi_{3_t})\,\lambda^4 s^3\xi^3 e^{-2s\alpha}\zeta\varphi_4 \,\mathrm{d}x\,\mathrm{d}t}_{K_1} \\ &- \underbrace{\iint_Q (\lambda_8\varphi_3)\,\lambda^4 s^3\xi^3 e^{-2s\alpha}\zeta\varphi_4 \,\mathrm{d}x\,\mathrm{d}t}_{K_2}. \end{aligned} \tag{3.45}$$

We start estimating

$$\begin{aligned} K_1 &= \iint_Q \varphi_{3_t}\lambda^4 s^3\xi^3 e^{-2s\alpha}\zeta\varphi_2 \,\mathrm{d}x\,\mathrm{d}t \\ &\overset{IBP}{=} \iint_Q \left[\varphi_3\lambda^4 s^3\xi^3 e^{-2s\alpha}\zeta\varphi_4\right]\Big|_0^T \,\mathrm{d}x\,\mathrm{d}t - \iint_Q \varphi_3\lambda^4 s^3 \frac{\partial}{\partial t}\left[\xi^3 e^{-2s\alpha}\zeta\varphi_4\right] \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.46}$$

We note from (3.24) that our weight function $e^{-2s\alpha}$ decays strongly at $t \to 0, T$. With this in mind, we notice the first integral disappears. Then, after applying the product rule and some estimation, we have

$$K_1 \leq 3 \iint_Q \varphi_3 \lambda^4 s^2 \xi^2 |\xi_t| e^{-2s\alpha} \zeta |\varphi_4| \text{ d}x \text{ d}t + \iint_Q |\varphi_3| \lambda^4 s^3 \xi^3 e^{-2s\alpha} \zeta |\varphi_{4_t}| \text{ d}x \text{ d}t$$
$$+ \iint_Q |\varphi_3| \lambda^4 s^3 \xi^3 \left(-2s|\alpha_t|\right) e^{-2s\alpha} \zeta |\varphi_4| \text{ d}x \text{ d}t.$$

We note that $|\xi_t|, |\alpha_t| \leq CT\xi^2$ for some $T > 0$, and split using the Peter-Paul inequality with epsilon to match the powers of s, ξ, φ_4 with the terms in $I_0(s, \lambda; \varphi_4)$. In doing so, we have

$$K_1 \leq \varepsilon_1 \frac{3CT}{2} \iint_Q \lambda^4 s^3 \xi^3 \zeta e^{-2s\alpha} |\varphi_4|^2 \text{ d}x \text{ d}t + C_{\varepsilon_1} \frac{3CT}{2} \iint_Q \lambda^4 s^3 \xi^5 \zeta e^{-2s\alpha} |\varphi_3|^2 \text{ d}x \text{ d}t \tag{3.47}$$
$$+ \varepsilon_1 \iint_Q \lambda^4 s^{-1} \xi^{-1} e^{-2s\alpha} \zeta |\varphi_{4_t}|^2 \text{ d}x \text{ d}t + C_{\varepsilon_1} \iint_Q \lambda^4 s^7 \xi^9 e^{-2s\alpha} \zeta |\varphi_3|^2 \text{ d}x \text{ d}t$$
$$+ \varepsilon_1 CT \iint_Q \lambda^4 s^3 \xi^3 e^{-2s\alpha} |\varphi_4|^2 \text{ d}x \text{ d}t + C_{\varepsilon_1} CT \iint_Q \lambda^4 s^5 \xi^7 e^{-2s\alpha} |\varphi_3|^2 \text{ d}x \text{ d}t$$
$$\leq \varepsilon_1 \tilde{C} I_0 \left(s, \lambda; \varphi_4\right) + C_{\varepsilon_1} \frac{3CT}{2} \iint_Q \lambda^4 s^3 \xi^5 \zeta e^{-2s\alpha} |\varphi_3|^2 \text{ d}x \text{ d}t$$
$$+ C_{\varepsilon_1} \iint_Q \lambda^4 s^7 \xi^9 e^{-2s\alpha} \zeta |\varphi_3|^2 \text{ d}x \text{ d}t + C_{\varepsilon_1} CT \iint_Q \lambda^4 s^5 \xi^7 e^{-2s\alpha} |\varphi_3|^2 \text{ d}x \text{ d}t$$
$$\leq \varepsilon_1 I_0 \left(s, \lambda; \varphi_4\right) + C_{\varepsilon_1} \iint_Q \lambda^4 s^9 \xi^9 e^{-2s\alpha} \zeta |\varphi_3|^2 \text{ d}x\text{d}t.$$

In a similar manner we have

$$
\begin{aligned}
K_2 &= -\iint_Q (\lambda_8 \varphi_3)\, \lambda^4 s^3 \xi^3 e^{-2s\alpha} \zeta \varphi_4 \ \mathrm{d}x\ \mathrm{d}t \\
&\le \frac{\lambda_8}{2} C_{\varepsilon_2} \iint_Q \lambda^4 s^3 \xi^3 e^{-2s\alpha} \zeta |\varphi_3|^2 \ \mathrm{d}x\ \mathrm{d}t + \varepsilon_2 \iint_Q \lambda^4 s^3 \xi^3 e^{-2s\alpha} \zeta |\varphi_4|^2 \ \mathrm{d}x\ \mathrm{d}t \\
&\le C_{\varepsilon_2} \iint_Q \lambda^4 s^3 \xi^3 e^{-2s\alpha} \zeta |\varphi_3|^2 \ \mathrm{d}x\ \mathrm{d}t + \varepsilon_2 I_0(s\lambda; \varphi_4).
\end{aligned} \tag{3.48}
$$

We plug estimates from (3.47), (3.48) into (3.45), and then use (3.44) to localize the integrals to ω_0. From this we have

$$
\begin{aligned}
&\iint_{Q_{\omega_0}} \lambda^4 s^3 \xi^3 e^{-2s\alpha} |\varphi_4|^2 \ \mathrm{d}x\ \mathrm{d}t \le (\varepsilon_1 + \varepsilon_2) I_0 \left(s, \lambda; \varphi_4\right) \\
&+ C_{\varepsilon_1} \iint_{Q_{\omega_0}} \lambda^4 s^9 \xi^9 e^{-2s\alpha} |\varphi_3|^2 \ \mathrm{d}x\mathrm{d}t + C_{\varepsilon_2} \iint_{Q_{\omega_0}} \lambda^4 s^3 \xi^3 e^{-2s\alpha} |\varphi_3|^2 \ \mathrm{d}x\ \mathrm{d}t.
\end{aligned} \tag{3.49}
$$

We update the estimate from (3.43) to

$$
\begin{aligned}
&I_0\left(s, \lambda; \varphi_1\right) + I_0\left(s, \lambda; \varphi_2\right) + I_0\left(s, \lambda; \varphi_4\right) \\
&\overset{(3.43)}{\le} \lambda^4 C \iint_{Q_{w_0}} s^3 \xi^2 e^{-2s\alpha} \left(|\varphi_1|^2 + |\varphi_2|^2 + |\varphi_4|^2\right) \\
&\le C \left(\iint_{Q_{w_0}} \lambda^4 s^3 \xi^2 e^{-2s\alpha} \left(|\varphi_1|^2 + |\varphi_2|^2\right) \ \mathrm{d}x\ \mathrm{d}t \right. \\
&+ C_{\varepsilon_1} \iint_{Q_{\omega_0}} \lambda^4 s^9 \xi^9 e^{-2s\alpha} |\varphi_3|^2 \ \mathrm{d}x\mathrm{d}t \\
&\left. + C_{\varepsilon_2} \iint_{Q_{\omega_0}} \lambda^4 s^3 \xi^3 e^{-2s\alpha} |\varphi_3|^2 \ \mathrm{d}x\ \mathrm{d}t \right) \\
&+ C(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) I_0\left(s, \lambda; \varphi_4\right).
\end{aligned} \tag{3.50}
$$

We then fix $(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \le \frac{1}{2C}$, so we may pull $I_0(s, \lambda; \varphi_4)$ to the left-hand-side, as we have argued previously. This produces the inequality

$$I_0(s,\lambda;\varphi_1)+I_0(s,\lambda;\varphi_2)+I_0(s,\lambda;\varphi_4)\le C\left(\iint_{Q_{w_0}}\lambda^4 s^3\xi^2 e^{-2s\alpha}\left(|\varphi_1|^2+|\varphi_2|^2\right)\,\mathrm{d}x\,\mathrm{d}t\right.$$
$$\left.+\iint_{Q_{\omega_0}}\lambda^4 s^9\xi^9 e^{-2s\alpha}|\varphi_3|^2\,\mathrm{d}x\mathrm{d}t+\iint_{Q_{\omega_0}}\lambda^4 s^3\xi^3 e^{-2s\alpha}|\varphi_3|^2\,\mathrm{d}x\,\mathrm{d}t\right). \tag{3.51}$$

□

We can now state the main result of the section, namely the null controllability of system (2.1). We again recall that establishing null controllability is equivalent to the observability inequality, as was outlined in section 3.1. Thus, this will be the end goal for our proof.

Theorem 3.6. For given $T>0$, the linear system (2.1) in $(0,T]\times(0,1)$ and associated boundary and initial conditions is null controllable with $\omega\subset\subset\Omega$ with controls in $\left[L^2(0,T;L^2(\Omega))\right]^3\times\{0\}$.

Proof. We remark that for this step, we will need to look at the estimates term-by-term. Following the argument in Proposition (3.3), one can obtain

$$I_0(s,\lambda;\varphi_1)\le C\lambda^4\iint_{Q_{\omega_0}}(s\xi)^3 e^{-2s\alpha}|\varphi_1|^2\,\mathrm{d}x\,\mathrm{d}t, \tag{3.52}$$

$$I_0(s,\lambda;\varphi_2)\le C\lambda^4\iint_{Q_{\omega_0}}(s\xi)^3 e^{-2s\alpha}|\varphi_2|^2\,\mathrm{d}x\,\mathrm{d}t \tag{3.53}$$

and

$$I_0(s,\lambda;\varphi_4)\le C\lambda^4\left(\iint_{Q_{\omega_0}}\lambda^4 s^9\xi^9 e^{-2s\alpha}|\varphi_3|^2\,\mathrm{d}x\mathrm{d}t+\iint_{Q_{\omega_0}}\lambda^4 s^3\xi^3 e^{-2s\alpha}|\varphi_3|^2\,\mathrm{d}x\,\mathrm{d}t\right). \tag{3.54}$$

From these, we may drop terms to obtain

$$\int_0^T \int_0^1 \lambda^2 s\xi e^{-2s\alpha} \left|\frac{\partial}{\partial x}\varphi_2\right|^2 \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_0^1 \lambda(s\xi)^3 e^{-2s\alpha} \left|\varphi_2\right|^2 \, \mathrm{d}x \, \mathrm{d}t \\ \leq C\lambda^4 \iint_{Q_{\omega_0}} (s\xi)^3 e^{-2s\alpha} |\varphi_2|^2 \, \mathrm{d}x \, \mathrm{d}t \tag{3.55}$$

and

$$\int_0^T \int_0^1 \lambda^2 s\xi e^{-2s\alpha} \left|\frac{\partial}{\partial x}\varphi_4\right|^2 \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_0^1 \lambda(s\xi)^3 e^{-2s\alpha} \left|\varphi_4\right|^2 \, \mathrm{d}x \, \mathrm{d}t \\ \leq C\lambda^4 \left(\iint_{Q_{\omega_0}} \lambda^4 (s\xi)^3 e^{-2s\alpha} |\varphi_2|^2 \, \mathrm{d}x \, \mathrm{d}t \right. \\ \left. + \iint_Q (s\xi)^3 e^{-2s\alpha} |\varphi_3|^2 \, \mathrm{d}x \, \mathrm{d}t + \iint_Q (s\xi)^9 e^{-2s\alpha} |\varphi_3|^2 \, \mathrm{d}x \, \mathrm{d}t \right). \tag{3.56}$$

We want to eliminate the weight functions from the above estimates, thereby getting our inequality of the form (3.3) as promised in the beginning of the chapter. Fix $s \geq s_0$ and $\lambda \geq \lambda_0$. Furthermore, notice by (3.24), for each we have constants $C_1(\Omega, \omega, T)$ and $C_2(\Omega, \omega, T)$ such that

$$C_1(\Omega, \omega, T) \leq \xi^n e^{-2s\alpha} \quad \text{on } (0,1) \times \left(\frac{T}{4}, \frac{3T}{4}\right) \\ \xi^n e^{-2s\alpha} \leq C_2(\Omega, \omega, T), \quad \text{on } (0,1) \times (0,T). \tag{3.57}$$

It's here that we use the fact that α forced exponential decay at $t = 0, T$ to balance the powers of ξ blowing up at the temporal boundary. Using the lower bound on the left and the upper bound on the right, one may see that there is a C, depending on Ω, ω, T and the system parameters, such that

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^1 \left|\frac{\partial}{\partial x}\varphi_1\right|^2 \, \mathrm{d}x \, \mathrm{d}t + \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^1 \left|\varphi_1\right|^2 \, \mathrm{d}x \, \mathrm{d}t \leq C \iint_{Q_{\omega_0}} |\varphi_1|^2 \, \mathrm{d}x \, \mathrm{d}t \tag{3.58}$$

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^1 \left|\frac{\partial}{\partial x}\varphi_2\right|^2 \, \mathrm{d}x \, \mathrm{d}t + \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^1 |\varphi_2|^2 \, \mathrm{d}x \, \mathrm{d}t \leq C \iint_{Q_{\omega_0}} |\varphi_2|^2 \, \mathrm{d}x \, \mathrm{d}t \tag{3.59}$$

and

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^1 \left|\frac{\partial}{\partial x}\varphi_4\right|^2 \, \mathrm{d}x \, \mathrm{d}t + \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^1 |\varphi_4|^2 \, \mathrm{d}x \, \mathrm{d}t \leq C \iint_{Q} |\varphi_3|^2 \, \mathrm{d}x \, \mathrm{d}t. \tag{3.60}$$

We now wish to estimate lower bounds for the time-restricted integrals on the left-hand-side of (3.58), (3.59) and (3.60). This will follow from elementary energy methods. Start by multiplying the φ_1 equation with φ_1, and integrate over $(0, t) \times (0, 1)$ for $t \in (0, T)$. This gives

$$\begin{aligned} \int_0^t \int_0^1 \varphi_{1_t}\varphi_1 \, \mathrm{d}x \, \mathrm{d}t + \int_0^t \int_0^1 \frac{\partial^2}{\partial x^2}[\varphi_1]\varphi_1 \, \mathrm{d}x \, \mathrm{d}t \\ - \lambda_3 \int_0^t \int_0^1 |\varphi_1|^2 \, \mathrm{d}x \, \mathrm{d}t = 0. \end{aligned} \tag{3.61}$$

Noticing that the first term is the integral of $\frac{\partial}{\partial t}\left[\frac{1}{2}\varphi_1^2\right]$ (Chain Rule) and using integration-by-parts (Green's Formula) on the second integral, we can estimate by dropping the nonpositive gradient term to obtain

$$\begin{aligned} \|\varphi_1(t)\|^2_{L^2(0,1)} - \|\varphi_1(0)\|^2_{L^2(0,1)} - \int_0^t \int_0^1 \frac{\partial}{\partial x}|\varphi_1|^2 \, \mathrm{d}x \, \mathrm{d}t \\ - \lambda_3 \int_0^t \int_0^1 |\varphi_1|^2 \, \mathrm{d}x \, \mathrm{d}t = 0 \\ \Longrightarrow \|\varphi_1(0)\|^2_{L^2(0,1)} \leq \|\varphi_1(t)\|^2_{L^2(0,1)} + \lambda_3 \int_0^t \int_0^1 |\varphi_1|^2 \, \mathrm{d}x \, \mathrm{d}t. \end{aligned} \tag{3.62}$$

We integrate the inequality again, this time over $(\frac{T}{4}, \frac{3T}{4})$ to connect with the estimate (3.58). This yields, after an application of Fubini-Tonelli,

$$\|\varphi_1(0)\|^2_{L^2(0,1)} \leq \left(\frac{2+2T\lambda_3}{T}\right) \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^1 |\varphi_1(t)|^2 \text{ d}x \text{ d}t. \tag{3.63}$$

We can similarly estimate the initial conditions of φ_2 and φ_4. This gives us

$$\|\varphi_2(0)\|^2_{L^2(0,1)} \leq \left(\frac{2+2T\lambda_{16}}{T}\right) \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^1 |\varphi_2(t)|^2 \text{ d}x \text{ d}t \tag{3.64}$$

and

$$\|\varphi_4(0)\|^2_{L^2(0,1)} \leq \left(\frac{2+2T\lambda_{15}}{T}\right) \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^1 |\varphi_4(t)|^2 \text{ d}x \text{ d}t. \tag{3.65}$$

We now turn our attention to incorporating the estimate for $\varphi_3(0)$. Multiply the φ_3 equation by φ_3 and, for $t \in (0,T)$, integrate over $(0,t) \times (0,1)$ to obtain

$$\int_0^t \int_0^1 \varphi_{3_t}\varphi_3 \text{ d}x \text{ d}t - \lambda_8 \int_0^t \int_0^1 |\varphi_3|^2 \text{ d}x \text{ d}t - \lambda_{14} \int_0^t \int_0^1 \varphi_3\varphi_4 \text{ d}x \text{ d}t = 0 \tag{3.66}$$

We then estimate

$$\begin{aligned} |\varphi_3(0)|^2_{L^2(0,1)} &\leq |\varphi_3(t)|^2_{L^2(0,1)} + \lambda_8 \int_0^t |\varphi_3|^2 \text{ d}x \text{ d}t \\ &+ \lambda_{14} \int_0^t \int_0^1 \varphi_3\varphi_4 \text{ d}x \text{ d}t \\ &\overset{\text{C.S.}}{\leq} |\varphi_3(t)|^2_{L^2(0,1)} + \left(\lambda_8 + \frac{1}{2}\lambda_{14}\right) \int_0^t |\varphi_3|^2 \text{ d}x \text{ d}t \\ &+ \frac{1}{2}\lambda_{14} \int_0^t \int_0^1 |\varphi_4|^2 \text{ d}x \text{ d}t \end{aligned} \tag{3.67}$$

Integrating again over a subinterval of $[0, T]$, say $\left(\frac{T}{4}, \frac{3T}{4}\right)$, and using Fubini-Tonelli, we then have

$$
\begin{aligned}
|\varphi_3(0)|^2_{L^2(0,1)} \leq \left(\frac{1 + 4\lambda_8 + 2\lambda_{14} T}{T}\right) \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^1 |\varphi_3|^2 \, \mathrm{d}x \, \mathrm{d}t \\
+ \lambda_{14} \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^1 |\varphi_4|^2 \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned}
\tag{3.68}
$$

Remark 3.7. It is here that we require the b control to act on all of $(0, 1)$. In the previous results, we have kept the restricted observation, since the estimates from φ_3 have entered via multiplication of the φ_4 equation by weight functions and truncating. However, in the energy estimate, we will necessary get the integral over all of $(0, 1)$. Since this ODE-type equation does not have the associated Carleman-type estimate, we are not able to connect the energy estimate with a restricted observation, meaning we must allow for a brutal control for the φ_3 variable.

By incorporating the estimates from (3.60) into (3.68), and allowing C to denote a generic constant, this gives

$$
\|\varphi_3(0)\|^2_{L^2(0,1)} \leq C \int_0^T \int_0^1 |\varphi_3|^2 \, \mathrm{d}x \, \mathrm{d}t. \tag{3.69}
$$

At long last, we may add the individual observability inequalities (3.63), (3.64), (3.65), and (3.69) together to obtain the desired observability inequality for the system:

$$
\begin{aligned}
&\|\varphi_1(0)\|^2_{L^2(0,1)} + \|\varphi_2(0)\|^2_{L^2(0,1)} + \|\varphi_3(0)\|^2_{L^2(0,1)} + \|\varphi_4(0)\|^2_{L^2(0,1)} \\
&\qquad \leq C \left(\|\varphi_1\|^2_{L^2(0,T;L^2(\omega))} + \|\varphi_2\|^2_{L^2(0,T;L^2(\omega))} + \|\varphi_3\|^2_{L^2(0,T;L^2(0,1))} \right),
\end{aligned}
\tag{3.70}
$$

where C depends on T, Ω, ω and the parameters of the system, including the

$L^\infty(0,T;L^\infty(0,1))$ norms of $\tilde{w},\tilde{n},\tilde{b},\tilde{c}$. As noted previously, establishing this inequality for the adjoint system is equivalent to establishing the null controllability for our state system, and so our proof is complete. □

3.3 Local Null Control for Nonlinear Problem with Localized Controllers

Now we consider controlling the nonlinear system

$$\begin{cases} w_t &= w_{xx} - \lambda_1 nw - \lambda_2 bw - \lambda_3 w + \lambda_4 + \chi_1 u_1 \\ n_t &= \lambda_5 n_{xx} - \lambda_6 \left(nc_x\right)_x + \lambda_7 bn\left(1-n\right) + \chi_2 u_2 \\ b_t &= \lambda_8 b(1-b) - b\dfrac{w}{\lambda_9 + w}\dfrac{\lambda_{10}+\lambda_{11}n}{\lambda_{12}b+1} + \chi_3 u_3 \\ c_t &= \lambda_{13} c_{xx} + \lambda_{14} b - \lambda_{15} c + \chi_4 u_4 \end{cases} \tag{3.71}$$

to the origin.

Our approach is to apply the inverse function theorem for mappings between Banach spaces. The original idea to use the inverse function theorem for nonlinear (ODE) control problems was first introduced by Larry Marcus in his 1965 paper "Controllability on nonlinear processes" [36]. This approach was adopted to systems governed by PDEs as well, for instance, in the work of Lagnese [28]. For a recollection of this method explicitly using the language of vector-valued functions, see Bradley [3]. We will recount this approach here as well. Introduce the variable $Y = [w,n,c,b] \in \mathcal{Z}$ and the nonlinear maps as before

$$\begin{cases} w_t &= w_{xx} - f_1(Y) + \chi_1 u_1 \\ n_t &= \lambda_5 n_{xx} + P(Y) + f_2(Y) + \chi_2 u_2 \\ b_t &= \lambda_8 b + f_3(Y) + \chi_3 u_3 \\ c_t &= \lambda_{13} c_{xx} + \lambda_{14} b - \lambda_{15} c + \chi_4 u_4, \end{cases} \tag{3.72}$$

where we consider the mappings $f_i, P : \mathcal{Z} \to L^2(0,T;L^2(0,1))$ for $i = 1,2$ and $f_3 : \mathcal{Z} \to L^2(0,T;H^1(0,1))$, much as we were doing in the proof of existence. Then, by the local existence theory of the nonlinear system, for the controls in the appropriate spaces, we may find a solution to our system $Y(U)$. This map can be expressed using the variation-of-parameters form of the solution:

$$Y(t) = e^{-At}Y_0 + \int_0^t e^{-A(t-s)}BU(s)\ \mathrm{d}s + \int_0^t e^{-A(t-s)}F(Y(U(s)))\ \mathrm{d}s, \tag{3.73}$$

where A is the generator of an strongly continuous semigroup and B is the restriction of the controls to the predefined spatial subdomains. We introduce the control-to-state map C_t via

$$\begin{aligned}
\mathcal{L}_t(Z) &= \int_0^t e^{\mathcal{A}(t-s)}Z(s)\ \mathrm{d}s \\
\mathcal{C}_t(U, Y(U)) &= \int_0^t e^{\mathcal{A}(t-s)}BU(s)\ \mathrm{d}s + \int_0^t e^{\mathcal{A}(t-s)}F(Y(U(s)))\ \mathrm{d}s \\
&= \mathcal{L}_t(BU) + \mathcal{L}_t(F(Y(U))). \\
C_t &: \mathcal{U} \to \mathcal{Z},
\end{aligned} \tag{3.74}$$

where $\mathcal{Z}$ is defined as in (2.32) and $\mathcal{U} = L^2(0,T;L^2(\omega_1)) \times L^2(0,T;L^2(\omega_2)) \times L^2(0,T;H^1(0,1)) \times \{0\}$. To establish null-controllability, we want to show that, at a predetermined T, we have

$$0 = e^{-AT}Y_0 + \mathcal{C}_T(U, Y(U)), \tag{3.75}$$

or that $\mathcal{C}_T$ maps into the image $e^{-AT}Y_0$ for Y_0 in $[H^1(0,1)]^4$. This is an implicit equation, which we wish to solve for the requisite control U. This requires using the inverse function theorem in Banach spaces. To this end, we want to show that

$C_T(U, Y(U)$ is Fréchet differentiable in a neighborhood of the origin in $\mathcal{U}$, and that $DC_T(0, Y(0))$ is continuously invertible in this neighborhood. Proceeding formally, the derivative $D\mathcal{C}_T$ is given by

$$[D\mathcal{C}_T(U)]\,(V) = \mathcal{L}_T(BV) + \mathcal{L}_T\left(DF(Y(U))\left[\frac{dY}{dU}(U)\right](V)\right). \tag{3.76}$$

It is evident that, to study $D\mathcal{C}_T(U)$, we must first study the smoothness of our nonlinearities comprising F. In particular, $[D\mathcal{C}_T(U)]$ is a sum of the linear map $\mathcal{L}_T B$ along with the nonlinear map $\mathcal{L}_T DF(Y(U))\left[\frac{dY}{dU}(U)\right]$. Surjectivity of the linear map $\mathcal{L}_T$ is the content of the previous section. The purpose of this section is to show that the second operator is surjective in a small neighborhood about the origin $0 \in \mathcal{U}$. we note that the derivative of this map at the origin will be of the form

$$\Big\{\mathcal{L}_T\left(DF(Y(0))\left[\tfrac{dY}{dU}(0)\right](V)\right. \tag{3.77}$$

We wish to show that this map is surjective, which will follow from an application of the inverse function theorem for Banach spaces.

We begin with the following lemma, which will justify the existence of the derivative of the derivative with respect to the control in (3.76).

Lemma 3.8. The state Y, given by equation (3.73) is Fréchet differentiable with respect to the control in a neighborhood of the origin $0 \in \mathcal{U}$, with derivative

$$\frac{\mathrm{d}Y}{\mathrm{d}U} = \left(I - \mathcal{L}_t[DF(Y(U))]\right)^{-1}\mathcal{L}_t B. \tag{3.78}$$

Proof. One can define $DF(Y)$ and show the appropriate remainder $F(Y+H) - F(Y) - DF(Y)$ go to zero in $\mathcal{W}$ faster than H goes to zero in $\mathcal{Z}$. Furthermore, the derivative satisfies $\|[DF(Y)](H)\|_{\mathcal{W}} \leq C(\|Y\|_{\mathcal{Z}})\|H\|_{\mathcal{Z}}$, where $C(\|Y\|_{\mathcal{Z}}) \to 0$ as $\|Y\|_{\mathcal{Z}} \to 0$. These arguments mimic the proofs of the existence and

Lipschitz behavior of our nonlinearities found in chapter 2, and hence we relegate the details to the appendix. Furthermore, the map $U \mapsto Y(U)$ is bounded from $\mathcal{U}$ to $\mathcal{M}$, which comes from the maximal regularity bounds of (2.6). As such, we have

$$\|[DF(Y(U))](H)\|_{\mathcal{W}} \leq C(\|U\|_{\mathcal{U}})\|H\|_{\mathcal{Z}}. \tag{3.79}$$

Then, differentiating (3.73) with respect to the control, we have

$$\begin{aligned} \frac{\mathrm{d}Y}{\mathrm{d}U}(U) &= \mathcal{L}_t B + \mathcal{L}_t[DF(Y(U))]\left[\frac{\mathrm{d}Y}{\mathrm{d}U}(U)\right] \\ \implies (I - \mathcal{L}_t[DF(Y(U))])&\left[\frac{\mathrm{d}Y}{\mathrm{d}U}(U)\right] = \mathcal{L}_t B. \end{aligned} \tag{3.80}$$

By choosing $\|U\|_{\mathcal{U}}$ sufficiently small, we have $C(\|U\|_{\mathcal{U}}) < R = \dfrac{1}{2\|\mathcal{L}_t\|_\infty}$, and then

$$\|I - (I - \mathcal{L}_t[DF(Y(U))])\|_\infty \leq \|\mathcal{L}_t\|_\infty R < \frac{1}{2}. \tag{3.81}$$

As such, $(I - \mathcal{L}_t[DF(Y(U))])$ is invertible $B_R(\mathcal{U}) \to B_r(\mathcal{Z})$, where r will be determined by the size of R and the maximal regularity constant. Inverting gives the desired formula (3.78). □

Proposition 3.9. The control to state map, $\mathcal{C}_T$, given by (3.74) is differentiable in a ball of the origin in $\mathcal{U}$, with derivative

$$[D\mathcal{C}_T(U)](V) = \mathcal{L}_T(BV) + \mathcal{L}_T DF(Y(U))\,(I - \mathcal{L}_t[DF(Y(U))])^{-1}\,\mathcal{L}_t B. \tag{3.82}$$

Moreover, $[D\mathcal{C}_T(0)]$ is surjective $\mathcal{U} \to \mathcal{R}(e^{-AT})$.

Proof. Now we know that $\mathcal{C}_T(U)$ is a composition of differentiable maps in $B_R(\mathcal{U})$, and therefore is differentiable $B_R(\mathcal{U})$ as well. Taking this derivative yields (3.82), where we have used identity (3.78) to replace $\dfrac{\mathrm{d}Y}{\mathrm{d}U}$. From this form it is apparent that

$$[D\mathcal{C}_T(0)]\,(V) = \mathcal{L}_T(BV) + \mathcal{L}_T DF(Y(0))\,(I - \mathcal{L}_t[DF(Y(0))])^{-1}\,\mathcal{L}_t B. \tag{3.83}$$

The operator $\mathcal{L}_T B$ is surjective onto $\mathcal{R}(e^{-AT})$ by the linear controllability of theorem (3.6). We now consider the map $\mathcal{L}_T DF(Y(0))\,(I - \mathcal{L}_t[DF(Y(0))])^{-1}\,\mathcal{L}_t B$. Similarly, the mapping $\mathcal{L}_t B$ is a bounded linear operator between $\mathcal{U}$ and $\mathcal{Z}$ by maximal regularity. Being defined as an inverse, $(I - \mathcal{L}_t[DF(Y(0))])^{-1}$ is also boundedly invertible from the ball $B_r(\mathcal{Z})$ into $B_R(\mathcal{U})$. As Fréchet derivatives are bounded linear operators by definition, we have that $DF(Y(0))$ is boundedly invertible as well. Finally, the mapping $\mathcal{L}_T$ is, again, a bounded linear operator into $\mathcal{R}(e^{-AT})$. As a consequence, we have that $[D\mathcal{C}_T(0)]$ is surjective onto $\mathcal{R}(e^{-AT})$, meaning that this map is invertible, as required. □

We now state the main result of the section, namely that equation (3.75) has a control U for each state Y_0 (giving rise to a state Y that forces $Y(T) = 0$ in $L^2(0,1)$.

Theorem 3.10. Given $T > 0$ and $Y_0 = [w_0, n_0, b_0, c_0] \in [H^1(0,1)]^4$ such that $\|Y_0\| \leq R_1$, there exists controllers $[u_1, u_2, u_3, 0]^T \in [L^2(0,T; L^2(\omega))]^2 \times L^2(0,T; H^1(0,1)) \times \{0\}$ such that the associated solutions $Y = [w, n, b, c] \in \mathcal{Z}$ to (3.71) satisfies $Y(T) = 0 \in L^2(0,1)$.

Proof. We apply inverse function theorem to the state-to-control map given in (3.75). With $U = 0$ in $\mathcal{W}$ and $Y_0 = 0$ in $[H^1(0,1)]^4$, we have that $e^{-AT}Y_0 = 0$ and $F(Y(0)) = 0$, which means that

$$e^{-AT}Y_0 + \mathcal{C}_T(0, Y(0)) = 0. \tag{3.84}$$

Furthermore, by proposition (3.9), we known that $[D\mathcal{C}_T(0)]$ is surjective $\mathcal{U} \to \mathcal{R}(e^{-AT})$. By application of implicit function theorem, we obtain a control U

for each $Y_0 \in B_R(\mathcal{R}(e^{-AT}))$ satifying (3.75), which is equivalent to the null controllability of system (3.71). $\square$

CHAPTER 4

RECAPITULATION AND FURTHER WORK

In this work, we began with a study of the well-posedness of system (2.1) under boundary conditions (2.2) and initial data in $H^1(0,1)$. This system, marked by the presence of a degenerate equation in b and a significant nonlinear coupling term $(nc_x)_x$, we shown to have unique, positive solutions for small time, or assuming the solutions are small. The argument was constructed around a fixed-point map and showing that the nonlinear terms generated locally-Lipschitz operators between the appropriate spaces. Maximal regularity for the linear heat equation was crucial in this argument, as we could allow the nonlinear maps to lose regularity and expect the maximal regularity to bring us back to the desired solution space. In the degenerate equation, however, we have no such results, and this leads us to choose controllers in a fairly restrictive class, namely $L^2(0,T;H^1(0,1))$.

The assumption of smallness of the solutions (or generally, nearness to the steady states) comes from the nonlinearities in two ways. First, the superlinear polynomial nonlinearities (quadratic and cubic terms) cause a potential for unbounded growth in the time derivatives unless the solutions remain close to the steady states. Specifically, these terms can only be shown to be locally Lipschitz, with the Lipschitz constant depending on the size of the solutions. The restriction on the size of the solutions translates into allowing the Lipschitz constant to be small enough to apply Banach's contraction principle. The rational nonlinearities compound this difficulty, since these lead to a sharp blow up if denominators become arbitrarily small. We are able to circumvent this issue by showing the system has a maximum-principle-type result, namely that nonnegative initial conditions lead to nonnegative solutions for some time. These nonlinearities are a novel change to the common models for chemotaxis, which often restrict source and sink terms to being bounded by polynomial-type growth.

For further directions, we propose several possible modifications. First, as is often the case with differential equations, one can always try to reduce the regularity of either the initial conditions or the controllers. An expected result, of course, is that we will no longer have solutions that can be interpreted as functions (or almost-everywhere equivalence classes or functions, to be precise). Instead, reducing the regularity of the data will lead our vector outputs to be distributions. Another modification would be to return the system to having mixed-boundary conditions, with zero Neumann conditions at the center (as is required to avoid cusps in the graphs of the solutions) as well as Dirichlet on the outside end. A further extension of this result would be to allow the controls to act along the boundary. This can be beneficial, for example, in the oxygen equation. Here the boundary control has an interpretation of implementing topical oxygen therapy. Although focus was given to solutions about zero, the nonlinear problem is capable of having nonzero equilibrium states. The stability and rate of convergence of the solution to these steady states is a common theme in the literature. Connected to the analytic solution of the equations is the numerical recovery of solutions.

Although not presented in the main body of this work, numerical solutions have been obtained using a finite-difference scheme, treating the chemotactic term using the Lax-Wendroff method for systems. An outline of this method can be found in Morton and Mayers [37]. In these solutions, parameter values chosen to produce smooth solutions to avoid difficulties with artificial smoothing. The method was written in Python and implemented in Spyder in the Anaconda Software Distribution [1], the results of this effort can be found in Appendix B. This method was inspired by the work done by the author under the directorship of Dr. James Nutaro at Oak Ridge National Lab while studying the use of chemotactic interactions in social sciences. Other methods, based on finite-volume or flux-limiting methods are been employed successfully for chemotactic models. For a positivity-preserving scheme utilizing flux limiters, see Chertock and Kurganov, [8].

A secondary problem we studied was the issue of driving the system to zero, the so-called null-controllability problem. We first studied the linear problem. There, null controllability was studied by introducing the appropriate Carleman estimates. These estimates allow us to drive the linear system to localize the action of the controls to a compact subdomain of the wound $(0, 1)$, at least in the case of the parabolic-like equations. For the degenerate ODE, one must exert the control over the entire domain. Moreover, we were able to use a truncation argument along with the Carleman estimates to eliminate one of the controllers acting on the linear system. This result is optimal, in the sense that one can not control to the null state with any fewer controllers. From there, we employ the implicit function theorem to show that, at least for small trajectories, the nonlinear system is also null controllable using two localized controllers and one distributed control.

For further work into the control problem, one may first look into controlling to states other than the state that is identically zero. Of course, this requirement is a mathematical nicety. In reality, one would not want to drive the oxygen or the body's defense cells to zero in the wound. Thus, driving the system to a steady state with zero bacteria and nonzero oxygen and neutrophils would be particularly interesting. Furthermore, one can look at the rate of blow up of the controls as the time to control goes to zero. For more information on problems in this direction, one may look at Seidman's 2005 paper [45]. More recently, numerical methods have been proposed for approximating the null controllers. For null controls of the the linear heat problem, consider the work by A. Münch and E. Zuazua, [38] and by E. Fernández-Cara and A. Münch [12].

APPENDIX A

DIFFERENTIABILITY OF THE NONLINEAR MAPPINGS

In this section, we show that the nonlinear maps, once given the appropriate structure, are differentiable in the sense of Fréchet. In the following,we let $Y = [w, n, b, c], H = [h_1, h_2, h_3, h_4] \in \mathcal{M}$. Much like in the text, we take

$$\begin{aligned} \mathcal{M} &= H^1(0,T;L^2(0,1)) \cap L^2(0,T;H^2(0,1)) \\ \mathcal{Z} &= [\mathcal{M}]^2 \times H^1(0,T;H^1(0,1)) \times \mathcal{M} \\ \mathcal{W} &= \left[L^2(0,T;L^2(0,1))\right]^2 \times L^2(0,T;H^1(0,1)) \times L^2(0,T;L^2(0,1)). \end{aligned} \tag{A.1}$$

Recall that we define the scaler nonlinearities by

$$\begin{aligned} f_{1,1} &= -\lambda_1 n w \\ f_{1,2} &= -\lambda_2 b w \\ f_1 &= f_{1,1} + f_{1,2} \\ f_{2,1} &= \lambda_7 b n \\ f_{2,2} &= -\lambda_7 b n^2 \\ f_2 &= f_{2,1} + f_{2,2} \\ p &= -\lambda_6 (n c_x)_x \\ f_{3,1} &= -\lambda_8 b^2 \\ f_{3,2} &= -\lambda_{10} \frac{b}{\lambda_{12} b + 1} \frac{w}{\lambda_9 + w} \\ f_{3,3} &= -\lambda_{11} n \frac{b}{\lambda_{12} b + 1} \frac{w}{\lambda_9 + w} \\ f_3 &= f_{3,1} + f_{3,2} + f_{3,3}, \end{aligned} \tag{A.2}$$

which lead to the nonlinear operators $F_i, P : \mathcal{Z} \to L^2(0,T; L^2(0,1))$, by

$$F_i(Y(t)) = f_i(w,n,b,c)(\cdot,t)$$
$$P(Y(t)) = p(n,c)(\cdot,t)$$

as well as

$$F_3(Y(t)) = f_3(w,n,b,c)(\cdot,t), F_3 : \mathcal{Z} \to L^2(0,1; H^1(0,1)).$$

We first note that the increment of $F_{1,1}$ is given by

$$\begin{aligned} F_{1,1}(Y+H) - F_{1,1}(Y) &= -\lambda_1(n+h_2)(w+h_1) + \lambda_1 nw \\ &= -\lambda_1 nh_1 - \lambda_1 wh_2 - \lambda_1 h_1 h_2. \end{aligned} \tag{A.3}$$

We then define, formally at first, the operator

$$[DF_{1,1}(Y)](H) = -\lambda_1 nh_1 - \lambda_1 wh_2. \tag{A.4}$$

To verify the candidate $DF_{1,1}$ is the Freéchet derivative of $F_{1,1}$, we must show that

$$\frac{\|F_{1,1}(Y+H) - F_{1,1}(Y) - DF_{1,1}(Y)H\|_{L^2(0,t;L^2((0,1))}}{\|H\|_{\mathcal{Z}}} \overset{\|H\|_{\mathcal{Z}}\to 0}{\longrightarrow} 0. \tag{A.5}$$

Following the arguments establishing estimates (2.13), we have that

$$\begin{aligned} &\frac{\|F_{1,1}(Y+H) - F_{1,1}(Y) - DF_{1,1}(Y)H\|^2_{L^2(0,t;L^2((0,1))}}{\|H\|^2_{\mathcal{Z}}} \\ &\leq \frac{C(\|n\|_{\mathcal{M}}, \|w\|_{\mathcal{M}})\|h_1\|^2_{\mathcal{M}}\|h_2\|^2_{\mathcal{M}}}{\|H\|^2_{\mathcal{Z}}} \\ &= \frac{C(\|n\|_{\mathcal{M}}, \|w\|_{\mathcal{M}})\|H\|^4_{\mathcal{Z}}}{\||H\|^2_{\mathcal{Z}}}. \end{aligned} \tag{A.6}$$

This quantity clearly goes to 0 as H vanishes in $\mathcal{Z}$. Removing the squares, which produces no meaningful change to the argument, gives the desired result.

Next we consider the regularity of the map $F_{1,2}$ We have

$$\begin{aligned} F_{1,2}(Y+H) - F_{1,2}(Y) &= \lambda_2(w+h_1)(b+h_2) - \lambda_2 wb \\ &= \lambda_2 w h_2 + \lambda_2 b h_1 + \lambda_2 h_2 h_2. \end{aligned} \tag{A.7}$$

Define

$$\left[(Df_2)(-Y)\right](H) = \lambda_2 b h_1 + \lambda_2 w h_2. \tag{A.8}$$

Then, as before,

$$\begin{aligned} &\|f_2(Y+H) - f_2(Y) - \left[(Df_2)(Y)\right](H)\|^2_{L^2(0,T;L^2(0,1))} \\ &\leq \frac{C(\|w\|_{\mathcal{M}}, \|b\|_{H^1(0,T;H^1(0,1))})\|h_1\|^2_{\mathcal{M}}\|h_2\|^2_{\mathcal{M}}}{\|H\|^2_{\mathcal{Z}}} \\ &= \frac{C(\|w\|_{\mathcal{M}}, \|b\|_{H^1(0,T;H^1(0,1))})\|H\|^4_{\mathcal{Z}}}{\|H\|^2_{\mathcal{Z}}} \end{aligned} \tag{A.9}$$

by virtue of the estimates in equation (2.15).

At this point it is evident that the map $F_{2,1}(Y) = \lambda_7 bn$ will be differentiable with $[DF_{2,1}(Y)](H) = \lambda_7 n h_3 + \lambda_7 b h_2$.

Next, define

$$D_{2,2}(Y) = \lambda_7 b n^2$$

. Then noting

$$\begin{aligned} DF_{2,2}(Y+H) - f_{2,2}(Y) &= \lambda_7(b+h_3)(n+h_2)^2 - \lambda_7 b n^2 \\ &= 2\lambda_7 n b h_2 + \lambda_7 b h_2 h_2 + \lambda_7 n^2 h_3 + 2\lambda_7 n h_2 h_3 + \lambda_7 h_2^2 h_3, \end{aligned} \tag{A.10}$$

we define

$$[DF_{2,2}(Y)]\,(H) = 2\lambda_7 n b h_2 + \lambda_7 n^2 h_3 \tag{A.11}$$

As such, we have

$$\begin{aligned}&\|F_{2,2}(Y+H) - F_{2,2}(Y) - [DF_{2,2}(Y)]\,(H)\|^2_{L^2(0,T;L^2(0,1))}\\ &\leq \|b\|^2_{L^2(0,T;H^1(0,1))}\|h_2\|^4_{\mathcal{M}} + \lambda_7\|h_2\|^2_{\mathcal{M}} + 2\lambda_7\|n\|^2_{\mathcal{M}}\|h_1\|^2_{\mathcal{M}}\|h_2\|^2_{\mathcal{M}}.\end{aligned} \tag{A.12}$$

similar to (2.17). By dividing by $\|H\|^2_{\mathcal{Z}}$ and estimating the above norms by their vector counterparts, we see that $DF_{2,2}(Y)$ does, in fact, exist.

Now let $P_1(Y) = \lambda_6 n_x c_x$. Then we have

$$P_1(Y+H) - P_1(Y) = \lambda_6 n_x c_x + \lambda_6 n_x h_{4x} + \lambda_6 c_x h_{2x} + \lambda_6 h_{2x} h_{4x}. \tag{A.13}$$

Define

$$[DP_1(Y)]\,(H) = \lambda_6 n_x h_{4x} + \lambda_6 c_x h_{2x}. \tag{A.14}$$

Then

$$\begin{aligned}&\|P_1(Y+H) - P_1(Y) - [DP_1(Y)]\,(H)\|^2_{L^2(0,T;L^2(0,1))}\\ &\leq C(\|n\|_{\mathcal{M}}, \|c\|_{\mathcal{M}})\left(\|h_4\|^2_{\mathcal{M}} + \|h_2\|^2_{\mathcal{M}}\right),\end{aligned} \tag{A.15}$$

analogous to (2.30). Again, we note that by dividing by $\|H\|^2_{\mathcal{Z}}$ one obtains P_1 is also Fréchet differentiable.

Similarly we define

$$\begin{aligned}P_2(Y) &= \lambda_6 n c_{xx}, \quad \text{and}\\ [DP_2(Y)]\,(H) &= \lambda_6 n h_{4xx} + \lambda_6 c_{xx} h_2,\end{aligned} \tag{A.16}$$

with

$$\|P_2(Y+H) - P_2(Y) - [(DP_2)(Y)]\,(H)\|^2_{L^2(0,T;L^2(0,1))} \leq C\|h_4\|^2_{\mathcal{M}}\|h_2\|^2_{\mathcal{M}} \tag{A.17}$$

by (2.31). Much like the previous terms, it is evident that this mapping is also Fréchet differentiable between the appropriate spaces.

For $F_{3,2}(Y) = -\lambda_{10}\dfrac{b}{\lambda_{12}b+1}\cdot\dfrac{w}{\lambda_9+w}$, we first consider the real-valued function $g(s) = \dfrac{s}{1+s}$ and the associated Nemytskii operator

$$g(u(s)) = \frac{u(s)}{1+u(s)}. \tag{A.18}$$

We know, by (2.9) that $g(u(s)) \in H^1(0,1)$ whenever $u \in H^1(0,1)$ is sufficiently small in norm. A similar argument shows that the map will also be locally lipschitz whenever u is nonnegative (pointwise). We wish to show that is a differentiable map into $H^1(0,1)$ for u nonnegative. We define, for $y(t,\cdot), h(t,\cdot) \in H^1(0,1)$, with $y(t,x), h(t,x) \geq 0$ for all $x \in (0,1)$,

$$[Dg(y)](h) = \frac{h}{(1+y)^2}. \tag{A.19}$$

Then, since y, h are continuous we may compute using pointwise values:

$$\begin{aligned}
&\|g(y+h) - g(y) - [Dg(y)](h)\|^2_{L^2(0,T;L^2(0,1))} \\
&= \int_0^T \int_0^1 \left(\frac{y(x,t)+h(x,t)}{1+y(x,t)+h(x,t)} - \frac{y(x,t)}{1+y(x,t)} - \frac{y}{(1+y)^2}\right)^2 \,\mathrm{d}x\ \mathrm{d}t \\
&= \int_0^T \int_0^1 \left(\frac{-h(x,t)^2}{(1+y(x,t)+h(x,t))(1+y(x,t))}\right)^2 \,\mathrm{d}x\ \mathrm{d}t \\
&\leq \int_0^T \int_0^1 h(x,t)^4 \,\mathrm{d}x\ \mathrm{d}t \\
&\leq C\|h\|^4_{L^2(0,T;H^1(0,1))}.
\end{aligned} \tag{A.20}$$

Arguing similarly, we define

$$[DF_{3,3}(Y)](H) = \frac{\lambda_{10}}{\lambda_{12}} \left(\frac{h_1}{(1+\lambda_{12}b)^2} \frac{w}{\lambda_9 + w} + \frac{\lambda_{12}b}{(1+\lambda_{12}b)^2} \frac{h_2}{(\lambda_9 + w)^2} \right) \tag{A.21}$$

We can then have

$$\|f_{3,2}(Y+H) - f_{3,2}(Y) - [Df_{3,2}(Y)](H)\|^2_{L^2(0,T;L^2(0,1))} \leq C\left(\|H\|^4\right), \tag{A.22}$$

where C depends on the norm of Y.

Of course, no significant difference is made with $F_{3,3}$, since we have $n \in \mathcal{M}$ and so we can pull out the $L^\infty(0,T;L^\infty(0,1))$ norm and then this reduces to a term similar to $F_{3,2}$.

Finally, we remark that we would also need the resulting operators to be continuous in the Y variables (continuity in H is given by the definition). However, almost all our terms are polynomial terms in Y, thus difference quotients of the derivatives will lead to estimates like those performed to show the original operators were Lipschitz. Now the output must be measured in the operator norm, i.e., $\|D(Y_1) - DF(Y_2)\|_{\mathcal{B}(\mathcal{Z},\mathcal{W})} \leq \|Y_1 - Y_2\|_{\mathcal{Z}}$. For the rational nonlinearities, we note that the arguments from (2.9) and (2.10) are still applicable.

APPENDIX B

NUMERICAL APPROXIMATION TO SOLUTIONS AND ASSOCIATED CODE

In this section we present the numerical solution to the system of equations in (2.1). The numerical method is an implementation of the Lax-Wendroff method as discussed in [37]. The coefficients are not readily available in the literature, which led to several choices made by the author in the previous numerical study of the system [14]. As these coefficients were taken from disparate sources, there's no guarantee the given model would produce realistic solutions. As such, we have dispensed of the requirement that our method produces quantitatively-accurate solutions, are search for coefficients that produces solutions whose qualitative behavior agrees with the intuition of the modeling. As such, the following results are meant to be taken as simple verification of feasibility, and should not be used to make any firmly-held conclusions about the model. The solutions are plotted over $0 \leq t \leq 7$ at four equal time points, $t = 0, t = 7/3, t = 14/3, t = 7$, from left-to-right, top-to-bottom.

In these figures, the initial conditions are plotted over the spatial region $(0, 1)$. The bacterial and chemoattractant are concentrated near the center of the wound $x = 0$, and the neutrophils are concentrated on the outside edge of the wound $x = 1$. The oxygen is assumed to be uniformly distributed throughout the wound. For the simulations, the more realistic mixed boundary conditions were kept, similar to the previous simulations in [14]. As time evolves, the neutrophils are seen to move into the wound after a period of diffusion, and then concentrate near the invasive bacteria. The bacteria grow and proliferate throughout the wound, and the chemoattractant follows the bacteria distribution. The oxygen level falls throughout the wound, indicating a hypoxic environment. With the given parameter values (found easily in the code below), the solutions remain positive and bounded, which

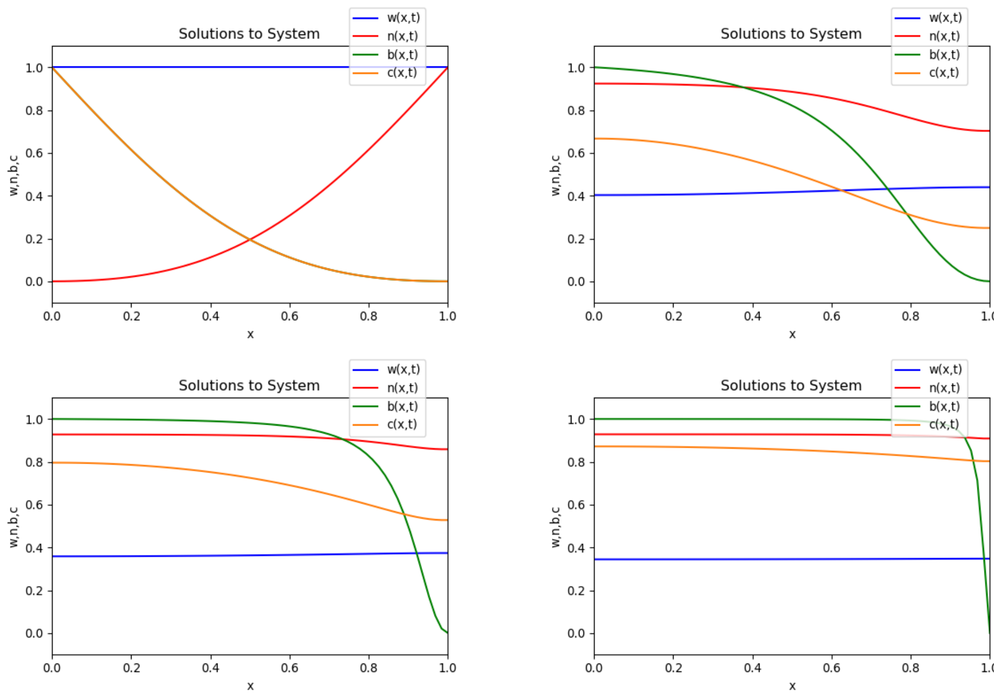

Figure 1: Solutions to the dissertation system (2.1). In the top left, the initial profiles are plotted. In the top right, solutions are plotted at $t = 7/3$. In the bottom left, solutions are given at $t = 14/3$. In the bottom right, solutions at the final time $t = 7$ are given.

indicated that the model might have a positivity principle with respect to the initial conditions.

```
#Created by Stephen Guffey on 7-20-20

#Import packages needed for calculation/visualization
import scipy.sparse
import numpy as np #imports numpy, sets abbreviations to np.(something)
from mpl_toolkits.mplot3d import Axes3D
import matplotlib.pyplot as plt
from matplotlib import cm
from matplotlib.ticker import LinearLocator, FormatStrFormatter

# In[34]:
#By iterating the coefficients over a list you can run multiple
#simulations#without manually restarting each time.
for coeff in [0.3]:

xf=1 #final x position
tf = 7 #final time position #0.0045

lambda1 = 1#0.1 #Coefficients being used
lambda2 = 1
lambda7 = 14#0.1
lambda3 = 1
lambda4 = 0
lambda5 = 0.2
```

```
lambda16 = 1

lambda8 =  1.26

lambda6 = 0.01

lambda13 = 0.1

lambda14 = .9

lambda15 = 1

lambda9 = .75

lambda10 = .7992

lambda11 = 2

lambda12 = 3.73

eps=1


Nx = 64  # Number of x spatial meshes

Nt = int(2*(1+lambda6)*tf*Nx**2/xf+Nx) # Number of temporal meshes


#Note below that Python 3.(anything) uses "/" as floating point

#division whereas Python version 2.# uses "/" as integer division


dx = (xf-0)/Nx # X direction Spatial Mesh Size

dt = (tf-0)/Nt # Temporal Mesh Size



#Initialize space,time and solution vectors

x = np.linspace(0,xf, Nx)

t = np.linspace(0,tf, Nt) #temporal mesh initialization


#Intialize the Initial States
```

```
W0=np.zeros(Nx)
N0=np.zeros(Nx)
B0=np.zeros(Nx)
C0=np.zeros(Nx)

#Initialize the Solution States
SolW = np.zeros((Nx, Nt))
SolN = np.zeros((Nx,Nt))
SolB = np.zeros((Nx,Nt))
SolC = np.zeros((Nx,Nt))

#Print to check the orientation of how information is stored in
#the Solution array.

#print('Sol=',Sol[:,:,0])




# ==============================================================
#Define the initial conditions

for i in range(0,Nx):
W0[i] =1
#print(u0)
SolW[:,0]=W0

for i in range(0,Nx):
```

```
N0[i] =x[i]**2*np.exp(-(1-x[i])**2)
#print(u0)
SolN[:,0]=N0

for i in range(0,Nx):
B0[i] =(1-x[i])**2*np.exp(-x[i]**2)
#print(u0)
SolB[:,0]=B0

for i in range(0,Nx):
C0[i] =(1-x[i])**2*np.exp(-x[i]**2)
#print(u0)
SolC[:,0]=C0



#==============================================================
#     Define the equations in Ax=b for the explicit scheme.
#    This is done sinceactually constructing the matrix doesn't
#seem necessary. The interior at new step is updated first and
#then the boundary conditions are updated. The diffusive terms
#are approximated using central differences, explicitly
#    updated. The chemotactic term is derived following the
#Lax-Wendroff method,see Morton & Myer's "Numerical Solution
# of Partial Differential Equations" for a detailed explaination
#of the method, if desired. The reaction terms
#    are taken explicitly to avoid use of a nonlinear solver.
```

```
#=============================================================
for n in range(1,Nt):
SolW[1,n] = SolW[1,n-1]+dt/(dx)**2*(SolW[2,n-1]-2*SolW[1,n-1]\
+SolW[0,n-1]) \
-dt*lambda1*SolN[1,n-1]*SolW[1,n-1]-dt*lambda2*\
SolB[1,n-1]*\
SolW[1,n-1]-dt*lambda3*SolW[1,n-1]+dt*1


SolN[1,n] = SolN[1,n-1]+lambda5*dt/(dx**2)*(SolN[2,n-1]-\
2*SolN[1,n-1]+SolN[0,n-1])+\
-0.5*dt/dx*(-lambda6/(2*dx)*(SolW[2,n-1]*\
(SolC[3,n-1]-SolC[1, n-1])))+\
0.5*(dt**2)/(2*dx**4)*lambda6**2*((SolC[2, n-1]\
-SolC[1, n-1])*\
(SolN[2, n-1]*(SolC[3, n-1]-SolC[1, n-1])-\
SolW[1, n-1]*(SolC[2, n-1]-SolC[0, n-1]))-\
(SolC[1, n-1]-SolC[0, n-1])*(SolN[1, n-1]*\
(SolC[2, n-1]-SolC[0, n-1])))\
+dt*lambda7*SolB[1,n-1]*SolN[1,n-1]*\
(1-SolN[1,n-1])-\
dt*lambda16*SolN[1,n-1]


SolB[1,n] = SolB[1,n-1]+dt*lambda8*SolB[1,n-1]*(1-SolB[1,n-1])


SolC[1,n] = SolC[1,n-1]+dt*lambda13/(dx)**2*(SolC[2,n-1]-\
2*SolC[1,n-1]+SolC[0,n-1])\
+dt*lambda14*SolB[1,n-1]-dt*lambda15*SolC[1,n-1]
```

```
for i in range(2,Nx-2):
SolW[i,n] = SolW[i,n-1]+dt/(dx)**2*(SolW[i+1,n-1]-\
2*SolW[i,n-1]+SolW[i-1,n-1]) \
-dt*lambda1*SolN[i,n-1]*SolW[i,n-1]-dt*\
lambda2*SolB[i,n-1]*SolW[i,n-1]\
-dt*lambda3*SolW[i,n-1]+dt*1

SolN[i,n] = SolN[i,n-1]+lambda5*dt/(dx**2)*(SolN[i+1,n-1]-\
2*SolN[i,n-1]+SolN[i-1,n-1])+\
-0.5*dt/dx*(-lambda6/(2*dx)*(SolN[i+1,n-1]*\
(SolC[i+2,n-1]-SolC[i, n-1])-SolN[i-1,n-1]*\
(SolC[i,n-1]-SolC[i-2, n-1])))+\
0.5*(dt**2)/(2*dx**4)*lambda6**2*((SolC[i+1, n-1]-\
SolC[i, n-1])*(SolN[i+1, n-1]*(SolC[i+2, n-1]-\
SolC[i, n-1])-SolN[i, n-1]*(SolC[i+1, n-1]-\
SolC[i-1, n-1]))\
-(SolC[i, n-1]-SolC[i-1, n-1])*(SolN[i, n-1]*\
(SolC[i+1, n-1]-SolC[i-1, n-1])-SolN[i-1, n-1]*\
(SolC[i, n-1]-SolC[i-2, n-1])))\
+dt*lambda7*SolB[i,n-1]*SolN[i,n-1]*(1-SolN[i,n-1])-\
dt*lambda16*SolN[i,n-1]

SolB[i,n] = SolB[i,n-1]+dt*lambda8*SolB[i,n-1]*(1-SolB[i,n-1])

SolC[i,n] = SolC[i,n-1]+dt*lambda13/(dx)**2*(SolC[i+1,n-1]-\
2*SolC[i,n-1]+SolC[i-1,n-1])\
```

```
+dt*lambda14*SolB[i,n-1]-dt*lambda15*SolC[i,n-1]


SolW[Nx-2,n] = SolW[Nx-2,n-1]+dt/(dx)**2*(SolW[Nx-1,n-1]-\
2*SolW[Nx-2,n-1]+SolW[Nx-3,n-1]) \
-dt*lambda1*SolN[Nx-2,n-1]*SolW[Nx-2,n-1]-\
dt*lambda2*SolB[Nx-2,n-1]*SolW[Nx-2,n-1]\
-dt*lambda3*SolW[Nx-2,n-1]+dt*1


SolN[Nx-2,n] = SolN[Nx-2,n-1]+lambda5*dt/(dx**2)*\
(SolN[Nx-2+1,n-1]-\
2*SolN[Nx-2,n-1]+SolN[Nx-2-1,n-1])+\
-0.5*dt/dx*(-lambda6/(2*dx)*(SolN[Nx-2+1,n-1]*\
(SolC[Nx-2,n-1]-SolC[Nx-2, n-1])-SolC[Nx-2-1,n-1]*\
(SolC[Nx-2,n-1]-SolC[Nx-2-2, n-1])))+\
0.5*(dt**2)/(2*dx**4)*lambda6**2*((SolC[Nx-2+1, n-1]-\
SolC[Nx-2, n-1])*(SolN[Nx-2+1, n-1]*(SolC[Nx-2, n-1]-\
SolC[Nx-2, n-1])-SolN[Nx-2, n-1]*(SolC[Nx-2+1, n-1]-\
SolC[Nx-2-1, n-1]))\
-(SolC[Nx-2, n-1]-SolC[Nx-2-1, n-1])*(SolN[Nx-2, n-1]*\
(SolC[Nx-2+1, n-1]-SolC[Nx-2-1, n-1])-\
SolN[Nx-2-1, n-1]*(SolC[Nx-2, n-1]-SolC[Nx-2-2, n-1])))-\
+dt*lambda7*SolB[Nx-2,n-1]*SolN[Nx-2,n-1]*\
(1-SolN[Nx-2,n-1])-dt*lambda16*SolN[Nx-2,n-1]


SolB[Nx-2,n] = SolB[Nx-2,n-1]+dt*lambda8*SolB[Nx-2,n-1]\
*(1-SolB[Nx-2,n-1])
```

```
SolC[Nx-2,n] = SolC[Nx-2,n-1]+dt*lambda13/(dx)**2*\
(SolC[Nx-1,n-1]-2*SolC[Nx-2,n-1]+SolC[Nx-3,n-1])\
+dt*lambda14*SolB[Nx-2,n-1]-dt*lambda15*SolC[Nx-2,n-1]


SolW[0,n]=SolW[1,n].copy()
SolW[Nx-1,n]=SolW[Nx-2,n].copy()
SolN[0,n]=SolN[1,n].copy()
SolN[Nx-1, n]=SolN[Nx-2,n].copy()
SolB[0,n] = SolB[0,n-1]+dt*lambda8*SolB[0,n-1]*(1-SolB[0,n-1])
SolB[Nx-1,n] = SolB[Nx-1,n-1]+dt*lambda8*SolB[Nx-1,n-1]*\
(1-SolB[Nx-1,n-1])
SolC[0,n]=SolC[1,n].copy()
SolC[Nx-1, n]=SolC[Nx-2,n].copy()


## 3D Plot for the solution
#fig = plt.figure()
#ax = fig.gca(projection='3d')
## Make data.
#Tgrid, Xgrid = np.meshgrid(t,x )
#
## Plot the surface.
#surf = ax.plot_surface( Tgrid, Xgrid, Sol,
#             cmap=cm.coolwarm,linewidth=0, antialiased=False)
#
## Customize the axes.
#ax.set_zlim(0,1)
#ax.zaxis.set_major_locator(LinearLocator(10))
```

```
#ax.zaxis.set_major_formatter(FormatStrFormatter('%.02f'))
#ax.set_xlabel("t")
#ax.set_ylabel("x")
#ax.set_zlabel('u(x,t)')
#
## Add a color bar which maps values to colors.
#fig.colorbar(surf, shrink=0.5, aspect=5)
#
#
#plt.gca()

# ==========================================================
# There's often not a reason to plot every single time step
#iteration of the solutions, so Plotnumber allows one to adjust
#the number of plots that are actually saved
Plotnumber=200
for n in range(0, Plotnumber):
fig=plt.figure()
plt.plot(x, SolW[:, n*int(Nt/Plotnumber)],'b-', x, \
SolN[:, n*int(Nt/Plotnumber)],'r-', x, \
SolB[:, n*int(Nt/Plotnumber)],\
'g-', x, SolC[:, n*int(Nt/Plotnumber)],'tab:orange')
plt.xlabel('x')
plt.ylabel('w,n,b,c')
plt.legend(('w(x,t)', 'n(x,t)', 'b(x,t)', 'c(x,t)'), \
loc=(0.75,0.85))
plt.title(f'Solutions to System')
```

```
plt.axis([0,xf,-0.1,1.1])

# Save the plot. By default, this outputs into the current

#working directory.

filename=f'Solution {n:03}.png'

plt.savefig(filename, dpi=96)

plt.gca()


#Code to save Solutions at a given time. With this, you can

#import the final initial conditions are reiterate the solver

#to larger times. The method we use is crude, and hence large

#mesh numbers can easily lead to storage

#   errors. This is, admittedly, an equally crude workaround.



#np.save('u0init'+str(Nx)+'x'+str(Nt), Sol1[:,Nt-1], \

#allow_pickle=False)

#np.save('v0init'+str(Nx)+'x'+str(Nt), Sol2[:,Nt-1], \

#allow_pickle=False)
```

APPENDIX C

COMBAT MODELING WITH CHEMOTAXIS

This purpose of this section is to describe the project and methods used during the author's 2020 NSF-MSGI internship, under the direction of Dr. James Nutaro. These methods were later applied to the system (2.1) to produce the results found in appendix B. The following is comprised, primarily of the end-of-appointment report submitted to the NSF.

This project focused on the modeling of movement of competing species via a system of nonlinear advection-diffusion-reaction partial differential equations (PDEs). The PDE system is motivated by a combat model developed by Protopopescu, Santoro, Dockery, Cox, and Barnes (1987) [2]. This model was a generalization to the classical Lancaster equations for warfare. The new model allowed for spatial movement of the forces as well as attrition effects. In the model of Protopopescu *et. al.*, troop movement was directed via a predetermined advective current. In the current work we modify the advective term to depend on gradient of the opponent. This modification is taken from the modeling of chemotaxis, and represents the attraction (or repulsion) of forces to (away from) the opponents position. In this way the movement is generated naturally from the solution itself rather than being hard-coded into the solver.

The purpose of this project was to develop a modification to the existing combat modeling literature and develop a finite difference solver for the resulting coupled system of advection-diffusion-reaction equations with a strong nonlinear coupling. Over the course of the project, the supplementary objectives were as follows:

1. Identify a potential modification to allow for spontaneous movement.

2. Identify an appropriate finite difference scheme for the system of equations.

3. Implement the finite difference scheme to solve the PDE system.

4. Qualitatively identify possible parameter values that will produce expected troop movements.

We consider a coupled system of PDEs, which models the movement of competitive forces on a unit interval $0 \le x \le x_l$ over a predetermined time interval $0 \le t \le T$. Derivatives are denoted with both subscript and Leibniz notation; in other words, u_t and $\dfrac{\partial u}{\partial t}$ denote the same derivative. The governing system of equations is

$$\begin{cases} u_t &= D_u u_{xx} + d_u(u_{xx} - v_{xx}) - \chi_u (u v_x)_x + a_u u(1 - b_u u) - p_u uv \\ v_t &= D_v v_{xx} + d_v(v_{xx} - u_{xx}) - \chi_v (v u_x)_x + a_v v(1 - b_v v) - p_v uv \end{cases} \tag{C.1}$$

considered in $Q = (0, x_l) \times (0, T)$, with boundary conditions

$$\begin{cases} \left.\dfrac{\partial u}{\partial x}\right|_{x=0} = \left.\dfrac{\partial u}{\partial x}\right|_{x=1} = 0, \\ \left.\dfrac{\partial v}{\partial x}\right|_{x=0} = \left.\dfrac{\partial v}{\partial x}\right|_{x=1} = 0, \end{cases} \tag{C.2}$$

and initial conditions

$$\begin{cases} u(x,0) = u_0(x) \\ v(x,0) = v_0(x) \end{cases} \tag{C.3}$$

taken to be Gaussian-type functions (or sums thereof) over the interval $0 \le x \le x_l$.

In this system the terms $D_u u_{xx}, D_v v_{xx}$ denote random motility of the troops,$D_u, D_v > 0$ are taken to be constant, although generally would depend on the local troop levels. The term $d_u(u_{xx} - v_{xx})$ and the corresponding term in the v equation represent a pressure of troops to minimize force discrepancies in regions where the force profiles overlap. In minimizing the discrepancies, troops will mitigate the possibility of being outflanked by the opposing forces. The terms like $\chi_u (u v_x)_x$ is is nonlinear chemotactic-type advective term, which causes a force advancement if $\chi_u > 0$ and retreat if $\chi_u < 0$ in areas where the opposing force is present. The logistic growth terms, $a_u(u - b_u u)$ and $a_v(v - b_v v)$, come from the Protopopescu model and represent recruitment of reinforcement troops up to a given density capacity. Here $a_u, b_u, a_v, b_v > 0$ in accordance with growth models. The final terms, $p_u uv$ and $p_v uv$, are representative of competition. In ecology or sociology, these represent predation or excommunication, when the coefficient is negative. In terms of the Lanchester combat models, this term represents the targeted attrition (via rifles or hand-to-hand combat) of forces. The coefficients p_u, p_v are also positive constants in the model. The homogeneous Neumann boundary conditions are taken to prevent troops/individuals from moving outside of the battle arena.

To solve the system we used finite differences, choosing to model each of the terms separately using appropriate methods and creating the numerical method as a superposition of solvers. An explicit scheme was used in solving the diffusion problems. We followed the Lax-Wendroff derivation to approximate the chemotactic terms [37]. Specifically, to model the diffusion, we used the standard explicit forward difference scheme. We discretize the spatial interval $0 \leq x \leq x_l$ into $N_x - 1$ subintervals, thereby obtaining spatial increment $\Delta x = \frac{x_l - 0}{N_x}$ and sample points $x_i = i\Delta x$. Similarly, given final time t_f, we discretize time into $N_t - 1$ subintervals with temporal increment $\Delta t = \dfrac{t_f - 0}{N_t}$.

$$\frac{\partial w}{\partial t}(x_i, t_j) \approx \frac{w_i^j - w_i^{j-1}}{\Delta t} \quad \text{and} \quad \frac{\partial^2 w}{\partial x^2}(x_i, t_j) \approx \frac{w_{i+1}^j - 2w_i^j + w_{i-1}^j}{(\Delta x)^2}.$$

After applying the Lax-Wendroff to the convective (chemotactic) term

$$F\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} -\chi_u(uv_x)_x \\ -\chi_v(vu_x)_x \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} f_1(u, v_x) \\ f_2(u_x, v) \end{bmatrix}, \tag{C.4}$$

whereby we obtain

$$\begin{aligned}
\left[-\chi_u(uv_x)_x\right](x_i, t_j) = & -\frac{1}{2}\frac{\Delta t}{\Delta x}\frac{-\chi_u}{2\Delta x}\left(U_{i+1}^{j-1}(V_{i+2}^{j-1} - V_i^{j-1}) - U_{i-1}^{j-1}(V_i^{j-1} - V_{i-2}^{j-1})\right) \\
& + \frac{1}{2}\frac{\Delta t^2}{2\Delta x^4}(\chi_u)\left((V_{i+1}^{j-1} - V_i^{j-1})\left(U_{i+1}^{j-1}(V_{i+2}^{j-1} - V_i^{j-1})\right.\right. \\
& \left. -U_i^{j-1}(V_{i+1}^{j-1} - V_{i-1}^{j-1})\right) \\
& \left. - (V_i^{j-1} - V_{i-1}^{j-1})\left(U_i^{j-1}(V_{i+1}^{j-1} - V_{i-1}^{j-1}) - U_{i-1}^{j-1}(V_i^{j-1} - V_{i-1}^{j-1})\right)\right),
\end{aligned}$$

and a symmetric expression for $-\chi_v(vu_x)_x$. Nonlinear source and sink terms are evaluated pointwise at (x_i, t_{j-1}).

Due to the limitations of Lax-Wendroff in reproducing sharp fronts, parameter values and initial conditions were taken to produce reasonably smooth solutions [18]. More sophisticated techniques for finite differences involving flux limiters, exponential fitting, or operator splitting could be used to improve the performance of the solver [23].

We make remarks of a few of the simulations produced to qualitatively verify the validity of the model. First, we consider the troop matching behavior modeled by the terms of the form $d_u(u_{xx} - v_{xx})$. We set $T = 5, xf = 10, d_u = d_v = 0.5$ and all other coefficients taken to be zero. The initial conditions taken are

$$\begin{aligned}
u_0(x) &= e^{-10(x-5)^2} \\
v_o(x) &= \frac{1}{2}e^{-10(x-3)^2} + \frac{1}{2}e^{-10(x-7)^2}.
\end{aligned}$$

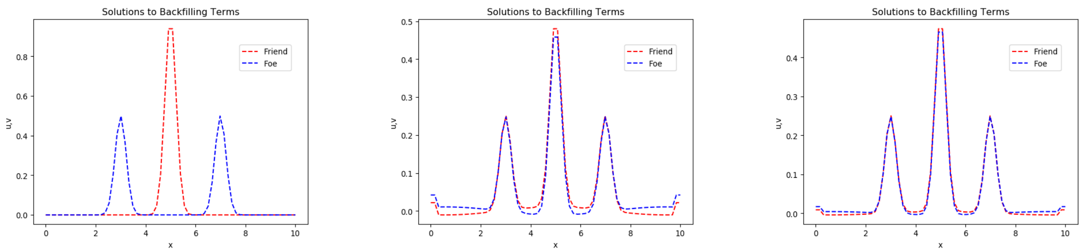

Figure 2: modeling the behavior of matching troop levels

From Figure (2) we see the profiles of the troops adjust so that any disparity of force profiles are eventually mitigated. We note that only shapes are preserved in absence of outside forces. In particular, if the initial force distributions are taken to be of unequal size (in terms of the L^1 norm) then the shapes would be similar however the levels would maintain the discrepancy. We conclude that this term acceptably accounts for the movement of troops as an effort to mitigate troop discrepancies, hence trying to eliminate the chance of the opponent to outflank one's forces.

Next we focus on the effects of the chemoattractant-type terms $\chi_u(uv_x)_x$. Depending on the sign of χ_u, this term models either attraction to the opposing force (as in the case of engagement) or repulsion (as when the force wishes to retreat). In Figure (3) below, we model both behaviors. We take the friendly forces to attack and the enemy forces to retreat. Here $xf = 10, T = 20$. The relevant chemotactic parameters are taken to be $\chi_u = 1, \chi_v = -1$. Other parameter values are $D_u = D_v = 0.1, d_u = d_v = 0, p_u = p_v = 0.01, a_u = b_u = a_v = b_v = 0.01$. The initial conditions are taken to be

$$u_0(x) = e^{-10(x-5)^2}$$
$$v_0(x) = e^{-10(x-4)^2}.$$

The initial forces are given, then the solutions at $T = 10, T = 20$. We see the

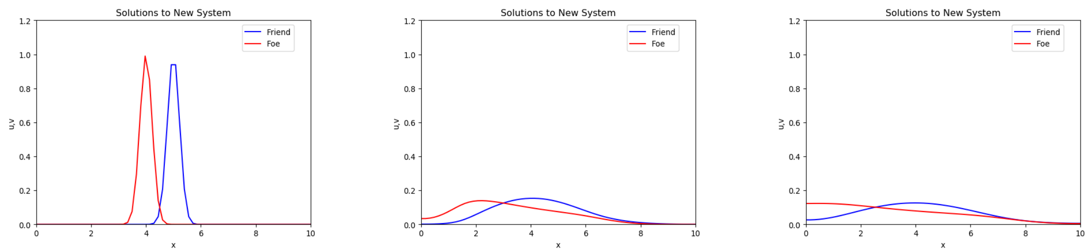

Figure 3: Attacking and Retreating behavior

force profiles of the friendly forces moving towards the enemy while the enemy retreats until they hit the left-hand boundary. Diffusion of the solution leads to a smoothing of the profiles are movement occurs. While limiting the diffusion further would be desirable for reproducing the results, this can cause sharp fronts to develop that undermine the validity of finite difference approximations. As such, we need solutions with nontrivial diffusion, which our method is taken to reproduce sufficiently accurately.

Next we note the possibility stable pattern formations when the forces and attrition abilities are equal, and diffusion is dominated by the affects of the chemotactic-type movement and attrition effects. Below we plot the solutions with values $xf = 10, T = 160, D_u = D_v = 0.3, d_u = d_v = 0, \chi_u = \chi_v = 1, p_u = p_v = 1, a_u = b_u = a_v = b_v = 0.1$. The initial conditions are taken to be

$$u_0(x) = \frac{1}{2}e^{-10(x-3)^2} + \frac{1}{2}e^{-10(x-7)^2}$$
$$v_0(x) = \frac{1}{2}e^{-10(x-4)^2} + \frac{1}{2}e^{-10(x-8)^2}.$$

The solutions are plotted at $T = 0, 80, 160$ from left to right. We can see the initial distributions smooth out until troops overlap, then attrition effects overtake to produce sharp fronts, which seem to persist as $T \to \infty$. We'll note that these solutions highlight a shortcoming of the Lax-Wendroff method: namely there are numerical overshoots where the solution has a steep gradient or discontinuity.

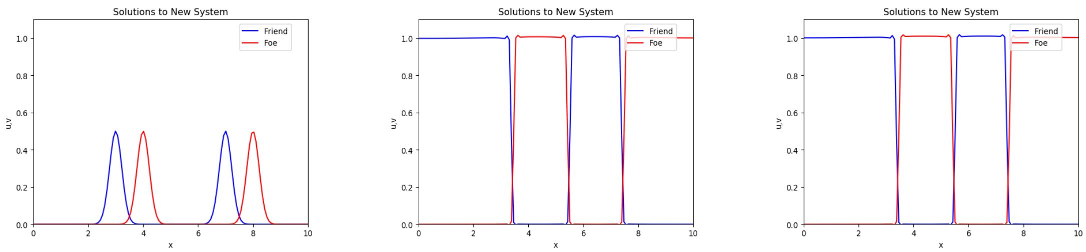

Figure 4: Pattern Formation with Stalemate

The last simulation we wish to remark on is similar to the last in the reproduction of pattern formation, however it is believed that this pattern formation comes from the reaction-diffusion interaction in the equations, since diffusion is taken to be relatively large. Due to the nonsymmetric initial troop placement, the enemy forces reach the boundary before the friendly forces, leading to the recruitment of more reinforcements which eventually allow for the enemy solution to dominate the friendly forces in the sense of the L^1 norm. Attempts to run the simulation for longer final times were met with memory errors. Batch processing the solution could lead to finding the solution over a longer interval, in which it is conjectured that we would see the enemy forces eradicate the friendly forces. The values for the simulation are $T = 180, xf = 10, D_u = D_v = 0.3, d_u = d_v = 0, \chi_u = \chi_v = 1, p_u = p_v = 0.1, a_u = b_u = a_v = b_v = 0.05$. The initial conditions are taken as

$$u_0(x) = \frac{1}{2}e^{-10(x-4)^2} + \frac{1}{2}e^{-10(x-7)^2}$$
$$v_0(x) = \frac{1}{2}e^{-10(x-3)^2} + \frac{1}{2}e^{-10(x-8)^2}.$$

Throughout the simulations, the model seems to predict three distinct outcomes for competitive behavior in one dimension, depending on the strength of the forces at work, given equal abilities and initial conditions that are almost equal in L^1 norm. Under diffusion-dominated conditions, neither force is able to outflank the

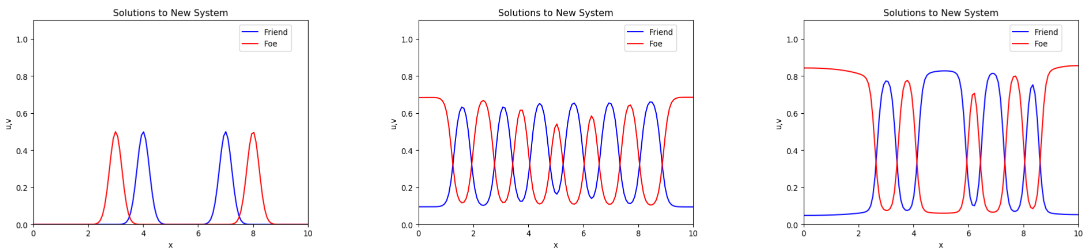

Figure 5: Pattern Formation with Winning Forces

other, and hence the solutions reach constant or near-constant equilibria. This leads to both troop levels going extinct, filling the space, or maintaining some other force level. When diffusion is reduced and the affects of the reinforcement and chemotactic-type advection becomes more apparent, we are likely to see solutions develop regions in which the solutions do not overlap. In terms of combat models, this seems to mimic entrenchments such as seen in the first world war.

More generally it is hoped this model contributes to modeling competitive species in an environment. At least from the current simulations the model seems to predict three basic outcomes. The first case is the extinction of one species, as is the case when one force has an advantage – either through larger initial conditions or through a better rate of attrition. In the second case, the species share the region in a stalemate with constant asymptotic behavior, perhaps with both species going extinct or reaching their carrying capacities. Finally, when diffusion is mitigated, the reaction and advection terms dominate, which give rise to the developments of niches. With further development of the model, it is hoped that more complex behavior such as chaotic or periodic solutions can be observed as well. One particularly interesting avenue to continue would be to modify the chemotactic coefficients χ_u and chi_v to depend on the relative local population densities. It seems apparent that when a species has a local population advantage, they would feel a greater conviction to attack the opposing force. Similarly, when outnumbered, a species should exhibit a desire to flee from conflict. The addition of these terms are conjectured to produced more realistic behavior which regards to spatial movement.

REFERENCES

[1] *Anaconda Software Distribution*. Anaconda, Inc. 2020. https://docs.anaconda.com/.

[2] Barnes, J. *et al. Combat modeling with partial differential equations*. Technical Report. Oak Ridge National Laboratory. (1987).

[3] Bradley, M. E. *Local Controllability of a Nonlinear Spherical Shallow Shell.* Journal of Mathematical Systems, Estimation, and Control. **8**, No. 2. (1998), 1—12.

[4] Brezis, H. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer. (2010).

[5] Bellomo, N. *et al. Toward a mathematical theory of Keller–Segel models of pattern formation in biological tissues.* Mathematical Models and Methods in Applied Sciences. **25**, No. 9. (2015) 1663—1763.

[6] Bensoussan, A. *et al. Representation and Control of Infinite Dimensional System*. Birkhäuser. (2006).

[7] Carleman, T. *Sur un Probléme d'unicité pour les systèmes d'équations aux dérivées partielles à deux variables indépendantes.* (1939).

[8] Chertock, A. and Kurganov, A. *A positivity preserving central-upwind scheme for chemotaxis and haptotaxis models*. Numer. Math. **111**. 169—205 (2008).

[9] Chaves-Silva, F.W. and Guerrero, S. *A uniform controllability result for the keller-segel system*. Asymptotic Analysis. **93** (2013).

[10] Fernández-Cara, E. *et al. Null controllability of the heat equation with boundary fourier conditions: the linear case*. ESAIM:COCV. **12** (2006). 442—465.

[11] Fernández-Cara, E. and Guerrero, S. *Global carleman inequalities for parabolic systems and applications to controllability.* SIAM J. Control Optim. **45** (2006) 1395—1446.

[12] Fernández-Cara, E. and Múnch, A. *Numerical null controllability of the 1D heat equation: primal algorithms*. (2009).

[13] Fursikov, A. V. and Imanuvilov, O.Y. *Controllability of Evolution Equations.* Lecture Note Series **40** (1996) Research Institute of Mathematics, Seoul National University, Seoul.

[14] Guffey, S. *Application of a Numerical Method and Optimal Control Theory to a Partial Differential Equation Model for a Bacterial Infection in a Chronic Wound*. Master's Thesis & Specialist Projects. **1494**. (2015) https://digitalcommons.wku.edu/theses/1494.

[15] Guo, B. and Zhang, L. *Local null controllability for a chemotaxis system of parabolic-elliptic type*. Systems & Control Letters. **65** (2014) 106—111.

[16] Herrero, M. A. and Valázquez, J. J. L. *Chemotactic collapse for the Keller-Segel model*. J. Math. Biol. **35** (1996) 177—194.

[17] Herrero, M. A. and Valázquez, J. J. L. *A blow up mechanism for a chemotaxis model*. Ann. Scoula Norm. Sup. Pisa IV **35** (1997) 633—683.

[18] Homes, M. *Introduction to numerical methods in differential equations.* Springer. (2007).

[19] Hörmander, L. *The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis*. Springer-Verlag. (1983).

[20] Hörmander, L. *The Analysis of Linear Partial Differential Operators II: Differential Operators with Constant Coefficients*. Springer-Verlag. (1983).

[21] Hörmander, L. *The Analysis of Linear Partial Differential Operators III: Pseudo-Differential Operators*. Springer-Verlag. (1985).

[22] Hörmander, L. *The Analysis of Linear Partial Differential Operators IV: Fourier Integral Operators*. Springer-Verlag. (1985).

[23] Hunsdorfer, W. and Verwer, J. *Numerical solution of time-dependent advection-diffusion-reaction equations.* Spinger. (2003).

[24] Imanuvilov, O.Y. *Boundary Controllability of parabolic equations*. Russian Acad. Sci. Sb. Math. **186** (1995) 109—132.

[25] Keller, E. F. and Segel, L. A. *Initiation of slide mold aggregation viewed as an instability*. J. Theor. Biol. **26**. (1970) 399—415.

[26] Keller, E. F. and Segel, L. A. *Model for Chemotaxis*. J. Theor. Biol. **30**. (1971) 225—234.

[27] Kuehn, B.M., *Chronic wound care guidelines issued.* JAMA (2007).

[28] Lagnese, J.E. *Boundary controllability of nonlinear beams to bounded states.* Proceedings of the Royal Society of Edinburgh. **119A**, 63—72. (1991).

[29] Lasiecka, I. and Triggiani, R. *Control Theory for Partial Differential Equations: Continuous and Approximation Theories. Vol. I: Abstract Parabolic Systems.* Cambridge University Press. (2000).

[30] Lasiecka, I. and Triggiani, R. *Control Theory for Partial Differential Equations: Continuous and Approximation Theories. Vol. II: Abstract Hyperbolic-like Systems Over a Finite Time Horizon.* Cambridge University Press. (2000).

[31] Lasiecka, I. and Triggiani, R. *Exact controllability of semilinear abstract systems with application to waves and plates boundary control problems*. Appl Math Optim **23** (1991) 109—154.

[32] Lebeau, G. and Robbiano, L. *Contrôle exact de l'équation de la chaleur*. Comm. P.D.E. **20** (1995). 335—356.

[33] Lions, J.L. and Magenes, E. *Non-homoheneous Boundary Value Problems and Applications. Vol. I*. Springer. (1972).

[34] Lions, J.L. and Magenes, E. *Non-homoheneous Boundary Value Problems and Applications. Vol. II*. Springer. (1972).

[35] Lions, J.L. and Magenes, E. *Non-homoheneous Boundary Value Problems and Applications. Vol. III*. Springer. (1972).

[36] Markus, L. *Controllability of non linear processes*. SIAM Control. 78—90. (1965).

[37] Morton, K.W. and Mayers, D.F. *Numerical Solution of Partial Differential Equations*. Cambridge University Press. (2005).

[38] Münch, A. and Zuazua,E. *Numerical approximation of null controls for the heat equation through transmutation*. (2009).

[39] Nagi, T. *et al. Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis*. Funkcial. Ekvac.**40** (1997), 411—433.

[40] Osaki, K. and Yagi, A. *Finite Dimensional Attractor for one-dimensional Keller-Segel Equations*. Funkcial. Ekvac. **44** (2001) 441—469.

[41] Okposo, N. and Willie, R. *Well-posedness, blow-up dynamics and controllability of the classical chemotaxis model*. Advances in Pure and Applied Mathematics. **10**. (2018).

[42] Pazy, A. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer-Verlag New York, Inc. Applied Mathematical Sciences. **44**. (1983).

[43] Painter, K. *Mathematical models for chemotaxis and their applications in self-organisation phenomena*. J. Theor. Biol. **481**. (2019) 162—182.

[44] Russell,D. L. *A unified boundary controllability theory for hyperbolic and parabolic partial differential equations*. Int. Math. Res. Not. **52** (1973) 189—221.

[45] Seidman, T. *On Uniform Null Controllability and Blowup Estimates*. (2005).

[46] Schugart, R. *et al. Wound angiogenesis as a function of tissue oxygen tension: a mathematical model*. PNAS. (2008).

[47] Yagi, A. *Norm behavior of solutions to the parabolic system of chemotaxis.* Math. Japonica. **45** (1997) 241—265.

[48] Yagi, A. *Abstract Parabolic Evolution Equations and their Applications.* Springer (2010).

[49] Zabczyk, J. *Mathematical Control Theory.* Birkhäuser. (1985).